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TRUNCATION OF THE BECHHOFFER-KIEFER-SOBEL
SEQUENTIAL PROCEDURE FOR SELECTING
THE NORMAL POPULATION WHICH HAS THE
LARGEST MEAN (II): SUPPLEMENTARY
TRUNCATION NUMBERS AND RESULTING
PERFORMANCE CHARACTERISTICS

by

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TRUNCATION OF THE BECHHOFFER–KIEFER–SOBEL SEQUENTIAL
PROCEDURE FOR SELECTING THE NORMAL POPULATION WHICH
HAS THE LARGEST MEAN (III): SUPPLEMENTARY TRUNCATION
NUMBERS AND RESULTING PERFORMANCE CHARACTERISTICS

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ABSTRACT

In this paper we provide a supplementary set of truncation numbers (n_0) for δ^* -values not studied in Bechhofer and Goldsman (1987). These are necessary to implement the truncated sequential selection procedure ($\mathcal{P}_{B_T}^*$). Performance characteristics are given for $\mathcal{P}_{B_T}^*$ used with the new n_0 -values.

A by-product of this study was the discovery of a remarkable and completely unexpected pattern among the estimated n_0 -values. The practical implications of this finding are indicated.

1. INTRODUCTION AND SUMMARY

In a recent article, Bechhofer and Goldsman (B–G) (1987), the authors studied the performance characteristics of the open sequential procedure ($\mathcal{P}_B^*$) of Bechhofer, Kiefer and Sobel (1968) for selecting the normal population which has the largest population mean when the k populations have a common known variance. They showed that Procedure

$\mathcal{P}_B^*$ could be truncated in such a way that the indifference-zone probability requirement would be guaranteed; at the same time the expected number and variance of the number of stages to terminate sampling, $E\{n|\mu\}$ and $\text{Var}\{n|\mu\}$, respectively, are dramatically reduced for all $\mu = (\mu_1, \dots, \mu_k)$. In a more recent paper, B-G (1988), the authors compared the performance of this truncated procedure ($\mathcal{P}_{B_T}^*$) with that of eight other competing procedures, including a single-stage procedure, a closed two-stage procedure, and six closed multi-stage procedures, all of which were designed to guarantee the same probability requirement as $\mathcal{P}_{B_T}^*$. In that paper $\mathcal{P}_{B_T}^*$ was shown to have highly desirable performance characteristics relative to those of all of the procedures included in the study, and in fact to dominate all of them in most of the parameter configurations of interest. Thus $\mathcal{P}_{B_T}^*$ was recommended for use in most settings for which truly sequential sampling is feasible.

In order to implement $\mathcal{P}_{B_T}^*$ it is necessary to have available predetermined truncation numbers (n_0) which depend on k , δ^* and P^* . Such numbers are tabulated in B-G (1987) for all combinations (k, δ^*, P^*) with $k = 2(1)5$, $\delta^* = 0.2(0.2)0.8$ and $P^* = 0.75, 0.90, 0.95$ and 0.99 . To capitalize more fully on the findings concerning the above noted superior performance of $\mathcal{P}_{B_T}^*$, it was decided to extend the range of usefulness of that procedure by determining supplementary n_0 -values for δ^* -values not included in the original study.

Accordingly, in the present paper n_0 -values were determined by Monte Carlo (MC) sampling (exactly as in B-G (1987)) for all

combinations (k, δ^*, P^*) with $k = 2(1)5$, $\delta^* = 0.3(0.2)0.7$ and $P^* = 0.75, 0.90, 0.95$ and 0.99 . In addition, the performance of $\mathcal{S}_B^*$ and of $\mathcal{S}_{B_T}^*$ in terms of $E\{n|\mu\}$ was estimated for every combination noted above not only for the least-favorable (LF) and equal means (EM) configurations but also for the configuration in which all of the population means are equally-spaced δ^* apart ($ES(\delta^*)$). Finally, the achieved probability of a correct selection ($P\{CS|\mu\}$) was estimated for every (k, δ^*, P^*) combination noted above for the $ES(\delta^*)$ - and LF-configurations. Although our main focus was on $\mathcal{S}_{B_T}^*$ and its performance, we included $\mathcal{S}_B^*$ in our study to obtain comparability with results reported in B-G(1987). Also, building on our findings in B-G(1988), the $ES(\delta^*)$ results allowed us to show how responsive $\mathcal{S}_B^*$ is to favorable configurations of the population means, resulting in early termination of sampling even without truncation. Thus, between B-G(1987) and the present article the experimenter will have easily available the n_0 -values necessary to implement $\mathcal{S}_{B_T}^*$, and a comprehensive picture of its most important performance characteristics, over a broad range of the (k, δ^*, P^*) -values of principal interest.

