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ON INFINITE PERFECT GRAPHS AND
RANDOMIZED STOPPING POINTS ON THE PLANE*

By

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On Infinite Perfect Graphs and Randomized Stopping Points on the Plane

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Abstract. We show that perfection of polytopes defined by infinite zero-one matrices can be characterized as in the finite case through an appropriate definition of infinite perfect graphs. We examine the relationship between certain such polytopes and a convex set of random measures on the discrete plane, termed randomized stopping points, that appear in the context of the optimal stopping problem for two-parameter processes. Using probabilistic techniques, we show that under a conditional qualitative independence hypothesis on the underlying filtration, which we express as a commutation property of a "conditional supremum" operator, the only extremal elements of this set of random measures are ordinary stopping points.

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§I. Introduction

To a filtration indexed by a directed set one can associate the convex set U of randomized stopping points, into which the set of (ordinary) stopping points can be embedded. This set was first introduced by Baxter and Chacon [BC] for linear filtrations (i.e. single parameter filtrations indexed by $\mathbb{N}$ or $\mathbb{R}_+$). Stopping points are always extremal elements of U , and in the linear case, U has the following property:

(P) *every extremal element of U is a stopping point*

(see Ghoussoub [Gh]). However, for non-linear filtrations, simple examples show that this is generally not the case (see Mazziotto and Millet [MM1]).

Interest in this property of U comes initially from the optimal stopping problem for processes indexed by two or more parameters. It was shown by the author in [D] (where an incorrect statement concerning the general validity of (P) was made) that a straightforward solution to the question of existence of optimal stopping points is possible when (P) holds (the existence question for the general multiparameter optimal stopping problem has recently been solved by Mazziotto and Millet [MM2], who introduced the notion of "randomized stopping system"). So one would like to determine what conditions on the filtration imply this property.

This question was addressed by Dalang, Trotter and de Werra [DTW] when the set of randomized stopping points is finite-dimensional, by using techniques from the theory of perfect graphs. In particular, it was shown that in this case, the conditional qualitative independence hypothesis (CQI) of Krengel and Sucheston [KS] is sufficient to guarantee that stopping points are the only extremal elements of U .

The object of this paper is twofold. First, to extend a now classical result due to Fulkerson [F] and Chvátal [C], who gave necessary and sufficient conditions for integrality of all extremal elements of a polytope defined by a zero-one matrix. This result was used in [DTW] in the finite setting, and we show here that an analogous statement can be made for polytopes defined by infinite matrices. The necessary and sufficient conditions are now expressed in terms of the notion of infinite perfect graph, which we define as a graph such that each finite vertex-induced subgraph is perfect (see [G]). This result is obtained from the finite statement through a topological argument akin to that presented in Kendall [K] (in an addendum by J.

C. Kiefer) for countably infinite doubly stochastic matrices, which it contains as a special case. It also contains an infinite-dimensional Dilworth-type theorem, and gives the solution to an extreme point question closely related to that for randomized stopping points on $\mathbb{N}^2$. These results are presented in Section III.

Our second objective is to show that for filtrations indexed by $\mathbb{N}^2$, hypothesis (CQI) is sufficient to imply that the only extremal elements of the set of randomized stopping points are stopping points. For this we use an approach that is different from that of Section III: we give an explicit procedure for expressing a randomized stopping point as a convex combination of two other elements of U . This is achieved through the use of a "conditional supremum" operator, which has many properties of conditional expectation, and furthermore is invariant under a change of equivalent measure. In particular, it allows us to give an equivalent definition of conditional qualitative independence: hypothesis (CQI) becomes a commutation property of conditional suprema analogous to hypothesis (F4) of Cairoli and Walsh [CW].

§II. The set of randomized stopping points on $\mathbb{N}^2$

Let $\leq$ be the order relation on $\mathbb{N}^2$ defined by

$$s = (s_1, s_2) \leq t = (t_1, t_2) \Leftrightarrow s_1 \leq t_1 \text{ and } s_2 \leq t_2 ,$$

and denote by $\overline{\mathbb{N}^2}$ the set obtained by adding to $\mathbb{N}^2$ the element ∞ .

On a complete probability space $(\Omega, \mathcal{F}, P)$, let $(\mathcal{F}_t; t \in \mathbb{N}^2)$ be a (two-parameter) filtration, i.e. a family of complete sub σ -algebras of $\mathcal{F}$ such that $s \leq t$ implies $\mathcal{F}_s \subseteq \mathcal{F}_t$. A stopping point will be a random variable $T: \Omega \rightarrow \overline{\mathbb{N}^2}$ such that $\{T \leq t\} \in \mathcal{F}_t$, for all $t \in \mathbb{N}^2$. Let $\mathcal{T}$ denote the set of all stopping points.

In this context, a randomized stopping point (see [BC], [M], [Gh]) can be defined as a process $a = (a_t; t \in \mathbb{N}^2)$ such that

$$(2.1) \quad a_t \text{ is } \mathcal{F}_t\text{-measurable, } \forall t \in \mathbb{N}^2,$$

$$(2.2) \quad a_t \geq 0 \text{ a.s., } \forall t \in \mathbb{N}^2 ,$$

$$(2.3) \quad \sum_{t \in \mathbb{N}^2} a_t \leq 1 \text{ a.s.}$$

This definition is slightly different from those given in the references above, where a randomized stopping point is identified with a random probability measure on $\mathbb{N}^2$. Here, $a_t(\bullet)$ would be the (random) mass of $\{t\}$ under this measure, so there is an obvious equivalence of these definitions.

Let U denote the set of all randomized stopping points. This set is clearly convex. Furthermore, if $T \in T$, one can define $a = (a_t: t \in \mathbb{N}^2) \in U$ by setting $a_t = I_{\{T=t\}}$. This randomized stopping point has the property that $a_t \in \{0,1\}$ a.s., $\forall t \in \mathbb{N}^2$, and it is not difficult to see that every $\{0,1\}$ -valued randomized stopping point arises in this way. This association provides a means for embedding T into U .

It is clear that the set of extremal elements of U contains T . As pointed out in the introduction, this inclusion is generally strict. When U is finite-dimensional, techniques from the theory of perfect graphs were used in [DTW] to show that the hypothesis (CQI) of [KS] is sufficient to guarantee that $T = \text{ext } U$.

Given the success of graph-theoretic techniques in the finite case, it is natural to try to develop infinite-dimensional results of this kind. This is the object of the next section.

§III. Infinite perfect graphs

The graph-theoretic terminology used throughout will be that of Golumbic [G].

We prove here a generalization of a theorem due to Fulkerson [F] and Chvátal [C] that gives necessary and sufficient conditions for the polyhedron $P(A) = \{x \in \mathbb{R}^n: Ax \leq \underline{1}, x \geq 0\}$ to have only integer-valued extreme points. (A denotes an $m \times n$ matrix with coefficients in $\{0,1\}$ and $\underline{1}$ denotes a column vector of m ones.)

In this section we consider a matrix $A = (a_{i,j}: i \in I, j \in J)$, where I and J are two arbitrary sets, and $a_{i,j} \in \{0,1\}$, $\forall i \in I, j \in J$. The set $P(A)$ is now defined by

$$P(A) = \{x = (x_j: j \in J): \sum_{j \in J} a_{i,j} x_j \leq 1, \forall i \in I, x_j \geq 0, \forall j \in J\},$$

which is a (possibly) infinite-dimensional convex set (for a given $x \in P(A)$ and $i \in I$, there will of course be at most countably many $j \in J$ such that $a_{i,j} x_j > 0$).

The main result of this section is that the necessary and sufficient condition for this "infinite-dimensional polyhedron" to have integer-valued extreme points is that every subpolyhedron supported by finitely many columns of A have this property. This leads to a natural definition of infinite perfect graphs.

