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COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853-7501

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MULTIVARIATE RECORDS AND SHAPE

by

Keizo Kinoshita* and Sidney I. Resnick**

* Department of Statistics, Colorado State University, Fort Collins, CO 80523 USA. Partially supported by NSF Grant DMS-8501673 at Colorado State University. The hospitality of the School of Operations Research and Industrial Engineering, Cornell University during the academic year 1987-1988 is gratefully acknowledged.

** School of Operations Research and Industrial Engineering, Upson Hall, Cornell University, Ithaca, NY 14853 USA. Partially supported by the Mathematical Sciences Institute, Cornell University and by NSF Grant DMS-8801034 at Cornell University.

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K. Kinoshita^{*}
Department of Statistics
Colorado State University
Fort Collins, CO 80523 USA

S.I. Resnick^{**}
School of Operations Research and Industrial Engineering
Cornell University
Upson Hall
Ithaca, NY 14853 USA

ABSTRACT

For one dimensional data, the notion of a record is natural and universally agreed upon but this is not the case when the data is in two or more dimensions. We outline several plausible definitions of a record and settle on one induced by a partial ordering: For observations $\mathbf{X}_1, \dots, \mathbf{X}_n$ we say $\mathbf{X}_n$ is a record if each component of $\mathbf{X}_n$ is bigger than the corresponding components of previous observations. Some properties of such records of iid observations are reviewed and used to study the shape of the convex hull of the first n observations, as $n \rightarrow \infty$.

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1. INTRODUCTION

The notion of a record in a one-dimensional series of observations is familiar, natural and unambiguous. It means of course an observation bigger (or smaller in appropriate contexts) than previous observations and is commonly used in connection with data describing sports, temperature, rainfall, stress and reliability.

When analyzing multivariate data there is no longer a commonly agreed upon definition and the simplicity resulting from the fact that $\mathbb{R}$ is totally ordered is lost. Consider $\mathbb{R}^2$ -valued observations $\{\mathbf{X}_n\} = \{(X_n^{(1)}, X_n^{(2)})'\}$. For such data, we can define the notion of record in many ways and the following are all plausible and interesting possibilities:

$$(a) \mathbf{X}_n \text{ is a record if } X_n^{(p)} > \bigvee_{i=1}^{n-1} X_i^{(p)}, \quad p = 1 \text{ and } 2.$$

$$(b) \mathbf{X}_n \text{ is a record if } X_n^{(p)} > \bigvee_{i=1}^{n-1} X_i^{(p)}, \quad p = 1 \text{ and } 2.$$

$$(c) \mathbf{X}_n \text{ is a record if } \mathbf{X}_n \text{ falls outside the convex hull of } \mathbf{X}_1, \dots, \mathbf{X}_{n-1}.$$

Definitions (b) and (c) are related as (c) is the infinite dimensional version of (b). One can see this by expressing the definition (c) in terms of the support function of a convex set. Mathematical analysis of (b) and (c) is more difficult than (a) because the definition (a) can be expressed in terms of a partial order, namely that $\mathbf{X}_n > \mathbf{X}_i$,

$i = 1, \dots, n-1$, where the inequality between vectors is interpreted componentwise.

In section 2 we review some results about records defined according to (a). This material is taken from Goldie and Resnick (1989) where records on general partially ordered sets are considered. For simplicity in this paper we only consider $\mathbb{R}^2$ -valued observations. In particular we need to review when an iid sequence has a finite or infinite number of records.

If $\mathbf{X}_n$ is a record, then previous observations must be located southwest of $\mathbf{X}_n$ in the plane and in this sense $\mathbf{X}_n$ sticks out. Another interpretation is that if a right angled template is lowered from ∞ until it contacts the sample, $\mathbf{X}_1, \dots, \mathbf{X}_n$, then this template contacts the sample in one point iff $\mathbf{X}_n$ is a record. Thus we arrive at the idea of probing the sample with various geometric objects (cf. Groeneboom, 1987) and relating the shape of the convex hull of the sample to records. This idea is explored in the last sections where the

fact that a multivariate normal sequence has a finite number of records is used to prove that the boundary of the convex hull of a normal sample becomes smooth as $n \rightarrow \infty$ in the sense that the exterior angles at vertices of the convex hulls converge to π uniformly.