A remarkable and completely unexpected pattern among the estimated n_0 -values was noted. This finding is not only of considerable theoretical interest but also serves as a partial check on the correctness of our already-determined truncation numbers, and simplifies the determination of new ones.

2. ESTIMATES OF PERFORMANCE CHARACTERISTICS OF $\mathcal{P}_B^*$ AND OF $\mathcal{P}_{B_T}^*$ WITH ESTIMATED TRUNCATION NUMBERS

The format of this paper parallels that of B–G(1987) although the present paper is much abbreviated. It is recommended that the earlier paper be read for background, motivation, and further details.

The performance characteristics studied for $\mathcal{P}_B^*$ were $P\{CS|ES(\delta^*)\}$, $E\{n|ES(\delta^*)\}$, $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EM\}$. Two estimates of $P\{CS|LF\}$ are given, namely, the observed proportion of correct selections ($\bar{p}$), and the observed average value ($\bar{w}$) of W_n (see (3.6) of B–G(1987)). (Both $\bar{p}$ and $\bar{w}$ are unbiased estimates of $P\{CS|LF\}$, the latter having much smaller variance than the former.) The same characteristics were studied for $\mathcal{P}_{B_T}^*$ when used with the estimated smallest truncation number (n_0) necessary to implement the procedure and guarantee the $P\{CS\}$ requirement. The results for $k = 2, 3, 4, 5$ are given in TABLES I, II, III, IV, respectively; these are supplementary to corresponding results given in TABLES II, III, IV, V, respectively, of B–G(1987). (For $k = 2$ the $ES(\delta^*)$ – and LF –configurations are the same.) It would have been extremely costly and time consuming to extend the results to $k > 5$; in fact, the results for $k = 5$ (particularly for the small δ^* – and high P^* –values) were quite costly.

All results for $\mathcal{P}_B^*$ in the $ES(\delta^*)$ –, LF – and EM –configurations are based on 4,000, 24,000 and 4,000 independent replications, respectively. Each result for $\mathcal{P}_{B_T}^*$ in the LF –configuration is based on at least 24,000 replications while some were based on as many as 192,000 replications when

TABLE I

Estimated⁺ $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EM\}$ for $\mathcal{P}_B^*$ and for $\mathcal{P}_{B_T}^*$ with associated truncation number n_0 and selected $\{\delta^*, P^*\}$ when $k = 2$

P^*	δ^*	n_0	Least Favorable			Equal Means
			$\bar{p}$	$\bar{w}$	$\bar{n}$	$\bar{n}$
0.75	0.7		0.8431 (0.0023)	0.8426 (0.0004)	2.60 (0.01)	3.16 (0.04)
		2	0.7517 (0.0028)	0.7542 (0.0008)	1.67 (<0.01)	1.74 (<0.01)
	0.5		0.8167 (0.0025)	0.8186 (0.0003)	4.12 (0.02)	4.82 (0.06)
		4	0.7571 (0.0028)	0.7533 (0.0007)	2.94 (<0.01)	3.08 (0.01)
	0.3		0.7950 (0.0026)	0.7933 (0.0002)	9.05 (0.05)	10.44 (0.13)
		12	0.7500 (0.0028)	0.7523 (0.0006)	7.38 (0.02)	7.90 (0.03)
0.90	0.7		0.9413 (0.0015)	0.9409 (0.0002)	5.26 (0.02)	8.16 (0.10)
		8	0.9075 (0.0019)	0.9082 (0.0006)	4.63 (0.01)	5.67 (0.02)
	0.5		0.9330 (0.0016)	0.9316 (0.0001)	9.31 (0.04)	13.90 (0.18)
		16	0.9043 (0.0019)	0.9045 (0.0005)	8.34 (0.03)	10.51 (0.04)
	0.3		0.9215 (0.0017)	0.9204 (<0.0001)	23.39 (0.11)	33.46 (0.43)
		47	0.9017 (0.0019)	0.9020 (0.0005)	21.49 (0.08)	27.28 (0.13)

TABLE I (continued)

(k = 2)