We now introduce the graph-theoretic terminology we will be needing. A couple $G = (V, E)$ is a graph if V is a (not necessarily finite) set whose elements are termed vertices, and E is a set of pairs of elements of V , called edges. A clique in G is a set $K \subseteq V$ such that for all $v_1, v_2 \in K$, $\{v_1, v_2\} \in E$. Two elements $v_1, v_2 \in V$ are neighbors if $\{v_1, v_2\} \in E$.

If G is a finite graph (i.e. V is finite), let $w(G)$ be the cardinality of the largest clique in G , and let $\chi(G)$ denote the fewest number of colors needed to color the vertices of G in such a way that no two neighbors receive the same color.

Given $W \subseteq V$, we define the subgraph induced by W to be $G_W = (W, E_W)$, where $E_W = \{\{v_1, v_2\} \in E : v_1, v_2 \in W\}$. A finite graph $G = (V, E)$ is termed perfect if $w(G_W) = \chi(G_W)$ for each $W \subseteq V$.

3.1 Definition. An infinite graph is perfect if the subgraph induced by each finite set of vertices is perfect.

We will say that $(x_j : j \in J) \in \mathbb{R}^J$ dominates $y = (y_j : j \in J) \in \mathbb{R}^J$ if $y_j \leq x_j, \forall j \in J$. Given a matrix $A = (a_{i,j} : i \in I, j \in J)$ with $\{0,1\}$ -valued entries, the derived graph of A has vertices $v_j, j \in J$, corresponding to the columns of A , with an edge $\{v_j, v_{j'}\}$ when there is some $i \in I$ such that $a_{i,j} = a_{i,j'} = 1$. Given a graph $G = (V, E)$, an augmented clique matrix of G is a matrix A with $\{0,1\}$ -valued entries such that G is the derived graph of A , and such that the indicator function of each finite (not necessarily maximal) clique in G is dominated by some row of A . These definitions coincide with the usual ones when I and J are finite.

3.2 Theorem. All extreme points of $P(A)$ are integer-valued if and only if (a) and (b) below hold:

- (a) A is an augmented clique matrix of its derived graph;
- (b) the derived graph of A is perfect.

The starting point to the proof is the theorem of Fulkerson and Chvátal that states that 3.2 holds when I and J are finite (see [G], Chap. 3.4). The extension to general I and J is similar to that contained in [K] for countably infinite doubly stochastic matrices, and uses the following form of the Krein-Milman theorem (see [B], II.7.1).

3.3 In a locally convex Hausdorff vector space, consider a compact convex set P and a closed subset T of P . Then (a) and (b) are equivalent:

- (a) P is the closed convex hull of T ;
- (b) T contains all extreme points of P .

Let τ be the product topology on $\mathbb{R}^J$. This topology makes $\mathbb{R}^J$ a locally convex Hausdorff vector space.

For a finite sequence $(j_1, \dots, j_n)$ of elements of J , let $A_{(j_1, \dots, j_n)}$ be the submatrix of A corresponding to columns $j_1, \dots, j_n$ of A . Note that $A_{(j_1, \dots, j_n)}$ only has finitely many distinct rows. Set

$$P_{(j_1, \dots, j_n)} = \{x \in \mathbb{R}^n : A_{(j_1, \dots, j_n)} x \leq \mathbf{1}, x \geq 0\}.$$

3.4 Lemma. If A contains no zero-column, then:

- (a) $P(A)$ is τ -compact;
- (b) Suppose $P_{(j_1, \dots, j_n)}$ has only integer-valued extreme points for all $n \in \mathbb{N}$, $(j_1, \dots, j_n) \in J^n$. Then $P(A)$ is the closed convex hull of $P(A) \cap \{0,1\}^J$.

Proof. If A contains no zero-column, then $[0,1]^J \supseteq P$, so for (a) we only need to show that P is τ -closed.

Observe that

$$P = \left(\bigcap_{i \in I} \bigcap_{n \in \mathbb{N}} \bigcap_{(j_1, \dots, j_n) \in J^n} \left\{ \sum_{k=1}^n a_{ij_k} x_{j_k} \leq 1 \right\} \right) \cap \left(\bigcap_{j \in J} \{x_j \geq 0\} \right),$$

so P , as the intersection of closed sets, is closed.

To prove (b), consider $x \in P(A)$, and let O be a τ -open set containing x . Without loss of generality we may suppose

$$O = \prod_{j \in J} \sigma_j,$$

where $O_j \neq \mathbb{R}$ only for j in some finite subset $\{j_1, \dots, j_n\}$ of J . Note that $(x_{j_1}, \dots, x_{j_n}) \in P_{(j_1, \dots, j_n)}$, and so by hypothesis $(x_{j_1}, \dots, x_{j_n}) \in \text{conv}(\{0, 1\}^n \cap P_{(j_1, \dots, j_n)})$. It follows that $y \in \mathbb{R}^J$ defined by $y_j = x_{j_k}$ for $j = j_k$, $y_j = 0$ if $j \notin \{j_1, \dots, j_n\}$ is an element of $O \cap \text{conv}(\{0, 1\}^J \cap P(A))$, concluding the proof. $\square$

Proof of 3.2. Suppose $P(A)$ has only integer-valued extreme points. Then for $n \in \mathbb{N}$, $(j_1, \dots, j_n) \in J^n$, $P_{(j_1, \dots, j_n)}$ does too. But then the finite-dimensional theorem implies that (a) and (b) hold.

Suppose (a) and (b) hold, and A contains no zero column. Again, the finite statement of 3.2 implies that $P_{(j_1, \dots, j_n)}$ has integer-valued extreme points. But since $P(A) \cap \{0, 1\}^J$ is τ -compact, 3.3 and 3.4 imply that all extreme points of $P(A)$ are integer-valued. If A has some zero columns, then coordinates corresponding to these columns will be zero for each extreme point of P , so we can reduce to the previous situation. $\square$

Since the set M of countably infinite doubly stochastic matrices can be expressed as the nonnegative solutions to a linear system whose coefficients are $\{0, 1\}$ -valued, and which is known to satisfy the conditions in 3.2, 3.2 implies the now well-known result that the extreme points of M are permutation matrices (see [K]).

Another immediate corollary of 3.2 is the following result for ordered sets, which for finite ordered sets immediately gives Dilworth's theorem (see [G], Chap. 5.7).

3.5 Corollary. Let $(J, \leq)$ be an ordered set (of arbitrary cardinality). Let C^1 be the family of all maximal totally ordered subsets of J , and let C^2 be the family of all maximal sets of pairwise non-comparable elements of J . Define $A^i = (a_{c,j}^i : c \in C^i, j \in J)$ by $a_{c,j}^i = 1$ if $j \in c$ and $a_{c,j}^i = 0$ otherwise.

Then each extremal element of

$$P^i = \{x = (x_j : j \in J) \in \mathbb{R}^J : \sum_{j \in J} a_{c,j}^i x_j \leq 1, \forall c \in C^i, x_j \geq 0, \forall j \in J\}$$

is the indicator function of a set of pairwise non-comparable elements of J when $i = 1$, and the indicator function of a totally ordered subset of J when $i = 2$.

In the above, the derived graph of A^1 is transitively orientable, hence perfect (see [G], Th. 5.34), and condition (a) of 3.2 is clearly satisfied since C^1 contains all maximal totally ordered subsets of J . The derived graph of A^2 is a cocomparability graph, hence perfect (see [G], Th. 5.35), and it is again easy to see that condition (a) of 3.2 is satisfied.

Let us look at how theorem 3.2 relates to the set of randomized stopping points on $\mathbb{N}^2$.

