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TECHNICAL REPORT NO. 722

October 1986

LEAVING AN INTERVAL IN
LIMITED PLAYING TIME

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*Research supported by the U.S. Army Research Office through the Mathematical Sciences Institute at Cornell University and by a grant from the AT&T Foundation.

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AMS 1980 Subject Classifications: Primary 60G40, 60J60, 93E20
Secondary 60J15, 60J65, 90C99
49A99

Key Words and Phrases: Optimal stochastic control, gambling problems, bold play, excessive functions, exit times, limited playing time, Brownian motion, random walk

*Partly supported by the U.S. Army Research Office through the Mathematical Sciences Institute at Cornell University and by a grant from the AT&T Foundation.

**Supported in part by National Science Foundation grants DMS-84-01604 and DMS-86-01153.

Abstract

A player starts at x in $(-G, G)$ and attempts to leave the interval in a limited playing time. In the discrete time problem, G is positive integer and the position is described by a random walk starting at integer x , with mean increments zero, and variance increment chosen by the player from $[0, 1]$ at each integer playing time. In the continuous time problem, the player's position is described by an Ito diffusion process with infinitesimal mean parameter zero and infinitesimal diffusion parameter chosen by the player from $[0, 1]$ at each time instant of play. To maximize the probability of leaving the interval $(-G, G)$ in a limited playing time, the player should play boldly by always choosing largest possible variance increment in the discrete-time setting and largest possible diffusion parameter in the continuous-time setting, until the player leaves the interval. In the discrete time setting, this result affirms a conjecture of Spencer. In the continuous time setting, the value function of play is also identified.

1. Introduction

In this paper the best strategies are identified for analogous stochastic control problems in discrete-time and continuous-time settings.

The discrete-time problem is stated and solved in the setting of Dubins and Savage [1] in section two. In this problem, the player's initial position is integer x in interval $(-G, G)$, with positive integer G given; as time evolves, the player's position is described by a random walk $X_0 = x, X_1, X_2, \dots$ with increments $X_{n+1} - X_n = s_n Y_n$, where $Y_1, Y_2, \dots$ are independent random variables with $P(Y_n = 1) = 1/2 = P(Y_n = -1)$ and $s_n = s_n(X_0, \dots, X_n)$ is chosen by the player from $[0, 1]$ at time n . The player's goal is to maximize the probability that he leaves the interval $(-G, G)$ within N plays (or equivalently, to maximize the probability that he is outside the interval after N plays). It is shown that an optimal strategy for the player is bold play, that is, to always choose $s_n = 1$ until he leaves the interval (Theorem 2.1). The technique of proof consists of examining two related stochastic control problems, one in which the player has fewer options at each instant and the other in which he has more options, and showing that bold play is optimal for each of these problems and hence also for the original problem. In gambling terminology, this result says that a gambler, betting on the outcome of a fair coin toss, restricted to making bets

of size 1 or smaller and attempting to win or lose at least G in N or fewer bets, should always bet 1 until he achieves $\pm G$.

The result that bold play is optimal when starting at 0 and attempting to leave $(-G, G)$ in limited playing time was conjectured by Joel Spencer [7], in the context of a problem in combinatorial optimization. In [4], Maria Klawe considered a similar problem in which the interval $(-G, G)$ is replaced by $(-\infty, G)$, and showed that if G is a positive integer and the player starts at 0 and is limited to i plays, then always betting 1 is optimal. Klawe used recursion to prove her result; the technique used to prove Theorem 2.1 in this paper can also be applied to Klawe's problem to yield her main result. Klawe also showed that if G is not an integer, then a natural extension of the above bold strategy (the gambler bets 1 or the smallest amount required to reach G with a move to the right, whichever is smaller) is not in general optimal. If, in our problem, the above bold strategy is extended to that strategy in which the gambler bets 1 or the smallest amount required to have some chance of hitting G or $-G$, whichever is smaller, then it is easy to find examples in our setting also to show that if G is not an integer the bold strategy need not be optimal.

The continuous-time stochastic control problem is stated and solved in the setting of Pestien and Sudderth [6] in section three. In this problem, the player's initial

position is x in an interval (A,B) , with $-\infty < A < B < \infty$ given; as time evolves, the player's position is described by an Ito diffusion process with $X_0 = x$, and $dX_t = s(t)dB_t$ for $t \geq 0$, where $\{B_t; t \geq 0\}$ is a standard Brownian motion and $s(t)$ is a non-anticipative function chosen by the player with values in $[0,1]$. The player's goal is to maximize the probability that he leaves the interval (A,B) within time $\tau > 0$ (or equivalently, to maximize the probability that he is outside the interval (A,B) after τ units of playing time). It is shown that an optimal strategy for the player is bold play, that is, to choose $s(t) = 1$ at all times t until he leaves the interval (Theorem 3.1). The value function for this control problem is also identified. The technique of proof is to apply a verification lemma for identifying optimal strategies of Pestien and Sudderth [6], through use of probabilistic representations and properties of solutions of heat equation, initial-boundary value problems. This method of proof can also be used to prove a continuous-time stochastic control analogue for the discrete-time problem of M. Klawe (see Remark 3.4 and Theorem 3.5).