P^*	δ^*	n_0	Least Favorable			Equal Means
			$\bar{p}$	$\bar{w}$	$\bar{n}$	$\bar{n}$
0.95	0.7		0.9718 (0.0011)	0.9711 (<0.0001)	7.00 (0.03)	13.11 (0.17)
		13	0.9547 (0.0013)	0.9537 (0.0004)	6.57 (0.02)	10.09 (0.03)
	0.5		0.9670 (0.0011)	0.9663 (<0.0001)	12.76 (0.06)	22.88 (0.29)
		26	0.9500 (0.0010)	0.9515 (0.0003)	12.11 (0.03)	16.96 (0.07)
	0.3		0.9633 (0.0012)	0.9606 (<0.0001)	32.77 (0.15)	57.20 (0.73)
		78	0.9514 (0.0010)	0.9506 (0.0002)	31.62 (0.09)	46.51 (0.22)
0.99	0.7		0.9950 (0.0005)	0.9943 (<0.0001)	10.65 (0.04)	27.59 (0.36)
		26	0.9908 (0.0006)	0.9910 (0.0002)	10.54 (0.04)	18.73 (0.07)
	0.5		0.9933 (0.0005)	0.9933 (<0.0001)	20.05 (0.08)	50.21 (0.66)
		52	0.9907 (0.0004)	0.9904 (0.0001)	19.75 (0.05)	35.91 (0.14)
	0.3		0.9920 (0.0006)	0.9922 (<0.0001)	53.37 (0.21)	130.83 (1.71)
		154	0.9899 (0.0003)	0.9902 (<0.0001)	52.74 (0.10)	98.35 (0.43)

⁺All $P\{CS\}$ and $E\{n\}$ results and their standard errors are estimated by Monte Carlo sampling. The numbers in parentheses are the estimated standard errors of the estimates above them.

TABLE II

Estimated⁺ $P\{CS|ES(\delta^*)\}$, $E\{n|ES(\delta^*)\}$, $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EM\}$ for $\mathcal{P}_B^*$ and for $\mathcal{P}_{B_T}^*$ with associated truncation number n_0 and selected $\{\delta^*, P^*\}$ when $k = 3$

P^*	δ^*	n_0	Equally-spaced δ^* apart		Least Favorable			Equal Means
			$\bar{p}$	$\bar{n}$	$\bar{p}$	$\bar{w}$	$\bar{n}$	$\bar{n}$
0.75	0.7		0.8805 (0.0051)	3.64 (0.03)	0.8360 (0.0024)	0.8342 (0.0004)	4.34 (0.02)	6.02 (0.07)
		5	0.8471 (0.0033)	3.15 (0.01)	0.7640 (0.0027)	0.7645 (0.0008)	3.60 (0.01)	4.02 (0.01)
	0.5		0.8613 (0.0055)	6.07 (0.06)	0.8127 (0.0025)	0.8131 (0.0003)	7.67 (0.03)	9.97 (0.11)
		10	0.8378 (0.0034)	5.62 (0.02)	0.7605 (0.0028)	0.7613 (0.0007)	6.52 (0.02)	7.42 (0.03)
	0.3		0.8395 (0.0058)	14.74 (0.14)	0.7901 (0.0026)	0.7898 (0.0002)	18.43 (0.08)	23.42 (0.26)
		28	0.8324 (0.0034)	13.96 (0.07)	0.7539 (0.0020)	0.7527 (0.0004)	16.39 (0.04)	18.82 (0.07)
0.90	0.7		0.9545 (0.0033)	5.99 (0.06)	0.9382 (0.0016)	0.9385 (0.0002)	7.58 (0.03)	13.67 (0.16)
		12	0.9342 (0.0023)	5.76 (0.03)	0.9068 (0.0019)	0.9075 (0.0006)	6.99 (0.02)	9.21 (0.03)
	0.5		0.9475 (0.0035)	10.94 (0.11)	0.9293 (0.0017)	0.9292 (0.0001)	13.91 (0.06)	23.95 (0.28)
		24	0.9377 (0.0022)	10.45 (0.05)	0.9020 (0.0019)	0.9030 (0.0006)	12.99 (0.04)	17.59 (0.06)
	0.3		0.9380 (0.0038)	27.43 (0.27)	0.9157 (0.0018)	0.9187 (<0.0001)	35.59 (0.14)	58.91 (0.68)
		70	0.9296 (0.0023)	27.25 (0.14)	0.9008 (0.0007)	0.9003 (0.0002)	34.01 (0.04)	47.04 (0.19)

TABLE II (Continued)

(k = 3)