Let $\Omega = \mathbb{R}^{\mathbb{N}^2}$ be the canonical space for discrete time two-parameter processes, $X_t: \Omega \rightarrow \mathbb{R}$ be the t^{th} coordinate mapping, $F_t = \sigma(X_s, s \leq t) : t \in \mathbb{N}^2$ be the canonical filtration,

$$F = \bigvee_{t \in \mathbb{N}^2} F_t$$

be the standard σ -algebra on Ω . Let P be some probability measure on (Ω, F) , and let $\omega = (\omega_s : s \in \mathbb{N}^2)$ denote the generic element of Ω .

In this setting, the set of randomized stopping points is the set

$$U = \{(a_t : t \in \mathbb{N}^2) : \quad (3.6) \quad a_t : \Omega \rightarrow \mathbb{R} \text{ is a Borel function that only depends on} \\ \text{coordinates } \omega_s, s \leq t;$$

$$(3.7) \quad a_t \geq 0 \text{ a.s.};$$

$$(3.8) \quad \sum_{t \in \mathbb{N}^2} a_t \leq 1 \text{ a.s.} \} .$$

Consider the set

$$V = \{ (f_t: t \in \mathbb{N}^2): \quad (3.9) \quad f_t: \Omega \rightarrow \mathbb{R} \text{ only depends on coordinates } \omega_s, s \leq t; \}$$

$$(3.10) \quad f_t \geq 0 ;$$

$$(3.11) \quad \sum_{t \in \mathbb{N}^2} f_t \leq 1 \} .$$

(The only difference between U and V is that there is no measurability requirement in (3.9), and the "a.s." have been dropped in (3.10) and (3.11)). Requiring that f_t only depends on coordinates $\omega_s, s \leq t$, is equivalent to requiring that f_t is constant on atoms of F_t .

3.12 Corollary. $(f_t: t \in \mathbb{N}^2)$ is an extremal element of V if and only if $f_t(\omega) \in \{0,1\}, \forall \omega \in \Omega, \forall t \in \mathbb{N}^2$.

Proof. Define a matrix $A = (a_{\omega, H}: \omega \in \Omega, H \in H)$, where H is the set of all atoms of all the F_t 's, $t \in \mathbb{N}^2$, and set $a_{\omega, H} = 1$ if $\omega \in H$, $a_{\omega, H} = 0$ otherwise. Then V can clearly be identified with the polytope $P(A)$.

If one considers the matrix $P_{(H_1, \dots, H_n)}$ associated with a finite set of atoms $H_1, \dots, H_n$ of H , it is easy to see that its derived graph is a two-parameter graph (see [DTW], Example 6.4), thus is perfect, and that all extreme points of $P_{(H_1, \dots, H_n)}$ are integer-valued. $\square$

One might expect from this result that extremal elements of U are $\{0,1\}$ -valued for any probability measure P . However, this depends on the null sets of P , which may be so large that U and V will be very different.

3.13 Example. Set

$$\omega^1 = (\omega_t^1 = I_{\{(0,2)\}}(t): t \in \mathbb{N}^2),$$

$$\omega^2 = (\omega_t^2 = I_{\{(1,1)\}}(t): t \in \mathbb{N}^2),$$

$$\omega^3 = (\omega_t^3 = I_{\{(2,0)\}}(t): t \in \mathbb{N}^2),$$

and let P be the probability measure on W defined by $P\{\omega^1\} = P\{\omega^2\} = P\{\omega^3\} = 1/3$. It is then easy to see that for any randomized stopping point $a = (a_t: t \in \mathbb{N}^2)$, there are constants $\beta, \beta', \beta'', \gamma, \gamma', \gamma''$, such that

$$a_{(0,2)} = \beta I_{\{\omega^1\}} + \gamma I_{\{\omega^2, \omega^3\}} \quad \text{P-a.s.}$$

$$a_{(1,1)} = \beta' I_{\{\omega^2\}} + \gamma' I_{\{\omega^1, \omega^3\}} \quad \text{P-a.s.}$$

$$a_{(2,0)} = \beta'' I_{\{\omega^3\}} + \gamma'' I_{\{\omega^1, \omega^2\}} \quad \text{P-a.s.}$$

Since the sets $\{\omega^2, \omega^3\}$, $\{\omega^1, \omega^3\}$, $\{\omega^1, \omega^2\}$ are pairwise intersecting but have an empty common intersection, they do not satisfy Helly's property. Thus there is a non-integral extremal element in U (see [DTW], section 6). In fact, setting $\beta = \beta' = \beta'' = 0$ and $\gamma = \gamma' = \gamma'' = 1/2$, and $a_t \equiv 0$ if $t \notin \{(0,2), (1,1), (2,0)\}$ gives an extremal element of U . $\square$

It is possible to use 3.2 to check whether extremal elements of the set of randomized stopping points are stopping points in the following special case:

- (3.14) each F_t is generated by a countable partition $\{H_t^1, H_t^2, \dots\}$ of Ω ,
such that $P(H_t^i) > 0$, for $t \in \mathbb{N}^2$, $i \geq 1$.

In this case, proceeding as in Section 2 of [DTW] one can show that U can be identified with the set $\{x: Ax \leq 1, x \geq 0\}$, where $A = (a_{\omega, H}: \omega \in \Omega, H \in H)$, H is the set of all the atoms of all the F_t 's and $a_{\omega, H} = 1$ if $\omega \in H$, $a_{\omega, H} = 0$ otherwise. Stopping points correspond to x with zero-one valued components. Contrary to the general case (see [BC], [M]), this set is closed both in the product topology and the Baxter-Chacon topology (see [BC]). So we can apply 3.2 and the result of [DTW, Th. 3.3 and 6.1], to get:

- (3.15) if $(F_t: t \in \mathbb{N}^2)$ satisfies hypothesis (CQI) of [KS] and (3.13) holds,
then the only extremal elements of U are stopping points.

In the following section, we shall show that this result remains valid even when the F_t are not discrete.

§IV. Randomized stopping points on $\mathbb{N}^2$

The techniques of §III, while giving conditions for extremal elements of certain polytopes to be integral, do not prescribe how to express a non-extremal element of the polytope as a convex combination

of two other distinct elements of the polytope. This is the approach we shall now adopt to show that for any filtration that satisfies hypothesis (CQI) of Krengel and Sucheston [KS], all extremal elements of the associated set of randomized stopping points on $\mathbb{N}^2$ are stopping points.

In order to do this, we need some properties of conditional probabilities. In particular, we define a "conditional supremum" operator which has properties similar to those of conditional expectation. The study of the properties of this operator is the object of section IV.1. In section IV.2, we show that conditional qualitative independence can be expressed using the conditional supremum operator in the same way that conditional independence is expressed using conditional expectation. In section IV.3 we use the equivalent form of (CQI) to express a randomized stopping point as a convex combination of two others, proving our main theorem.

§IV.1 The conditional supremum operator

We begin by stating some elementary properties of conditional expectations. G will denote a complete sub σ -algebra of F . $C \supset B$ a.s. (we will often omit the "a.s.") means $P(B \cap C^c) = 0$. Letters X , Y and Z will denote random variables. $X \in G$ will mean that X is G -measurable.

