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**NODE-PACKING PROBLEMS
WITH INTEGER ROUNDING PROPERTIES¹**

by

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Abstract

We consider an integer programming formulation of the node-packing problem, $\max\{1 \cdot x: Ax \leq w, x \geq 0, x \text{ integral}\}$, and its linear programming relaxation, $\max\{1 \cdot x: Ax \leq w, x \geq 0\}$, where A is the edge-node incidence matrix of a graph G and w is a nonnegative integral vector. We give an excluded subgraph characterization quantifying the difference between the values of these two programs. One consequence of this characterization is an explicit description for the "integer rounding" case. Specifically, we characterize those graphs G with the property that for every subgraph of G and for any choice of w , the optimum objective function values of these two problems differ by less than unity.

1. Introduction

Let $G = (V, E)$ be a graph with node set V and edge set E . We restrict attention throughout to finite, undirected, loopless graphs. An *independent (stable)* set of nodes is a subset $S \subseteq V$ such that no two nodes of S are joined by an edge of G . Let $\alpha(G)$ denote the size of a largest stable set in G . A subset $F \subseteq E$ is an *edge-cover* if every node of G is the endpoint of some edge in F . Denoting $\delta(G)$ as the size of a smallest edge-cover in G , G is called a *König graph* when $\alpha(G) = \delta(G)$. A well-known theorem of König states that bipartite graphs without isolated nodes are König graphs. It is not difficult, though, to see that the converse is false, even for graphs without isolated nodes. Characterizations of König graphs are given in Deming [1979], Sterboul [1979], Korach [1982] and Lovász and Plummer [1986], the latter reference containing a thorough discussion of the subject.

One may model the problem of determining $\alpha(G)$ as follows. The *edge-node incidence matrix* of G is the $|E| \times |V|$ matrix $A = [a_{ev}]$ given by

$$a_{ev} = \begin{cases} 1, & \text{if edge } e \text{ is incident to node } v; \\ 0, & \text{otherwise.} \end{cases}$$

For w a nonnegative, integral vector whose components correspond to the edges of G , we consider

$$\alpha_w(G) \equiv \max\{1 \cdot x: Ax \leq w, x \geq 0, x \text{ integral}\},$$

with 1 denoting an appropriately dimensioned vector of ones. We denote the corresponding linear programming relaxation by

$$\bar{\alpha}_w(G) \equiv \max\{1 \cdot x: Ax \leq w, x \geq 0\}.$$

Then the quantities $\alpha_w(G)$ and $\bar{\alpha}_w(G)$ are finite if and only if G has no isolated nodes. When $\alpha_w(G)$ is finite, we have $\alpha(G) = \alpha_1(G)$ and $\delta(G) = \min\{1 \cdot y: yA \geq 1, y \geq 0, y \text{ integral}\}$.

A *subdivision* of G is obtained by replacing the edges of G by simple paths, i.e., by inserting new nodes of degree two into the edges. An *even subdivision* results when the number of new nodes inserted into each edge is even. Thus a graph is bipartite if and only if it contains no subgraph that is an even subdivision of K_3 , the complete graph on three nodes. It is well-known that when G is bipartite, $\alpha_w(G) = \bar{\alpha}_w(G)$ for all nonnegative, integral w ; this follows from total unimodularity of the matrix A for bipartite graphs. Thus $\alpha_w(H) = \bar{\alpha}_w(H)$ for all subgraphs H of G and each nonnegative, integral w if and only if G contains no even subdivision of K_3 . Here, $H = (W, F)$ is a subgraph of $G = (V, E)$ provided $W \subseteq V$ and $F \subseteq \{ \{u, v\} \in E : u, v \in W \}$. Note, in particular, that we do not exclude the case in which H contains an isolated node, as we still have $\alpha_w(H) = \bar{\alpha}_w(H) = +\infty$ in this case.