P^*	δ^*	n_0	Equally-spaced δ^* apart		Least Favorable			Equal Means
			$\bar{p}$	$\bar{n}$	$\bar{p}$	$\bar{w}$	$\bar{n}$	$\bar{n}$
0.95	0.7		0.9758 (0.0024)	7.71 (0.08)	0.9702 (0.0011)	0.9701 (<0.0001)	9.68 (0.04)	20.41 (0.23)
		17	0.9684 (0.0016)	7.40 (0.04)	0.9526 (0.0014)	0.9516 (0.0005)	9.14 (0.03)	13.46 (0.04)
	0.5		0.9728 (0.0026)	14.08 (0.14)	0.9648 (0.0012)	0.9653 (<0.0001)	17.90 (0.07)	37.42 (0.43)
		35	0.9687 (0.0016)	13.84 (0.07)	0.9487 (0.0010)	0.9508 (0.0003)	17.27 (0.04)	26.13 (0.09)
	0.3		0.9682 (0.0028)	36.67 (0.36)	0.9598 (0.0013)	0.9598 (<0.0001)	46.72 (0.18)	95.87 (1.07)
		106	0.9617 (0.0018)	36.20 (0.19)	0.9513 (0.0010)	0.9512 (0.0002)	45.76 (0.11)	72.49 (0.28)
0.99	0.7		0.9960 (0.0010)	11.01 (0.10)	0.9942 (0.0005)	0.9941 (<0.0001)	13.74 (0.05)	41.12 (0.49)
		31	0.9944 (0.0007)	11.04 (0.05)	0.9904 (0.0004)	0.9906 (0.0001)	13.53 (0.03)	25.26 (0.07)
	0.5		0.9945 (0.0012)	20.90 (0.18)	0.9936 (0.0005)	0.9932 (<0.0001)	25.89 (0.09)	76.62 (0.91)
		62	0.9921 (0.0008)	20.85 (0.10)	0.9903 (0.0004)	0.9904 (0.0001)	25.80 (0.06)	48.98 (0.15)
	0.3		0.9935 (0.0013)	54.73 (0.49)	0.9921 (0.0006)	0.9921 (<0.0001)	69.74 (0.25)	198.76 (2.32)
		188	0.9926 (0.0008)	55.48 (0.29)	0.9906 (0.0004)	0.9904 (<0.0001)	69.44 (0.16)	140.18 (0.48)

[†]All $P\{CS\}$ and $E\{n\}$ results and their standard errors are estimated by Monte Carlo sampling. The numbers in parentheses are the estimated standard errors of the estimates above them.

TABLE III

Estimated⁺ $P\{CS|ES(\delta^*)\}$, $E\{n|ES(\delta^*)\}$, $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EM\}$ for $\mathcal{P}_B^*$ and for $\mathcal{P}_{B_T}^*$ with associated truncation number n_0 and selected $\{\delta^*, P^*\}$ when $k = 4$

P^*	δ^*	n_0	Equally-spaced δ^* apart		Least Favorable			Equal Means
			$\bar{p}$	$\bar{n}$	$\bar{p}$	$\bar{w}$	$\bar{n}$	$\bar{n}$
0.75	0.7		0.8795 (0.0051)	3.81 (0.03)	0.8335 (0.0024)	0.8327 (0.0004)	5.73 (0.02)	8.16 (0.09)
		7	0.8700 (0.0031)	3.64 (0.02)	0.7684 (0.0027)	0.7699 (0.0009)	4.86 (0.01)	5.64 (0.02)
	0.5		0.8690 (0.0053)	6.64 (0.06)	0.8115 (0.0025)	0.8113 (0.0003)	10.05 (0.04)	14.15 (0.15)
		13	0.8613 (0.0032)	6.36 (0.03)	0.7500 (0.0028)	0.7533 (0.0008)	8.71 (0.02)	10.09 (0.03)
	0.3		0.8518 (0.0056)	16.02 (0.14)	0.7863 (0.0026)	0.7880 (0.0002)	25.01 (0.10)	33.85 (0.36)
		39	0.8490 (0.0033)	15.91 (0.08)	0.7568 (0.0020)	0.7518 (0.0005)	22.93 (0.05)	27.20 (0.10)
0.90	0.7		0.9555 (0.0033)	6.22 (0.06)	0.9371 (0.0016)	0.9377 (0.0002)	9.18 (0.03)	17.05 (0.19)
		14	0.9465 (0.0021)	6.05 (0.03)	0.9068 (0.0019)	0.9034 (0.0007)	8.45 (0.02)	11.35 (0.03)
	0.5		0.9505 (0.0034)	11.08 (0.11)	0.9298 (0.0016)	0.9284 (0.0001)	16.80 (0.06)	30.48 (0.33)
		29	0.9414 (0.0021)	10.95 (0.06)	0.9034 (0.0019)	0.9025 (0.0006)	15.88 (0.05)	22.11 (0.07)
	0.3		0.9390 (0.0038)	28.73 (0.27)	0.9190 (0.0018)	0.9182 (<0.0001)	44.03 (0.17)	78.63 (0.83)
		86	0.9358 (0.0022)	28.52 (0.15)	0.9024 (0.0019)	0.9020 (0.0005)	42.30 (0.14)	61.00 (0.22)

TABLE III (Continued)

(k = 4)