It is rather well known that the shape of a normal sample becomes ellipsoidal (Geoffroy, 1961; Fisher, 1969; Davis, Mulrow, Resnick, 1987) and hence the boundary becomes smooth. Using only records, it does not seem possible to obtain detailed information on the shape of the sample beyond what happens on the boundary of the convex hull.

2. MULTIVARIATE RECORDS

As in the introduction let $\{X_n = (X_n^{(1)}, X_n^{(2)})', n \geq 1\}$ be iid $\mathbb{R}^2$ -valued random variables. X_n is a record if

$$X_n > X_j, \quad j = 1, \dots, n-1$$

or equivalently

$$X_n > \bigvee_{j=1}^{n-1} X_j$$

where $(x_1, x_2)' \vee (y_1, y_2)' = (x_1 \vee y_1, x_2 \vee y_2)'$. As a convention, make X_1 a record.

The first order of business is to decide when the sequence has an infinite number of records so define the counting variables

$$N(A) = \sum_{n=1}^{\infty} 1_{[X_n \in A, X_n \text{ is a record}]}$$

for $A \in \mathcal{B}(\mathbb{R}^2)$. Let $N = N(\mathbb{R}^2)$ be the total number of records in the sequence. From the Hewitt–Savage 0–1 Law, N is finite or infinite with probability 1 and either case can occur. For example, if the bivariate distribution of X_1 is a product of two continuous distributions so that for each n we have $X_n^{(1)}$ and $X_n^{(2)}$ are independent random variables then

$$EN = \sum_{n=1}^{\infty} P[X_n \text{ is a record}]$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} P[X_n^{(1)} > \bigvee_{j=1}^{n-1} X_j^{(1)}] P[X_n^{(2)} > \bigvee_{j=1}^{n-1} X_j^{(2)}] \\
&= \sum_{n=1}^{\infty} (1/n)(1/n) = \sum_{n=1}^{\infty} 1/n^2 < \infty
\end{aligned}$$

so that $P[N < \infty] = 1$. On the other hand if the distribution of $\mathbf{X}_1$ concentrates on the diagonal so that $P[X_n^{(1)} = X_n^{(2)}] = 1$ for $n \geq 1$, and the distribution of $X_n^{(1)}$ is continuous then $EN = \sum_{n=1}^{\infty} P[X_n^{(1)} > \bigvee_{j=1}^{n-1} X_j^{(1)}] = \sum_{n=1}^{\infty} (1/n) = \infty$ and from one dimensional considerations it is clear $P[N = \infty] = 1$.

In general we have for $A \in \mathcal{B}(\mathbb{R}^2)$

$$EN(A) = \sum_{n=1}^{\infty} P[\mathbf{X}_n \in A, \mathbf{X}_n \text{ is a record}]$$

and if we denote the distribution of $\mathbf{X}_1$ by F we obtain

$$\begin{aligned}
EN(A) &= \sum_{n=1}^{\infty} \int_A F(d\mathbf{x}) P\left\{ \bigcap_{j=1}^{n-1} [\mathbf{X}_j < \mathbf{x}] \right\} \\
&= \sum_{n=1}^{\infty} \int_A F(d\mathbf{x}) (F(-\infty, \mathbf{x}))^{n-1} \\
&= \sum_{n=1}^{\infty} \int_A \frac{F(d\mathbf{x})}{F((-\infty, \mathbf{x})^c)} =: H(A) .
\end{aligned}$$

The criterion for when the number of records is finite is in terms of H .

THEOREM 2.1. (Goldie and Resnick, 1989). We have $P[N(A) < \infty] = 1$ or $P[N(A) = \infty] = 1$ according as $H(A) < \infty$ or $H(A) = \infty$.

The converse half of this result is proved using a standard converse to the Borel–Cantelli lemma.