In this paper we give a similar characterization of graphs G for which $\bar{\alpha}_w(H) - \alpha_w(H) < 1$ holds for all subgraphs H of G and each nonnegative, integral w . Note that if G is an even subdivision of K_4 , then the solution $x_v = 1/2$ for all nodes v achieves the value $\bar{\alpha}_1(G) = 2$, but differs from $\alpha_1(G) = 1$ by unity. Similarly, if G consists of two node-disjoint even subdivisions of K_3 (i.e., two node-disjoint odd cycles), then again $\bar{\alpha}_1(G) - \alpha_1(G) = 1$. In Section 3 below we show that $\bar{\alpha}_w(H) - \alpha_w(H) < 1$ for all subgraphs H of G and each nonnegative, integral w if and only if G contains neither of the two configurations just mentioned. We thus obtain a graphical characterization of edge-node incidence matrices having certain "integer rounding" properties (see Trotter [1985]). Linear programming duality shows that the equality $\alpha(G) = \delta(G)$ implies the equality $\alpha_1(G) = \bar{\alpha}_1(G)$ and hence the relation $\bar{\alpha}_w(G) - \alpha_w(G) < 1$ may be viewed as a "near-König" property.

2. α -Critical Graphs

A graph is *α -critical* if, for each edge, its deletion increases the stability number by one. The following important property of α -critical graphs was given by Hajnal [1965].

Theorem 1 (Hajnal): If $G = (V, E)$ is an α -critical graph with no isolated nodes, then the degree of each vertex is at most $|V| - 2\alpha(G) + 1$. $\square$

This theorem can be used to characterize α -critical graphs with small values of the parameter $p = |V| - 2\alpha(G)$. Let Γ^p be the collection of all α -critical graphs such that $p = |V| - 2\alpha(G)$ and let Γ_c^p be the collection of all connected graphs in Γ^p . Let $G \in \Gamma_c^p$. If $p = 0$, then the degree of each node of G is at most one and hence the graph defined by a single edge is the only member of Γ_c^0 . If $p = 1$, then the degree of each node of G is at most two. The only connected graphs of this type are simple paths and cycles. Now, simple paths and cycles of even length are not α -critical. Hence Γ_c^1 consists of odd cycles. Andrásfai [1967] proved the following theorem for $p = 2$, thus characterizing members of Γ_c^2 .

Theorem 2 (Andrásfai): If $G = (V, E)$ is a connected α -critical graph with $|V| - 2\alpha(G) = 2$, then G is an even subdivision of K_4 . $\square$

The collection of all graphs in Γ^2 without single edges or isolated nodes as components is, by Theorem 2, the collection of all even subdivisions of K_4 or graphs consisting of two components, each an odd cycle. Lovász [1978] has established that for larger values of p , as well, there is a "finite basis" characterization of the members of Γ_c^p . That is, for each nonnegative integer p , there exists a finite collection of graphs (a *basis*) whose even subdivisions are (precisely) the members of Γ_c^p . (see Lovász and Plummer [1986], Chapter 12). The basis is known for $p = 3$ (Lovász [1983]) and we discuss this case in greater detail at the end of Section 3.

3. "Near-König" Graphs

Let A be the edge-node incidence matrix of graph $G = (V, E)$ and let $w = (w_e: e \in E)$ be a nonnegative, integral vector. We first state a well-known and useful

property (see, e.g., Nemhauser and Trotter [1974]) of the polyhedron of feasible solutions to the linear programming problem that determines $\bar{\alpha}_w(G)$.

Lemma 1: Let x be an extreme point of $\{x: Ax \leq w, x \geq 0\}$. Then $2x$ has integer-valued components. Thus, when $\bar{\alpha}_w(G) < +\infty$, there is an optimum solution x^* , i.e., $1 \bullet x^* = \bar{\alpha}_w(G)$, such that $2x^*$ is integer-valued. $\square$

By this lemma we know immediately that $\bar{\alpha}_w(G) - \alpha_w(G)$ is an integer divided by two; when G contains an isolated vertex, i.e., when $\bar{\alpha}_w(G) = \alpha_w(G) = +\infty$, we adopt the convention that $\bar{\alpha}_w(G) - \alpha_w(G) = 0$. Let Ω^p be the collection of all graphs $G = (V, E)$ such that $\bar{\alpha}_w(H) - \alpha_w(H) < p/2$ for all subgraphs H of G and each nonnegative, integral w . Recall that Γ^p is defined above to consist of all α -critical graphs with $|V| - 2\alpha(G) = p$. We have observed that $\bar{\alpha}_w(H) = \alpha_w(H)$ for all subgraphs H of G and for each nonnegative, integral w if and only if G is bipartite. That is, $G \in \Omega^1$ if and only if G contains no subgraph in Γ^1 . We now show that this relation remains valid for *all* $p > 0$.