P^*	δ^*	n_0	Equally-spaced δ^* apart		Least Favorable			Equal Means
			$\bar{p}$	$\bar{n}$	$\bar{p}$	$\bar{w}$	$\bar{n}$	$\bar{n}$
0.95	0.7		0.9732 (0.0026)	7.84 (0.07)	0.9679 (0.0011)	0.9698 (<0.0001)	11.31 (0.04)	25.23 (0.27)
		20	0.9741 (0.0015)	7.63 (0.04)	0.9532 (0.0010)	0.9518 (0.0003)	10.82 (0.02)	16.35 (0.05)
	0.5		0.9740 (0.0025)	14.37 (0.14)	0.9641 (0.0012)	0.9650 (<0.0001)	21.09 (0.07)	47.00 (0.53)
		41	0.9688 (0.0016)	14.12 (0.08)	0.9513 (0.0010)	0.9514 (0.0003)	20.46 (0.04)	42.14 (0.10)
	0.3		0.9618 (0.0030)	36.87 (0.35)	0.9590 (0.0013)	0.9596 (<0.0001)	55.37 (0.20)	118.11 (1.26)
		122	0.9639 (0.0017)	36.96 (0.20)	0.9504 (0.0010)	0.9507 (0.0002)	54.45 (0.13)	89.20 (0.31)
0.99	0.7		0.9952 (0.0011)	11.25 (0.10)	0.9936 (0.0005)	0.9941 (<0.0001)	15.62 (0.05)	50.82 (0.56)
		34	0.9942 (0.0007)	11.07 (0.06)	0.9909 (0.0004)	0.9908 (0.0001)	15.41 (0.03)	28.94 (0.07)
	0.5		0.9938 (0.0012)	21.02 (0.19)	0.9932 (0.0005)	0.9931 (<0.0001)	29.53 (0.09)	91.99 (1.04)
		68	0.9924 (0.0008)	20.80 (0.10)	0.9901 (0.0002)	0.9902 (<0.0001)	29.28 (0.03)	56.66 (0.15)
	0.3		0.9915 (0.0014)	55.95 (0.51)	0.9925 (0.0006)	0.9920 (<0.0001)	79.75 (0.26)	243.30 (2.64)
		200	0.9937 (0.0007)	56.12 (0.29)	0.9900 (0.0002)	0.9900 (<0.0001)	79.07 (0.07)	160.04 (0.46)

⁺All $P\{CS\}$ and $E\{n\}$ results and their standard errors are estimated by Monte Carlo sampling. The numbers in parentheses are the estimated standard errors of the estimates above them.

TABLE IV

Estimated⁺ $P\{CS|ES(\delta^*)\}$, $E\{n|ES(\delta^*)\}$, $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EM\}$ for $\mathcal{R}_B^*$ and for $\mathcal{R}_{B_T}^*$ with associated truncation number n_0 and selected $\{\delta^*, P^*\}$ when $k = 5$

P^*	δ^*	n_0	Equally-spaced δ^* apart		Least Favorable			Equal Means
			$\bar{p}$	$\bar{n}$	$\bar{p}$	$\bar{w}$	$\bar{n}$	$\bar{n}$
0.75	0.7		0.8815 (0.0051)	3.85 (0.03)	0.8307 (0.0024)	0.8306 (0.0004)	6.63 (0.02)	9.93 (0.10)
		8	0.8839 (0.0029)	3.76 (0.02)	0.7605 (0.0028)	0.7588 (0.0009)	5.71 (0.01)	6.62 (0.02)
	0.5		0.8685 (0.0053)	6.64 (0.06)	0.8052 (0.0026)	0.8098 (0.0003)	11.85 (0.04)	16.94 (0.17)
		16	0.8629 (0.0031)	6.63 (0.03)	0.7561 (0.0014)	0.7563 (0.0004)	10.55 (0.01)	12.45 (0.04)
	0.3		0.8552 (0.0056)	16.54 (0.14)	0.7898 (0.0026)	0.7873 (0.0002)	30.04 (0.11)	41.42 (0.42)
		47	0.8518 (0.0032)	16.54 (0.08)	0.7519 (0.0014)	0.7513 (0.0003)	27.74 (0.04)	33.55 (0.12)
0.90	0.7		0.9520 (0.0034)	6.14 (0.06)	0.9355 (0.0016)	0.9374 (0.0002)	10.19 (0.04)	20.28 (0.21)
		16	0.9493 (0.0020)	6.12 (0.03)	0.9082 (0.0019)	0.9049 (0.0007)	9.60 (0.03)	13.20 (0.03)
	0.5		0.9395 (0.0038)	11.14 (0.10)	0.9301 (0.0016)	0.9279 (0.0001)	19.03 (0.07)	36.39 (0.38)
		32	0.9433 (0.0021)	10.99 (0.06)	0.9015 (0.0014)	0.9014 (0.0004)	17.94 (0.04)	25.27 (0.07)
	0.3		0.9325 (0.0040)	28.63 (0.27)	0.9187 (0.0018)	0.9178 (<0.0001)	49.39 (0.18)	91.91 (0.95)
		96	0.9355 (0.0022)	28.50 (0.15)	0.9005 (0.0014)	0.9008 (0.0003)	48.19 (0.11)	70.20 (0.24)