4.1 Lemma.

$$(a) \ B \in F, C \in G, C \supset B \text{ a.s.} \Rightarrow C \supset \{P(B|G) > 0\} \text{ a.s.}$$

$$(b) \ B \in F, X \in G, X \geq I_B \text{ a.s.} \Rightarrow X \geq I_{\{P(B|G) > 0\}} \text{ a.s.}$$

Proof. (a) $C \supset B$ a.s. $\Rightarrow I_C \geq I_B$ a.s. $\Rightarrow I_C \geq P(B|G) = P(B|G) I_{\{P(B|G) > 0\}}$ a.s., so (a) follows.

$$(b) \ X \geq I_B \text{ a.s. implies } X \geq 0 \text{ a.s. and } \{X \geq 1\} \supset B \text{ a.s. By (a), } \{X \geq 1\} \supset \{P(B|G) > 0\}. \quad \square$$

4.2 Lemma. If $X \geq 0$ a.s., then

$$(a) \quad \{E(X|G) > 0\} \supset \{X > 0\}$$

$$(b) \quad \{E(X|G) > 0\} = \{P(X > 0|G) > 0\}.$$

Proof. (a) $E(X I_{\{E(X|G) = 0\}}) = E(E(X|G) I_{\{E(X|G) = 0\}}) = 0$, so $P(X > 0, E(X|G) = 0) = 0$.

$$(b) \quad 0 \leq E(E(X|G) I_{\{P(X > 0|G) = 0\}}) = E(X I_{\{P(X > 0|G) = 0\}}) \leq E(X I_{\{X=0\}}) = 0,$$

where the last inequality is a consequence of (a). Thus $\{P(X > 0|G) > 0\} \supset \{E(X|G) > 0\}$ a.s. Now by (a) and 4.1(a): $\{E(X|G) > 0\} \supset \{P(X > 0|G) > 0\}$ a.s., and (b) is proven. $\square$

4.3 Lemma. For $B, C, B_n \in \mathcal{F}$,

$$(a) \quad \{P(B|G) > 0\} \supset B \supset \{P(B^c|G) = 0\}$$

$$(b) \quad \{P(B \cup C|G) > 0\} = \{P(B|G) > 0\} \cup \{P(C|G) > 0\}$$

$$(c) \quad \bigcup_{n \in \mathbb{N}} B_n = B \Rightarrow \bigcup_{n \in \mathbb{N}} \{P(B_n|G) > 0\} = \{P(B|G) > 0\}.$$

Proof. (a) is a special case of 4.2(a).

(b) Suppose $B \cap C = \emptyset$. Then $P(B \cup C|G) = P(B|G) + P(C|G)$ a.s., so (b) holds in this case. If $B \cap C \neq \emptyset$, write $B \cup C = (B \cap C^c) \cup (B \cap C) \cup (B^c \cap C)$ to get the result.

(c) By (b),

$$\{P(\bigcup_{n \leq m} B_n|G) > 0\} = \bigcup_{n \leq m} \{P(B_n|G) > 0\} \uparrow \bigcup_{n \in \mathbb{N}} \{P(B_n|G) > 0\}$$

as $m \rightarrow \infty$. But then (c) follows from the fact that

$$P(\bigcup_{n \leq m} B_n|G) \uparrow P(B|G)$$

as $m \rightarrow \infty$. $\square$

Note that one cannot replace $\cup$ by $\cap$ in 4.3(b) or (c).

Recall the notion of "essential infimum" of a family of random variables (see Neveu [N], §VI-1).

We need the following property of "essinf".

4.4 Lemma. Let $(Z_i; i \in I)$ be random variables, and consider $Y \geq 0$. Then

$$\text{essinf}_{i \in I} (Y Z_i) = Y \text{essinf}_{i \in I} Z_i \text{ a.s.}$$

Proof. $Y \text{essinf}_{i \in I} Z_i \leq Y Z_i$ a.s., $\forall i \in I$,

so

$$Y \operatorname{essinf}_{i \in I} Z_i \leq \operatorname{essinf}_{i \in I} (YZ_i) \text{ a.s.}$$

Following [N, VI.6.1], let $(i_k; k \in \mathbb{N}) \subset I$ be a sequence such that

$$\operatorname{essinf}_{i \in I} Z_i = \inf_{k \in \mathbb{N}} Z_{i_k} \text{ a.s.}$$

Then

$$\operatorname{essinf}_{i \in I} (YZ_i) \leq YZ_{i_k} \text{ a.s., } \forall k \in \mathbb{N},$$

so

$$\begin{aligned} \operatorname{essinf}_{i \in I} (YZ_i) &\leq \inf_{k \in \mathbb{N}} (YZ_{i_k}) = Y \inf_{k \in \mathbb{N}} Z_{i_k} \\ &= Y \operatorname{essinf}_{i \in I} Z_i \text{ a.s. } \square \end{aligned}$$

We can now define our "conditional supremum" operator.

4.5 Definition. For bounded Y , define the conditional supremum of Y given G (denoted $S(Y|G)$) by

$$S(Y|G) = \operatorname{essinf}_{Z \in G, Z \geq Y} Z.$$

Similar to conditional expectation, this operator is defined on the space of equivalence classes of random variables yielding an equivalence class (though "conditional infimum" can be defined by analogy, we will have no need for it in this paper). Furthermore, it has many properties of conditional expectations. Before summarizing them, let us consider some examples.

4.6 Examples.

- (a) $Y = I_B, B \in F$. Then by 4.1, $S(Y|G) = I_{\{P(B|G) > 0\}}$.
- (b) G is the σ -algebra of null sets and their complements. Then $S(Y|G) = \|Y\|_\infty$ a.s.,

for all Y .

- (c) Let $\Omega = \{\omega_1, \dots, \omega_n\}$, $\Omega' = \{\omega'_1, \dots, \omega'_m\}$, $\Omega = \Omega \times \Omega'$, $F = 2^\Omega$, and let P be a probability on Ω such that $P\{(\omega_i, \omega'_j)\} > 0$, for all i, j . Set $G = 2^\Omega \otimes \{\emptyset, \Omega'\}$. Then for $Y \in F$,

$$S(Y|G)(\omega_i, \omega'_j) = \sup_{k=1, \dots, m} Y(\omega_i, \omega'_k), \quad \forall i, j.$$

The following lemma establishes basic properties of $S(\bullet|G)$.

4.7 Lemma.

- (a) $S(Y|G) \in G; \quad Y \in G \Rightarrow S(Y|G) = Y$
- (b) $Y \leq S(Y|G); \quad Z \in G, Z \geq Y \Rightarrow Z \geq S(Y|G)$
- (c) $Y_1 \leq Y_2 \Rightarrow S(Y_1|G) \leq S(Y_2|G)$
- (d) $G_2 \supseteq G_1 \Rightarrow S(Y|G_1) \geq S(Y|G_2)$ and $S(S(Y|G_2)|G_1) = S(Y|G_1)$
- (e) $\{S(Y|G) > k\} = \{P(Y > k|G) > 0\}$
- (f) $X \in G \Rightarrow S(X+Y|G) = X + S(Y|G)$
- (g) $X \in G, X, Y \geq 0 \Rightarrow S(XY|G) = X S(Y|G)$
- (h) $Y_n \uparrow Y \Rightarrow S(Y_n|G) \uparrow S(Y|G)$
- (i) $G_n \supseteq G_{n+1}, \quad \bigcap_{n \in \mathbb{N}} G_n = G \Rightarrow S(Y|G_n) \uparrow S(Y|G)$

(j) Suppose $(F_n: n \in \mathbb{N})$ is a standard filtration, $T: \Omega \rightarrow \mathbb{N}$ is a stopping time. Then for any bounded process $(X_n: n \in \mathbb{N})$, $S(X_T|F_T) = \sum_{t \in \mathbb{N}} I_{\{T=t\}} S(X_t|F_t)$ ($F_T = \{B \in F: B \cap \{T \leq t\} \in F_t, \forall t \in \mathbb{N}\}$).

Proof. (a), (b) and (c) follow immediately from the definition, as does the first inequality in (d). As for the second statement in (d), observe that $S(Y|G_2) \geq Y$, so by (c), $S(S(Y|G_2)|G_1) \geq S(Y|G_1)$. On the other hand, $S(Y|G_1) \geq Y$, $S(Y|G_1) \in G_2$, so by (b), $S(Y|G_1) \geq S(Y|G_2)$. But then (a) and (c) give $S(Y|G_1) \geq S(S(Y|G_2)|G_1)$, and equality follows.

