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SEQUENTIAL SELECTION PROCEDURES FOR
MULTI-FACTOR EXPERIMENTS INVOLVING
KOOPMAN-DARMOIS POPULATIONS
WITH ADDITIVITY¹

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1. Introduction and Summary

The benefit achieved by factorial experimentation for ranking and selection problems in which there is no interaction between the (qualitative) factor-levels was pointed out by Bechhofer [1954], Section 4. That section considered a single-stage procedure for ranking normal means when the factor-level effects on the normal means are additive, and the common variance of the populations is assumed to be known. This problem was studied further by Bawa [1972] who indicated the magnitude of savings in total sample size that could be achieved in such settings by employing one multi-factor experiment in place of the corresponding independent single-factor experiments. Bawa's result was asymptotic for the situation in which the specified minimum probability of a correct selection approaches unity.

The purpose of the present article is to generalize these ideas and results in two directions: First, to point out how the additive model for normal means can be extended in a natural and meaningful way to other members of the Koopman-Darmois (K-D), i.e., exponential, family. To that end we shall employ the notation and terminology used in Table 4.1.5 of Bechhofer, Kiefer and Sobel [1968] (referred to hereinafter as B-K-S). (In that table the normal means problem is referred to as Case A.) Second, to show how the basic sequential ranking procedure (Procedure ϕ_B^*) described in Section 12.3 of B-K-S, can be adapted to accommodate selection and ordering (ranking) goals for multi-factor experiments involving K-D and related frequency functions (fr.f.'s) when the additive model holds. We rely here heavily on the results in B-K-S, no new theoretical developments being required.

Although the procedures that we describe are applicable to p-factor experiments ($p \geq 2$) we shall illustrate the methods only with 2-factor experiments (the generalization to multi-factor experiments being

straightforward). Also, we shall focus our attention only on one goal for selection problems, and consider it for normal means, Bernoulli single-trial "success" probabilities, Poisson means and multinomial single-trial "cell" probabilities. Selection procedures for single-factor experiments involving these fr.f.'s are given in Sections 12.6.1.1, 12.6.1.4, 12.6.1.5 and 5.3.3, respectively, of B-K-S.

In Section 2 we set forth our K-D model for 2-factor experiments, and introduce our notation. We also state for arbitrary K-D fr.f.'s with "additivity" our selection goal and indifference-zone probability requirement. In Section 3 we describe our generalized sequential selection procedure for 2-factor experiments involving arbitrary K-D populations with "additivity." These procedures are then specialized in Section 4 to specific K-D and related fr.f.'s. Section 5 discusses two methods of sequential experimentation, namely one-factor-at-a-time vs. factorial experimentation, and demonstrates, via Monte Carlo (MC) sampling, the savings in expected total sample size that can result from using factorial experiments. Some concluding remarks are made in Section 6.

2. Two-factor Experiments for Koopman-Darmois fr.f.'s

For 2-factor experiments involving qualitative variables we assume that we have $a \cdot b$ fr.f.'s which are of the same univariate K-D or related family, i.e., for each (i, j) ($1 \leq i \leq a$, $1 \leq j \leq b$) the random variable has a fr.f. of the form

$$f(x_{ij}, \theta_{ij}) = \exp\{P(x_{ij})Q(\theta_{ij}) + R(x_{ij}) + S(\theta_{ij})\} \quad (2.1)$$

for θ_{ij} in an open interval Θ ; these fr.f.'s are assumed to differ at most from each other only in the values of the θ_{ij} . Here $P(x_{ij})$ and $R(x_{ij})$ do not depend on θ_{ij} , while $Q(\theta_{ij})$ and $S(\theta_{ij})$ do not depend on x_{ij} . We assume that Q is a continuous strictly increasing function on Θ , and that $Q(\theta_{ij})$ can be expressed as

$$Q(\theta_{ij}) = \tau + \xi_i + \eta_j \quad (1 \leq i \leq a, 1 \leq j \leq b) \quad (2.2)$$

for some $\{\xi_i\}$ and $\{\eta_j\}$ satisfying $\sum_{i=1}^a \xi_i = \sum_{j=1}^b \eta_j = 0$. If (2.2) holds we shall say that the factor-level "effects" on $Q(\theta_{ij})$ are additive. In this article we are concerned only with the additive model. We refer to ξ_i (η_j) as the "effect" of the ith (jth) level of the first (second) factor on $Q(\theta_{ij})$. The ordered values of the ξ_i and η_j are denoted by

$$\xi_{[1]} \leq \dots \leq \xi_{[a]}, \quad \eta_{[1]} \leq \dots \leq \eta_{[b]}. \quad (2.3)$$

It is assumed that the values of the $\xi_i, \xi_{[i]}$ (all i) and $\eta_j, \eta_{[j]}$ (all j) are completely unknown, and that the pairing of the $\xi_{[i]}$ and $\eta_{[j]}$ with the $a \cdot b$ fr.f.'s (2.1) is also completely unknown.

To illustrate the formulation of the problem we focus on the important goal of selecting the factor-level combination associated with $\xi_{[a]}$ and $\eta_{[b]}$. However, our methods apply equally well to the more general goals (and others) posed as Goals I and II for $t \geq 1$ in Section 3.6.2 of B-K-S. We now formally state the objective of our experiment, and give the associated probability requirement.

GOAL: To select that one of the $a \cdot b$ fr.f.'s (2.1) which is associated with $\xi_{[a]}$ and $\eta_{[b]}$. (2.4)

We limit consideration to procedures that guarantee the following indifference-zone requirement on the probability of a correct selection (CS):

PROBABILITY REQUIREMENT:

$$P\{CS\} \geq P^* \quad \text{whenever} \quad \begin{cases} \xi_{[a]} - \xi_{[a-1]} \geq \delta_{\xi}^* \\ \eta_{[b]} - \eta_{[b-1]} \geq \delta_{\eta}^* \end{cases} \quad (2.5)$$

The constants $\{\delta_{\xi}^*, \delta_{\eta}^*; P^*\}$ with $(0 < \delta_{\xi}^*, \delta_{\eta}^* < \infty, 1/ab < P^* < 1)$ are specified prior to the start of experimentation.

