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SUBEXPONENTIAL DISTRIBUTION TAILS AND
POINT PROCESSES

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ABSTRACT

The condition that a distribution function $F(x)$ be both subexponential and in the domain of attraction of $\exp\{-e^{-x}\}$ is natural when considering extreme value behavior of linear combinations of iid random variables with distribution F . We give interpretations of this condition by means of point processes.

Key words: subexponential distribution, extreme values, point processes, convolution.

1. INTRODUCTION.

We will give a probabilistic interpretation by means of point processes of what it means for iid random variables to have their common distribution both in the domain of attraction of the double exponential extreme value distribution and also in the subexponential class. These assumptions are made more precise as follows:

A distribution function F is in the domain of attraction of the extreme-value distribution $\Lambda(x) = \exp\{-e^{-x}\}$, $x \in \mathbb{R}$ if there exist $a_n > 0$, $b_n \in \mathbb{R}$ ($n \geq 1$) and

$$(1.1) \quad \lim_{n \rightarrow \infty} F^n(a_n x + b_n) = \Lambda(x), \quad x \in \mathbb{R}.$$

If $\{X_n, n \geq 1\}$ are iid with distribution F this can be rephrased

$$\lim_{n \rightarrow \infty} P\left[\bigvee_{i=1}^n X_i \leq a_n x + b_n\right] = \Lambda(x), \quad x \in \mathbb{R}.$$

In this paper it will always be the case that $x_0 := \sup\{x : F(x) < 1\} = \infty$. When (1.1) holds we write $F \in D(\Lambda)$. Cf. de Haan (1970), Resnick (1987).

A distribution F is in the class $S_r(\gamma)$ for $\gamma \geq 0$ if $F(x) < 1$ for all x and

$$(1.2) \quad \lim_{x \rightarrow \infty} (1 - F(x - y)) / (1 - F(x)) = e^{\gamma y}, \quad \text{for each } y \in \mathbb{R},$$

and

$$(1.3) \quad \lim_{x \rightarrow \infty} (1 - F * F(x)) / (1 - F(x)) = \lim_{x \rightarrow \infty} P[X_1 + X_2 > x] / P[X_1 > x] = d.$$

The constant d is known to equal $2E\exp\{\gamma X_1\}$ which was proved for the case $F(0) = 0$ by Chover *et al* (1973) (see also Cline, 1986b; Embrechts, 1984) and extended to the case that F concentrates on $\mathbb{R}$ by Willekens (1986). When $F(0) = 0$ and $d = 2$, (1.3) implies (1.2) with $\gamma = 0$ (Chistyakov, 1964), and in this case the class $S_r(0)$ is called in the literature S , the subexponential class. For our purposes it is not natural to restrict attention to distributions concentrating on $[0, \infty)$. Note that the exponential distribution does not satisfy (1.3).

So we consider distributions F satisfying

$$F \in D(\Lambda) \cap S_r(\gamma).$$

The condition $F \in D(\Lambda)$ is necessary for a sequence of point processes based on $\{(X_k - b_n)/a_n, k \geq 1\}$ to have good convergence properties. The condition $F \in S_r(\gamma)$ is necessary for the tail of $X_1 + X_2$ to be comparable to $1 - F$ and means that sequences of point processes based on various linear combinations of the $\{X_n, n \geq 1\}$ will have interesting convergence properties.

The condition $F \in D(\Lambda) \cap S_r(\gamma)$ was discussed in Goldie and Resnick (1988) where various sufficient conditions on F were summarized and examples discussed of $F \in D(\Lambda) \cap S_r(\gamma)$. See also the important paper by Cline (1986a).

We are interested in giving a probabilistic interpretation of the "2" appearing in $d = 2E\exp\{\gamma X_1\}$ given after (1.3). A second goal is to develop machinery for considering the asymptotic behavior of various functionals of possibly infinite moving average processes of the form $\{\sum_{j=-\infty}^{\infty} c_{n-j} X_j, n \geq 1\}$ where $\{c_j\}$ are given constants satisfying suitable summability conditions. Such processes receive attention in time series contexts and related problems arise in regression analysis (Boos, 1986; Cline, 1983) where the error variables have a distribution $F \in D(\Lambda)$ and one wishes to control probabilities of large errors in the spirit of Cline and Brockwell (1985). Results of the present paper are applied in Davis and Resnick (1988) to moving average processes. We emphasize that some of the results proven in Section 2 have fairly straightforward analytic proofs. However, the processes dealt with in Section 2 are a natural class to consider in order to build up methodology and experience in handling point processes based on linear combinations of iid random variables whose distribution satisfies (1.1) – (1.3). The point process approach has proven invaluable in studying a wide variety of functionals of random variables with regularly varying tails (Resnick, 1986; Davis and Resnick, 1985a,b, 1986; Rootzen 1978, 1983, 1986) and the same should be true for the exponential-like tail case.