It is not easy to calculate the measure H explicitly. Cases where this can be done are when (i) components are independent or (ii) the components are totally dependent so that effectively the data are one dimensional. In general, it is a helpful heuristic to think of

$$H(d\mathbf{x}) \approx P[\text{some record occurs at } d\mathbf{x}] .$$

Because H is difficult to calculate, examples in $\mathbb{R}^2$ are best checked by not computing H but rather using a notion of asymptotic independence. Earlier we saw that independent components resulted in a finite number of records but in fact asymptotic independence in the following sense is all that is necessary. Let the common distribution of $\{\mathbf{X}_n\}$ be $F(x, y)$. Then we say F is in the domain of attraction of a multivariate extreme value distribution G if there exist $a_n^{(i)} > 0$, $b_n^{(i)} \in \mathbb{R}$, $n \geq 1$, $i = 1, 2$ such that

$$P\left(\bigvee_{j=1}^n X_j^{(i)} - b_n^{(i)} / a_n^{(i)} \leq x_i, i = 1, 2\right) = F^n(a_n^{(1)}x_1 + b_n^{(1)}, a_n^{(2)}x_2 + b_n^{(2)}) \rightarrow G(x_1, x_2)$$

(cf. Resnick, 1987). Then

THEOREM 2.2. (Goldie and Resnick, 1989). N is finite or infinite with probability 1 according as G is or is not a product measure.

Criteria based on F which guarantee G is a product measure are well known (Resnick, 1987). Using these, there is a surprising result of Sibuya (1960) (cf. Resnick, 1987) that states that the bivariate normal distribution is in a domain of attraction of a bivariate extreme value distribution which is a product measure, provided the correlation ρ of the bivariate normal density is not equal to -1 . Thus we get the next result.

COROLLARY 2.3. (Goldie and Resnick, 1989). If $\{\mathbf{X}_n\}$ is iid from a bivariate normal whose correlation ρ satisfies $\rho \neq -1$ then $\{\mathbf{X}_n\}$ has a finite number of records:

$$P[N < \infty] = 1.$$

Further properties of multivariate records are given in Goldie and Resnick (1989) including expressions for the moments and Laplace functional of $N(A)$, $A \in \mathcal{B}(\mathbb{R}^2)$ and an expression for the fundamental quantity $Q(A) = P[N(A) = 0]$. In terms of Q , the Markovian structure of the records falling in the set A can be discussed and the transition probabilities characterized. A partial independent increment property for the point process $N(\cdot)$ is also given.

3. GEOMETRICAL INTERPRETATION

Let H_n be the convex hull of $X_1, \dots, X_n$. Whether or not X_n is a record (i.e., whether or not $X_n > \bigvee_{i=1}^{n-1} X_i$) can be examined graphically in the following way.

Consider a right angled template which has edges of infinite length and whose bisecting line has inclination $\pi/4$ with the positive x -axis. Suppose the template is moving down from $\infty = (\infty, \infty)$ to $-\infty = (-\infty, -\infty)$ without changing the inclination and that it cannot pass any point of H_n . The template stops when it hits H_n . If the sample is from a continuous distribution, there are only two possible outcomes: Either the template hits H_n at one point or two points. These possibilities are shown in Figure 3.1.

Template hits H_n at one point

Template hits H_n at two points

FIGURE 3.1

Notice that no matter how the template approaches H_n , it hits H_n at the same point (or points). Consider the situations in Figure 3.2 and 3.3. In Figure 3.2, the template cannot move down, but still can move to the left until it hits X_2 . Figure 3.3 shows the opposite case and the template eventually hits H_n at X_1 and X_2 in both cases.

Remember that the only restriction when we move the template is to keep the inclination $\pi/4$. Alternatively, this procedure may be viewed as moving a horizontal line down until it contacts the sample and then moving a vertical line to the left until the sample is encountered. Do the lines meet the sample at different points or at the same point?

FIGURE 3.2

FIGURE 3.3

It is then clear that

$$X_n \text{ is a record iff the template hits } H_n \text{ only at } X_n \quad (3.1)$$

and

$$[\text{Template hits } H_n \text{ at two points}] = \bigcap_{j=1}^n [X_j > \bigvee_{\substack{i=1 \\ i \neq j}}^n X_i]^c. \quad (3.2)$$

Now we consider more general templates. We denote by $\text{Temp}(\theta, \varphi)$ the template with the exterior angle θ ($\pi < \theta \leq 3\pi/2$) and whose bisecting line makes angle φ ($0 \leq \varphi < 2\pi$) with the positive x -axis. (See Figure 3.4). The template used in the previous discussion is now denoted $\text{Temp}(3\pi/2, \pi/4)$. Then bringing down (or up depending on φ) $\text{Temp}(\theta, \varphi)$ from the point at infinity in the direction φ , we again see that $\text{Temp}(\theta, \varphi)$ hits H_n at either one point or two points. We can use this fact to define another record property as follows.