(e) To check "c", let $M \in \mathbb{R}$ be such that $|Y| \leq M$ a.s. Using 4.3(a), one sees that

$$Y \leq k I_{\{P(Y > k|G) = 0\}} + M I_{\{P(Y > k|G) > 0\}}.$$

Thus $S(Y|G) \leq k$ a.s. on $\{P(Y > k|G) = 0\}$, proving

$$\{P(Y > k|G) > 0\} \supset \{S(Y|G) > k\} \text{ a.s.}$$

To see the converse inclusion, consider $Z \in G$, $Z \geq Y$. For $r > 0$, $\{Z \geq k+r\} \supset \{Y \geq k+r\}$, so by 4.1(a),

$\{Z \geq k+r\} \supset B = \{P(Y \geq k+r|G) > 0\}$. It follows by 4.4 that

$$\begin{aligned} I_B S(Y|G) &= I_B \operatorname{essinf}_{Z \in G, Z \geq Y} Z = \operatorname{essinf}_{Z \in G, Z \geq Y} (Z I_B) \\ &\geq \operatorname{essinf}_{Z \in G, Z \geq Y} ((k+r)I_B) = (k+r) I_B. \end{aligned}$$

Thus $S(Y|G) \geq k+r$ on B , meaning $\{S(Y|G) \geq k+r\} \supset \{P(Y \geq k+r|G) > 0\}$. This implies

$$\begin{aligned} \{S(Y|G) > k\} &= \bigcup_{r \in \mathbb{Q}_+} \{S(Y|G) \geq k+r\} \supset \bigcup_{r \in \mathbb{Q}_+} \{P(Y \geq k+r|G) > 0\} \\ &= \{P(Y > k|G) > 0\}, \end{aligned}$$

where the last equality follows from 4.3(c). (e) is proven.

(f) Consider $Z \in G$ such that $Z \geq X + Y$. Then $Z - X \geq Y$, $Z - X \in G$, so $Z - X \geq S(Y|G)$, or equivalently $Z \geq X + S(Y|G)$. Thus $S(X+Y|G) \geq X + S(Y|G)$. Since $X + S(Y|G)$ is G -measurable and greater than $X+Y$, (b) implies the desired equality.

(g) Using (e), we have for $k \geq 0$:

$$\begin{aligned} \{X S(Y|G) > k\} &= \bigcup_{r \in \mathbb{Q}_+} (\{X > r\} \cap \{P(Y > k/r | G) > 0\}) \\ &= \bigcup_{r \in \mathbb{Q}_+} \{I_{\{X > r\}} P(Y > k/r | G) > 0\} \\ &= \bigcup_{r \in \mathbb{Q}_+} \{P(X > r, Y > k/r | G) > 0\} \\ &= \{P(\bigcup_{r \in \mathbb{Q}_+} \{X > r, Y > k/r\} | G) > 0\} \\ &= \{S(XY|G) > k\}. \end{aligned}$$

The fourth equality follows from 4.3(c); (g) is proven.

(h) $S(Y_n|G)$ is increasing in n by (c). Furthermore, $S(Y_n|G) \leq S(Y|G)$. Now

$$\begin{aligned}
\{S(Y|G) > k\} &= \{P(Y > k|G) > 0\} = \bigcup_{n \in \mathbf{N}} \{P(Y_n > k|G) > 0\} \\
&= \bigcup_{n \in \mathbf{N}} \{S(Y_n|G) > k\}.
\end{aligned}$$

thus proving (h).

(i) Since $P(Y > k|G_n) \rightarrow P(Y > k|G)$ as $n \rightarrow \infty$, one sees that

$$\begin{aligned}
\{S(Y|G) > k\} &= \{P(Y > k|G) > 0\} \subset \bigcup_{n \in \mathbf{N}} \{P(Y > k|G_n) > 0\} \\
&= \bigcup_{n \in \mathbf{N}} \{S(Y|G_n) > k\}.
\end{aligned}$$

Since by (d), $S(Y|G) \geq S(Y|G_n)$, the above inclusion is in fact an equality, thus proving (i).

(j)

$$\begin{aligned}
\left\{ \sum_{t \in \mathbf{N}} I_{\{T=t\}} S(X_t|F_t) > k \right\} &= \bigcup_{t \in \mathbf{N}} \{T = t, S(X_t|F_t) > k\} \\
&= \bigcup_{t \in \mathbf{N}} \{T = t, P(X_t > k|F_t) > 0\} = \bigcup_{t \in \mathbf{N}} \{T = t, P(X_T > k|F_T) > 0\} \\
&= \{P(X_T > k|F_T) > 0\} = \{S(X_T|F_T) > k\}.
\end{aligned}$$

This proves (j). $\square$

4.8. Remarks. It is not possible to obtain a statement analogous to 4.7(h) for decreasing sequences, nor a statement analogous to 4.7(i) for increasing sequences of σ -algebras.

The operator S has interesting properties with respect to continuous time filtrations. We hope to come back to these elsewhere.

IV.2 Conditional qualitative independence

Conditional qualitative independence is a notion introduced by Krengel and Sucheston in [KS]. Recall that two σ -algebras F^1 and F^2 are said to be conditionally qualitatively independent given a σ -algebra G (denoted $F^1 \perp F^2 | G$) if

$$B^1 \in F^1, B^2 \in F^2 \Rightarrow \{P(B^1|G) > 0\} \cap \{P(B^2|G) > 0\} = \{P(B^1 \cap B^2 | G) > 0\}$$

(in [KS], there was an "c" instead of an "=", but the two forms are equivalent since the right-hand side is always contained in the left).

In this section, we express this condition in terms of the operator $S(\bullet|G)$. It is interesting to observe that in contrast to hypothesis (F4) of Cairoli and Walsh [CW], this condition remains valid if P is replaced by an equivalent measure (i.e. a measure with the same null sets). Clearly $S(\bullet|G)$ is invariant under such a change of measure, and as the proposition below shows, this operator is to conditional qualitative independence what conditional expectation is to conditional independence.

4.9 Proposition. The following statements are equivalent:

- (a) $F^1 \perp F^2 | G$
- (b) $X^1 \in F^1, X^2 \in F^2, X^1, X^2 \geq 0 \Rightarrow S(X^1 X^2 | G) = S(X^1 | G) S(X^2 | G)$
- (c) $X^1 \in F^1, X^2 \in F^2, Z \in G, X^1 + X^2 \leq Z \Rightarrow S(X^1 | G) + S(X^2 | G) \leq Z$.

Proof.

(a) $\Rightarrow$ (b). Using 4.7(e), we see that

$$\begin{aligned} \{S(X^1 | G) S(X^2 | G) > k\} &= \bigcup_{r \in \mathbb{Q}_+} (\{S(X^1 | G) > r\} \cap \{S(X^2 | G) > k/r\}) \\ &= \bigcup_{r \in \mathbb{Q}_+} (\{P(X^1 > r | G) > 0\} \cap \{P(X^2 > k/r | G) > 0\}). \end{aligned}$$

By (a), this is equal to

$$\bigcup_{r \in \mathbb{Q}_+} \{P(X^1 > r, X^2 > k/r | G) > 0\},$$

which by 4.3(c) is

$$\{P(X^1 X^2 > k | G) > 0\} = \{S(X^1 X^2 | G) > k\},$$

proving the desired implication.

(b) $\Rightarrow$ (a). Set $X^i = I_{B^i}$, where $B^i \in F^i$, $i = 1, 2$. By 4.6(a),

$$S(X^i | G) = I_{\{P(B^i | G) > 0\}}, \quad i = 1, 2, \quad S(X^1 X^2 | G) = I_{\{P(B^1 \cap B^2 | G) > 0\}},$$

so (a) is really a special case of (b) .