Work related to ours, and one of the strong stimuli for our investigations is by Rootzen (1983, 1986), who considers a special class of densities in $D(\Lambda) \cap S_r(\gamma)$. See also Finster (1982).

2. POINT-PROCESS INTERPRETATIONS

We now consider point-process interpretations of the condition $F \in D(\Lambda) \cap S_r(\gamma)$.

For background on point processes see Neveu (1976), Kallenberg (1983), Resnick (1987). See Resnick (1986, 1987) for discussions emphasizing the relevance of point processes to limit theory. Applications of the point-process method also appear in Davis and Resnick (1985a,b, 1986, 1988). We set $M_p(E)$ equal to the space of point measures on the LCCB space E and metrize $M_p(E)$ by the vague metric. A point measure on E is a Radon measure on E of the form $\sum_i \varepsilon_{x_i}$ where $x_i \in E$ and for a Borel subset $B \subset E$ we have $\varepsilon_x(B) = 1$ if $x \in B$; 0 otherwise. A Poisson process on E with mean measure μ will be denoted $PRM(\mu)$; i.e. Poisson random measure with mean measure μ .

Let F and G be two distributions on $\mathbb{R}$ with

$$(2.1) \quad F \in D(\Lambda) \cap S_r(\gamma)$$

and

$$(2.2) \quad \lim_{x \rightarrow \infty} \bar{G}(x)/\bar{F}(x) = c, \quad 0 \leq c < \infty.$$

where $\bar{F} = 1 - F$, $\bar{G} = 1 - G$. We will compare $1 - F * G$ with $1 - F$ using point-process methods. Direct analytic methods have been considered (Embrechts and Goldie, 1980; Cline, 1986a,b), but the following provides alternative insights.

Suppose $\{X_n, n \geq 1\}$ are iid with common distribution F , $\{Y_n, n \geq 1\}$ are iid with common distribution G and $\{X_n\}$ and $\{Y_n\}$ are independent. Suppose a_n, b_n satisfy (1.1) so that

$$(2.3) \quad n\bar{F}(a_n x + b_n) \rightarrow e^{-x}, \quad x \in \mathbb{R},$$

$$(2.4) \quad n\bar{G}(a_n x + b_n) \rightarrow ce^{-x}, \quad x \in \mathbb{R}.$$

We have $b_n \rightarrow \infty$ and from Lemma 2.1 of Goldie and Resnick (1988) we have $a_n \sim \gamma^{-1}$.

For what follows, we have need of compact spaces punctured by the removal of one point. In such spaces, the compact sets are those sets closed in the original topology before puncturing and bounded away from the deleted point. For instance in the punctured

compactified plane $[-\infty, \infty]^2 \setminus \{(-\infty, -\infty)\}$, the compact sets are the closed sets away from $(-\infty, -\infty)$. Similarly in one dimension $(-\infty, \infty] = [-\infty, \infty] \setminus \{-\infty\}$ and compact sets are closed subsets of $[a, \infty]$ for some $a > -\infty$.

We proceed by means of a series of lemmas supposing for each that (2.1) – (2.3) hold. The approach is to get a basic result about convergence of a sequence of point processes and then get to the desired result by means of a series of transformations on the basic result. To get the basic convergence result we rely on Proposition 3.21 of Resnick (1987) for which the following lemma is necessary.