DEFINITION 3.1. X_n is a (θ, φ) -record iff $\text{Temp}(\theta, \varphi)$ hits H_n only at X_n .

FIGURE 3.4

FIGURE 3.5

Let $\pi < \theta_1 < \theta_2 \leq 3\pi/2$, then it follows from the definition that if X_n is a (θ_2, φ) -record, then it is always a (θ_1, φ) -record and that X_n can possibly be a (θ_1, φ) -record even though it is not a (θ_2, φ) -record. For example, in Figure 3.5, X_n is not a $(3\pi/2, \pi/4)$ -record (or simply a record) since $\text{Temp}(3\pi/2, \pi/4)$ hits H_n at X_j and X_k , but it is a $(\theta, \pi/4)$ -record since $\text{Temp}(\theta, \pi/4)$ hits H_n only at X_n .

The following proposition explains the relation between these two records.

PROPOSITION 3.2. X_n is a (θ, φ) -record ($\pi < \theta \leq 3\pi/2$, $0 \leq \varphi < 2\pi$) iff Y_n is a record among $Y_1, \dots, Y_n$ where $Y_j = S_\theta \cdot T_\varphi \cdot X_j$, $j = 1, 2, \dots, n$,

$$T_\varphi = \begin{bmatrix} \cos(\frac{\pi}{4} - \varphi) & -\sin(\frac{\pi}{4} - \varphi) \\ \sin(\frac{\pi}{4} - \varphi) & \cos(\frac{\pi}{4} - \varphi) \end{bmatrix},$$

$$S_\theta = \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix},$$

$$\alpha = \tan\left(\frac{3}{4}\pi - \frac{\theta}{2}\right).$$

PROOF. Observe by trigonometry that

$$T_{\varphi} \begin{bmatrix} r \cos t \\ r \sin t \end{bmatrix} = \begin{bmatrix} r \cos (t + \pi/4 - \varphi) \\ r \sin (t + \pi/4 - \varphi) \end{bmatrix}$$

so that the mapping T_{φ} rotates points in the plane through an angle $\pi/4 - \varphi$. Let

$$\mathbf{Z}_j = T_{\varphi} \mathbf{X}_j.$$

It follows that $\mathbf{X}_n$ is a (θ, φ) -record iff $\mathbf{Z}_n$ is a $(\theta, \pi/4)$ -record. Thus it suffices to show that

$$\mathbf{Z}_n \text{ is a } (\theta, \pi/4)\text{-record iff } \mathbf{Y}_n \text{ is a record.} \quad (3.3)$$

Note (3.3) holds when $\theta = 3\pi/2$ since in this case $\mathbf{Y}_j = (Z_j^{(2)}, Z_j^{(1)})'$, $j = 1, 2, \dots, n$.

So assume $\pi < \theta < 3\pi/2$, which implies $0 < \alpha < 1$. The two edges of Temp $(\theta, \pi/4)$ are given by the lines (see Figure 3.6)