2. Discrete-Time Stochastic Control

In this section the discrete-time stochastic control problem of the introduction is formulated in the setting of Dubins and Savage [1]. The main result, Theorem 2.1, is then stated. For the proof of this result, two modifications of this control problem are introduced; in the first the player has less choice and in the second he has more choice than in the original problem. It is shown that bold play is optimal for each of these modifications. Because starting at an integer the player can achieve the same return in each of these problems, bold play is optimal for the original problem.

The stochastic control problem is described by the gambling problem (F, Γ, u) . The state space (fortune space) F is given by

$$F = \{(x, i) : x \in \mathbb{R}, i = 0, 1, \dots\}, \quad (2.1)$$

where x is the player's current position and i is the number of plays remaining. The gambling house Γ is a set mapping defined on F for $f = (x, i)$ by

$$\begin{aligned} \Gamma(f) &= \{\gamma : \gamma = 1/2 \delta(x+s, i-1) + 1/2 \delta(x-s, i-1) & (2.2) \\ &\quad \text{for some } s \in [0, 1]\} \text{ if } i > 0, \text{ and} \\ &= \{\delta(f)\} & \text{if } i = 0, \end{aligned}$$

where $\delta(f)$ denotes the probability measure concentrated at f (thus, if $i > 0$, the player's choice of $s \in [0, 1]$ moves him from x to $x+s$ or $x-s$, with probability $1/2$ each, and reduces the number of plays remaining from i to $i-1$). Let

G be any positive integer. The utility function u is defined on F for $f = (x, i)$ by

$$\begin{aligned} u(f) &= 1 && \text{if } |x| \geq G \text{ and } i = 0, \text{ and} \\ &= 0 && \text{otherwise.} \end{aligned} \quad (2.3)$$

The player constructs strategies σ by piecing-together gambles from the gambling house Γ in the usual way (as described in [1]); these strategies are the possible laws of motion for processes $f_0 = (x, i), f_1, f_2, \dots$. For this gambling problem, it suffices to consider those strategies under which motion ceases (and number of remaining plays decrease to zero) if and when the fortune process exits the interval $(-G, G)$; such strategies σ have payoff under utility u and initial state $f = (x, i)$ of P_σ (player reaches $\pm G$ from x in i plays). The utility U of the gambling house Γ is then given as that function on F defined for initial state $f = (x, i)$ by

$$U(f) = \sup \{ P_\sigma(\text{player reaches } \pm G \text{ from } x \text{ in } i \text{ plays}) : \sigma \text{ is a strategy from } \Gamma \text{ starting at } f \}. \quad (2.4)$$

The player's goal is to find an optimal strategy, i.e., a strategy σ which achieves the value of the utility of the gambling house $U(f)$, for every initial state $f \in F$. The *bold strategy* (bold play) is defined to be that strategy for which the gambler bets 1 or the smallest amount required to have positive probability of hitting G or $-G$, whichever is smaller, until the player leaves $(-G, G)$, and ceases motion after

exiting the interval. The main result of this section is the following.

Theorem 2.1. In the gambling problem (F, Γ, u) , bold play achieves the utility of the house $U(f)$, for each $f = (x, i)$ with integer-valued starting position x ; i.e., bold play is optimal.

The proof of Theorem 2.1 is based on the following two propositions.

First, consider the modified control problem in which the player has the reduced options of either staking 0 (no motion) or staking 1 (maximum displacement ± 1) on each play. That is, consider the gambling problem (F, Γ_*, u) , where F and u are as in (2.1) and (2.3) respectively, and Γ_* is defined for $f = (x, i)$ by

$$\begin{aligned} \Gamma_*(f) &= \{\delta(f), 1/2 \delta(x-1) + 1/2 \delta(x+1)\} & \text{if } i > 0, \text{ and} \\ &= \{\delta(f)\} & \text{if } i = 0. \end{aligned} \quad (2.5)$$

For this problem, the result is straightforward.

Proposition 2.2. For the gambling problem (F, Γ_*, u) bold play is an optimal strategy.