Lemma 2.1. The sequence of measures

$$nP[a_n^{-1}(X_1 - b_n, Y_1 - b_n, Y_1, X_1) \in (dx, dy, dz, dw)]$$
 converges vaguely on $([-\infty, \infty]^2 \setminus \{(-\infty, -\infty)\}) \times ([-\infty, \infty]^2)$ to the limit measure

$$m_\infty(dx, dy, dz, dw) = \int e^{-x} dx \varepsilon_\infty(dy) G(\gamma^{-1} dz) \varepsilon_\infty(dw) + \varepsilon_\infty(dx) c e^{-y} dy \varepsilon_\infty(dz) F(\gamma^{-1} dw)$$

which concentrates on

$$(\mathbb{R} \times \{-\infty\} \times \mathbb{R} \times \{\infty\}) \cup (\{-\infty\} \times \mathbb{R} \times \{\infty\} \times \mathbb{R}).$$

If $c = 0$, the second term in the limit measure vanishes and the limit measure concentrates on $\mathbb{R} \times \{-\infty\} \times \mathbb{R} \times \{\infty\}$. If $\gamma = 0$ interpret $G(\gamma^{-1} dz)$ and $F(\gamma^{-1} dz)$ as $\varepsilon_0(dz)$.

Proof. Observe first that for any $(x, y) \in \mathbb{R}^2$ we have

$$nP[X_1 > a_n x + b_n, Y_1 > a_n y + b_n] \sim e^{-x} \bar{G}(a_n y + b_n) \rightarrow 0$$

as $n \rightarrow \infty$ and furthermore for any $w \in \mathbb{R}$

$$nP[X_1 > a_n x + b_n, X_1 \leq a_n w] = 0$$

for n large as a consequence of $b_n/a_n \rightarrow \infty$, the latter convergence resulting from $F \in D(\Lambda)$ (de Haan, 1970; Resnick, 1987). This indicates why the support of any limit of the sequence of measures in the statement of the lemma must be as described. For any $(x, y, z, w) \in \mathbb{R}^4$ we have

$$\begin{aligned} & nP[X_1 > a_n x + b_n, X_1 > a_n w, Y_1 \leq a_n y + b_n, Y_1 \leq a_n z] \\ &= nP[X_1 > a_n x + b_n, Y_1 \leq a_n z] \end{aligned}$$

for large n . However

$$\begin{aligned} nP[X_1 > a_n x + b_n, Y_1 \leq a_n z] &= n\bar{F}(a_n x + b_n)G(a_n z) \\ &\sim e^{-x}G(\gamma^{-1}z) = m_\infty\{(x, \infty] \times [-\infty, y] \times [-\infty, z] \times (w, \infty]\} \end{aligned}$$

from (2.3) and $a_n = \gamma^{-1}$. In a similar way, for large n ,

$$\begin{aligned} nP[X_1 \leq a_n x + b_n, X_1 \leq a_n w, Y_1 > a_n y + b_n, Y_1 > a_n z] \\ = nP[X_1 \leq a_n w, Y_1 > a_n y + b_n] = F(a_n w) n\bar{G}(a_n y + b_n) \\ \rightarrow ce^{-y} F(\gamma^{-1} w) = m_\infty\{[-\infty, x] \times (y, \infty] \times (z, \infty] \times [-\infty, w]\}. \end{aligned}$$

Likewise one can show convergence on regions of the form

- (1) $(x, \infty] \times [-\infty, y] \times [-\infty, z] \times [-\infty, w]$
- (2) $(x, \infty] \times [-\infty, y] \times (z, \infty] \times (w, \infty]$
- (3) $[-\infty, x] \times (y, \infty] \times [-\infty, z] \times [-\infty, w]$
- (4) $[-\infty, x] \times (y, \infty] \times (z, \infty] \times (w, \infty]$
- (5) $(x, \infty] \times (y, \infty] \times [-\infty, \infty]^2$

for $(x, y, z, w) \in \mathbb{R}^4$; the limit measure assigns zero mass to each of these regions.

□

Consider two independent identically distributed Poisson processes on $(-\infty, \infty]$ with mean measure $e^{-x} dx$ which we denote by $\sum_k \varepsilon_{j_k^{(i)}}$ for $i = 1, 2$. (No ordering of the points is

implied by this notation.) If $c > 0$,

$$\sum_k \varepsilon_{j_k^{(2)}} + \log c$$

is PRM($ce^{-y} dy$); if $c = 0$ we take this point process to be identically 0. Suppose $\{X(k)\}$, $\{Y(k)\}$ are two independent iid sequences which are also independent of the two Poisson processes just specified. The common distribution of $\{X(k)\}$ is F and that of $\{Y(k)\}$ is G .