The basic sequential selection procedure (Procedure $\mathcal{P}_B^*$) for K-D and related fr.f.'s and Goals I and II, which is given for single-factor experiments in Chapter 5 of B-K-S, takes observations a vector-at-a-time. Our corresponding generalized procedure (Procedure $\mathcal{P}_{B(2,+)}^*$) for 2-factor experiments with additivity takes observations a matrix-at-a-time, i.e., $\{X_{ijs} \mid (1 \leq i \leq a, 1 \leq j \leq b)\}$, $s = 1, 2, \dots$. Except in Section 4.4 which deals with multinomial populations, the elements X_{ijs} ($1 \leq i \leq a, 1 \leq j \leq b$) in each $a \cdot b$ matrix are independent for each $s = 1, 2, \dots$, and the matrices are i.i.d.

At the mth stage of experimentation ($m = 1, 2, \dots$) the X_{ijs} enter our procedure only through the statistics

$$A_{im} = \sum_{j=1}^b \sum_{s=1}^m P(X_{ijs}) \quad (1 \leq i \leq a)$$

and

(2.6)

$$B_{jm} = \sum_{i=1}^a \sum_{s=1}^m P(X_{ijs}) \quad (1 \leq j \leq b).$$

We denote the ordered values of the A_{im} and B_{jm} by

$$A_{[1]m} \leq \dots \leq A_{[a]m}, \quad B_{[1]m} \leq \dots \leq B_{[b]m}. \quad (2.7)$$

Equal A_{im} -values and equal B_{jm} -values can occur with positive probabilities for discrete r.v.'s; when this happens they should be arranged in any definite order.

In the next section we describe our generalization of Procedure $\mathcal{P}_B^*$ for 2-factor experiments involving K-D populations with additivity.

3. Sequential Selection Procedure (Procedure $\mathcal{P}_{B(2,+)}^*$) for 2-factor K-D fr.f.'s with Additivity

BASIC SELECTION PROCEDURE (Procedure $\mathcal{P}_{B(2,+)}^*$)

The sampling rule: At the m th stage of experimentation ($m = 1, 2, \dots$)

observe the random matrix $\{X_{ijm} \mid (1 \leq i \leq a, 1 \leq j \leq b)\}$. (3.1a)

The stopping rule: After the m th stage of experimentation, compute

the statistic

$$Z_m = U_m + V_m + U_m \cdot V_m$$

$$\text{where } \begin{cases} U_m = \sum_{i=1}^{a-1} \exp\{-\delta_{\xi}^*(A_{[a]m} - A_{[i]m})\} \\ V_m = \sum_{j=1}^{b-1} \exp\{-\delta_{\eta}^*(B_{[b]m} - B_{[j]m})\} \end{cases} \quad (3.1b)$$

Stop at the first value of m (call it n) for which

$$Z_n \leq (1-P^*)/P^*.$$

The terminal decision rule: Select the population associated with

$A_{[a]n}$ and $B_{[b]n}$. If, because of ties between the A_{in}
and/or B_{jn} , two or more populations are associated with (3.1c)

$A_{[a]n}$ and/or $B_{[b]n}$, then select one such population at random.

Remark 3.1: The stopping rule (3.1b) and the terminal decision rule (3.1c) can be derived by paralleling the development of Section 5.2.1 of B-K-S as follows: Consider the associated identification problem when

$$\xi_{[1]}^0 = \dots = \xi_{[a-1]}^0 = \xi_{[a]}^0 - \delta_{\xi}^* \quad \text{and} \quad (3.2)$$

$$\eta_{[1]}^0 = \dots = \eta_{[b-1]}^0 = \eta_{[b]}^0 - \delta_{\eta}^*.$$

After cancelling out the common factors involving R and S , we obtain as our stopping rule: "Stop at the first value of m for which

$$\max_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b}} \left[\frac{\exp\{\xi_{[a]}^0 A_{im} + \eta_{[b]}^0 B_{jm} + \xi_{[a-1]}^0 \sum_{r \neq i} A_{rm} + \eta_{[b-1]}^0 \sum_{s \neq j} B_{sm}\}}{\sum_{p=1}^a \sum_{q=1}^b \exp\{\xi_{[a]}^0 A_{pm} + \eta_{[b]}^0 B_{qm} + \xi_{[a-1]}^0 \sum_{r \neq p} A_{rm} + \eta_{[b-1]}^0 \sum_{s \neq q} B_{sm}\}} \right] \geq P^*, \quad (3.3)$$

and as our terminal decision rule: "Select the population associated with the cell (i^*, j^*) which yields the maximum in the l.h.s. of (3.3)." It is readily verified that the maximum of the l.h.s. of (3.3) is given by the population yielding $A_{[a]n}$ and $B_{[b]n}$. Furthermore, after dividing the

denominator of the l.h.s. of (3.3) by the numerator, replacing $\xi_{[a]}^0 - \xi_{[a-1]}^0$ by δ_{ξ}^* and $\eta_{[b]}^0 - \eta_{[b-1]}^0$ by δ_{η}^* , and then inverting both sides of equation (3.3) we obtain

$$\begin{aligned}
 1 + \sum_{i=1}^{a-1} \exp\{-\delta_{\xi}^*(A_{[a]n} - A_{[i]n})\} + \sum_{j=1}^{b-1} \exp\{-\delta_{\eta}^*(B_{[b]n} - B_{[j]n})\} \\
 + \sum_{i=1}^{a-1} \exp\{-\delta_{\xi}^*(A_{[a]n} - A_{[i]n})\} \times \sum_{j=1}^{b-1} \exp\{-\delta_{\eta}^*(B_{[b]n} - B_{[j]n})\} \leq 1/P^*
 \end{aligned} \tag{3.4}$$

which is equivalent to (3.1b).