(a) $\Rightarrow$ (c). Use 4.7(e) to observe that

$$\begin{aligned}
 \{S(X^1|G) + S(X^2|G) > Z\} &= \bigcup_{k \in \mathbb{Q}} (\{S(X^1|G) + S(X^2|G) > k\} \cap \{Z \leq k\}) \\
 &= \bigcup_{k \in \mathbb{Q}} \left(\bigcup_{r \in \mathbb{Q}} (\{S(X^1|G) > r\} \cap \{S(X^2|G) > k-r\}) \cap \{Z \leq k\} \right) \\
 &= \bigcup_{k \in \mathbb{Q}} \left(\bigcup_{r \in \mathbb{Q}} (\{P(X^1 > r|G) > 0\} \cap \{P(X^2 > k-r|G) > 0\}) \cap \{Z \leq k\} \right).
 \end{aligned}$$

By (a), this set is equal to

$$\bigcup_{k \in \mathbb{Q}} \left(\bigcup_{r \in \mathbb{Q}} (\{P(X^1 > r, X^2 > k-r|G) > 0\}) \cap \{Z \leq k\} \right),$$

which by 4.3(c) is equal to

$$\bigcup_{k \in \mathbb{Q}} (\{P(X^1 + X^2 > k|G) > 0\} \cap \{Z \leq k\}).$$

Since $X^1 + X^2 \leq Z$, this last set is included in

$$\bigcup_{k \in \mathbb{Q}} (\{P(Z > k|G) > 0\} \cap \{Z \leq k\}),$$

which, since $Z \in G$, is equal to

$$\bigcup_{k \in \mathbb{Q}} (\{Z > k\} \cap \{Z \leq k\}),$$

which is a.s. the empty set, proving the desired implication.

(c) $\Rightarrow$ (a). Consider $B^i \in F^i$, $i = 1, 2$. Then, where Δ denotes symmetric difference,

$$\begin{aligned}
 I_{B^1} + I_{B^2} &= 2I_{B^1 \cap B^2} + I_{B^1 \Delta B^2} \\
 &\leq 2I_{\{P(B^1 \cap B^2|G) > 0\}} + I_{\{P(B^1 \Delta B^2|G) > 0\}}.
 \end{aligned}$$

This last expression is G -measurable, so by (c) and 4.6(a),

$$I_{\{P(B^1|G) > 0\}} + I_{\{P(B^2|G) > 0\}} \leq 2 I_{\{P(B^1 \cap B^2|G) > 0\}} + I_{\{P(B^1 \Delta B^2|G) > 0\}}$$

This inequality clearly implies (a). $\square$

We will mainly be interested in conditional qualitative independence for two-parameter filtrations.

Recall that a two-parameter filtration $(F_t; t \in \mathbb{N}^2)$ is said to satisfy hypothesis (CQI) if

$$s, t \in \mathbb{N}^2, s \triangle t \Rightarrow F_s \perp F_t | F_{\min(s,t)}$$

(see [KS]). In the above, $\triangle$ is the order relation on $\mathbb{N}^2$ defined by

$$s \triangle t \Leftrightarrow s_1 \leq t_1, s_2 \geq t_2,$$

def

and $\min(s, t) = (\min(s_1, t_1), \min(s_2, t_2))$. We will write $s \wedge t$ when $s_1 < t_1, s_2 > t_2$.

Observe that (CQI) is equivalent to $F_s \perp F_t | F_u$ if $s \triangle u \triangle t$.

Just as the conditional independence hypothesis (F4) of Cairoli and Walsh (see [CW]) can be expressed as the commutation of certain conditional expectations, hypothesis (CQI) can be expressed as the commutation of conditional supremum operators. Let $S(Y|G_1|G_2)$ denote the iterated conditional supremum $S(S(Y|G_1)|G_2)$.

4.10 Proposition. Hypothesis (CQI) is equivalent to

$$s, t \in \mathbb{N}^2, s \triangle t \Rightarrow (F_{\min(s,t)} = F_s \cap F_t \text{ and } S(\bullet|F_s|F_t) = S(\bullet|F_t|F_s)).$$

Proof. Fix $Y \in F$, and $s, t \in \mathbb{N}^2$ such that $s \triangle t$.

$\Rightarrow$: We first show that $F_{\min(s,t)} = F_s \cap F_t$. Since $(F_t; t \in \mathbb{N}^2)$ is increasing for $\leq$, $F_s \cap F_t \supseteq F_{\min(s,t)}$.

As for the converse inclusion, we prove the general statement

$$(4.11) \quad F^1 \perp F^2 | G \Rightarrow G \supseteq F^1 \cap F^2.$$

Let $B \in F^1 \cap F^2$. Then by (CQI),

$$\{P(B|G) > 0\} \cap \{P(B^c|G) > 0\} = \{P(B \cap B^c | G) > 0\} = \emptyset \text{ a.s.,}$$

so $B \supset \{P(B^c|G) = 0\} \supset \{P(B|G) > 0\} \supset B$. The first and last inclusions come from 4.3(a)). This implies that $B = \{P(B|G) > 0\}$, so $B \in G$, proving (4.11).

To show that $S(\bullet|F_s|F_t) = S(\bullet|F_t|F_s)$, it is sufficient by symmetry to show that $S(\bullet|F_s|F_t) = S(\bullet|F_{\min(s,t)})$. By 4.7(d), $S(Y|F_{\min(s,t)}) = S(S(Y|F_s)|F_{\min(s,t)})$. But by 4.12(a) and (b) (see below), this is equal to $S(S(Y|F_s)|F_t)$, and the implication is proven.

≡: Observe that $S(\bullet|F_s|F_t) = S(\bullet|F_t|F_s)$ implies $S(Y|F_s|F_t) \in F_s \cap F_t = F_{\min(s,t)}$. Since by 4.7(b) and (c), $Y \leq S(Y|F_t) \leq S(S(Y|F_s)|F_t)$, we get $S(Y|F_{\min(s,t)}) \leq S(Y|F_s|F_t)$. Furthermore, $S(Y|F_{\min(s,t)}) \geq Y$, so by 4.7(d), $S(Y|F_{\min(s,t)}) \geq S(Y|F_s|F_t)$, proving $S(Y|F_{\min(s,t)}) = S(Y|F_s|F_t)$.

We now prove $F_s \perp F_t | F_{\min(s,t)}$ in the equivalent form 4.9(c). Consider $X^1 \in F_s$, $X^2 \in F_t$ and $Z \in F_{\min(s,t)}$ such that $X^1 + X^2 \leq Z$. By 4.7(f) and (c),

$$X^1 + S(X^2|F_s) = S(X^1 + X^2|F_s) \leq S(Z|F_s) = Z.$$

But since $S(X^2|F_s) = S(S(X^2|F_t)|F_s) = S(X^2|F_{\min(s,t)})$, applying $S(\bullet|F_t)$ to the above inequality gives $S(X^1|F_t) + S(X^2|F_{\min(s,t)}) \leq Z$, which can be rewritten as

$$S(X^1|F_{\min(s,t)}) + S(X^2|F_{\min(s,t)}) \leq Z,$$

thus proving (CQI). $\square$

The fact that the σ -algebras F_t are increasing with respect to $\leq$ and that they satisfy (CQI) gives monotonicity properties for $S(\bullet|F_t)$ with respect to $\leq$ and $\triangle$.

