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**A GENERALIZATION OF
ROBACKER'S THEOREM**

by

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Abstract

Let $\mathcal{P}$ be the polyhedron given by $\mathcal{P} = \{x \in \mathbb{R}^n: Nx = 0, a \leq x \leq b\}$, where N is a totally unimodular matrix and a and b are any integral vectors. For $x \in \mathbb{R}^n$ let $(x)^+$ denote the vector obtained from x by changing all its negative components to zeros. Let $x^1, \dots, x^p$ be the integral points in $\mathcal{P}$ and let $\mathcal{P}^+$ be the convex hull of $(x^1)^+, \dots, (x^p)^+$. In this paper we derive the blocking polyhedron for $\{x \in \mathbb{R}_+^n: Mx \geq 1\}$, where the rows of M are integral points in $\mathcal{P}^+$. We also show that the optimum objective function values of the integer programming problem $\max\{1 \cdot y: yM \leq w, y \geq 0 \text{ and integral}\}$ and its linear programming relaxation differ by less than one for any nonnegative integral vector w . As a special case of this result we derive Robacker's Theorem, stating that for a directed graph with two distinguished nodes s and t , the maximum value of an integral packing of forward edges in (s,t) -cuts into a nonnegative integral weighting w of the edges is equal to the minimum w -weight of an (s,t) -path.

1. Introduction and Definitions

Let $G = (V, E)$ be a directed graph with node set V and edge set E and with two distinguished nodes, s and t . Let $X \subseteq V$ with $s \in X$ and $t \notin X$. The set of edges G with one end in X and the other end in $\bar{X} = V \setminus X$ is called an (s, t) -cut and is denoted by $[X, \bar{X}]$. The set of edges $\{e \in E: e = (u, v), u \in X, v \in \bar{X}\}$ is called the set of *forward* edges in the (s, t) -cut $[X, \bar{X}]$. Define

$$\mathcal{C} = \{S \subseteq E: \text{edges in } S \text{ are forward edges in a minimal } (s, t)\text{-cut in } G\}$$

and $\bar{\mathcal{C}} = \{S \subseteq E: \text{edges in } S \text{ are a directed, minimal } (s, t)\text{-path in } G\}$.

Let M be the cut-edge incidence matrix for $\mathcal{C}$ and $\bar{M}$ be the path-edge incidence matrix for $\bar{\mathcal{C}}$. It is well-known that M and $\bar{M}$ form a "blocking" pair of matrices (defined later in this section—see Fulkerson(2)). Also, the following two combinatorial max-min relations are well-known, where w is any nonnegative integral vector with $|E|$ components,

$$\max\{1 \cdot y: yM \leq w, y \geq 0 \text{ integral}\} = \min\{w \cdot \bar{\mu}: \bar{\mu} \text{ a row of } \bar{M}\} \quad (1.1)$$

$$\max\{1 \cdot y: y\bar{M} \leq w, y \geq 0 \text{ integral}\} = \min\{w \cdot \mu: \mu \text{ a row of } M\}. \quad (1.2)$$

Since M and $\bar{M}$ are a blocking pair of matrices, (1.1) and (1.2) are equivalent to the assertion that the corresponding linear programming problems in (1.1) and (1.2) have integral optimum solutions for all nonnegative integral right-hand side vectors w . (1.2) is the max-flow min-cut theorem of Ford and Fulkerson and (1.1) is a theorem of Robacker (5).

Trotter and Weinberger (7) have obtained a generalization of (1.2) by treating the problem in the setting of packing into a nonnegative integral vector w , the integral solutions to the system $Nx = 0, a \leq x \leq b$, where N is a totally unimodular matrix and a and b are nonnegative integral vectors. In this paper we extend this approach to obtain a generalization of (1.1). We define the *positive part* of $x \in \mathbb{R}^n$, denoted by x^+ , as the vector in $\mathbb{R}^n$ obtained from x by changing all its negative components to zeros. We consider incidence vectors of forward edges in (s, t) -cuts from the general viewpoint of positive parts

of integral vectors in $\mathcal{P} = \{x \in \mathbb{R}^n: Nx = 0, a \leq x \leq b\}$, where N is a totally unimodular matrix and a, b are integral vectors.