TABLE IV (Continued)

(k = 5)

P^*	δ^*	n_0	Equally-spaced δ^* apart		Least Favorable			Equal Means
			$\bar{p}$	$\bar{n}$	$\bar{p}$	$\bar{w}$	$\bar{n}$	$\bar{n}$
0.95	0.7		0.9768 (0.0024)	7.73 (0.07)	0.9710 (0.0011)	0.9694 (<0.0001)	12.48 (0.04)	29.64 (0.32)
		22	0.9749 (0.0014)	7.63 (0.04)	0.9506 (0.0010)	0.9517 (0.0003)	12.03 (0.02)	18.45 (0.05)
	0.5		0.9758 (0.0024)	14.10 (0.13)	0.9655 (0.0012)	0.9647 (<0.0001)	23.33 (0.08)	53.88 (0.57)
		44	0.9694 (0.0016)	14.29 (0.07)	0.9498 (0.0007)	0.9505 (0.0002)	22.65 (0.03)	35.62 (0.10)
	0.3		0.9688 (0.0028)	37.53 (0.36)	0.9591 (0.0013)	0.9595 (<0.0001)	62.08 (0.21)	139.16 (1.47)
		132	0.9680 (0.0016)	37.22 (0.20)	0.9499 (0.0010)	0.9504 (0.0002)	60.84 (0.14)	100.51 (0.32)
0.99	0.7		0.9958 (0.0010)	11.29 (0.10)	0.9941 (0.0005)	0.9940 (<0.0001)	16.97 (0.05)	56.47 (0.61)
		36	0.9940 (0.0007)	11.17 (0.06)	0.9905 (0.0004)	0.9904 (0.0001)	16.76 (0.03)	31.53 (0.07)
	0.5		0.9950 (0.0011)	21.14 (0.18)	0.9925 (0.0006)	0.9931 (<0.0001)	32.31 (0.10)	106.17 (1.16)
		73	0.9933 (0.0007)	20.92 (0.11)	0.9907 (0.0003)	0.9904 (<0.0001)	31.82 (0.05)	62.28 (0.15)
	0.3		0.9938 (0.0012)	55.42 (0.50)	0.9916 (0.0006)	0.9920 (<0.0001)	86.56 (0.27)	279.21 (2.98)
		216	0.9917 (0.0008)	55.67 (0.29)	0.9901 (0.0003)	0.9902 (<0.0001)	86.34 (0.13)	177.00 (0.48)

⁺All $P\{CS\}$ and $E\{n\}$ results and their standard errors are estimated by Monte Carlo sampling. The numbers in parentheses are the estimated standard errors of the estimates above them.

it was necessary to estimate $P\{CS|LF\}$ to a very high degree of precision. All results for $\mathcal{P}_{B_T}^*$ used with the appropriate estimated n_0 -value in the ES(δ^*)- and EM- configurations are based on 12,000 replications. The numbers in parentheses in the tables are the estimated standard errors of the estimates above them.

3. STUDY OF ESTIMATED TRUNCATION NUMBERS

In the course of summarizing the estimated truncation numbers n_0 for $\mathcal{P}_{B_T}^*$ reported in TABLES I–IV, we discovered a remarkable and completely unexpected pattern among them. This can be seen in TABLE V. For any fixed pair of δ^* -values the associated n_0 -values are in essentially the same proportion for all (k,P*)-combinations. For example, the ratios of n_0 -values associated with $\delta^* = 0.3$ and $\delta^* = 0.5$, namely (12/4, 47/16, 78/26, 154/52, 28/10, 70/24,...,216/73), are all approximately equal to 3.

We did not notice the pattern displayed in TABLE V until roughly three-fourths of the estimates of n_0 -values in TABLES I–IV had been determined. After the pattern was noted we were able to capitalize on the fact that the empirically determined ratios were approximately constant for each pair of δ^* -values. This enabled us to determine a desired n_0 -value after only two or three computer runs with different trial n_0 -values. (See Remark 3.1.)

TABLE VI was prepared by pooling the estimated n_0 -values for $\delta^* = 0.2(0.2)0.8$ from TABLES II–V of B–G(1987) with the corresponding n_0 -values for $\delta^* = 0.3(0.2)0.7$ from TABLES I–IV of the present article.

TABLE V

Estimated truncation numbers⁺ (n_0) for $\mathcal{P}_{B_T}^*$
for selected (k, δ^*, P^*)

P^*	δ^*	k			
		2	3	4	5
0.75	0.7	2	5	7	8
	0.5	4	10	13	16
	0.3	12	28	39	47
0.90	0.7	8	12	14	16
	0.5	16	24	29	32
	0.3	47	70	86	96
0.95	0.7	13	17	20	22
	0.5	26	35	41	44
	0.3	78	106	122	132
0.99	0.7	26	31	34	36
	0.5	52	62	68	73
	0.3	154	188	200	216

⁺Abstracted from Tables I–IV.