Remark 3.2: If (2.2) does not hold, i.e., if

$$Q(\theta_{ij}) = \tau + \xi_i + \eta_j + \gamma_{ij} \quad (1 \leq i \leq a, 1 \leq j \leq b) \tag{3.5}$$

where $\sum_{i=1}^a \xi_i = \sum_{j=1}^b \eta_j = \sum_{i=1}^a \gamma_{ij} = \sum_{j=1}^b \gamma_{ij} = 0$ but $\gamma_{ij} \neq 0$ (all i, j), then the factor-level "effects" on $Q(\theta_{ij})$ are not additive. An alternative objective which can be considered in such a case is:

GOAL: To select that one of the $a \cdot b$ fr.f.'s which is associated with $\max_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b}} Q(\theta_{ij})$. (3.6)

The problem is thus reduced to one involving a single factor with $a \cdot b$ levels, and Procedure $\mathcal{P}_B^*$ described in Section 12.3 of B-K-S can be used.

4. Application of Procedure $\mathcal{P}_{B(2,+)}^*$ to Specific K-D and Related fr.f.'s with Additivity

As pointed out in B-K-S, univariate fr.f.'s of the K-D form are widely used in applied studies. A list of six of these fr.f.'s is given in Table 4.1.5 of B-K-S, these six being referred to as Cases A-F; additional univariate fr.f.'s for which Procedure $\mathcal{P}_B^*$ can be used to solve ranking goals are given in B-K-S, Table 5.1.1. Three k-variate ($k \geq 2$) fr.f.'s for which Procedure $\mathcal{P}_B^*$ can be used to solve ranking goals are given in Table 5.1.2 of B-K-S. The sequential selection procedure of our Section 3 is applicable to all of these fr.f.'s.

To illustrate our procedure we consider in some detail three univariate cases of Table 4.1.5, i.e., normal density functions with unknown means and a common known variance (Case A), Bernoulli fr.f.'s with unknown single-trial "success" probabilities (Case D), and Poisson fr.f.'s with unknown means (Case E), respectively. We also consider one multivariate case of Table 5.1.2, i.e., a single multinomial fr.f. with unknown single-trial "event" probabilities which we refer to here as Case M. To indicate that the additivity model is being assumed for these 2-factor experiments, we use the notation Case $A_2^+, \dots, \text{Case } M_2^+$.

4.1 The normal means case (Case A_2^+)

Let $\{X_{ijs} \mid (1 \leq i \leq a, 1 \leq j \leq b)\}$ ($s = 1, 2, \dots$) denote independent matrix-observations from $a \cdot b$ independent normal populations Π_{ij} with unknown means θ_{ij} ($1 \leq i \leq a, 1 \leq j \leq b$) and a common known variance σ^2 which without loss of generality we take to be unity. The fr.f. of X_{ij} is

$$f(x_{ij}, \theta_{ij}) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} (x_{ij} - \theta_{ij})^2\right\}.$$

Here $P(x_{ij}) = x_{ij}$, $Q(\theta_{ij}) = \theta_{ij}$, $R(x_{ij}) = -\frac{1}{2} [\log 2\pi + x_{ij}^2]$ and $S(\theta_{ij}) = -\theta_{ij}^2/2$. Let $\mu = \tau$, $\alpha_i = \xi_i$ ($1 \leq i \leq a$) and $\beta_j = \eta_j$ ($1 \leq j \leq b$). Then (2.2) becomes

$$\theta_{ij} = \mu + \alpha_i + \beta_j \quad (4.2)$$

which is the standard additive (no interaction) fixed-effects ANOVA model (qualitative variables). Since $\sigma^2 = 1$, the probability requirement can be stated as

$$P\{CS\} \geq P^* \text{ whenever } \begin{cases} \alpha_{[a]} - \alpha_{[a-1]} \geq \delta_\alpha^* \\ \beta_{[b]} - \beta_{[b-1]} \geq \delta_\beta^* \end{cases} \quad (4.3)$$

where $\delta_\alpha^* = \delta_\xi^*$, $\delta_\beta^* = \delta_\eta^*$ and $\alpha_{[1]} \leq \dots \leq \alpha_{[a]}$, $\beta_{[1]} \leq \dots \leq \beta_{[b]}$ are the ordered values of the "effects" α_i ($1 \leq i \leq a$) and β_j ($1 \leq j \leq b$), respectively. The constants $\{\delta_\alpha^*, \delta_\beta^*; P^*\}$ with $(0 < \delta_\alpha^*, \delta_\beta^* < \infty, 1/ab < P^* < 1)$ are specified prior to the start of experimentation.

If observations are taken a matrix-at-a-time, then $A_{im} = \sum_{j=1}^b \sum_{s=1}^m X_{ijs}$, $B_{jm} = \sum_{i=1}^a \sum_{s=1}^m X_{ijs}$, and $A_{[i]m}$ ($1 \leq i \leq a$), $B_{[j]m}$ ($1 \leq j \leq b$) are defined by (2.7). The stopping rule is then given by (3.1b) and the terminal decision rule by (3.1c).

Remark 4.1: For Case A_2^+ the common variance σ^2 is assumed to be known. If the additive model (4.2) is valid but σ^2 is unknown then Procedure $\mathcal{P}_{B(2,+)}^*$ will not guarantee (4.3). A two-stage procedure that will guarantee (4.3) when (4.2) is valid and σ^2 is unknown has been proposed by Bechhofer and Dunnett [1986].

4.2 The Bernoulli case (Case D_2^+)

Let $\{X_{ijs} \ (1 \leq i \leq a, 1 \leq j \leq b)\}$ ($s = 1, 2, \dots$) denote independent matrix-observations from $a \cdot b$ independent Bernoulli populations Π_{ij} with unknown single-trial "success" probabilities θ_{ij} ($1 \leq i \leq a, 1 \leq j \leq b$). When observations are taken a matrix-at-a-time, then $X_{ijs} = 1$ or 0 according as Π_{ij} yields a success or a failure on the s th matrix observation. The fr.f. of X_{ij} is

$$f(x_{ij}, \theta_{ij}) = \theta_{ij}^{x_{ij}} (1 - \theta_{ij})^{1-x_{ij}} = \exp\{x_{ij} \log[\theta_{ij}/(1-\theta_{ij})] + \log(1-\theta_{ij})\}. \quad (4.4)$$

Here $P(x_{ij}) = x_{ij}$, $Q(\theta_{ij}) = \log[\theta_{ij}/(1-\theta_{ij})]$, $R(x_{ij}) = 0$ and $S(\theta_{ij}) = \log(1-\theta_{ij})$.