Let $V \subseteq \mathbb{R}^n$ be a linear subspace. The *support* of a vector $x \in V$, denoted by $S(x)$, is defined by $S(x) = \{j: x_j \neq 0\}$. The *positive support* of a vector $x \in V$, denoted by $S^+(x)$, is defined by $S^+(x) = \{j: x_j > 0\}$. The *negative support* of a vector $x \in V$, denoted by $S^-(x)$, is similarly defined. A vector $x \in V$ is said to be an *elementary vector* in V if $x \neq 0$ and has minimal support in V , i.e., provided $0 \neq y \in V$ implies $S(y)$ is not a proper subset of $S(x)$. We denote by $\mathcal{F}(V)$ the *frame* of V , defined as the set of all elementary vectors of V . For a detailed development of elementary vectors the reader is referred to Fulkerson (1). We say that the vector y *conforms to* the vector x if $S^+(y) \subseteq S^+(x)$ and $S^-(y) \subseteq S^-(x)$. It is well-known that if V has dimension m then there exists a matrix $A = (I_m | L)$ where I_m denotes the $m \times m$ identity matrix and L is an $m \times (n-m)$ matrix such that, after a suitable permutation of the coordinates, V is the row space of A . Matrix A is called a *standard representative matrix* of V .

Let A be an $m \times n$ matrix with nonnegative, real entries and with no zero rows. Let $\mathcal{B}$ be the polyhedron $\mathcal{B} = \{x \in \mathbb{R}_+^n: Ax \geq 1\}$, where 1 denotes an m -vector whose components are all equal to one. The *blocking polyhedron* of $\mathcal{B}$ is defined by $\hat{\mathcal{B}} = \{b \in \mathbb{R}_+^n: bx \geq 1, \text{ for every } x \in \mathcal{B}\}$. The following theorem of Fulkerson (3) shows that $\hat{\mathcal{B}}$ is a polyhedron and that blocking polyhedra occur in dual pairs.

Theorem 1.1. Let the rows of matrix $\bar{A}$ be the extreme points of $\mathcal{B}$ and let $\bar{\mathcal{B}} = \{x \in \mathbb{R}_+^n: \bar{A}x \geq 1\}$. Then $\hat{\mathcal{B}} = \bar{\mathcal{B}}$ and $\bar{\bar{\mathcal{B}}} = \mathcal{B}$. □

We call any nonnegative matrix $\bar{A}$ such that $\mathcal{B}$ and $\bar{\mathcal{B}}$ as in Theorem 1.1 form a blocking pair of polyhedra, a *blocking matrix* of A . If we restrict to matrices A having only rows that are *essential* in defining $\mathcal{B}$ (a row of A is said to be *inessential* for $\mathcal{B}$ if its corresponding inequality may be omitted when defining $\mathcal{B}$), we obtain unique pairs of

blocking matrices. The next theorem of Fulkerson (3) shows the relationship of blocking theory to the optimal objective function values of certain linear programming problems.

Theorem 1.2. Let A and $\bar{A}$ be nonnegative matrices, each with n columns and without zero rows. Then $\max\{1 \cdot y: yA \leq w, y \geq 0\} = \min\{w \cdot \bar{A}_j: \bar{A}_j \text{ some row of } \bar{A}\}$ for every $w \in \mathbb{R}_+^n$ if and only if A and $\bar{A}$ are a blocking pair of matrices. $\square$

For $x^1, x^2, \dots, x^p \in \mathbb{R}^n$ we denote by $\text{conv}\{x^1, x^2, \dots, x^p\}$ the convex hull of the vectors $x^1, x^2, \dots, x^p$. Let $x^1, x^2, \dots, x^p$ be the integral points in polyhedron $\mathcal{P}$ defined above. Let $\mathcal{P}^+ = \text{conv}\{(x^1)^+, (x^2)^+, \dots, (x^p)^+\}$. In Section 2 we derive the blocking polyhedron for $\{x \in \mathbb{R}_+^n: Mx \geq 1\}$, where the rows of M are integral points in $\mathcal{P}^+$. In Section 3 we prove that the optimum objective function values of the integer programming problem $\max\{1 \cdot y: yM \leq w, y \geq 0 \text{ integral}\}$, where w is any nonnegative integral vector, and its linear programming relaxation differ by at most one. Finally, we point out that a particular choice of N , a and b leads to relation (1.1) as a corollary.

2. A Blocking Relationship

Let N be an $m \times n$ matrix and let $[a_j, b_j]$, $1 \leq j \leq n$, be nonempty, closed intervals. We will be concerned with the polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n: Nx = 0, x_j \in [a_j, b_j], 1 \leq j \leq n\}$. The following theorem [see Fulkerson (1), Rockafeller (6)] will be useful.