Proof. We claim that there is an optimal strategy for which: if the gambler stakes 0 at any stage, he stakes 0 from then on. Clearly there is an optimal strategy (because there are essentially only finitely many strategies available from each state). Pick any such strategy. If this strategy

violates our claim then there must be some stage at which the strategy bets 0 and at the next stage bets 1. It is clear that simply interchanging these bets (betting 1 first and then betting 0) yields the same distribution of fortunes after two plays, and hence gives the gambler the same payoff. Moreover, this interchange, by pushing bets of 1 earlier and bets of 0 later, produces an optimal strategy more nearly satisfying the claim. A finite number of such interchanges results in an optimal strategy satisfying our claim. Now if we consider strategies which once having bet 0 must always thereafter bet 0, it is clear that bold is as good as any. $\square$

Now, define function R on F as follows:

for $f = (x, i)$ with integer-valued x , $R_i(x) = U_*(x, i) =$ the utility of the reduced gambling house Γ_* (defined analogously to (2.4)) at initial state (x, i) ; and for all other real x , $R_i(x)$ is defined by linear interpolation inside $[-G, G]$, and by 1 outside this interval. (2.6)

It follows that

- (a) $R_i(x) \geq R_{i-1}(x)$ for all $x \in \mathbb{R}$, $i > 0$, and (2.7)
- (b) R_i is convex on $[-G, G]$, for each $i = 0, 1, \dots$.

It suffices to verify (2.7) for (x, i) with integer x in $(-G, G)$ and $i > 0$; in this case (2.7)(a) is immediate and

(2.7) (b) follows from (2.7) (a) and Proposition 2.2 in that $R_{i-1}(x) \leq R_i(x) = (R_{i-1}(x-1) + R_{i-1}(x+1))/2$. This function R is precisely the utility function of a gambling house which is larger than the original house Γ , as the next proposition demonstrates.

As the second modification of the original control problem, consider the enlargement of (F, Γ, u) in which the player has the additional option starting at (x, i) for any non-integer $x \in [-G, G]$ of choosing a gamble which allows him to go to one of the two nearest integers with probabilities that make the gamble fair, and which does not decrease the number of plays of the original kind that he is allowed. That is, consider the gambling problem (F, Γ^*, u) , where F and u are as in (2.1) and (2.3) respectively, and Γ^* is defined for $f = (x, i)$ by

$$\begin{aligned} \Gamma^*(f) &= \Gamma(f) \cup \{(1-\varepsilon)\delta(j, i) + \varepsilon\delta(j+1, i)\} \\ &\quad \text{if } x = j+\varepsilon \text{ where } j \text{ is an integer and } \varepsilon \in (0, 1), \\ &\quad \text{with } x \in [-G, G], \text{ and} \\ &= \Gamma(f) \text{ otherwise.} \end{aligned} \tag{2.8}$$

Let U^* to be the utility of the enlarged gambling house Γ^* (defined analogously to (2.4)).

Proposition 2.3. For the gambling problem (F, Γ^*, u) , $U^* = R$, where R is defined in (2.6).

Proof. Since a player can achieve R by taking the extra

gamble on day 1 and then playing boldly, it clearly follows that $R \leq U^*$. It follows from Theorem 2.12.1 of [1] that also $R \geq U^*$, once it is shown that

(a) $R \geq u$, and (2.9)

(b) R is excessive for Γ^* , i.e., for each state

$f = (x, i)$ and each gamble γ in $\Gamma^*(f)$,

$R_i(x) \geq$ the expected value of R under γ .

Clearly, $R \geq u$; so it suffices to show (2.9)(b).

If x is an integer, then R is excessive at (x, i) . In particular, if $x \in (-G, G)$, $i > 0$, and $s \in [0, 1]$, then

$$R_i(x) = (R_{i-1}(x-1) + R_{i-1}(x+1))/2 \geq (R_{i-1}(x-s) + R_{i-1}(x+s))/2,$$

where the inequality follows from (2.7)(b). Note that since x is an integer, there is no extra gamble available in $\Gamma^*(x, i)$.

If x is not an integer, the desired inequality holds for the extra gamble, because R is linear between the integers.

For the remaining cases, note first that it is not necessary to consider gambles which give positive probability to achieving cash outside $[-G, G]$. The reason for this is the following. Each R_i is convex on $[-G, G]$ and is an even function; therefore it is non-increasing on $[-G, 0]$ and non-decreasing on $[0, G]$. Assuming $G > 1$ (since the case $G = 1$ is trivial), if s is so large that the resulting cash might be outside the interval, reducing s to just hit the endpoint

of the interval will not decrease the expected value of R . Thus we can restrict attention to gambles which give probability 1 to $[-G, G]$.