Lemma 2: Suppose p is a positive integer and G is a graph. Then $G \in \Omega^p$ if and only if G contains no subgraph in Γ^p .

Proof: Suppose $G' = (V', E')$ is a subgraph of G and $G' \in \Gamma^p$. Since $G' \in \Gamma^p$ we have $|V'| - 2\alpha(G') = p$, and we may remove isolated vertices (if any exist) from G' to obtain a subgraph $H = (W, F)$ of G for which $|W| - 2\alpha(H) \geq |V'| - 2\alpha(G') = p$. Now $\bar{\alpha}_1(H) < +\infty$ and clearly $\bar{\alpha}_1(H) \geq |W|/2$, as the solution $x_v = 1/2$ for all $v \in W$ is feasible for the problem defining $\bar{\alpha}_1(H)$. Hence $2(\bar{\alpha}_1(H) - \alpha_1(H)) \geq |W| - 2\alpha_1(H) \geq p$, and it follows that $G \notin \Omega^p$.

To establish the converse, suppose $G \notin \Omega^p$ and select a subgraph $H = (W, F)$ of G and a nonnegative, integral vector w so that $2(\bar{\alpha}_w(H) - \alpha_w(H)) \geq p$ with the quantity $|W| + \sum(w_e: e \in F)$ as small as possible. We will show that $H \in \Gamma^p$. Let x^* denote an optimum linear programming solution yielding value $1 \bullet x^* = \bar{\alpha}_w(H)$; Lemma 1 assures that we may select x^* so that $2x^*$ is integer-valued.

We first argue that $x_v^* = 1/2$, $\forall v \in W$, and $w_e = 1$, $\forall e \in F$. Consider $\bar{x}$ defined by $\bar{x}_v = \lfloor x_v^* \rfloor$, $\forall v \in W$, and define $\bar{w} = w - A\bar{x}$, $\hat{x} = x^* - \bar{x}$, where A is the edge-vertex

incidence matrix of H . It is straightforward to see that $\hat{x}$ solves $\max\{1 \bullet x: Ax \leq \bar{w}, x \geq 0\}$. Thus $\bar{\alpha}_{\bar{w}}(H) = \bar{\alpha}_w(H) - 1 \bullet \bar{x}$. Similarly, since $\bar{x}$ is integral, $\alpha_{\bar{w}}(H) + 1 \bullet \bar{x} \leq \alpha_w(H)$, and hence

$$p/2 \leq \bar{\alpha}_w(H) - \alpha_w(H) \leq \bar{\alpha}_{\bar{w}}(H) - \alpha_{\bar{w}}(H).$$

Minimality of $|W| + \sum(w_e: e \in F)$ and the fact that $\bar{w} \leq w$ imply that $\bar{w} = w$. Thus $A\bar{x} = 0$ and, as $\bar{\alpha}_w(H) > \alpha_w(H)$ implies H has no isolated nodes, we conclude that $\bar{x} = 0$. It follows that $x_v^* = \hat{x}_v < 1, \forall v \in W$, which then implies that $w_e \leq 1, \forall e \in F$. Furthermore, if $x_v^* = 0$ for some $v \in W$, then $\bar{\alpha}_w(H \setminus v) = \bar{\alpha}_w(H)$ and $\alpha_w(H \setminus v) \leq \alpha_w(H)$. Thus

$$p/2 \leq \bar{\alpha}_w(H) - \alpha_w(H) \leq \bar{\alpha}_w(H \setminus v) - \alpha_w(H \setminus v),$$

again contradicting minimality of $|W| + \sum(w_e: e \in F)$. We must thus have $x_v^* = 1/2, \forall v \in W$. This also forces $w_e = 1, \forall e \in F$.