Theorem 2.1. Suppose N , $\mathcal{P}$ and $[a_j, b_j]$, $1 \leq j \leq n$, are as defined above. Then $\mathcal{P}$ is nonempty if and only if:

$$\sum_{j \in S^+(k)} k_j b_j + \sum_{j \in S^-(k)} k_j a_j \geq 0 \text{ for all } k \in \mathcal{F}(V),$$

where V is the row space of N . $\square$

For the rest of this paper, let N be a totally unimodular matrix and suppose a and b are integral vectors. Consider the polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n: Nx = 0, a \leq x \leq b\}$. Let $x^1, \dots, x^p$ be the integral vectors in $\mathcal{P}$. Note that since N is totally unimodular and a and b are integral vectors, the extreme points of $\mathcal{P}$ are among $x^1, \dots, x^p$. Let $\mathcal{P}^+ = \text{conv}\{(x^1)^+, \dots, (x^p)^+\}$ and let the integral vectors in $\mathcal{P}^+$ be the rows of the matrix M . We wish to derive the blocking polyhedron for $\mathcal{B} = \{x \in \mathbb{R}^n: Mx \geq 1\}$. Towards this end we first establish the following two lemmas.

Lemma 2.1. If $x \in \mathcal{P}$, then $x^+ \in \mathcal{P}^+$, for $\mathcal{P}$ and $\mathcal{P}^+$ as defined above.

Proof: Notice that it suffices to show that for every x in $\mathcal{P}$, there exist integral vectors $x^1, \dots, x^k$ in $\mathcal{P}$, conforming to x and scalars $\lambda_1, \dots, \lambda_k$ such that

$$x = \sum_{i=1}^k \lambda_i x^i, \quad \sum_{i=1}^k \lambda_i = 1, \quad \lambda_i \geq 0, 1 \leq i \leq k.$$

This will suffice, since we then have

$$x^+ = \left(\sum_{i=1}^k \lambda_i x^i \right)^+ = \sum_{i=1}^k \lambda_i (x^i)^+,$$

which in turn belongs to $\mathcal{P}^+$.

Consider any $x \in \mathcal{P}$. Let $\mathcal{P}' = \mathcal{P} \cap \{y \in \mathbb{R}^n: y_j \geq 0 \forall j \in S^+(x), y_j \leq 0 \forall j \in S^-(x), y_j = 0 \forall j \notin S(x)\}$. Then, clearly $\mathcal{P}' \subseteq \mathcal{P}$. Also, the constraint matrix defining $\mathcal{P}'$ is totally unimodular, since N is. Hence $\mathcal{P}'$ has integral extreme points. By the way $\mathcal{P}'$ is defined, each vector y in $\mathcal{P}'$ (and in particular any extreme point of $\mathcal{P}'$) conforms to x . Also, $x \in \mathcal{P}'$ and hence there exist integral extreme points $x^1, \dots, x^k$ of $\mathcal{P}'$ that conform to x and scalars $\lambda_1, \dots, \lambda_k$ such that

$$x = \sum_{i=1}^k \lambda_i x^i, \quad \sum_{i=1}^k \lambda_i = 1, \quad \lambda_i \geq 0, 1 \leq i \leq k.$$

These x^i are integral vectors in $\mathcal{P}$, as observed, and so the proof is complete. $\square$

Lemma 2.2. There exists $x \in \mathcal{P}^+$ such that $x \leq w$ if and only if there exists $y \in \mathcal{P}$ such that $y \leq w$, where w is any nonnegative vector and $\mathcal{P}$ and $\mathcal{P}^+$ are as in Lemma 2.1.

Proof: If x is any vector in $\mathcal{P}^+$ such that $w \geq x$ then we have $x = \sum_{i=1}^p \lambda_i (x^i)^+$, where

$$\sum_{i=1}^p \lambda_i = 1, \lambda_i \geq 0, 1 \leq i \leq p. \text{ Hence } x \geq \sum_{i=1}^p \lambda_i x^i = y, \text{ which in turn is in } \mathcal{P}. \text{ Hence there}$$

exists y in $\mathcal{P}$ such that $y \leq w$. Conversely, let y be a vector in $\mathcal{P}$ such that $y \leq w$. Then, since $w \geq 0$, $y^+ \leq w$. But from Lemma 2.1, $y^+ \in \mathcal{P}^+$ and hence there exists $x = y^+$ in $\mathcal{P}^+$ such that $x \leq w$. $\square$

We now derive the blocking polyhedron for $\mathcal{B} = \{x \in \mathbb{R}_+^n: Mx \geq 1\}$, where the matrix M is as defined above.