If we assume the log linear (linear logistic) model

$$\log[\theta_{ij}/(1-\theta_{ij})] = \tau + \xi_i + \eta_j \quad (\text{all } i, j) \quad (4.5)$$

where $\sum_{i=1}^a \xi_i = \sum_{j=1}^b \eta_j = 0$, then the factor-level "effects" are additive.

Define $\alpha = (\alpha_1, \dots, \alpha_a)$ and $\beta = (\beta_1, \dots, \beta_b)$ by $\xi_i = \log[\alpha_i/(1-\alpha_i)]$ and $\eta_j = \log[\beta_j/(1-\beta_j)]$, respectively. The model (4.5) has been employed by among others, Cox [1970], Sections 2.3-2.7, 6.1-6.5, Feinberg [1977], Sections 6.2-6.5, Haberman [1978], Chapter 5, and Breslow and Day [1980], Sections 6.1-6.4.

If observations are taken a matrix-at-a-time, then $A_{im} = \sum_{j=1}^b \sum_{s=1}^m X_{ijs}$, $B_{jm} = \sum_{i=1}^a \sum_{s=1}^m X_{ijs}$, and $A_{[i]m}$ ($1 \leq i \leq a$), $B_{[j]m}$ ($1 \leq j \leq b$) are defined by (2.7), and $\xi_{[i]}$ ($1 \leq i \leq a$) and $\eta_{[j]}$ ($1 \leq j \leq b$) are defined by (2.3).

For this model the distance function can be expressed as

$$\begin{aligned}\xi_{[a]} - \xi_{[a-1]} &= \log \left\{ \frac{\alpha_{[a]}}{1-\alpha_{[a]}} \right\} - \log \left\{ \frac{\alpha_{[a-1]}}{1-\alpha_{[a-1]}} \right\} \\ \eta_{[b]} - \eta_{[b-1]} &= \log \left\{ \frac{\beta_{[b]}}{1-\beta_{[b]}} \right\} - \log \left\{ \frac{\beta_{[b-1]}}{1-\beta_{[b-1]}} \right\}.\end{aligned}\tag{4.6}$$

Since Q is a strictly increasing function, the goal (2.4), probability requirement (2.5), and procedure (3.1) can be stated equivalently as

GOAL: To select that one of the $a \cdot b$ fr.f.'s which is associated with $\alpha_{[a]}$ and $\beta_{[b]}$. (4.7)

PROBABILITY REQUIREMENT:

$$P\{CS\} \geq P^* \text{ whenever } \begin{cases} \frac{\alpha_{[a]}}{1-\alpha_{[a]}} / \frac{\alpha_{[a-1]}}{1-\alpha_{[a-1]}} \geq \theta_{\alpha}^* \\ \frac{\beta_{[b]}}{1-\beta_{[b]}} / \frac{\beta_{[b-1]}}{1-\beta_{[b-1]}} \geq \theta_{\beta}^* \end{cases} \tag{4.8}$$

where $\theta_{\alpha}^* = e^{\delta_{\xi}^*}$ and $\theta_{\beta}^* = e^{\delta_{\eta}^*}$. The constants $\{\theta_{\alpha}^*, \theta_{\beta}^*; P^*\}$ with $(1 < \theta_{\alpha}^*, \theta_{\beta}^* < \infty, 1/ab < P^* < 1)$ are specified prior to the start of experimentation. Note that the probability requirement is now stated in terms of the "odds ratios" of the α_i 's and of the β_j 's. The stopping rule is then given by (3.1b) with

$$U_m = \sum_{i=1}^{a-1} \left[\frac{1}{\theta_{\alpha}^*} \right]^{A_{[a]m} - A_{[i]m}}, \quad V_m = \sum_{j=1}^{b-1} \left[\frac{1}{\theta_{\beta}^*} \right]^{B_{[b]m} - B_{[j]m}}. \tag{4.9}$$

4.3 The Poisson case (Case E_2^+)

Let $\{X_{ijs} \mid (1 \leq i \leq a, 1 \leq j \leq b)\} \quad s = 1, 2, \dots$ denote independent matrix-observations from $a \cdot b$ independent Poisson populations Π_{ij} with unknown means $\theta_{ij} \quad (1 \leq i \leq a, 1 \leq j \leq b)$. The fr.f. of X_{ij} is

$$f(x_{ij}, \theta_{ij}) = \frac{\theta_{ij}^{x_{ij}}}{x_{ij}!} \exp\{-\theta_{ij}\} = \exp\{x_{ij} \log \theta_{ij} - \log(x_{ij}!) - \theta_{ij}\}. \quad (4.10)$$

Here $P(x_{ij}) = x_{ij}$, $Q(\theta_{ij}) = \log \theta_{ij}$, $R(x_{ij}) = -\log(x_{ij}!)$ and $S(\theta_{ij}) = -\theta_{ij}$.

If we assume the log linear model

$$\log \theta_{ij} = \tau + \xi_i + \eta_j \quad \text{all } (i, j) \quad (4.11)$$

where $\sum_{i=1}^a \xi_i = \sum_{j=1}^b \eta_j = 0$, then the factor-level effects are additive.

Define α and β by $\xi_i = \log \alpha_i$ and $\eta_j = \log \beta_j$, respectively. The model (4.11) was employed by, among others, Rasch [1960], pp. 13-23 and Haberman [1978], Section 2.5.

If observations are taken a matrix-at-a-time, then A_{im} , $A_{[i]m}$ ($1 \leq i \leq a$), B_{jm} , $B_{[j]m}$ ($1 \leq j \leq b$) are defined as for Case D_2^+ , and $\xi_{[i]}$ ($1 \leq i \leq a$), $\eta_{[j]}$ ($1 \leq j \leq b$) are defined as in (2.3). The goal is given by (4.7), and the probability requirement (2.5) can be stated equivalently as

PROBABILITY REQUIREMENT:

$$P\{CS\} \geq P^* \quad \text{whenever} \quad \begin{cases} \frac{\alpha_{[a]}}{\alpha_{[a-1]}} \geq \theta_{\alpha}^* \\ \frac{\beta_{[b]}}{\beta_{[b-1]}} \geq \theta_{\beta}^* \end{cases} \quad (4.12)$$

where θ_{α}^* and θ_{β}^* are defined as below (4.8). The stopping rule is then given by (3.1b) with U_m and V_m given by (4.9).