$$\begin{aligned} y - Z_n^{(2)} &= -\alpha(x - Z_n^{(1)}) && \text{for } x \geq Z_n^{(1)}, \text{ and} \\ y - Z_n^{(2)} &= -\frac{1}{\alpha}(x - Z_n^{(1)}) && \text{for } x < Z_n^{(1)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\{\mathbf{Z}_n \text{ is a } (\theta, \pi/4)\text{-record}\} \\ &= \bigcap_{j=1}^{n-1} \{Z_j^{(2)} - Z_n^{(2)} < -\alpha(Z_j^{(1)} - Z_n^{(1)}), Z_j^{(2)} - Z_n^{(2)} < -\frac{1}{\alpha}(Z_j^{(1)} - Z_n^{(1)})\} \\ &= \bigcap_{j=1}^{n-1} \{\alpha Z_j^{(1)} + Z_j^{(2)} < \alpha Z_n^{(1)} + Z_n^{(2)}, Z_j^{(1)} + \alpha Z_j^{(2)} < Z_n^{(1)} + \alpha Z_n^{(2)}\} \\ &= \bigcap_{j=1}^n \left\{ S_{\theta} \begin{bmatrix} Z_j^{(1)} \\ Z_j^{(2)} \end{bmatrix} < S_{\theta} \begin{bmatrix} Z_n^{(1)} \\ Z_n^{(2)} \end{bmatrix} \right\} \\ &= \bigcap_{j=1}^{n-1} \{Y_j^{(1)} < Y_n^{(1)}, Y_j^{(2)} < Y_n^{(2)}\} \\ &= \{\mathbf{Y}_n \text{ is a record}\} \end{aligned}$$

so that (3.3) holds. □

FIGURE 3.6

4. APPLICATION TO A RANDOM SAMPLE FROM A BIVARIATE NORMAL DISTRIBUTION

Let $\mathbf{X}_n = (X_n^{(1)}, X_n^{(2)})'$ be an iid sequence of random vectors in $\mathbb{R}^2$ from the bivariate normal distribution. We assume $E\mathbf{X}_n = 0$ and that the correlation ρ of $\mathbf{X}_1$ satisfies $|\rho| < 1$. Define events as follows:

$$\begin{aligned} A_n &= \{\mathbf{X}_n \text{ is a record}\} \\ &= \{X_n^{(i)} > \bigvee_{j=1}^{n-1} X_j^{(i)}, i = 1, 2\} \end{aligned}$$

$$\begin{aligned} A_n^{(i)} &= \{X_n^{(i)} \text{ is a record}\} \\ &= \{X_n^{(i)} > \bigvee_{j=1}^{n-1} X_j^{(i)}\}, i = 1, 2. \end{aligned}$$

$$B_n = \{\text{Temp}(3\pi/2, \pi/4) \text{ hits } H_n \text{ at one point}\}$$

$$B_n(\theta, \varphi) = \{\text{Temp}(\theta, \varphi) \text{ hits } H_n \text{ at one point}\}.$$

Notice that $A_n = A_n^{(1)} \cap A_n^{(2)}$ and $A_n \subset B_n$. We also define

$$N = \sum_{n=1}^{\infty} 1_{A_n}, \text{ and}$$

$$N_i = \sum_{n=1}^{\infty} 1_{A_n^{(i)}}, \quad i = 1, 2.$$

It follows from Corollary 2.3 and one dimensional theory (Resnick, 1987, p. 169) that

$$P(N < \infty) = 1 \quad (4.1)$$

$$P(N_1 = \infty) = P(N_2 = \infty) = 1. \quad (4.2)$$

LEMMA 4.1. Temp $(3\pi/2, \pi/4)$ hits H_n at two points for all large n , i.e., $P(B_n \text{ i.o.}) = 0$.

PROOF. We have (4.1) and (4.2) are respectively equivalent to

$$P(A_n \text{ i.o.}) = 0, \text{ and} \quad (4.3)$$

$$P(A_n^{(1)} \text{ i.o.}) = P(A_n^{(2)} \text{ i.o.}) = 1. \quad (4.4)$$

These imply that

$$1 = P(\liminf_{n \rightarrow \infty} A_n^c \cap \limsup_{n \rightarrow \infty} A_n^{(1)}).$$

Now pick $\omega \in \liminf_{n \rightarrow \infty} A_n^c \cap \limsup_{n \rightarrow \infty} A_n^{(1)}$ and there exists some m such that

$\omega \in A_m^{(1)} (A_m^{(2)})^c$ and $\omega \in A_n^c$ for any $n \geq m$. If there exists some $n \geq m$ such that for some $\ell \leq n$

$$X_\ell > \bigvee_{\substack{i=1 \\ i \neq \ell}}^n X_i(\omega)$$

then if $\ell < m$, $X_m^{(1)}(\omega)$ would not be a record (a contradiction to $\omega \in A_m^{(1)}$) and if