H must also be α -critical, for if not, $\alpha_1(H \setminus e) = \alpha_1(H)$ for some edge $e \in F$. Since $\bar{\alpha}_1(H \setminus e) \geq \bar{\alpha}_1(H)$, we would have that

$$p \leq 2(\bar{\alpha}_1(H) - \alpha_1(H)) \leq 2(\bar{\alpha}_1(H \setminus e) - \alpha_1(H \setminus e)),$$

in contradiction with the minimality assumption on H .

Finally, we show that $|W| - 2\alpha(H) = p$. Since $x_v^* = 1/2, \forall v \in W$, we have $\bar{\alpha}_1(H) = |W|/2$. Hence

$$p \leq 2(\bar{\alpha}_1(H) - \alpha_1(H)) = 2(|W|/2 - \alpha_1(H)) = |W| - 2\alpha_1(H).$$

To see that $p \geq |W| - 2\alpha_1(H)$, pick $v \in W$ for which $\alpha_1(H \setminus v) = \alpha_1(H)$. Now $x_v^* = 1/2, \forall v \in W$, implies $\bar{\alpha}_1(H \setminus v) \geq \bar{\alpha}_1(H) - 1/2$, from which it follows that

$$p > 2(\bar{\alpha}_1(H \setminus v) - \alpha_1(H \setminus v)) \geq 2(\bar{\alpha}_1(H) - 1/2 - \alpha_1(H)) = |W| - 2\alpha_1(H) - 1.$$

Thus $p \geq |W| - 2\alpha_1(H)$.

We have thus shown that H is an α -critical graph for which $|W| - 2\alpha_1(H) = p$. That is, $H \in \Gamma^p$ and the proof is complete. $\square$

Applying Lemma 2 in the case $p = 2$ now gives our main result.

Theorem 3: The inequality $\bar{\alpha}_w(H) - \alpha_w(H) < 1$ holds for all subgraphs H of G and each nonnegative, integral w if and only if G contains neither of the following two types of subgraphs: (i) two node-disjoint odd cycles; (ii) an even subdivision of K_4 .

Proof: Set $p = 2$ in the definition of Ω^p above. Then $\Omega^p = \Omega^2$ is exactly the class of graphs such that $\bar{\alpha}_w(H) - \alpha_w(H) < 1$ for all subgraphs H of G and each nonnegative, integral w . Thus when $G \in \Omega^2$, Lemma 2 implies that G contains no subgraph in Γ^2 and hence no subgraph as in (i) and (ii). On the other hand, if $G \notin \Omega^2$, then, as in the proof of Lemma 2, G contains a subgraph in Γ^2 without single edges or isolated nodes. By the development of the previous section, this subgraph is as in (i) or (ii). $\square$

One may use Theorem 3 to obtain a combinatorial max-min statement relating node-packings and edge-covers in graphs containing neither of the configurations forbidden by the theorem. Specifically, let G be a graph containing neither two node-disjoint odd cycles nor an even subdivision of K_4 and let A be the edge-node incidence matrix of G . Then, for any nonnegative, integral vector w ,

$$\begin{aligned}
& \max\{1 \cdot x: Ax \leq w, x \geq 0, x \text{ integral}\} \\
&= \lfloor \max\{1 \cdot x: Ax \leq w, x \geq 0\} \rfloor && \text{(by Theorem 3)} \\
&= \lfloor \frac{1}{2} \max\{2 \cdot x: Ax \leq w, x \geq 0\} \rfloor && \text{(where } 2 = (2, \dots, 2)) \\
&= \lfloor \frac{1}{2} \min\{w \cdot y: yA \geq 2, y \geq 0\} \rfloor && \text{(by linear programming duality)} \\
&= \lfloor \frac{1}{2} \min\{w \cdot y: yA \geq 2, y \geq 0, y \text{ integral}\} \rfloor,
\end{aligned}$$

where the last equality follows from the fact that the polyhedron $\{y: yA \geq 1, y \geq 0\}$ has

only $(0, \frac{1}{2}, 1)$ -valued extreme points. Thus, e.g., when G has no isolated nodes (else both sides of the max-min relation are $+\infty$), the stability number of G is equal to the "floor" of one-half the value of a 2-cover of nodes by edges in G .