We need thus only check that at each (x, i) with non-integer $x \in (-G, G)$ and $i > 0$ and each $s \in [0, 1]$ (and small enough that $x \pm s$ lies in $[-G, G]$) we have: $R_i(x) \geq (R_{i-1}(x-s) + R_{i-1}(x+s))/2$. To simplify the arithmetic, let α, β, γ , and δ be consecutive integers, and suppose $x = \beta + \epsilon$, where $\epsilon \in (0, 1)$.

Case 1: $s < \min(\epsilon, 1-\epsilon)$. Then $x+s$ and $x-s$ both belong to $[\beta, \gamma]$ so

$$(R_{i-1}(x-s) + R_{i-1}(x+s))/2 = R_{i-1}(x) \leq R_i(x).$$

Case 2: $\min(\epsilon, 1-\epsilon) \leq s \leq \max(\epsilon, 1-\epsilon)$. Then one of $x \pm s$ lies in $[\beta, \gamma]$ and (at most) one outside. Without loss of generality (use symmetry of R about 0) we suppose that $x-s \in [\alpha, \beta]$. Then $R_{i-1}(x-s) = (s-\epsilon)R_{i-1}(\alpha) + (1+\epsilon-s)R_{i-1}(\beta)$. Moreover, $R_{i-1}(x+s) = (1-\epsilon-s)R_{i-1}(\beta) + (\epsilon+s)R_{i-1}(\gamma)$. Hence the expected value of R under this gamble is

$$((s-\epsilon)R_{i-1}(\alpha) + (2-2s)R_{i-1}(\beta) + (\epsilon+s)R_{i-1}(\gamma))/2.$$

We need to show that this does not exceed

$$\begin{aligned} R_i(x) &= (1-\epsilon)R_i(\beta) + \epsilon R_i(\gamma) \\ &= ((1-\epsilon)/2)R_{i-1}(\alpha) + ((1-\epsilon)/2)R_{i-1}(\gamma) + \epsilon R_i(\gamma) \\ &\geq ((1-\epsilon)/2)R_{i-1}(\alpha) + ((1+\epsilon)/2)R_{i-1}(\gamma). \end{aligned}$$

To this end, we subtract the expected value of R for a bet of s from our lower bound for $R_i(x)$ just computed. This yields $((1-s)/2)R_{i-1}(\alpha) - (1-s)R_{i-1}(\beta) + ((1-s)/2)R_{i-1}(\gamma)$, which is non-negative by the convexity of R_{i-1} .

Case 3: $\max(\epsilon, 1-\epsilon) \leq s$. Then both $x+s$ and $x-s$ lie outside (β, γ) . The same sort of calculation as in Case 2 works here too. □

Proof of Theorem 2.1. Because R equals the utility function U^* of the gambling house Γ^* which is larger than the original gambling house Γ , it exceeds the utility function U for the original house. Since starting at any integer one can achieve R in the original house, R on the integers cannot exceed the utility U . Hence at integer points, R must be equal to the utility U of the original house, and bold play is optimal starting from an integer. □

3. Continuous-Time Stochastic Control

In this section we present the continuous-time stochastic control problem of the introduction, formulated in the setting of Pestien and Sudderth [6]. We then state our main result, Theorem 3.1, and prove it using a verification lemma (analogous to the application of Theorem 2.12.1 of [1] in Proposition 2.3) of Pestien and Sudderth [6]. To set up this application, we use probabilistic representations and properties of solutions of heat equation, initial-boundary value problems.

The stochastic control problem is described by the continuous-time gambling problem (F, Σ, u) . The state space F is given by $F = \mathbb{R} \times [0, \infty)$, where, for $f = (x, \tau) \in F$, x is the player's current position and τ is the remaining playing time. The global approach to defining strategies is taken as the gambling house Σ is defined as the mapping which assigns to each $f = (x, \tau) \in F$, the non-empty collection $\Sigma(f)$ of