4.4 The multinomial case (Case M_2^+)

This example does not have the structure (2.1). Nevertheless Procedure $\mathcal{P}_{B(2,+)}^*$ of Section 3 is applicable. Let $\{X_{ijs} \ (1 \leq i \leq a, 1 \leq j \leq b)\}$ $s = 1, 2, \dots$ denote independent matrix-observations from a single $(a \cdot b)$ -variate multinomial population having an unknown probability matrix $\{\theta_{ij}\} = \{\theta_{ij} \ (1 \leq i \leq a, 1 \leq j \leq b)\}$. Here $\theta_{ij} \ (0 \leq \theta_{ij} \leq 1, \sum_{i=1}^a \sum_{j=1}^b \theta_{ij} = 1)$ denotes the probability that $X_{ijs} = 1 \ (s = 1, 2, \dots)$ on the sth matrix observation and 0 otherwise with $\sum_{i=1}^a \sum_{j=1}^b X_{ijs} = 1$ for all s .

The $a \cdot b$ variate fr.f. which is bivariate K-D is

$$f(\{x_{ij}\}, \{\theta_{ij}\}) = \prod_{i=1}^a \prod_{j=1}^b \theta_{ij}^{x_{ij}} = \prod_{i=1}^a \prod_{j=1}^b e^{x_{ij} \log \theta_{ij}}. \quad (4.13)$$

Here $P(x_{ij}) = x_{ij}$, $Q(\theta_{ij}) = \log \theta_{ij}$ and $R(x_{ij}) = S(\theta_{ij}) = 0$. Define $\alpha_i = \sum_{j=1}^b \theta_{ij} \geq 0 \ (1 \leq i \leq a)$ and $\beta_j = \sum_{i=1}^a \theta_{ij} \geq 0 \ (1 \leq j \leq b)$; then $\sum_{i=1}^a \alpha_i = \sum_{j=1}^b \beta_j = 1$. If we assume the log linear model

$$\log \theta_{ij} = \xi_i + \eta_j \quad (\text{all } i, j) \quad (4.14)$$

where $\xi_i = \log \alpha_i$, $\eta_j = \log \beta_j$, then $\{\sum_{j=1}^b X_{ijs} \ (1 \leq i \leq a)\}$, $(s = 1, 2, \dots)$ can be regarded as independent vector-observations from a single a -variate $(a \geq 2)$ multinomial population with an unknown probability vector α . A similar interpretation can be given to $\{\sum_{i=1}^a X_{ijs} \ (1 \leq j \leq b)\}$, $(s = 1, 2, \dots)$ and β . Then (4.14) is equivalent to assuming the multiplicative model

$$\theta_{ij} = \alpha_i \cdot \beta_j \quad (\text{all } i, j) \quad (4.15)$$

wherein the factor-levels for each of the two factors are independent. This model is being studied by Bechhofer and Goldsman [1986b] in the context of single-stage multinomial selection procedures for multi-factor experiments. It had been studied earlier by, among others, Feinberg [1977], Chapter 2, Upton [1978], Sections 3.1-3.7, and Plackett [1974], Sections 5.1-5.3, mostly from the viewpoint of estimation and hypothesis testing.

Proceeding as in Sections 4.2 and 4.3, we see that the goal (2.4) and probability requirement (2.5) can be stated equivalently as (4.7) and (4.12), respectively, where the α_i ($1 \leq i \leq a$) and β_j ($1 \leq j \leq b$) are defined as above, and the $\alpha_{[i]}$ and $\beta_{[j]}$ have the usual interpretation with θ_α^* and θ_β^* defined as below (4.8). The U_m and V_m of (3.1b) are defined as in (4.9).

5. Savings in Expected Total Sample Size Resulting from Factorial Experimentation

The analysis in this section wherein the emphasis is on expected total sample size is relevant for Cases A_p^+ , D_p^+ and E_p^+ but not for Case M_p^+ ; for the latter the corresponding emphasis would be on expected total number of stages.

5.1 One-factor-at-a-time vs. factorial experimentation

In this section we compare the expected total sample sizes resulting from two possible methods of experimentation both of which guarantee the same probability requirement (2.5).

1) In the first method of experimentation two independent single-factor experiments are conducted, one of which guarantees the probability requirement

$$P\{CS\} \geq P_1^* \quad \text{whenever} \quad \xi_{[a]} - \xi_{[a-1]} \geq \delta_\xi^*, \quad (5.1a)$$

while the second guarantees

$$P\{CS\} \geq P_2^* \quad \text{whenever} \quad \eta_{[b]} - \eta_{[b-1]} \geq \delta_\eta^*. \quad (5.1b)$$

The constants $\{\delta_\xi^*, P_1^*\}$, $\{\delta_\eta^*, P_2^*\}$ with $(0 < \delta_\xi^* < \infty, 1/a < P_1^* < 1)$, $(0 < \delta_\eta^* < \infty, 1/b < P_2^* < 1)$ and $P_1^* \cdot P_2^* = P^*$ with $(1/ab < P^* < 1)$ are specified prior to the start of experimentation.

Remark 5.1: This method of experimentation was suggested by the results reported for Case A_2^+ in Section 4 of Bechhofer [1954]. There the $P\{CS\}$ of our (2.5) factored into the product of the $P\{CS\}$ of our (5.1a) and the $P\{CS\}$ of our (5.1b). Of course this factorization is a consequence of the independence present in Case A_2^+ .