If A is any matrix with nonnegative, integral entries, we say that the *integer round-down property* holds for A provided

$$\max\{1 \bullet x: Ax \leq w, x \geq 0, x \text{ integral}\} = \lfloor \max\{1 \bullet x: Ax \leq w, x \geq 0\} \rfloor,$$

for all vectors w with nonnegative, integral components (see Baum [1977], Chandrasekaran [1981], Marcotte [1982], Orlin [1982], Trotter [1985], Tipnis [1986]). Thus Theorem 3 characterizes those graphs for which each subgraph has an edge-node incidence matrix satisfying the integer round-down property. Similarly, the *integer round-up property* holds for A when $\min\{1 \bullet x: Ax \geq w, x \geq 0, x \text{ integral}\} = \lceil \min\{1 \bullet x: Ax \geq w, x \geq 0\} \rceil$ for all nonnegative, integral w . For $G = (V, E)$ with edge-node incidence matrix A and $w = (1, \dots, 1)$, the integer programming problem here is known as the *node-covering problem* for G . We denote by $\tau(G)$ the value of the node-covering problem for G and say that G is τ -critical when the deletion of each edge of G leads to a reduction in $\tau(G)$. Since the (node)-complement of any node-packing in G is a node-cover, and vice-versa, we have that $\alpha(G) + \tau(G) = |V|$ and consequently that G is α -critical if and only if G is τ -critical. Thus the characterization given by Andrásfai [1967] in Theorem 2 along with an analysis similar to that provided above leads to a node-covering analogue of Theorem 3 and the corollary that (precisely) for the class of graphs specified in Theorem 3, each subgraph has an edge-node incidence matrix satisfying the integer round-up property.

We also point out that Chandrasekaran [1981] and Orlin [1982] have given polynomial-time algorithms for solving $\max\{1 \bullet x: Ax \leq w, x \geq 0, x \text{ integer}\}$, where A is a matrix with nonnegative, integral entries and w is any nonnegative, integral vector, provided that A satisfies the integer round-down property. Hence when the graph G is of the type specified in Theorem 3, the node-packing problem for G can be solved in

polynomial-time. The related recognition question remains open, i.e., whether one can determine in polynomial-time that G meets (or fails) the stipulation of Theorem 3.

Finally, we return to the result of Lovász [1983] on α -critical graphs for $p = 3$ alluded to at the end of Section 2. This characterization leads, in fact, to a result similar to that of Vizing [1964] on the edge-chromatic number of a simple, loopless graph. First consider Vizing's Theorem. A k -edge-coloring of a loopless graph $G = (V, E)$ is an assignment of k colors to the edges of G such that no two adjacent edges have the same color. The *edge-chromatic number*, $\chi(G)$, of a graph G is the minimum k such that G has a k -edge-coloring. Let Δ be the maximum degree of a node in G . Then Vizing's Theorem asserts that $\chi(G) = \Delta$ or $\chi(G) = \Delta + 1$ for any simple, loopless graph G . Now let the rows of M be the (edge-)incidence vectors of matchings in G . Then,

$$\chi(G) = \min\{1 \cdot y: yM \geq 1, y \geq 0, y \text{ integral}\},$$

and it follows from linear programming duality theory that

$$\chi(G) \geq \min\{1 \cdot y: yM \geq 1, y \geq 0\} = \max\{1 \cdot x: Mx \leq 1, x \geq 0\}.$$

Furthermore, $\max\{1 \cdot x: Mx \leq 1, x \geq 0\} \geq \Delta$, since the incidence vector of edges adjacent at a node gives an x satisfying $Mx \leq 1, x \geq 0$. Thus, Vizing's Theorem implies that the difference between the optimum objective function values of $\min\{1 \cdot y: yM \geq 1, y \geq 0, y \text{ integral}\}$ and its linear programming relaxation is at most one.

Now, Lovász [1983] has proved that Γ_c^3 is exactly the collection of all graphs that are even subdivisions of one of the four graphs shown in Figure 1. Since we know that $\bar{\alpha}_w(G) - \alpha_w(G)$ is an integer divided by two, Γ^3 can be used in Lemma 2 (i.e., the case $p = 3$) to obtain a characterization of graphs G , for which $\bar{\alpha}_w(H) - \alpha_w(H) \leq 1$ for all subgraphs H of G , and each nonnegative, integral w .

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Figure 1. Basic graphs in Γ_c^3 .

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