Ito processes $X = \{X(t) : t \geq 0\}$ with state space F satisfying

$X(0) = f = (x, \tau)$, $X(t) = (X_1(t), X_2(t))$, $t \geq 0$, for which

if $\tau > 0$, then

$$\begin{aligned} X(t) &= (x + \int_0^t s(r) dW(r), \tau - t) && \text{for } 0 \leq t \leq \tau, \\ &= X(\tau) && \text{for } \tau < t, \text{ and} \end{aligned}$$

if $\tau = 0$, then $X(t) = (x, 0)$ for $t \geq 0$, (3.1)

where $X, W = \{W(t): t \geq 0\}$, and $s = \{s(t): t \geq 0\}$ are defined on some probability space $(\Omega, \mathcal{F}, P)$ with increasing filtration $\{\mathcal{F}_t: t \geq 0\}$ of complete sub σ -fields of $\mathcal{F}$, so that $W = \{W(t): t \geq 0\}$ is a standard Brownian motion adapted to $\{\mathcal{F}_t: t \geq 0\}$ and $s = \{s(t): t \geq 0\}$ is adapted to $\{\mathcal{F}_t: t \geq 0\}$, with $s(t)$ taking values in $[0,1]$ for each $t \geq 0$ (for Ito processes, see e.g., Chapter 4 of [8]). Let $-\infty < A < B < \infty$ be given. The utility function u is defined on F for $f = (x, \tau)$ by

$$\begin{aligned} u(f) &= 1 && \text{if } x \notin (A, B) \text{ and } \tau = 0, \text{ and} \\ &= 0 && \text{otherwise.} \end{aligned}$$

In this approach strategies for the player correspond to the Ito processes $X = (X_1, X_2)$ available in the gambling house $\Sigma(f)$, for each initial state $f \in F$. For process $X = (X_1, X_2)$ chosen from $\Sigma(f)$ define

$$\begin{aligned} T(X_1) &= \inf\{t \geq 0: X_1(t) \notin (A, B)\} \text{ if this set is non-empty, and} \\ &= +\infty && \text{otherwise.} \end{aligned} \tag{3.2}$$

For this gambling problem, for any initial state $f = (x, \tau) \in F$, it suffices to consider those processes $X = (X_1, X_2)$ in $\Sigma(f)$ for which the motion ceases (and remaining playing time runs out) after the process exits the interval (A, B) ; such processes have payoff under utility u of $P^X(T(X_1) \leq \tau)$. The value function V is defined on F for $f = (x, \tau)$ by

$$V(f) = \sup\{P^X(T(X_1) \leq \tau): X = (X_1, X_2) \text{ is a process in } \Sigma(f)\}. \tag{3.3}$$

The player's objective is to identify V and to find an optimal process; i.e., for each $f = (x, \tau) \in F$, to find a process $X = (X_1, X_2)$ for which $V(f) = P^X(T(X_1) \leq \tau)$. The bold strategy (bold play) at $f = (x, \tau) \in F$ is that process X of form (3.1) with $s = \{s(t) : t \geq 0\}$ chosen so that, for $x \in (A, B)$, $s(t) = 1$ if $x + W(t) \in (A, B)$ and $= 0$ otherwise; and for $x \notin (A, B)$, $s(t) \equiv 0$. If we denote $\hat{W} = \{\hat{W}(t) : t \geq 0\}$ as a Brownian motion with state space $[A, B]$ and absorption at endpoints A and B (as defined, e.g., by Knight in Chapter 4 of [5]), then the payoff to the player using bold play at initial state $f = (x, \tau) \in F$ is

$$\begin{aligned} Q(x, \tau) &= P^X(T(\hat{W}) \leq \tau) && \text{if } x \in (A, B) \\ &= 1 && \text{otherwise.} \end{aligned} \quad (3.4)$$

The main result of this section is the following.

Theorem 3.1. For the gambling problem (F, Σ, u) , bold play is optimal.

Proof. It is shown that the payoff function $Q(f)$ for bold play at f (in (3.4)) equals the value function $V(f)$ (in (3.3)), for all $f \in F$. By definition of bold play, we have $Q \leq V$, and $Q(f) = V(f)$ for all $f = (x, \tau)$ with $x \notin (A, B)$ or $\tau = 0$. Clearly, state space F can be replaced by $\tilde{F} = [A, B] \times [0, \infty)$. It follows from Lemma 1 of [6] that $V(f) \geq Q(f)$ for all $f \in \tilde{F}$, once it is shown that

- (a) $Q \geq u$ on $\tilde{F}$, and
 (b) Q is excessive for Σ ; that is, for each $f \in \tilde{F}$,
 $X \in \Sigma(f)$, and (a.s. finite) $\{F_t\}$ -stopping time
 τ , $Q(f) \geq E(Q(X(\tau)))$. (3.5)

Now, (3.5) (a) is immediate. Let $\tilde{F}^0 = (A, B) \times (0, \infty)$. Property (3.5) (b) will follow from Lemma 3 of [6], once we exhibit a sequence of functions $\{Q_n: n = 1, 2, \dots\}$ on $\tilde{F}$ satisfying the following properties:

- (a) there is an open set G containing $\tilde{F}$, on which Q_n is defined and has continuous second-order partial derivatives;
 (b) $\lim_{n \rightarrow \infty} Q_n(f) = Q(f)$ for every $f \in \tilde{F}$;
 and for each $f = (x, \tau) \in \tilde{F}^0$ and each $X \in \Sigma(f)$ (of form (3.1)),
 (c) $\limsup_{n \rightarrow \infty} -(Q_n)_t(f) + (1/2)s^2(Q_n)_{xx}(f) \leq 0$
 for every $s \in [0, 1]$;
 (d) $E \int_0^t |(Q_n)_x(X(r))s(r)|^2 dr < \infty$ for all $n = 1, 2, \dots$
 and $t \geq 0$;
 (e) there is an integrable random variable Y such that
 $Q_n(X(t)) \geq Y$ for all $n = 1, 2, \dots$ and $t \geq 0$, and
 (f) there is a measurable process $C = \{C(r): r \geq 0\}$ such
 that $-(Q_n)_t(X(r)) + (1/2)(s(r))^2(Q_n)_{xx}(X(r)) \leq C(r)$
 for all $n = 1, 2, \dots$ and $r \geq 0$, and $E \int_0^t |C(r)| dr < \infty$
 for all $t \geq 0$. (3.6)

In order to identify functions $\{Q_n: n = 1, 2, \dots\}$ satisfying the properties of (3.6), we introduce the following

functions and initial-boundary value problems and prove a lemma. Define functions g and g_n , $n = 1, 2, \dots$, on $[A, B]$ by

$$\begin{aligned} g(x) &= Q(x, 0) = 0 \text{ if } x \in (A, B), \text{ and } = 1 \text{ if } x = A \text{ or } x = B; \\ \text{and for } n &= 1, 2, \dots, g_n \text{ is any function continuous on} \\ [A, B] &\text{ taking values in } [0, 1] \text{ with } g_n(A) = 1 = g_n(B), \\ \text{which has continuous second derivative } g_n'' &\text{ satisfying} \\ g_n'' &\geq 0 \text{ and } g_n''(A) = 0 = g_n''(B), \text{ and require that} \\ g_n(x) &\rightarrow g(x) \text{ as } n \rightarrow \infty \text{ for all } x \in [A, B]. \end{aligned} \quad (3.7)$$

Also, let δ_n , $n = 1, 2, \dots$, be (strictly) positive real numbers with $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. For each $n = 1, 2, \dots$, let (P_n) denote the initial-boundary value problem

$$z_t = (1/2)z_{xx} \quad \text{for } (x, t) \in (A, B) \times (-\delta_n, \infty) \quad (3.8)$$

$$z(A, t) = 1 = z(B, t) \quad \text{for } t \geq -\delta_n \quad (3.9)$$

$$z(x, -\delta_n) = g_n(x) \quad \text{for } A \leq x \leq B \quad (3.10)$$

By a solution of (P_n) we mean a function $z_n = z_n(x, t)$ which is defined and continuous on $[A, B] \times [-\delta_n, \infty)$; has continuous first partials and satisfies (3.8) in $(A, B) \times (-\delta_n, \infty)$, and satisfies (3.9)-(3.10). Properties of solutions of these initial-boundary value problems are collected in the following.

Lemma 3.2. For $n = 1, 2, \dots$, let $Z_n = Z_n(x, t)$ on $[A, B] \times [-\delta_n, \infty)$ denote the unique solution of (P_n) satisfying (3.8)-(3.10). Then Z_n has the following properties:

- (a) $0 \leq Z_n \leq 1$ on $[A, B] \times [-\delta_n, \infty)$, and $(Z_n)_t \geq 0$ on $(A, B) \times (-\delta_n, \infty)$;
- (b) Z_n is continuous on $[A, B] \times [-\delta_n, \infty)$, and there is a function $\tilde{Z}_n$ defined on $\mathbb{R} \times [-\delta_n, \infty)$ for which $\tilde{Z}_n = Z_n$ on $[A, B] \times [-\delta_n, \infty)$, and $\tilde{Z}_n$ has continuous second partials on $\mathbb{R} \times (-\delta_n, \infty)$;
- (c) Z_n has representation

$$Z_n(x, t) = P^x(T(\hat{W}) \leq t + \delta_n) + E^x[g_n(\hat{W}(t + \delta_n))I(T(\hat{W}) > t + \delta_n)],$$

where $\hat{W} = \{\hat{W}(t) : t \geq 0\}$ is Brownian motion on $[A, B]$ with absorption at both endpoints and $T(\hat{W})$ is defined in (3.2) ($I(A)$ denotes the indicator function of set A).