Lemma 2.2. We have

$$\begin{aligned} (2.5) \quad & \left[\sum_{k=1}^n \varepsilon_{(a_n^{-1}(X_k - b_n, Y_k))}, \sum_{k=1}^n \varepsilon_{(a_n^{-1}(Y_k - b_n, X_k))} \right] \\ & \Rightarrow \left[\sum_k \varepsilon_{(j_k^{(1)}, \gamma Y(k))}, \sum_k \varepsilon_{(j_k^{(2)} + \log c, \gamma X(k))} \right] \end{aligned}$$

in $(M_p((-\infty, \infty] \times [-\infty, \infty]))^2$. If $c = 0$ the second component of the limit is identically 0.

Proof: Let $f_i: (-\infty, \infty] \times [-\infty, \infty] \rightarrow [0, \infty]$ be continuous with compact support for $i = 1, 2$. Then

(2.5) is proven by means of joint Laplace functionals if we prove

$$(2.6) \quad E \exp \left\{ - \sum_{k=1}^n (f_1(a_n^{-1}((X_k - b_n), Y_k)) + f_2(a_n^{-1}(Y_k - b_n, X_k))) \right\} \\
= \exp \left\{ - \int_{(-\infty, \infty] \times [-\infty, \infty]} (1 - e^{-f_1(x, z)}) e^{-x} dx G(\gamma^{-1} dz) \right. \\
\left. + \int_{(-\infty, \infty] \times [-\infty, \infty]} (1 - e^{-f_2(y, w)}) c e^{-y} dy F(\gamma^{-1} dw) \right\}.$$

Define $h: ([-\infty, \infty]^2 \setminus \{(-\infty, -\infty)\}) \times [-\infty, \infty]^2 \rightarrow [0, \infty)$ by

$$h(x, y, z, w) = \begin{cases} f_1(x, z) + f_2(y, w) & \text{if } (x, z) \in (-\infty, \infty] \times [-\infty, \infty], (y, w) \in (-\infty, \infty] \times [-\infty, \infty] \\ f_1(x, z) & \text{if } x > -\infty, y = -\infty \\ f_2(y, w) & \text{if } x = -\infty, y > -\infty, \end{cases}$$

so that h is continuous with compact support. The left side of (2.6) is

$$E \exp \left\{ - \sum_{k=1}^n h(a_n^{-1}(X_k - b_n, Y_k - b_n, Y_k, X_k)) \right\}$$

which is the Laplace functional at the function h of the point process

$$\sum_{k=1}^n \epsilon_{(a_n^{-1}(X_k - b_n, Y_k - b_n, Y_k, X_k))}.$$

We now need a minor variant of Proposition 3.1 of Resnick (1986): If $\{X_{n,j}, j \geq 1\}$ are iid random elements for each n of a LCCB space E and if μ is a Radon measure on E , then

$$\sum_{j=1}^n \epsilon_{X_{n,j}} \rightarrow \text{PRM}(\mu) \text{ iff } nP[X_{n,1} \in \cdot] \xrightarrow{v} \mu.$$

(Cf. Resnick, 1986, p. 77 or see the "warm-up" to the proof of Proposition 3.21 in Resnick, 1987.) Using this and Lemma 2.1 we find

$$E \exp \left\{ - \sum_{k=1}^n h(a_n^{-1}(X_k - b_n, Y_k - b_n, Y_k, X_k)) \right\} \\
= \exp \left\{ - \int (1 - e^{-h(x, y, z, w)}) m_\infty(dx, dy, dz, dw) \right\} \\
= \exp \left\{ - \left(\int (1 - e^{-f_1(x, z)}) e^{-x} dx G(\gamma^{-1} dz) \right. \right. \\
\left. \left. + \int (1 - e^{-f_2(y, w)}) c e^{-y} dy F(\gamma^{-1} dw) \right) \right\}$$

and the desired result follows.

□

Because addition is vaguely continuous (Kallenberg, 1983), we get from Lemma 2.2

that

$$(2.7) \quad \sum_{k=1}^n \left[\varepsilon_{(a_n^{-1}(X_k - b_n, Y_k))} + \varepsilon_{(a_n^{-1}(Y_k - b_n, X_k))} \right] \\ \Rightarrow \sum_k \left[\varepsilon_{(j_k^{(1)}, \gamma Y(k))} + \varepsilon_{(j_k^{(2)} + \log c, \gamma X(k))} \right]$$

in $M_p((-\infty, \infty] \times [-\infty, \infty])$ where the second term in the limit vanishes if $c = 0$. Restrict the domain of the point measures to the compact set $[-M, \infty] \times [-\infty, \infty]$ for $M > 0$, and the restricted version of (2.7) is