$n \geq \ell \geq m$, $X_\ell(\omega)$ is a record (a contradiction to $\omega \in A_\ell^c$). From (3.2) we conclude

$\omega \in B_n^c$, $n \geq m$, whence

$$1 = P[\liminf_{n \rightarrow \infty} A_n^c \cap \limsup_{n \rightarrow \infty} A_n^{(1)}] \leq P[\liminf_{n \rightarrow \infty} B_n^c]$$

which gives the desired result. $\square$

LEMMA 4.2. For any θ ($\pi < \theta \leq 3\pi/2$) and φ ($0 \leq \varphi < 2\pi$), $\text{Temp}(\theta, \varphi)$ hits H_n at two points for all large n a.s.; i.e.,

$$P(B_n(\theta, \varphi) \text{ i.o.}) = 0 .$$

PROOF. Let X_n, Y_n, Z_n be as in Proposition 3.2. Let $H(Y_1, \dots, Y_n)$ be the convex hull of the indicated points. As in Proposition 3.2

$$[\text{Temp}(\theta, \varphi) \text{ hits } H_n \text{ at two points}]$$

$$= [\text{Temp}(3\pi/2, \pi/4) \text{ hits } H(Y_1, \dots, Y_n) \text{ at two points}].$$

Since $\{Y_n\}$ are iid normal, the desired result is a direct application of Lemma 4.1 provided $\rho_Y = \text{Corr}(Y_1^{(1)}, Y_1^{(2)}) < 1$. This is readily checked as follows: Since $\alpha < 1$, $\det S_\theta = \alpha^2 - 1 \neq 0$ so S_θ is invertible. The rotation T_φ is also invertible. Since $|\rho| = |\text{corr}(X_1^{(1)}, X_1^{(2)})| < 1$ the same is true for ρ_Y , else there is an a.s. linear relationship between $Y_1^{(1)}$ and $Y_1^{(2)}$ (see Brockwell and Davis, 1987, p. 43) which translates into an a.s. linear relationship between $X_1^{(1)}$ and $X_1^{(2)}$ which violates $|\rho| < 1$. $\square$

For any sequences of events $\{C_n\}, \{D_n\}$ we have

$$\limsup_{n \rightarrow \infty} C_n \cup D_n = \limsup_{n \rightarrow \infty} C_n \cup \limsup_{n \rightarrow \infty} D_n$$

and we therefore obtain from Lemma 4.2 the following result.

LEMMA 4.3. For θ_j ($\pi < \theta_j \leq 3\pi/2$) and φ_j ($0 \leq \varphi_j < 2\pi$), $j = 1, 2, \dots, m$, all the m templates $\text{Temp}(\theta_j, \varphi_j)$, $j = 1, 2, \dots, m$ eventually hit H_n at two points for all large n a.s.; i.e.,

$$P\left(\bigcup_{j=1}^m B_n(\theta_j, \varphi_j) \text{ i.o.}\right) = 0 .$$

Next consider a vertex v of H_n and $\text{Temp}(\theta, \varphi)$. Let ψ be the exterior angle of v and ϕ be the inclination between the bisecting line of v and the x -axis. (See Figure 4.1). Then we see that

$$\text{Temp}(\theta, \varphi) \text{ hits } H_n \text{ only at } v \text{ iff } \psi > \theta > \pi \text{ and } |\phi - \varphi| < \frac{\psi - \theta}{2}. \quad (4.5)$$

We denote by V_n the number of vertices of H_n and further define $v_{n,j}$, $\psi_{n,j}$, and $\phi_{n,j}$, $j = 1, 2, \dots, V_n$ as the vertices of H_n , the exterior angles, and the inclinations, respectively. (See Figure 4.2). The next proposition shows that the exterior angles converge to π uniformly.