Proof of Lemma 3.2. Existence and uniqueness (as a solution bounded on finite time intervals) of solution Z_n to (P_n) follows, for example, as in ([3]; page 123). For series representations of the solution see Widder ([9]; Sections II.5 and VIII.6) and Knight ([5]; Sections 4.1 and 5.2). The probabilistic representation of part (c) follows from Knight (see [5]; Section 4.1).

The maximum principle for (P_n) (as stated for example by Friedman ([3]; p. 34) or Widder ([9]; Section II.3) is used with (3.7) to obtain inequalities $0 \leq Z_n \leq 1$. This maximum principle for initial-boundary value problems can also be used to obtain the inequality $(Z_n)_t \geq 0$. For example, apply the maximum principle to obtain $v \geq 0$, where v satisfies

$$v_t = (1/2)v_{xx} \quad \text{for } (x,t) \in (A,B) \times (-\delta_n, \infty)$$

$$v(A,t) = 0 = v(B,t) \quad \text{for } t \geq -\delta_n$$

$$v(x, -\delta_n) = (1/2)g_n''(x) \quad \text{for } A \leq x \leq B;$$

show that for each $\varepsilon > -\delta_n$, $\phi(x,t) := \int_{\varepsilon}^t v(x,s)ds + Z_n(x,\varepsilon)$ satisfies

$$z_t = (1/2)z_{xx} \quad \text{for } (x,t) \in (A,B) \times (\varepsilon, \infty)$$

$$z(A,t) = 0 = z(B,t) \quad \text{for } t \geq \varepsilon$$

$$z(x, \varepsilon) = z_n(x, \varepsilon) \quad \text{for } A \leq x \leq B$$

and hence $Z_n(x,t) = \int_{\varepsilon}^t v(x,s)ds + Z_n(x,\varepsilon)$ for $(x,t) \in [A,B] \times [\varepsilon, \infty)$ for each $\varepsilon > -\delta_n$, and conclude $(Z_n)_t = v \geq 0$ on $(A,B) \times (-\delta_n, \infty)$.

To obtain the extension $\tilde{Z}_n$ of part (b), 'embed' (P_n) in a Cauchy problem as follows. Recall the definition of $g_n(x)$ for $x \in [A,B]$ given in (3.7) and define functions h_n , $\tilde{h}_n$, and w_n by

$$\begin{aligned} h_n(y) &= 1 - g_n(x) \quad \text{where } y = (x-A)/(B-A), \quad 0 \leq y \leq 1; \\ \tilde{h}_n(y) &= h_n(y) \text{ for } 0 \leq y \leq 1, \quad \tilde{h}_n(y) = -h_n(-y) \text{ for } -1 \leq y \leq 0, \\ &\text{and } \tilde{h}_n(y) = \tilde{h}_n(y+2) = \tilde{h}_n(y-2) \text{ for } y \in \mathbb{R}; \text{ and} \\ w_n(y, \tau) &= (2\pi\tau)^{-1/2} \int_{-\infty}^{\infty} \tilde{h}_n(\xi) \exp\{-(y-\xi)^2/2\tau\} d\xi \\ &\quad \text{for } (y, \tau) \in \mathbb{R} \times (0, \infty) \\ &= \tilde{h}_n(y) \quad \text{for } y \in \mathbb{R} \text{ and } \tau = 0. \end{aligned} \quad (3.11)$$

Then w_n is the solution of the Cauchy problem

$$w_\tau = (1/2)w_{yy} \quad \text{for } (y, \tau) \in \mathbb{R} \times (0, \infty) \quad (3.12)$$

$$w(y, 0) = \tilde{h}_n(y) \quad \text{for } y \in \mathbb{R}$$

which is unique as a solution bounded on finite time intervals and satisfies

$$w_n(0, \tau) = 0 = w_n(1, \tau) \quad \text{for } \tau \geq 0 \quad (3.13)$$

(see, for example ([3]; Sections I.8 and I.9)). Finally, define $\tilde{Z}_n$ on $\mathbb{R} \times [-\delta_n, \infty)$ by $\tilde{Z}_n(x, t) = 1 - w_n(y, \tau)$ where $y = (x-A)/(B-A)$ and $\tau = (B-A)^{-2}t + \delta_n$, and observe that $\tilde{Z}_n$ satisfies the extension properties of part (b) by use of (3.11)-(3.13). $\square$