$$(2.8) \quad \sum_{k=1}^n \varepsilon_{(a_n^{-1}(X_k - b_n, Y_k))} (\cdot \cap [-M, \infty] \times [-\infty, \infty]) \\ + \sum_{k=1}^n \varepsilon_{(a_n^{-1}(Y_k - b_n, X_k))} (\cdot \cap [-M, \infty] \times [-\infty, \infty]) \\ \Rightarrow \sum_k \varepsilon_{(j_k^{(1)}, \gamma Y(k))} 1_{[j_k^{(1)} \geq -M]} \\ + \sum_k \varepsilon_{(j_k^{(2)} + \log c, \gamma X(k))} 1_{[j_k^{(2)} + \log c \geq -M]}.$$

Define $T: (-\infty, \infty] \times [-\infty, \infty] \rightarrow (-\infty, \infty]$ by

$$T(x, y) = \begin{cases} x + y & \text{if } y > -\infty \\ 0 & \text{if } y = -\infty. \end{cases}$$

T induces a transformation $\tilde{T}: M_p([-M, \infty] \times [-\infty, \infty]) \rightarrow M_p((-\infty, \infty])$

defined by

$$\tilde{T}(\sum_i \varepsilon_{(x_i, y_i)}) = \sum_i \varepsilon_{T(x_i, y_i)}.$$

$\tilde{T}$ is a.s. continuous with respect to the limit point process in (2.8) because it is continuous on $\{\mu \in M_p([-M, \infty] \times [-\infty, \infty]): \mu([-M, \infty] \times \{-\infty\}) = 0\}$. This follows by a minor variant of Proposition 1.1 in Resnick (1986); cf. Proposition 2.3 in Davis and Resnick (1988). Applying T to (2.8) yields as $n \rightarrow \infty$

$$(2.9) \quad \sum_{k=1}^n \varepsilon_{a_n^{-1}(X_k + Y_k - b_n)} \left[1_{[a_n^{-1}(X_k - b_n) \geq -M]} + 1_{[a_n^{-1}(Y_k - b_n) \geq -M]} \right] \\ \Rightarrow \sum_k \varepsilon_{j_k^{(1)} + \gamma Y(k)} 1_{[j_k^{(1)} \geq -M]}$$

$$+ \sum_k \varepsilon_{j_k}^{(2)} + \log c + \gamma X(k) \quad 1_{j_k}^{(2)} + \log c > -M]$$

in $M_p((-\infty, \infty])$. Call the two indicator functions on the left side of (2.9) $1_k^{(1)}, 1_k^{(2)}$.

Theorem 2.3. Given (2.1), (2.2), (2.3) we have

$$(2.10) \quad \sum_{k=1}^n \varepsilon_{a_n^{-1}(X_k + Y_k - b_n)} \Rightarrow \sum_k \varepsilon_{j_k}^{(1)} + \gamma Y(k) + \sum_k \varepsilon_{j_k}^{(2)} + \log c + \gamma X(k)$$

in $M_p((-\infty, \infty])$ where the limit is PRM $((Ee^{\gamma Y_1} + cEe^{\gamma X_1})e^{-y}dy)$. Consequently

$$(2.11) \quad nP[X_1 + Y_1 > a_n x + b_n] \rightarrow (Ee^{\gamma Y_1} + cEe^{\gamma X_1})e^{-x}$$

and

$$(2.12) \quad (1 - F^*G(t))/\bar{F}(t) = P[X_1 + Y_1 > t]/P[X_1 > t] \\ \rightarrow Ee^{\gamma Y_1} + cEe^{\gamma X_1}.$$

(Cf. Cline, 1986a,b; Embrechts and Goldie (1980).)

Proof. The mean measure of $(x, \infty]$ in the limit PRM in (2.10) is

$$\int_{-\infty}^{\infty} G(dy)e^{-(x-\gamma y)} + \int_{-\infty}^{\infty} F(dy)e^{-(x-\gamma y - \log c)}$$

which gives the asserted mean measure.