For each of the two single-factor experiments we employ Procedure $\mathcal{P}_B^*$ as described in Section 12.3 of B-K-S using $\{\delta_\xi^*, P_1^*\}$ for the first experiment and $\{\delta_\eta^*, P_2^*\}$ for the second. Since the experiments are independent, it is clear that the two, regarded as a combined experiment, will guarantee (2.5). Let $n_1'(a, \delta_\xi^*, P_1^*)$ and $n_1''(b, \delta_\eta^*, P_2^*)$ denote the random number of stages to terminate sampling for the first and second experiment, respectively. These depend on the $\xi_{[a]} - \xi_{[i]}$ ($1 \leq i \leq a-1$) and the $\eta_{[b]} - \eta_{[j]}$ ($1 \leq j \leq b-1$); in our present study we shall restrict consideration to the so-called least-favorable (LF) configurations, i.e., $\xi_{[a]} - \xi_{[i]} = \delta_\xi^*$ ($1 \leq i \leq a-1$) and $\eta_{[b]} - \eta_{[j]} = \delta_\eta^*$ ($1 \leq j \leq b-1$). For these configurations we will be interested in $a \cdot E\{n_1'(a, \delta_\xi^*, P_1^*) | LF\} + b \cdot E\{n_1''(b, \delta_\eta^*, P_2^*) | LF\}$, the expected total number of observations to terminate sampling using Procedure $\mathcal{P}_B^*$ with the two independent single-factor experiments. The optimal choice of (P_1^*, P_2^*) to

minimize this quantity depends on the form of the fr.f., and on $(a, \delta_{\xi}^*; b, \delta_{\eta}^*)$. For the completely symmetric setup in which $a = b = k$ (say) and $\delta_{\xi}^* = \delta_{\eta}^* = \delta^*$ (say) it is clear that one should set $P_1^* = P_2^* = \sqrt{P^*}$. Then the expected total number of observations reduces to $2kE\{n_1'(k, \delta^*, \sqrt{P^*}) | LF\} = 2kE\{n_1 | LF\}$ (say).

2) In the second method of experimentation one 2-factor experiment is conducted as described in Section 3. Let $n_2(a, b, \delta_{\xi}^*, \delta_{\eta}^*; P^* | LF)$ denote the random number of stages to terminate sampling for this two-factor experiment when $\xi_{[a]} - \xi_{[i]} = \delta_{\xi}^*$ ($1 \leq i \leq a-1$) and $\eta_{[b]} - \eta_{[j]} = \delta_{\eta}^*$ ($1 \leq j \leq b-1$). To obtain direct comparability with the setup described in 1), above, we again consider the situation in which $a = b = k$ and $\delta_{\xi}^* = \delta_{\eta}^* = \delta^*$. Here we will be interested in the expected total number of observations,

$k^2 E\{n_2(k, k; \delta^*, \delta^*; P^*) | LF\} = k^2 E\{n_2 | LF\}$ (say). Then

$$RE(k, \delta^*, P^* | LF) = \frac{k^2 E\{n_2 | LF\}}{2kE\{n_1 | LF\}} \quad (5.2)$$

can be regarded as a relative efficiency (RE) measure associated with the two methods of experimentation.

Remark 5.2: Bawa [1972] showed that if the single-stage procedures of Bechhofer [1954] for Case A_2^+ are employed in the completely symmetric setup, then asymptotically ($P^* \rightarrow 1$), the RE (5.2) is equal to $1/2$, uniformly in (k, δ^*) , i.e., a 2-factor experiment which guarantees (4.3) requires one-half as many total observations on-the-average as do two independent single-factor experiments when the combined single-factor experiments also guarantee the probability requirement (4.3). More generally, in the completely symmetric setup for Case A_p^+ , the asymptotic RE is equal to $1/p$ where p is the number of factors.

5.2 Monte Carlo sampling results

We used Monte Carlo (MC) sampling to obtain estimates of $RE(k, \delta^*, P^* | LF)$ for Case A_2^+ of Section 4.1 for selected (k, P^*) with $\delta^* = 0.2$. These estimates which are given in Table 5.3 are based on the corresponding estimates of $E\{n_1 | LF\}$ and $E\{n_2 | LF\}$ in Tables 5.1 and 5.2, respectively. Both of these latter tables also contain estimates of the achieved probability of a correct selection, $P\{CS_1 | LF\}$ and $P\{CS_2 | LF\}$, for the corresponding procedures. As in Tables 18.4.1-18.4.6 of B-K-S, two estimates of these probabilities are given, the first being simply the observed proportion of correct selections while the second is the average value of $W_{[k]n_1}$ (or $W_{[k^2]n_2}$) at termination (defined by (12.1.10) of B-K-S for single-factor experiments); both are unbiased estimates of the achieved probability of a correct selection, the estimate based on $\bar{W}$ having much smaller variance than the one based on the observed proportion. All of the entries in a given row of Table 5.1 (Table 5.2) are based on the same 4000 independent replications of Procedure $\mathcal{P}_B^*$ (Procedure $\mathcal{P}_{B(2,+)}^*$) for the single-factor (two-factor) experiment. The number in parentheses below each entry is the estimated standard error of the estimate above it. The entries in Table 5.1 for $k = 2, 3, 5$ and 10 are directly comparable to the corresponding entries for $P^* = 0.75, 0.90, 0.95$ and 0.99 given in Tables 18.4.1, 18.4.2, 18.4.4 and 18.4.5, respectively, of B-K-S; these provided a partial check on the correctness of our present computer simulation program. The B-K-S entries are less precise being based on only 800 replications. The entries in Table 5.3 are derived from those in Tables 5.1 and 5.2.