Theorem 2.2. The blocking polyhedron for $\mathcal{B}$ is given by (where V denotes the row space of N)

$$\{x \in \mathbb{R}_+^n: \sum_{j \in J} k_j x_j \geq \sum_{j \in S^-(k)} (-k_j) a_j - \sum_{j \in S^+(k) \setminus J} k_j b_j, k \in \mathcal{F}(V), J \subseteq S^+(k)\}.$$

Proof: The blocking polyhedron for $\mathcal{B}$ is given by

$$\begin{aligned} \bar{\mathcal{B}} &= \{x \in \mathbb{R}_+^n: x \geq z \text{ for some } z \in \text{conv}\{\text{rows of } M\}\} \\ &= \{x \in \mathbb{R}_+^n: x \geq z \text{ for some } z \in \mathcal{P}^+\}. \end{aligned}$$

Using Lemma 2.2, we have that

$$\begin{aligned} \bar{\mathcal{B}} &= \{x \in \mathbb{R}_+^n: x \geq z \text{ for some } z \in \mathcal{P}\} \\ &= \{x \in \mathbb{R}_+^n: Nz = 0, a_j \leq z_j \leq \min\{x_j, b_j\}, 1 \leq j \leq n \text{ for some } z\}. \end{aligned}$$

Using Theorem 2.1 with V as the row space of N , we have

$$\bar{\mathbf{B}} = \{x \in \mathbb{R}_+^n: \sum_{j \in S^+(k)} k_j \min\{x_j, b_j\} + \sum_{j \in S^-(k)} k_j a_j \geq 0, \forall k \in \mathcal{F}(V)\}.$$

Hence, we have $\bar{\mathbf{B}}$ as required. $\square$

As a special case of Theorem 2.2, we can easily show that positive parts of minimal (s,t) -cuts in a network and directed (s,t) -paths form a blocking pair of clutters. (See (8).)

3. Packing Positive Parts of Integral Vectors in $\mathcal{P}$

As above we let N be a totally unimodular matrix and suppose a and b are integral vectors. Consider the polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n: Nx = 0, a \leq x \leq b\}$. Let $x^1, \dots, x^p$ be the integral vectors in $\mathcal{P}$ and let $\mathcal{P}^+ = \text{conv}\{(x^1)^+, \dots, (x^p)^+\}$. Let the integral vectors in $\mathcal{P}^+$ be the rows of matrix M . In this section we show that the objective function values of the integer programming problem, $\max\{1 \cdot y: yM \leq w, y \geq 0 \text{ and integral}\}$ and its linear programming relaxation differ by less than one for all nonnegative integral vectors w . We first prove a conformal decomposition theorem for $\mathcal{P}$. For any integer, $k \geq 1$, let $k\mathcal{P} = \{x: x = ky \text{ for some } y \in \mathcal{P}\}$. We note here that the following decomposition theorem is a direct generalization of a decomposition theorem by Trotter and Weinberger (7) to the case where a, b are integral vectors, as opposed to both a and b being in $\mathbb{Z}_+^n$. In the cases where a, b are integral vectors we get a conformal decomposition theorem. This theorem may also be deduced from results of McDiarmid (4), where it is shown (Section 4) that the decomposition $x = x^1 + \dots + x^k$ may be presumed equitable, i.e., $x^i - x^j \leq 1$ for all $i, j = 1, \dots, k$.

Theorem 3.1. Let N be a totally unimodular matrix. Suppose a, b are integral vectors. Let $\mathcal{P} = \{x \in \mathbb{R}^n: Nx = 0, a \leq x \leq b\}$. Then, for each positive integer k , and each integral vector x in $k\mathcal{P}$ there exist integral vectors $x^1, \dots, x^k$ in $\mathcal{P}$ and conforming to x such that $x = x^1 + \dots + x^k$.