We have (2.11) and (2.12) equivalent and (2.11) is equivalent to (2.10) by Proposition 3.1 of Resnick (1986), so we must prove (2.10). If we let $M \rightarrow \infty$ in the right side of (2.9) we clearly get the right side of (2.10) as the almost sure vague limit. The result will follow from Billingsley (1968), Theorem 4.2, if we show for any δ :

$$(2.13) \quad \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} d(\delta, n, M) = 0$$

where

$$d(\delta, n, M) = P \left[\rho \left[\sum_{k=1}^n \varepsilon_{a_n^{-1}(X_k + Y_k - b_n)} (1_k^{(1)} + 1_k^{(2)}), \sum_{k=1}^n \varepsilon_{a_n^{-1}(X_k + Y_k - b_n)} \right] > \delta \right]$$

and ρ is the vague metric. Because of the definition of the vague metric (Kallenberg, 1983) it suffices to show for any $h: (-\infty, \infty] \rightarrow [0, \infty)$, continuous with compact support in $[\theta, \infty]$

$$(2.14) \quad \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left[\left| \sum_{k=1}^n h(a_n^{-1}(X_k + Y_k - b_n)) (1 - 1_k^{(1)} - 1_k^{(2)}) \right| > \delta \right] = 0.$$

Note θ is fixed in $(-\infty, \infty]$ and we are free to suppose $M > |\theta|$. The probability in (2.14) is bounded by

$$P \left\{ \bigcup_{k=1}^n [h(a_n^{-1}(X_k + Y_k - b_n)) > 0, 1 - 1_k^{(1)} - 1_k^{(2)} \neq 0] \right\} \\ \leq nP[h(a_n^{-1}(X_1 + Y_1 - b_n)) > 0, 1 - 1_1^{(1)} - 1_1^{(2)} \neq 0],$$

and taking into account the compact support of h yields the bound

$$nP[a_n^{-1}(X_1 + Y_1 - b_n) \geq \theta, 1_1^{(1)} = 1_1^{(2)} = 0] \\ + nP[a_n^{-1}(X_1 + Y_1 - b_n) \geq \theta, 1_1^{(1)} = 1_1^{(2)} = 1] =: I + II.$$

Now

$$II \leq nP[a_n^{-1}(X_1 - b_n) \geq -M, a_n^{-1}(Y_1 - b_n) \geq -M] \\ \sim e^M P[Y_1 > a_n(-M) + b_n] \rightarrow 0$$

as $n \rightarrow \infty$.

For I we have

$$I = nP[a_n^{-1}(X_1 + Y_1 - b_n) \geq \theta, a_n^{-1}(X_1 - b_n) \leq -M, a_n^{-1}(Y_1 - b_n) \leq -M].$$

The strength of the assumption $F \in S_r(\gamma)$ is needed to kill I and we develop the machinery to do this in Lemma 2.4.

Lemma 2.4. We have

$$(2.15) \quad \lim_{n \rightarrow \infty} nP[X_1 + Y_1 > a_n \theta + b_n, M_1 \leq a_n^{-1} Y_1 \leq M_2] = e^{-\theta \int_{\gamma^{-1} M_1}^{\gamma^{-1} M_2} e^{\gamma y} G(dy)}$$

for $-\infty \leq M_1 < M_2 < \infty$.

$$(2.16) \quad \lim_{n \rightarrow \infty} nP[X_1 + Y_1 > a_n \theta + b_n, M_1 \leq a_n^{-1} X_1 \leq M_2] = ce^{-\theta \int_{\gamma^{-1} M_1}^{\gamma^{-1} M_2} e^{\gamma y} F(dy)}$$

for $-\infty \leq M_1 < M_2 < \infty$.

$$(2.17) \quad \lim_{n \rightarrow \infty} nP[X_1 + Y_1 > a_n \theta + b_n, a_n^{-1}(X_1 - b_n) \geq -M] \\ = e^M \bar{G}(\gamma^{-1}(\theta + M)) + e^{-\theta \int_{-\infty}^{\gamma^{-1}(\theta + M)} e^{\gamma y} G(dy)}$$

for $M \in \mathbb{R}$.

$$\begin{aligned}
(2.18) \quad & \lim_{n \rightarrow \infty} nP[X_1 + Y_1 > a_n \theta + b_n, a_n^{-1}(Y_1 - b_n) \geq -M] \\
& = ce^M \bar{F}(\gamma^{-1}(\theta + M)) + ce^{-\theta} \int_{-\infty}^{\gamma^{-1}(\theta+M)} e^{\gamma y} F(dy)
\end{aligned}$$

for $M \in \mathbb{R}$.