4.12 Proposition.

- (a) $s \leq t \Rightarrow S(\bullet|F_s) \geq S(\bullet|F_t)$
- (b) $s \triangle u \triangle t, Y \in F_s \Rightarrow S(Y|F_u) \leq S(Y|F_t)$

(c) Suppose $(s_n: n \in \mathbb{N}) \subset \mathbb{N}^2$ is such that for all $n \in \mathbb{N}$, $s_n \leq s_{n+1}$, $s_n \triangle u \triangle t$, and suppose $Y = \lim \uparrow Y_n$, where $Y_n \in F_{s_n}$. Then $S(Y|F_u) \leq S(Y|F_t)$.

Proof. (a) is an immediate consequence of 4.7(d) and the fact that $F_t \supseteq F_s$. To see (b), observe that

$$\{S(Y|F_u) > S(Y|F_t)\} = \bigcup_{k \in \mathbb{Q}} (\{S(Y|F_u) > k\} \cap \{S(Y|F_t) \leq k\}).$$

By 4.7(e) and 4.3(a), this is included in

$$\bigcup_{k \in \mathbb{Q}} (\{P(Y > k|F_u) > 0\} \cap \{P(S(Y|F_t) \leq k|F_u) > 0\}).$$

By (CQI), this is equal to

$$\bigcup_{k \in \mathbb{Q}} (\{P(Y > k, S(Y|F_t) \leq k|F_u) > 0\}),$$

which is a.s. the empty set, and (b) is proven.

As for (c), it is a straightforward consequence of (b) and 4.7(h). $\square$

IV.3 Extremal elements of the set of randomized stopping points on $\mathbb{N}^2$

We shall suppose throughout this section that the filtration $(F_t: t \in \mathbb{N}^2)$ satisfies hypothesis (CQI).

Let $a = (a_t: t \in \mathbb{N}^2)$ be a randomized stopping point. We shall construct two randomized stopping points $a = (a_t^i: t \in \mathbb{N}^2)$, $i = 1, 2$, such that

$$(4.13) \quad a = 1/2 a^1 + 1/2 a^2 \quad (\text{i.e. } a_t = 1/2 a_t^1 + 1/2 a_t^2 \text{ a.s., } \forall t \in \mathbb{N}^2)$$

and

$$(4.14) \quad P\{\omega: \text{there exists } t \in \mathbb{N}^2 \text{ with } 0 < a_t(\omega) < 1\} > 0 \Rightarrow a^1 \neq a \neq a^2.$$

a^1 and a^2 will be defined by induction. To this end, let α be the lexicographic order on $\mathbb{N}^2$, i.e.

$$s \alpha t \Leftrightarrow s_1 < t_1 \text{ or } (s_1 = t_1 \text{ and } s_2 \leq t_2).$$

$s \alpha t$ will mean $s \alpha t$, $s \neq t$.

In increasing order for α , define successively

$$(4.15) \quad C_{(0,0)} \equiv 0$$

$$(4.16) \quad \begin{aligned} (a) \quad & a_t^1 = 2a_t, \quad a_t^2 = 0 && \text{if } C_t + 2a_t < 1 \\ (b) \quad & a_t^1 = 1 - C_t, \quad a_t^2 = 2a_t - 1 + C_t && \text{if } C_t < 1, C_t + 2a_t \geq 1 \\ (c) \quad & a_t^1 = 0, \quad a_t^2 = 2a_t && \text{if } C_t \geq 1 \end{aligned}$$

$$(4.17) \quad C_t = S\left(\sum_{s \leq t} a_s^1 \mid F_t\right).$$

Observe that a_t^i is F_t -measurable, $a_t^i \geq 0$, $i = 1, 2$, and $a_t = (1/2) a_t^1 + (1/2) a_t^2$ a.s., $\forall t \in \mathbb{N}^2$. The following lemma shows that when the third case in (4.16) holds, $C_t = 1$.

4.18 Lemma. $C_t \leq 1$ a.s., $\forall t \in \mathbb{N}^2$.

Proof. Clearly $C_{(0,0)} \equiv 0 \leq 1$ a.s. So suppose $C_t \leq 1$ a.s., and show that $C_{(t_1, t_2+1)} \leq 1$ a.s. Using 4.7(f) gives

$$\begin{aligned} C_{(t_1, t_2+1)} &= S\left(\sum_{s \leq t} a_s^1 \mid F_{(t_1, t_2+1)}\right) = S\left(\sum_{s \leq t} a_s^1 + a_t^1 \mid F_{(t_1, t_2+1)}\right) \\ &= a_t^1 + S\left(\sum_{s \leq t} a_s^1 \mid F_{(t_1, t_2+1)}\right) \end{aligned}$$

Using 4.12(a), we see that this expression is less than

$$a_t^1 + S\left(\sum_{s \leq t} a_s^1 \mid F_t\right) = a_t^1 + C_t.$$

This last expression is less than 1 by the definition of a_t^1 and the fact that $C_t \leq 1$ a.s.

We now suppose that $C_{(t_1, t_2)} \leq 1$ a.s. for all $t_2 \in \mathbb{N}$, and show that $C_{(t_1+1, 0)} \leq 1$. $C_{(t_1, t_2)} \leq 1$ implies that $\sum_{s \propto (t_1, t_2)} a_s^1 \leq 1$ a.s. Since this holds for all t_2 , $\sum_{s \propto (t_1+1, 0)} a_s^1 \leq 1$, giving $C_{(t_1+1, 0)} \leq 1$ by the definition of C . This concludes the proof. $\square$

In order to prove that $a^i = (a_t^i: t \in \mathbb{N}^2)$, $i = 1, 2$, defined by (4.15), (4.16) and (4.17) are randomized stopping points, it only remains to be shown that $\sum_{s \in \mathbb{N}^2} a_s^i \leq 1$ a.s., $i = 1, 2$. Before doing this, we examine the relationship between the supports of a^1 , a^2 and a . In fact, we show that there is an unbounded optional increasing path $Z = (Z_p: p \in \mathbb{N})$ (see Walsh [W]) such that the support of a^1 is to the left of Z , whereas the support of a^2 will be to its right. The fact that this path can be built in an optional way will be a direct consequence of (CQI).

For our purpose, the following definition of an optional increasing path will be sufficient.

4.19 Definition. $Z = (Z_p: p \in \mathbb{N})$ is an optional increasing path if:

Z_p is a stopping point for all $p \in \mathbb{N}$,

$Z_0 \equiv 0, Z_{p+1} \in \{Z_p + (1, 0), Z_p + (0, 1)\}$ a.s.,

Z is F_{Z_p} -measurable.

Observe that if we set $|t| = t_1 + t_2$, then $|Z_p| = p$ a.s.

Let us define the following sets:

$$U_t = \{C_t + 2a_t \geq 1\}, \quad V_t = \bigcup_{k \geq 1} U_{(t_1, t_2+k)}, \quad t \in \mathbb{N}^2$$

and an optional increasing path $Z = (Z_p: p \in \mathbb{N})$ by $Z_0 \equiv (0, 0)$,

$$(4.20) \quad Z_{p+1} = \begin{cases} (t_1, t_2+1) & \text{on } \{P(V_t|F_t) > 0\} \cap \{Z_p = t\} \\ (t_1+1, t_2) & \text{on } \{P(V_t|F_t) = 0\} \cap \{Z_p = t\} \end{cases}, \quad p \geq 0.$$

For $t \in \mathbb{N}^2$, set $R_t = \{s \in \mathbb{N}^2: s \leq t\}$, $R_t^- = \{s \in \mathbb{N}^2: s_1 < t_1, s_2 > t_2\}$, $R_t^+ = \{s \in \mathbb{N}^2: s_1 > t_1, s_2 \leq t_2\}$ (note the lack of symmetry in the definitions of R_t^- and R_t^+).

4.21 **Lemma.** If $t \in \mathbb{R}_s^+$, then

- (a) $P(U_s \cap \{C_t < 1\}) = 0$
 (b) $\{P(V_s|F_s) > 0\} \cap \{C_t < 1\} = \emptyset$ a.s.