Proof: We prove the theorem by induction on k . The case $k = 1$ is trivially true. We assume the result to be true for $k = 1, 2, \dots, p-1$. Consider an integral vector x in $p\mathcal{P}$. Then,

$$a_j \leq x_j/p \leq b_j, \quad 1 \leq j \leq n. \quad (3.1)$$

Multiplying both sides of (3.1) by $(p-1)$ and rearranging, we have

$$x_j - (p-1)a_j \geq x_j/p \geq x_j - (p-1)b_j, \quad 1 \leq j \leq n. \quad (3.2)$$

From (3.1) and (3.2) we have that x/p is in $\{x \in \mathbb{R}^n: Nx = 0\} \cap \{J_1 \times \dots \times J_n\}$, where

$$J_j = [\max\{a_j, x_j - (p-1)b_j\}, \min\{b_j, x_j - (p-1)a_j\}] = [c_j, d_j]$$

and c_j, d_j are integers, $1 \leq j \leq n$. Since N is totally unimodular, there exists an $m \times n$ totally unimodular matrix A whose rows are a basis for the subspace $\{x \in \mathbb{R}^n: Nx = 0\}$. Thus there exists a vector $y \in \mathbb{R}^m$, such that $x/p = yA$. Also, $c \leq yA \leq d$. Since p is a positive integer, x/p conforms to x , with $x_j/p \leq x_j$ if $x_j \geq 0$ and $x_j/p \geq x_j$ if $x_j \leq 0$. Hence there exists $y \in \mathbb{R}^m$ such that

$$\begin{aligned} c &\leq yA \leq d \\ x_j &\geq yA_j \geq 0 \text{ if } x_j \geq 0 \\ x_j &\leq yA_j \leq 0 \text{ if } x_j \leq 0 \\ yA_j &= 0 \text{ if } x_j = 0, \end{aligned} \quad (3.3)$$

where A_j is the j^{th} column of A . Since A is totally unimodular, so is the matrix defining the inequality system in (3.3). Hence, because c, d, x are integral, there exists an integral vector y^1 satisfying (3.3). Hence, $x^1 = y^1A$ is an integral vector satisfying $c \leq x^1 \leq d$, $Nx^1 = 0$ and conforming to x .

Now, $x^1 \in J_1 \times \dots \times J_n$ and so we have that $(p-1)a_j \leq x_j - x_j^1 \leq (p-1)b_j$, $1 \leq j \leq n$,

and it is clear that we also have $x - x^1 \in \{z: Nz = 0, (p-1)a \leq z \leq (p-1)b\} = (p-1)\mathcal{P}$. From

(3.3), $x - x^1$ conforms to x . Hence by the induction hypothesis, $x - x^1 = x^2 + \dots + x^p$ where $x^2, \dots, x^p$ are integral vectors in $\mathcal{P}$ which conform to x . $\square$

Lemma 3.1. Let $k \in \mathbb{Z}_+$ and $w \in \mathbb{Z}_+^n$. Then,

$$k\mathcal{P}^+ \cap \{x: x \leq w\} \neq \emptyset \Rightarrow k\mathcal{P} \cap \{x: x \leq w\} \cap \mathbb{Z}^n \neq \emptyset.$$

Proof: First note that for every $x' \in \mathcal{P}^+$, there exists $y' \in \mathcal{P}$ such that $x' \geq (y')^+$, since $x' = \sum_{i=1}^p \lambda_i (x^i)^+$, $\sum_{i=1}^p \lambda_i = 1$, $\lambda_i \geq 0$, $1 \leq i \leq p$ and hence $x' \geq (\sum_{i=1}^p \lambda_i x^i)^+ = (y')^+$, where $y' \in \mathcal{P}$. Now, if $x \in k\mathcal{P}^+$ and $x \leq w$, then $x/k \in \mathcal{P}^+$ and $x/k \leq w/k$. Hence, by the argument above, there exists $y' \in \mathcal{P}$ such that $(y')^+ \leq x/k \leq w/k$. Taking $z' = ky'$, we obtain $z' \in k\mathcal{P}$ and $(z')^+ \leq w$. Since $z' \leq (z')^+$, we have $z' \in k\mathcal{P}$ and $z' \leq w$. But since $\mathcal{P} \neq \emptyset$ is defined by a totally unimodular matrix and since w is an integral vector, there exists an integral vector $z \in k\mathcal{P}$ such that $z \leq w$. $\square$

We are now ready to prove the main theorem of this paper.

Theorem 3.2. Where the rows of M are integral vectors in $\mathcal{P}^+$ and w is any nonnegative integral vector, the objective function values of the integer programming problem $\max\{1 \cdot y: yM \leq w, y \geq 0 \text{ integral}\}$ and its linear programming relaxation differ by less than one.