$$\begin{aligned}
(2.19) \quad & \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} nP[X_1 + Y_1 > a_n \theta + b_n, a_n M < X_1 < a_n(-M) + b_n] \\
& = \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} nP[X_1 + Y_1 > a_n \theta + b_n, a_n M < Y_1 < a_n(-M) + b_n] = 0.
\end{aligned}$$

Proof. The probability in (2.15) is

$$\int_{M_1}^{M_2} G(a_n dy) n\bar{F}(a_n(\theta - y) + b_n)$$

and because (2.3) holds locally uniformly on $(-\infty, \infty]$ this is

$$\begin{aligned}
& \int_{M_1}^{M_2} G(a_n dy) e^{-(\theta-y)} + o(1) \\
& = e^{-\theta} \int_{a_n M_1}^{a_n M_2} e^{a_n^{-1} y} G(dy) + o(1) = e^{-\theta} \int_{\gamma^{-1} M_1}^{\gamma^{-1} M_2} e^{\gamma y} G(dy)
\end{aligned}$$

by dominated convergence. The proof of (2.16) is the same. Write the left side of (2.17) as

$$\begin{aligned}
& nP[a_n^{-1}(X_1 + Y_1 - b_n) - \theta \wedge (a_n^{-1}(X_1 - b_n) + M) > 0] \\
& = nP[a_n^{-1}(X_1 - b_n) + ((a_n^{-1}Y_1 - \theta) \wedge M) > 0] \\
& = nP[a_n^{-1}(X_1 - b_n) > -M, a_n^{-1}Y_1 > \theta + M] \\
& \quad + nP[a_n^{-1}(X_1 + Y_1 - b_n) > \theta, a_n^{-1}Y_1 \leq \theta + M] \\
& = n\bar{F}(a_n(-M) + b_n) \bar{G}(a_n(\theta + M)) + e^{-\theta} \int_{-\infty}^{\gamma^{-1}(\theta+M)} e^{\gamma y} G(dy) + o(1)
\end{aligned}$$

(where the second term we applied (2.15) with $M_1 := -\infty$, $M_2 := \theta + M$)

$$= e^M \bar{G}(\gamma^{-1}(\theta + M)) + e^{-\theta} \int_{-\infty}^{\gamma^{-1}(\theta+M)} e^{\gamma y} G(dy).$$

The proof of (2.18) is similar.

Now consider (2.19). Since we assume $F \in D(\Lambda) \cap S_r(\gamma)$ we have that

$$nP[X_1 + X_2 > a_n \theta + b_n] \sim 2Ee^{\gamma X_1} nP[X_1 > a_n \theta + b_n] - 2Ee^{\gamma X_1} e^{-\theta}.$$

However

$$\begin{aligned}
n\mathbb{P}[X_1 + X_2 > a_n \theta + b_n] &= n\mathbb{P}[X_1 + X_2 > a_n \theta + b_n, X_1 \leq a_n M] \\
&\quad + n\mathbb{P}[X_1 + X_2 > a_n \theta + b_n, a_n M < X_1 < a_n(-M) + b_n] \\
&\quad + n\mathbb{P}[X_1 + X_2 > a_n \theta + b_n, a_n(-M) + b_n < X_1] \\
&= A + B + C.
\end{aligned}$$

Replacing Y_1 by X_2 in (2.15), (2.17) yields

$$\begin{aligned}
\lim_{n \rightarrow \infty} A &= e^{-\theta} \int_{-\infty}^{\gamma^{-1}M} e^{\gamma y} F(dy), \\
\lim_{n \rightarrow \infty} C &= e^M \bar{F}(\gamma^{-1}(\theta + M)) + e^{-\theta} \int_{-\infty}^{\gamma^{-1}(\theta+M)} e^{\gamma y} F(dy).
\end{aligned}$$

If $E e^{\gamma X_1} < \infty$ we have $e^M \bar{F}(\gamma^{-1}(\theta + M)) \rightarrow 0$ as $M \rightarrow \infty$ and thus when $F \in D(\Lambda) \cap S_r(\gamma)$

$$\lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} A = \lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} C = e^{-\theta} E e^{\gamma X_1}$$

and so

$$\begin{aligned}
&\lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} n\mathbb{P}[X_1 + X_2 > a_n \theta + b_n, a_n M < X_1 < a_n(-M) + b_n] \\
&= \lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{a_n M}^{a_n(-M)+b_n} F(ds) n\bar{F}(a_n \theta + b_n - s) = 0.
\end{aligned}$$