Proof. Concerning (a), observe that by 4.18

$$1 \geq C_t = S \left(\sum_{u \alpha t} a_u^1 |F_t \right) \geq S \left(\sum_{u \alpha s} a_u^1 |F_t \right).$$

By 4.12(c), the last expression is greater than $S \left(\sum_{u \alpha s} a_u^1 |F_s \right)$, which by 4.7(f) is equal to

$$a_s^1 + S \left(\sum_{u \alpha s} a_u^1 |F_t \right) = a_s^1 + C_s = 1 \text{ a.s. on } U_s.$$

This proves (a).

As for (b), observe that (a) implies that $U_{(s_1, s_2+k)} \cap \{C_t < 1\} = \emptyset$ a.s. for all $k \in \mathbb{N}$, so by (CQI),

$$\{P(U_{(s_1, s_2+k)} > 0\} \cap \{C_t < 1\} = \emptyset \text{ a.s., } \forall k \in \mathbb{N},$$

so taking the union over $k \in \mathbb{N}$ of these sets and using 4.3(c) and the definition of V_s , we get the desired result. $\square$

For $t \in \mathbb{N}^2$ set $L_t = \{\omega: t \wedge Z_{|t|}(\omega)\}$, $N_t = \{\omega: Z_{|t|}(\omega) \wedge t\}$, $M_t = \{\omega: Z_{|t|}(\omega) = t\}$. Clearly $\Omega = L_t \cup M_t \cup N_t$, $\forall t \in \mathbb{N}^2$, and these sets are disjoint. Looking back to the definition of Z and using 4.3(a), we see that $C_t + 2a_t < 1$ a.s. on L_t , and by 4.21(b), $C_t = 1$ a.s. on N_t . But then the definition of a^1 and a^2 implies

$$(4.22) \quad \begin{array}{lll} a_t^1 = 2a_t, & a_t^2 = 0 & \text{a.s. on } L_t, \\ a_t^1 = 0, & a_t^2 = 2a_t & \text{a.s. on } N_t. \end{array}$$

So the path $p \rightarrow Z_p(\omega)$ defined by (4.20) splits $\mathbb{N}^2$ into three parts: the left of the path, where (4.16)(a) always holds, the right of the path where (4.16)(c) holds, and the path itself, where either of the three cases are possible.

4.23 Theorem. a^1 and a^2 defined by (4.15), (4.16) and (4.17) are randomized stopping points.

Proof. We have already observed that a^1 and a^2 satisfy conditions (2.1) and (2.2). Note that $\sum_s \alpha_t a_s^1 \leq C_t \leq 1$ a.s. by 4.18, for all $t \in \mathbb{N}^2$, so $\sum_{s \in \mathbb{N}^2} a_s^1 \leq 1$ a.s. and thus a^1 is a randomized stopping point. So it only remains to be shown that

$$(4.24) \quad \sum_{s \in \mathbb{N}^2} a_s^2 \leq 1 \text{ a.s.}$$

Suppose we have proved the following inequality for all $t \in \mathbb{N}^2$:

$$(4.25) \quad \sum_{s \in R_t \cup R_t^+} a_s^2 \leq 1 \text{ a.s. on } U_t.$$

Then there would be a null set N such that

$$\omega \in U_t \setminus N \Rightarrow \sum_{s \in R_t \cup R_t^+} a_s^2(\omega) \leq 1, \forall t \in \mathbb{N}^2.$$

Set $U(\omega) = \{t: \omega \in U_t\}$. By the definition of a^2 , $U(\omega) \supset \{t: a_t^2(\omega) > 0\}$, so

$$\sum_{s \in \mathbb{N}^2} a_s^2(\omega) = \sum_{s \in U(\omega)} a_s^2(\omega) = \sup_{t \in U(\omega)} \sum_{s \in R_t \cup R_t^+} a_s^2(\omega) \leq 1 \text{ a.s.}$$

So we only need to prove (4.25).

To this end, observe that since $a \in U$,

$$\sum_{s \in R_t} a_s + \sum_{s \in R_t^-} a_s + \sum_{s \in R_t^+} a_s \leq 1 \text{ a.s.}$$

Using 4.7(h) (because the sums are infinite), applying (CQI) in the equivalent form 4.9(c) and doubling the above inequality gives

$$(4.26) \quad \sum_{s \in R_t} 2a_s + S\left(\sum_{s \in R_t^-} 2a_s | F_t\right) + S\left(\sum_{s \in R_t^+} 2a_s | F_t\right) \leq 2 \text{ a.s.}$$

Since $a_t^i \leq 2a_t$ a.s., $\forall t \in \mathbb{N}^2$, $i = 1, 2$, this implies

$$(4.27) \quad \sum_{s \in R_t} 2a_s + S\left(\sum_{s \in R_t^-} a_s^1 | F_t\right) + S\left(\sum_{s \in R_t^+} a_s^2 | F_t\right) \leq 2 \text{ a.s.}$$

Now on U_t , $C_t + a_t^1 = 1$ a.s., so

$$(4.28) \quad \sum_{s \in R_t} a_s^1 + S\left(\sum_{s \in R_t^-} a_s^1 | F_t\right) = 1 \text{ a.s. on } U_t.$$

Subtracting (4.28) from (4.27) gives

$$(4.28) \quad \sum_{s \in R_t} (2a_s - a_s^1) + S\left(\sum_{s \in R_t^+} a_s^2 | F_t\right) \leq 1 \text{ a.s. on } U_t.$$

Since $1/2(a_s^1 + a_s^2) = a_s$, $\forall s \in \mathbb{N}^2$, this implies (4.25). $\square$

We can now prove our main theorem.

4.29. Theorem. If $(F_t; t \in \mathbb{N}^2)$ satisfies (CQI), then every extremal element of the set U of randomized stopping points is a stopping point.

Proof. By 4.23, if $a = (a_t; t \in \mathbb{N}^2)$ is a randomized stopping point, then formulas (4.15), (4.16) and (4.17) define two randomized stopping points a^1 and a^2 such that $a = (1/2)a^1 + (1/2)a^2$. It remains to be shown that (4.14) holds.

Suppose $P\{\omega: \text{there exists } t \in \mathbb{N}^2 \text{ with } 0 < a_t(\omega) < 1\} > 0$. In this case, we can define $t \in \mathbb{N}^2$ as the smallest $t \in \mathbb{N}^2$ (with respect to α) such that $P(0 < a_t < 1) > 0$.

For $s \in \mathbb{N}^2$, $s \alpha t$, there is $B_s \in \mathcal{F}_s$ such that $a_s = I_{B_s}$ a.s. Let $B = \{0 < a_t < 1\}$. Since $\sum_{s \in \mathbb{N}^2} a_s \leq 1$ a.s., $B^c \supset B_s$ a.s., implying by 4.1(a) that $\{P(B_s | \mathcal{F}_t) > 0\} \cap B = \emptyset$ a.s. Using 4.3(c) we get

$$(4.30) \quad \{P(\bigcup_{s \alpha t} B_s | \mathcal{F}_t) > 0\} \cap B = \emptyset \text{ a.s.}$$

Observe that

$$\bigcup_{s \alpha t} B_s = \left\{ \sum_{s \alpha t} a_s = 1 \right\} = \left\{ \sum_{s \alpha t} a_s > 0 \right\},$$

so by 4.7(e), (4.30) is equivalent to

$$\left\{ S\left(\sum_{s \alpha t} a_s | \mathcal{F}_t \right) > 0 \right\} \cap B = \emptyset \text{ a.s.}$$

Thus $C_t = 0$ a.s. on B , so $a_t^1 = \min(2a_t, 1) \neq a_t \neq \max(2a_t - 1, 0) = a_t^2$ on B , and we are done. $\square$

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