Proof: If $0 \in \mathcal{P}$, then 0 is a row of M and both programming problems are unbounded. Also when M is vacuous (i.e., $\mathcal{P} = \emptyset$), there is nothing to show; thus we suppose y^* solves $\max\{1 \cdot y: yM \leq w, y \geq 0\}$ and $1 \cdot y^* = r^*$. If $0 \leq r^* < 1$, the theorem holds trivially. So, let $r^* \geq 1$. Let $x' = \lfloor r^* \rfloor y^* M / r^*$. Then $x' \in \lfloor r^* \rfloor \mathcal{P}^+$ and $x' \leq w$. Hence by Lemma 3.1, there exists an integral vector $z \in \lfloor r^* \rfloor \mathcal{P}$ such that $z \leq w$. Thus by Theorem 3.1 there exist integral vectors $z^1, \dots, z^{\lfloor r^* \rfloor}$ in $\mathcal{P}$ and conforming to z such that $z = z^1 + \dots + z^{\lfloor r^* \rfloor} \leq w$. Hence by Lemma 2.1, there exist integral vectors $(z^1)^+, \dots, (z^{\lfloor r^* \rfloor})^+$ in $\mathcal{P}^+$ such that

$(z^1)^+ + \dots + (z^{\lfloor r^* \rfloor})^+ \leq w$. This defines a solution to the integer programming problem $\max\{1 \cdot y: yM \leq w, y \geq 0 \text{ and integral}\}$ of value $\lfloor r^* \rfloor$, which completes the proof. $\square$

We now indicate how Robacker's Theorem (See (1.1) in Section 1.) can be obtained as a corollary of Theorem 3.2. Let $G' = (V, E')$ be a directed graph with $|V| = m+1$, $|E'| = n-1$ and let s and t be distinguished nodes of G' . Add the special edge from t to s to E' to get $G = (V, E)$. Then there exists (possibly after edge permutation) a standard representative matrix $A' = (I_m | L)$ for the row space of A , where A is the node-edge incidence matrix of G . Let $N = (-L^t | I_{n-m})$. Then, N is a totally unimodular matrix. Let $\mathcal{P} = \{x \in \mathbb{R}^n: Nx = 0, -1 \leq x \leq 1, x_{ts} = -1\}$. Integral vectors in $\mathcal{P}$ correspond to unions of disjoint cocycles of G , exactly one cocycle containing the edge from t to s . This follows because if x is in $\mathcal{P}$ and integral (i.e., has components in $\{0, +1, -1\}$), then x is in the null space of N and hence in the row space of A . Thus, there exist $x^1, \dots, x^k$, nonproportional elementary vectors in the row space of A which conform to x such that $x = x^1 + \dots + x^k$. Since elementary vectors in the row space of A correspond to incidence vectors of cocycles in G and since $x_{ts} = -1$, exactly one x^i is the incidence vector of a cocycle containing the edge from t to s . Furthermore, due to conformality and nonproportionality, the x^i must be incidence vectors of disjoint cocycles. Conversely, if x is the incidence vector of a disjoint union of cocycles of G with exactly one cocycle containing the edge from t to s , then, after a sign adjustment, if necessary, x is in the row space of A and all components of x are in $\{0, +1, -1\}$ with $x_{ts} = -1$. Hence x is an integral vector in $\mathcal{P}$.

Let the rows of M correspond to positive parts of integral vectors in $\mathcal{P}$ and the rows of M' correspond to positive parts of cocycles of G containing the edge from t to s (with negative sign). Since there exists an optimum solution y^* to the integer programming problem $\max\{1 \cdot y: yM \leq w, y \geq 0 \text{ and integral}\}$ with $y_j^* = 0$ if row j of M is not minimal in support, and since by Theorem 3.2, the optimum objective function values of this integer

programming problem and its linear programming relaxation differ by less than one, we have that for any nonnegative integral vector w ,

$$\begin{aligned}
 & \max\{1 \cdot y: yM' \leq w, y \geq 0 \text{ and integral}\} \\
 &= \max\{\lfloor 1 \cdot y \rfloor: yM' \leq w, y \geq 0\} \\
 &= \min\{\lfloor w \cdot \mu \rfloor: \mu \text{ the incidence vector of a minimal } (s,t)\text{-path}\} \\
 &\quad \text{(since } M' \text{ and the matrix whose rows are incidence vectors of} \\
 &\quad \text{minimal } (s,t)\text{-paths form a blocking pair of matrices)} \\
 &= \min\{w \cdot \mu: \mu \text{ the incidence vector of a minimal } (s,t)\text{-path}\}.
 \end{aligned}$$

This is precisely Robacker's Theorem as stated in (1.1) in Section 1.

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