Table 5.1

Monte Carlo Estimates of $P\{CS_1|LF\}$ and $E\{n_1|LF\}$
 obtained with Procedure $\mathcal{P}_B^*$ for single-factor Case A_1^+ experiments
 with selected $k = a = b$ and P^* when $\delta^* = 0.2$

k	P*	Estimate of $P\{CS_1 LF\}$		Estimate of $E\{n_1 LF\}$
		Observ. Proportion of Correct Selections	$\bar{w}_{[k]n_1}$	$\bar{n}_1(LF)$
2	0.75	0.7762 (0.0066)	0.7794 (0.0004)	17.6 (0.2)
	0.90	0.9100 (0.0045)	0.9138 (0.0002)	48.8 (0.6)
	0.95	0.9538 (0.0033)	0.9572 (0.0001)	70.6 (0.8)
	0.99	0.9925 (0.0014)	0.9915 (0.00002)	117.0 (1.2)
	0.999	0.9988 (0.0005)	0.99915 (<0.00001)	178.4 (1.5)
3	0.75	0.7650 (0.0067)	0.7766 (0.0003)	38.9 (0.4)
	0.90	0.9020 (0.0047)	0.9129 (0.0002)	77.6 (0.8)
	0.95	0.9568 (0.0032)	0.9569 (0.0001)	102.6 (1.0)
	0.99	0.9898 (0.0016)	0.9914 (0.00002)	153.9 (1.3)
	0.999	0.9988 (0.0005)	0.99915 (<0.00001)	218.8 (1.6)

Table 5.1 (continued)

k	P*	Estimate of $P\{CS_1 LF\}$		Estimate of $E\{n_1 LF\}$
		Observ. Proportion of Correct Selections	$\bar{w}_{[k]n_1}$	$\bar{n}_1(LF)$
5	0.75	0.7762 (0.0066)	0.7759 (0.0003)	64.3 (0.6)
	0.90	0.9135 (0.0044)	0.9124 (0.0002)	109.5 (1.0)
	0.95	0.9580 (0.0032)	0.9566 (0.0001)	136.5 (1.2)
	0.99	0.9898 (0.0016)	0.9914 (0.00002)	196.0 (1.5)
	0.999	0.9995 (0.0004)	0.99914 (<0.00001)	262.1 (1.7)
10	0.75	0.7665 (0.0067)	0.7742 (0.0003)	97.2 (0.8)
	0.90	0.9075 (0.0046)	0.9117 (0.0002)	149.8 (1.2)
	0.95	0.9585 (0.0032)	0.9564 (0.0001)	178.0 (1.3)
	0.99	0.9913 (0.0015)	0.9913 (0.00002)	238.4 (1.7)
	0.999	0.9990 (0.0005)	0.99913 (<0.00001)	313.8 (1.9)

All estimates in this table are based on 4,000 replications. The numbers in parentheses are the estimated standard errors of the estimates above them.

Table 5.2

Monte Carlo Estimates of $P\{CS_2|LF\}$ and $E\{n_2|LF\}$
 obtained with Procedure $\mathcal{P}_{B(2,+)}^*$ for 2-factor Case A_2^+ experiments
 with selected $k = a = b$ and P^* when $\delta^* = 0.2$

k	P^*	Estimate of $P\{CS_2 LF\}$		Estimate of $E\{n_2 LF\}$
		Observ. Proportion of Correct Selections	$\bar{w}_{[k^2]n_2}$	$\bar{n}_2(LF)$
2	0.5625 = $(0.75)^2$	0.6028 (0.0077)	0.6102 (0.0006)	11.1 (0.1)
	0.8100 = $(0.90)^2$	0.8388 (0.0058)	0.8383 (0.0004)	31.5 (0.3)
	0.9025 = $(0.95)^2$	0.9185 (0.0043)	0.9188 (0.0002)	45.0 (0.4)
	0.9801 = $(0.99)^2$	0.9825 (0.0021)	0.9838 (0.00004)	73.5 (0.6)
	0.998001 = $(0.999)^2$	0.9990 (0.0005)	0.99838 (<0.00001)	107.9 (0.7)
3	0.5625 = $(0.75)^2$	0.6200 (0.0077)	0.6155 (0.0007)	16.3 (0.1)
	0.8100 = $(0.90)^2$	0.8390 (0.0058)	0.8424 (0.0004)	32.4 (0.3)
	0.9025 = $(0.95)^2$	0.9270 (0.0041)	0.9213 (0.0002)	43.2 (0.3)
	0.9801 = $(0.99)^2$	0.9815 (0.0021)	0.9843 (0.00005)	63.6 (0.4)
	0.998001 = $(0.999)^2$	0.9983 (0.0007)	0.99844 (<0.00001)	88.8 (0.5)

Table 5.2 (continued)

k	P^*	Estimate of $P\{CS_2 LF\}$		Estimate of $E\{n_2 LF\}$
		Observ. Proportion of Correct Selections	$\bar{w}_{[k^2]n_2}$	$\bar{n}_2(LF)$
5	0.5625 = $(0.75)^2$	0.6328 (0.0076)	0.6291 (0.0009)	16.4 (0.1)
	0.8100 = $(0.90)^2$	0.8525 (0.0056)	0.8511 (0.0005)	27.6 (0.2)
	0.9025 = $(0.95)^2$	0.9245 (0.0042)	0.9252 (0.0003)	34.1 (0.2)
	0.9801 = $(0.99)^2$	0.9830 (0.0020)	0.9853 (0.0001)	47.5 (0.3)
	0.998001 = $(0.999)^2$	0.9973 (0.0008)	0.99853 (<0.00001)	63.0 (0.3)
10	0.5625 = $(0.75)^2$	0.6553 (0.0075)	0.6540 (0.0011)	12.7 (0.1)
	0.8100 = $(0.90)^2$	0.8520 (0.0056)	0.8640 (0.0006)	19.0 (0.1)
	0.9025 = $(0.95)^2$	0.9363 (0.0039)	0.9332 (0.0003)	22.6 (0.1)
	0.9801 = $(0.99)^2$	0.9880 (0.0017)	0.9867 (0.0001)	29.4 (0.2)
	0.998001 = $(0.999)^2$	0.9973 (0.0008)	0.99870 (<0.00001)	37.8 (0.2)

All estimates in this table are based on 4,000 replications. The numbers in parentheses are the estimated standard errors of the estimates above them.