FIGURE 4.1

FIGURE 4.2

PROPOSITION 4.4. We have for a bivariate normal sample whose correlation ρ satisfies $|\rho| < 1$:

$$\lim_{n \rightarrow \infty} \frac{1}{V_n} \sum_{j=1}^{V_n} \psi_{n,j} = \pi \text{ a.s.}$$

PROOF. Pick any $\epsilon > 0$. Then since $\psi_{n,j} > \pi$ for any j , it suffices to show

$$\frac{1}{V_n} \sum_{j=1}^{V_n} \psi_{n,j}(\omega) \leq \pi + \epsilon, \quad n \geq n_0(\omega). \quad (4.6)$$

Choose m large enough so that $m > 4\pi/\epsilon$. Then for any $\phi_{n,j}$, there exists some $\ell(j) \in \{1, 2, \dots, m\}$ such that

$$|\phi_{n,j} - \frac{2\pi \cdot \ell(j)}{m}| < \frac{\epsilon}{4}. \quad (4.7)$$

For m so chosen, we take m templates

$$\{\text{Temp}(\pi + \frac{\epsilon}{2}, \frac{2\pi}{m}j), 1 \leq j \leq m\}$$

and it follows from Lemma 4.3 that

$$P(\bigcup_{N \leq n} \bigcap_{j=1}^m \{\text{Temp}(\pi + \frac{\epsilon}{2}, \frac{2\pi}{m}j) \text{ hits } H_n \text{ at two points}\}) = 1. \quad (4.8)$$

Suppose for $n \geq N(\omega)$

$$\omega \in \bigcap_{j=1}^m [\text{temp}(\pi + \frac{\epsilon}{2}, \frac{2\pi}{m}j) \text{ hits } H_n \text{ at two points}].$$

For any $j \leq V_n$ there exists $\ell(j, \omega)$ as in (4.7), whence from (4.5) either

$$\left\{ \begin{array}{l} \psi_{n,j} \leq \pi + \frac{\epsilon}{2} \quad \text{or} \\ |\phi_{n,j}(\omega) - \frac{2\pi \ell(j, \omega)}{m}| > \frac{\psi_{n,j} - (\pi + \frac{\epsilon}{2})}{2} \end{array} \right.$$

and therefore from (4.7)

$$\frac{\psi_{n,j} - (\pi + \frac{\epsilon}{2})}{2} < \frac{\epsilon}{4},$$

whence

$$\psi_{n,j} < \pi + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \pi + \epsilon$$

giving (4.6) as desired. $\square$

COROLLARY 4.5. For a bivariate normal sample whose correlation ρ satisfies $|\rho| < 1$ we have

$$V_n \rightarrow \infty \text{ a.s.}$$

PROOF. We have by Proposition 4.4 that for any $\epsilon > 0$, there exists n_0 such that for any $n > n_0$,

$$\max_{1 \leq j \leq V_n} \psi_{n,j} < \pi + \epsilon \text{ a.s.} \quad (4.9)$$

We also have

$$\sum_{j=1}^{V_n} \psi_{n,j} = (V_n + 2)\pi \text{ a.s.} \quad (4.10)$$

It follows from (4.9) and (4.10) that

$$(V_n + 2)\pi \leq V_n(\pi + \epsilon) \text{ a.s.}$$

This implies

$$V_n > \frac{2\pi}{\epsilon} \text{ for any } n \geq n_0 \text{ a.s.} \quad \square$$

REFERENCES

- Brockwell, P. and Davis, R. (1987). Time Series: Theory and Methods. Springer-Verlag, New York.
- Davis, R., Mulrow, E. and Resnick, S. (1988). Almost sure limit sets of random samples in $\mathbb{R}^d$. Adv. Appl. Prob.
- Fisher, L. (1969). Limiting sets and convex hulls of samples from product measures. Ann. Math. Statist. 40, 1824–1832.
- Geffroy, J. (1961). Localisation asymptotique du polyedre d'appui d'un echantillon laplacien a k dimensions. Publ. Inst. Statist. Univ. Paris. X, 212–228.
- Goldie, C.M. and Resnick, S. (1989). Records in a partially ordered set. Forthcoming: Ann. Prob.
- Groeneboom, P. (1987). Limit theorems for convex hulls. Report 87–07, Mathematics Institute, University of Amsterdam. Forthcoming: Probability Theory and Related Fields.
- Resnick, S. (1987). Extreme Values, Regular Variation, and Point Processes. Springer-Verlag, New York.
- Sibuya, M. (1960). Bivariate extreme statistics. Ann. Inst. Stat. Math. 11, 195–210.