TABLE VI

Ratio $R = \frac{n_0(\delta^*=0.2)}{n_0(\delta^*)}$ of estimated truncation numbers⁺ for $\mathcal{P}_{B_T}^*$

as a function of $\delta^* = 0.3(0.1)0.8$ for selected (k, P^*)

δ^*	P^*	k			
		2	3	4	5
0.8	0.75	28/2=14.00	68/4=17.00	94/5=18.80	114/6=19.00
	0.90	114/6=19.00	170/9=18.89	205/11=18.64	230/12=19.17
	0.95	190/10=19.00	255/13=19.62	290/15=19.33	320/17=18.82
	0.99	370/19=19.47	440/23=19.13	480/25=19.20	520/27=19.06
0.7	0.75	28/2=14.00	68/5=13.60	94/7=13.43	114/8=14.25
	0.90	114/8=14.25	170/12=14.17	205/14=14.64	230/16=14.38
	0.95	190/13=14.62	255/17=15.00	290/20=14.50	320/22=14.55
	0.99	370/26=14.23	440/31=14.19	480/34=14.12	520/36=14.44
0.6	0.75	28/3=9.33	68/7=9.71	94/9=10.44	114/11=10.36
	0.90	114/11=10.36	170/16=10.63	205/19=10.79	230/22=10.45
	0.95	190/18=10.56	255/24=10.62	290/28=10.36	320/30=10.67
	0.99	370/35=10.57	440/42=10.48	480/46=10.43	520/49=10.61
0.5	0.75	28/4=7.00	68/10=6.80	94/13=7.23	114/16=7.13
	0.90	114/16=7.12	170/24=7.08	205/29=7.07	230/32=7.19
	0.95	190/26=7.31	255/35=7.29	290/41=7.07	320/44=7.27
	0.99	370/52=7.12	440/62=7.10	480/68=7.06	520/73=7.12
0.4	0.75	28/7=4.00	68/15=4.53	94/21=4.48	114/25=4.56
	0.90	114/25=4.56	170/38=4.47	205/46=4.46	230/51=4.51
	0.95	190/42=4.52	255/56=4.55	290/65=4.46	320/71=4.51
	0.99	370/84=4.40	440/100=4.40	480/110=4.36	520/116=4.48
0.3	0.75	28/12=2.33	68/28=2.43	94/39=2.41	114/47=2.43
	0.90	114/47=2.43	170/70=2.43	205/86=2.38	230/96=2.40
	0.95	190/78=2.44	255/106=2.41	290/122=2.38	320/132=2.42
	0.99	370/154=2.40	440/188=2.34	480/200=2.40	520/216=2.41

⁺The estimated n_0 -values for $\delta^* = 0.2(0.2)0.8$ were abstracted from TABLES II-V of B-G(1987) while those for $\delta^* = 0.3(0.2)0.7$ were abstracted from TABLES I-IV of the present paper.

For each (k, δ^*, P^*) -combination the numerator of the tabulated ratio is the n_0 -value for $\delta^* = 0.2$. This n_0 -value was used as the reference point since it is the largest for each (k, P^*) -combination, and a small error in its determination would have produced only a modest change in the ratio. It is to be noted that for every fixed $\delta^* = 0.3(0.1)0.8$ in TABLE VI, the ratio $R = \frac{n_0(\delta^*=0.2)}{n_0(\delta^*)}$ for each (k, P^*) is essentially constant for all combinations (k, P^*) with $k = 2(1)5$ and $P^* = 0.75, 0.90, 0.95, 0.99$, i.e., R for fixed δ^* is essentially independent of (k, P^*) although R changes with δ^* . For fixed δ^* , the failure of R to be "exactly" constant for all combinations (k, P^*) is presumably a consequence of the discreteness of the n_0 . This discreteness plays a minor role when n_0 is large (which occurs when δ^* is small). There is almost certainly a theoretical explanation of this constant ratio phenomenon. Its study will be the subject of a future article.