Thus if we write $K := \sup_{s \geq 0} \bar{G}(s)/\bar{F}(s)$ we have

$$\begin{aligned}
&\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} n\mathbb{P}[X_1 + Y_1 > a_n \theta + b_n, a_n M < X_1 < a_n(-M) + b_n] \\
&= \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{a_n M}^{a_n(-M)+b_n} F(ds) n\bar{G}(a_n \theta + b_n - s) \\
&\leq \lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} K \int_{a_n M}^{a_n(-M)+b_n} F(ds) n\bar{F}(a_n \theta + b_n - s) = 0.
\end{aligned}$$

The other relation in (2.19) is checked similarly making use of an integration by parts.

□

Continuation of the proof of Theorem 2.3: For I we have the upper bound

$$\begin{aligned}
&n\mathbb{P}[X_1 + Y_1 > a_n \theta + b_n, a_n M < X_1 < a_n(-M) + b_n] \\
&\quad + n\mathbb{P}[X_1 + Y_1 > a_n \theta + b_n, a_n M < Y_1 < a_n(-M) + b_n]
\end{aligned}$$

$$+ nP[X_1 + Y_1 > a_n \theta + b_n, X_1 \vee Y_1 \leq a_n M].$$

The first two terms are dealt with by (2.19), and for the last term we have 0 for large n since $X_1 \vee Y_1 \leq a_n M$ implies $X_1 + Y_1 \leq 2a_n M < a_n \theta + b_n$ for any fixed M and n sufficiently large.

Corollary 2.5. Suppose $\{X_k^{(1)}, k \geq 1\}, \{X_k^{(2)}, k \geq 1\}$ are two independent iid sequences with common distribution F . Suppose $F \in D(\Lambda)$ so that (2.3) holds. Then $F \in D(\Lambda) \cap S_r(\gamma)$ iff

$$\sum_{k=1}^n \varepsilon_{a_n^{-1}(X_k^{(1)} + X_k^{(2)} - b_n)} \Rightarrow \text{PRM} \left[2Ee^{\gamma X_1^{(1)}} e^{-x} dx \right]$$

in $M_p((-\infty, \infty])$.

Proof. If $F \in D(\Lambda) \cap S_r(\gamma)$ then apply Theorem 2.3 with $G = F, c = 1$. Conversely if the point process convergence holds we have by Proposition 3.21 of Resnick (1987) that

$$nP[a_n^{-1}(X_1^{(1)} + X_1^{(2)} - b_n) > x] \rightarrow 2E\exp\{\gamma X_1^{(1)}\}e^{-x}$$

and membership in $S_r(\gamma)$ is assured by Proposition 2.19 of Resnick (1987).

□

Corollary 2.6. Let $\{X_k^{(1)}, k \geq 1\}, \{X_k^{(2)}, k \geq 1\}$ be two independent iid sequences with common distribution $F \in D(\Lambda) \cap S_r(\gamma)$. If $c_1 > c_2 > 0$ and (2.3) holds we have

$$nP[(c_1 X_1^{(1)} + c_2 X_1^{(2)} - c_1 b_n)/a_n > x] \sim e^{-c_1^{-1}x} Ee^{\gamma X_1^{(1)}}$$

or equivalently

$$\lim_{t \rightarrow \infty} \frac{P[c_1 X_1^{(1)} + c_2 X_1^{(2)} > t]}{P[c_1 X_1^{(1)} > t]} = Ee^{\gamma X_1^{(1)}},$$

and also, in $M_p((-\infty, \infty])$,

$$\sum_{k=1}^n \varepsilon_{a_n^{-1}(c_1 X_k^{(1)} + c_2 X_k^{(2)} - b_n)} \Rightarrow \text{PRM} \left[Ee^{\gamma X_1 c_1^{-1}} e^{-xc_1^{-1}} dx \right]$$

Proof. In Theorem 2.1 put $X_k := c_1 X_k^{(1)}, Y_k := c_2 X_k^{(2)}$ so that $c = 0$ as a result of $F \in D(\Lambda)$:

$$\lim_{x \rightarrow \infty} \frac{P[Y_1 > x]}{P[X_1 > x]} = \lim_{x \rightarrow \infty} \frac{\bar{F}(c_2^{-1}x)}{\bar{F}(c_1^{-1}x)} = 0 = c.$$

□

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