Continuation of Proof of Theorem 3.1. For $n = 1, 2, \dots$, define $Q_n(x, \tau) = Z_n(x, \tau)$ for $(x, \tau) \in [A, B] \times [0, \infty)$, where Z_n is given in Lemma 3.2. We verify that $\{Q_n: n = 1, 2, \dots\}$ satisfy the properties of (3.6). Extension property (3.6)(a) follows from Lemma 3.2(b). Convergence property (3.6)(b) follows for $(x, \tau) \in (A, B) \times [0, \infty)$ by

$$\begin{aligned} \lim_{n \rightarrow \infty} Q_n(x, \tau) &= \lim_{n \rightarrow \infty} Z_n(x, \tau) \\ &= \lim_{n \rightarrow \infty} P^X(T(\hat{W}) \leq \tau + \delta_n) + E^X[g_n(\hat{W}(\tau + \delta_n)) I(T(\hat{W}) > \tau + \delta_n)] \\ &= P^X(T(\hat{W}) \leq \tau) \\ &= Q(x, \tau) \end{aligned}$$

where the second equality follows from Lemma 3.2(c); the third

equality follows from continuity of the distribution function of $T(\hat{W})$, continuity of the paths $\hat{W}(\cdot)$, and uniform convergence of $\{g_n: n = 1, 2, \dots\}$ to g (see (3.7)); and the fourth equality from (3.4) (for $x = A$ or B , (3.6)(b) is immediate).

Inequality (3.6)(c) follows since, for $f = (x, \tau) \in \tilde{F}^0$,

$$-(Q_n)_t(f) + (1/2)s^2(Q_n)_{xx}(f) = (-1+s^2)(Q_n)_t \leq 0,$$

where we have used that $Q_n = Z_n$ satisfies (3.8) and $(Z_n)_t \geq 0$ on $\tilde{F}^0$, and $0 \leq s \leq 1$. The inequality of (3.6)(f) follows similarly, by using $C(r) \equiv 0$. With $Y \equiv 0$, inequality (3.6)(e) follows from Lemma 3.2(a).

Finally, for the finiteness property (3.6)(d), observe that for X in $\Sigma(x, \tau)$ of form (3.1),

$$\begin{aligned} E \int_0^t |(Q_n)_x(X(r))s(r)|^2 dr \\ \leq E^X \int_0^{\tau \wedge T(X_1)} |(Q_n)_x(X_1(r), \tau-r)|^2 dr \\ \leq \max\{|(Q_n)_x(x, r)|^2 : (x, r) \in [A, B] \times [0, \tau]\} < \infty. \quad \square \end{aligned}$$

Remark 3.3. Series representations for function Q of (3.4) are given by D. Freedman ([2]; Section 1.3) and F. Knight ([5]; Theorem 4.1.1).

Remark 3.4. Consider the continuous-time gambling problem (F, Σ, u^*) , where F and Σ are defined as in the beginning of this section and utility function u^* is defined on F for $f = (x, \tau)$ by $u^*(f) = 1$ if $x \geq B$ and $\tau = 0$, and $= 0$ otherwise.

This stochastic control problem is a continuous-time analogue of the discrete-time gambling problem of M. Klawe mentioned in the introduction. If one defines for $X = (X_1, X_2)$ in $\Sigma(f)$, the stopping time

$$\begin{aligned} T^*(X_1) &= \inf\{t \geq 0: X_1(t) \geq B\} && \text{if this set is nonempty} \\ &= +\infty && \text{otherwise,} \end{aligned}$$

then one can represent the value function for this gambling problem for $f = (x, \tau) \in F$ by

$$V^*(f) = \sup\{P^X(T^*(X_1) \leq \tau): X = (X_1, X_2) \text{ is a process in } \Sigma(f)\}. \quad (3.14)$$

Now, define the bold strategy (bold play) in this context as the process X of (3.1) so that for $x < B$, then $s(t) = 1$ if $x + W(t) \in (-\infty, B)$, and $= 0$ otherwise; and for $x \geq B$, $s(t) \equiv 0$. Then the payoff under bold play is given by

$$\begin{aligned} Q^*(x, \tau) &= P^X(T(W^*) \leq \tau) && \text{if } x < B \\ &= 1 && \text{otherwise,} \end{aligned} \quad (3.15)$$

where W^* is a Brownian motion with state space $(-\infty, B]$ and with absorption at endpoint B . Then the method of proof of Theorem 3.1 carries over with few modifications to give the following.

Theorem 3.5. For the gambling problem (F, Σ, u^*) , bold play is optimal. In particular, the value function V^* of (3.14) equals the payoff function Q^* under bold play of (3.15).

Acknowledgement.

The authors wish to thank V. Pestien, G. Meyer, and N. Chafee for useful discussions concerning material in Section 3. This work was initiated at the Institut für Mathematische Stochastik of the Georg-August-Universität Göttingen. The authors wish to thank Prof. Krengel and the Institute for their hospitality.

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