Remark 3.1: The fact that R is essentially independent of (k, P^*) for fixed δ^* suggests a shortcut for the process of finding a good first approximation to the required n_0 -values. First, for (say) $k = 2$, $P^* = 0.99$, use MC sampling to estimate the required n_0 -values for $\delta^* = 0.2(0.1)0.8$; for the data of TABLE VI these yield $R = n_0(\delta^* = 0.2)/n_0(\delta^*) = 2.40, 4.40, \dots, 19.47$ for $\delta^* = 0.3, 0.4, \dots, 0.8$, respectively. Then for $k = 3, 4, 5$ with $P^* = 0.99$ use MC sampling to estimate the required n_0 -values for $\delta^* = 0.2$; for the data of TABLE VI this yields $n_0 = 440, 480, 520$ for $k = 3, 4, 5$, respectively. These data are sufficient to yield a good first approximation to the n_0 -values for all

combinations of $k = 3, 4, 5$; $\delta^* = 0.3(0.1)0.8$ and $P^* = 0.75, 0.90, 0.95, 0.99$. For example, suppose that we seek such an approximation for $\delta^* = 0.5, k = 4, P^* = 0.90$. We know that $n_0(0.2)/n_0(0.5) \sim 7.1$ for $\delta^* = 0.5$ and all combinations of (k, P^*) . Also, when $\delta^* = 0.2$ we have $n_0 \sim 205$ for $k = 4, P^* = 0.90$. Thus, an approximation for the required n_0 -value is given by $205/7.1 = 28.9$ or 29. The value $n_0 = 29$ is in fact the one that was obtained directly; see TABLE V. This technique would be particularly helpful for estimating n_0 -values for $k \geq 6$.

4. ABILITY OF $\mathcal{S}_B^*$ AND $\mathcal{S}_{B_T}^*$ TO CAPITALIZE ON FAVORABLE MEANS CONFIGURATIONS

It was noted in Section 3.1 of B-G(1987) that $\mathcal{S}_B^*$ is highly effective in capitalizing on favorable configurations of the population means, resulting in decreases in both $E\{n|\mu\}$ and $\text{Var}\{n|\mu\}$. In the present paper we have included $\bar{n}$ -results not only for the LF- and EM-configurations but also for the $ES(\delta^*)$ -configurations so that quantitative estimates of the decreases in $E\{n|\mu\}$ can be obtained. TABLE VII summarizes this $\bar{n}$ -information for $k = 5$. Using the $\bar{n}$ -results in TABLES II and III a corresponding table could be prepared for $k = 3$ and 4, respectively. (Similar tables could be prepared for the estimates of the achieved $P\{CS|\mu\}$ using $\bar{p}$ and $\bar{w}$ for the $ES(\delta^*)$ - and LF-configurations, respectively.) We note from TABLE VII the substantial decreases in $\bar{n}$ which occur, especially for $\mathcal{S}_B^*$ but also for $\mathcal{S}_{B_T}^*$, as the configuration changes from EM- to LF- to $ES(\delta^*)$, each time becoming more favorable to the experimenter.

TABLE VII

Estimates⁺ of $E\{n|\mu\}$ for $\mathcal{P}_B^*$ and $\mathcal{P}_{B_T}^*$
in three configurations of the population means
for $k = 5$ and selected (δ^*, P^*)

P^*	δ^*	$\bar{n}$ for $\mathcal{P}_B^*$			$\bar{n}$ for $\mathcal{P}_{B_T}^*$		
		ES(δ^*)	LF	EM	ES(δ^*)	LF	EM
0.75	0.7	3.85	6.63	9.93	3.76	5.71	6.62
	0.5	6.64	11.85	16.94	6.63	10.55	12.45
	0.3	16.54	30.04	41.42	16.54	27.74	33.55
0.90	0.7	6.14	10.19	20.28	6.12	9.60	13.20
	0.5	11.14	19.03	36.39	10.99	17.94	25.27
	0.3	28.63	49.39	91.91	28.50	48.19	70.20
0.95	0.7	7.73	12.48	29.64	7.63	12.03	18.45
	0.5	14.10	23.33	53.88	14.29	22.65	35.62
	0.3	37.53	62.08	139.16	37.22	60.84	100.51
0.99	0.7	11.29	16.97	56.47	11.17	16.76	31.53
	0.5	21.14	32.31	106.17	20.92	31.82	62.28
	0.3	55.42	86.56	279.21	55.67	86.34	177.00

⁺Abstracted from TABLE IV.

5. DIRECTIONS OF FUTURE RESEARCH

As pointed out in Section 3, a theoretical explanation of the constant proportionality phenomenon for the estimated n_0 -values is presently unknown to us. This result is fascinating and we plan to seek such an explanation.

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BIBLIOGRAPHY

Bechhofer, R.E. and Goldsman, D.M. (1987). Truncation of the Bechhofer–Kiefer–Sobel procedure for selecting the normal population which has the largest mean. Comm. Statist.—Simula. Computa. B16(4), 1067–1092.

Bechhofer, R.E. and Goldsman, D.M. (1988). A comparison of the performances of procedures for selecting the normal population having the largest mean when the variances are known and equal. Tech. Report No. 789 (May 1988), School of Operations Research and Industrial Engineering, Cornell University.

Bechhofer, R.E., Kiefer, J. and Sobel, M. (1968). Sequential Identification and Ranking Procedures (with special reference to Koopman–Darmois populations), Chicago: The University of Chicago Press.