Table 5.3

Monte Carlo Estimates of $P\{CS|(k, \delta^*, P^*|LF)\}$
 and $RE(k, \delta^*, P^*|LF)$ for Case A_2^+ with selected
 $k = a = b$ and P^* when $\delta^* = 0.2$

k	P*	Estimates of $P\{CS (k, \delta^*, P^* LF)\}$		Estimate of $RE(k, \delta^*, P^* LF)$
		$(\bar{W}_{[k]n_1})^2$	$\bar{W}_{[k^2]n_2}$	$k^2\bar{n}_2(LF)/2k\bar{n}_1(LF)$
2	0.5625	0.6075	0.6102	0.631
	0.8100	0.8350	0.8383	0.645
	0.9025	0.9162	0.9188	0.637
	0.9801	0.9831	0.9838	0.628
	0.998001	0.9983	0.9984	0.605
3	0.5625	0.6031	0.6155	0.629
	0.8100	0.8334	0.8424	0.626
	0.9025	0.9157	0.9213	0.632
	0.9801	0.9829	0.9843	0.620
	0.998001	0.9983	0.9984	0.609
5	0.5625	0.6020	0.6291	0.638
	0.8100	0.8325	0.8511	0.630
	0.9025	0.9151	0.9252	0.625
	0.9801	0.9829	0.9853	0.606
	0.998001	0.9982	0.9985	0.601
10	0.5625	0.5994	0.6540	0.653
	0.8100	0.8312	0.8640	0.634
	0.9025	0.9147	0.9332	0.635
	0.9801	0.9827	0.9867	0.617
	0.998001	0.9982	0.9987	0.602

We note from Tables 5.1 and 5.2 that the achieved $P\{CS|LF\}$ is estimated with very high precision by $\bar{W}_{[k]n_1}$ and $\bar{W}_{[k^2]n_2}$; for both methods of experimentation the $\bar{W}$ estimate always exceeds P^* , and the magnitude of the "excess" (which is quite large for $P^* = 0.75$ and $(0.75)^2$) decreases with increasing P^* . From Table 5.3 we see that for every (k, P^*) combination we have $\bar{W}_{[k^2]n_2} > (\bar{W}_{[k]n_1})^2$, i.e., the 2-factor experiment achieves a higher $P\{CS|LF\}$ than does the combined independent single-factor experiment. This superiority in achieved $P\{CS|LF\}$ is purchased at the cost of an increase in expected total sample size for Procedure $\phi_{B(2,+)}^*$.

We also point out that for fixed $P^* = 0.9025, 0.9801$ or 0.998001 , the expected number of stages to termination $(E\{n_2|LF\})$ decreases rapidly with increasing k for Procedure $\phi_{B(2,+)}^*$ (Table 5.2). This consequence of matrix-at-a-time sampling could have practical advantages in certain situations.

From Table 5.3 we see that for each k the MC estimate of $RE(k, \delta^*, P^*|LF)$ tends to decrease (very slowly) with increasing P^* ; it can be shown analytically (by using mean path approximations as in Section 14.2.2 of B-K-S) that $\lim_{P^* \rightarrow 1} RE(k, \delta^*, P^*|LF) = 1/2$, the result reported by Bawa [1972] for the Case A_2^+ single-stage procedures in the symmetric setup.

Remark 5.3: We have conducted additional MC samplings (the detailed results of which are not reported herein) in order to study the rate of approach of $RE(k, \delta^*, P^*|LF)$ to $1/2$ for P^* approaching unity. These were carried out for $k = 2$ and can be regarded as extensions of the results given in Table 5.3. For $P^* = (1-10^{-5})^2, (1-10^{-7})^2$ and $(1-10^{-9})^2$ we obtained 0.587, 0.574 and 0.564, respectively.

If the achieved $P\{CS|LF\}$ for the two methods of experimentation were equal (which could be accomplished by adjusting downward the P^* used on the r.h.s. of the equation for Z_n in (3.1b), then this would decrease $E\{n_2|LF\}$ and hence make the RE measure (5.2) somewhat closer to $1/2$. However, even without any adjustment, we see from Table 5.3 that for Case A_2^+ over the range of (k, P^*) studied, the expected total sample size using Procedure $\mathcal{P}_{B(2,+)}^*$ is only approximately 60 to 65 percent of that which would be required for two independent single-factor experiments each of which used Procedure $\mathcal{P}_B^*$. We would anticipate analogous results for Cases D_2^+ and E_2^+ although the rate of approach to $1/2$ with increasing P^* can be anticipated to depend on the fr.f.

6. Concluding Remarks

We have proposed a generalized version of Procedure $\mathcal{P}_B^*$ of B-K-S which can be used for selection and ordering (ranking) problems arising from multi-factor experiments involving K-D (and related) fr.f.'s which possess a certain additivity property. This procedure can be used for p-factor ($p \geq 2$) experiments having complete additivity, and in particular for normal means with a common known variance (Case A_p^+) and many of the standard log linear models arising in categorical data analysis, e.g., the Bernoulli (Case D_p^+), Poisson (Case E_p^+) and multinomial (Case M_p^+).

We have demonstrated conclusively using MC sampling experiments that this generalized procedure, Procedure $\mathcal{P}_{B(p,+)}^*$, for p-factor experiments with additivity when applied to Case A_p^+ for $p = 2$, requires a substantially smaller expected total number of observations than that which is required by

two independent single-factor experiments when both methods of experimentation guarantee the same requirement on the probability of a correct selection. In addition, the 2-factor experiment achieves a higher probability of a correct selection than do the combined two independent single-factor experiments.

Although our MC studies were only for Case A_2^+ in symmetric settings we anticipate savings (although not as great) in non-symmetric settings, with the same qualitative results for $p > 2$ and other K-D fr.f.'s with additivity.

Procedure ϕ_B^* and Procedure $\phi_{B(p,+)}^*$ are open procedures. The values of $E\{n_1\}$ and $E\{n_2\}$ as well as $\text{Var}\{n_1\}$ and $\text{Var}\{n_2\}$ can be decreased by truncation of the procedures for Case A_p^+ ($p \geq 1$). Truncation of Procedure ϕ_B^* for single-factor experiments involving the multinomial has been carried out by Bechhofer and Goldsman [1985], [1986a] with the objective of decreasing $E\{\text{Stages}\}$ and $\text{Var}\{\text{Stages}\}$. Such truncation studies can also be carried out for Case M_p^+ ($p > 1$). These problems are currently under investigation by the authors.

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