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**OPTIMAL REPLACEMENT FOR FAULT
TOLERANT SYSTEMS**

by

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1. **Introduction and summary.** Consider a mail order retailer who purchases a list of names and addresses to whom catalogs are sent. As time passes and people move, the number of valid addresses in the list decreases. If undeliverable catalogs are returned to the sender, it is possible to monitor the "quality" of the list, i.e., the current number of active addresses in it. At what point is it most profitable for the retailer to purchase a fresh address list?

Such a problem is a typical **optimal replacement model**. A system deteriorates over time, and based on its current state, one wishes to decide in an optimal manner whether or not to replace the system with a statistically identical new one. Let us suppose that there are N addresses in the file and that they become obsolete in a Markov manner with individual failure rate μ . Then $X(t)$, the number of active addresses at time t , is a pure death process with infinitesimal parameters $\mu_k = k\mu$ for $k = N, N-1, N-2, \dots, 1, 0$. ([18], pp. 217-221). We consider **control limit policies** that call for replacing the file at the first time $t = T(k)$ that $X(t) \leq k$ for some control limit k . Such rules are known to contain an optimal policy (among the class of all Markov times) [8]. If i is the income rate per active address per unit time, and c is the total cost to replace the file, then the long run average net income per unit time associated with control limit k is

$$\varphi_k = \frac{E[i \int_0^{T(k)} X(s) ds] - c}{E[T(k)]} \quad (1.1)$$

where it is easily calculated that

$$E[T(k)] = \frac{1}{\mu} \left(\frac{1}{N} + \frac{1}{N-1} + \dots + \frac{1}{k+1} \right), \quad (1.2)$$

and

$$E\left[i \int_0^{T(k)} x(s) ds\right] = \frac{i}{\mu} \left(\frac{N}{N} + \frac{N-1}{N-1} + \dots + \frac{k+1}{k+1} \right) = \frac{i}{\mu} (N-k) \quad (1.3)$$

whence

$$\varphi_k = \frac{\frac{i}{\mu} (N-k) - c}{\frac{1}{\mu} \left(\frac{1}{N} + \dots + \frac{1}{k+1} \right)} \approx \frac{\frac{i}{\mu} (N-k) - c\mu}{\ln(N/k)} \quad (1.4)$$

and the $k = k^*$ that maximizes φ_k may be found numerically or by analytical approximation: Set $x = k/N$ and equate the derivative of φ_k with respect to x to zero, whence $x^* = k^*/N$ solves $x(1 - \ln x) = 1 - c\mu/iN$.

Optimal replacement problems are the special class of sequential decision models that are distinguished by having only two actions available at any state: (i) To replace the system with a new and statistically identical one, or (ii) To continue operation with the current system. Often there is a notion of **total failure**, at which point replacement is mandated. (Some models allow so-called **minimal failures**, where a minimal repair is made and the operation continues. Minimal repairs can be modeled in the framework given here as simply a reduction in the mean rate of income equal to the mean rate of minimal repair

cost.)

In the address file model, the individual address records are assumed to be interchangeable; What is important is the total number of active records at any time, and not exactly which records they are. Such replacement models having interchangeable or indistinguishable components were treated by Gertsbakh in [8]. Our contribution is to develop the model in which the individual system components are distinguishable so that it is important to know exactly which components are operating at any given time, and not merely their total number. Moreover, we allow the individual component failure rates to depend on the state of the entire system. Such a feature is important in studying **load sharing systems** ([6],[10]) in which the load that a component had been carrying prior to its failure is, upon that component's failure, shifted to other components, thereby possibly increasing their failure rates. We are motivated to study this problem in part by the importance in modern practice of so-called **fault tolerant systems**, systems that can continue to operate even though some of their components have failed. In addition to simple redundancy, many such systems feature **function migration** in which the function served by a component is shifted to another component upon the first component's failure.

Over the past several decades, the theory of optimal replacement has been rather well developed, and we sketch the current state of this theory in Section 2. Our main contribution

begins in Section 3 where we use this theory to develop an algorithm that determines an optimal replacement policy for a system having distinguishable components whose failure rates are system dependent. Our algorithm is of the so-called **greedy** type. It begins with the policy that continues to operate the system, (that is, does not replace it), only if all components are operational. The algorithm then enlarges this continuation set in a monotonic manner, state by state, to arrive at the optimal such continuation set. If there are N system components, then there are 2^N system states. The optimal policy will be specified by a class of these system states, the class of states in which the system is allowed to continue to operate. There are, then, 2^{2^N} such possible continuation sets of states.

2. Optimal replacement theory. An optimal replacement model has the following structure: (1) A probability space $(\Omega, \mathcal{F}, P)$; (2) A nonnegative random variable $\zeta \leq \infty$, the system failure time at which point replacement is mandated; (3) An observable history $(\mathcal{F}_t)$; (4) A cost structure; and (5) A class of replacement times (policies) that is contained in the class of all Markov times $T \leq \zeta$.

All replacement models to date assume that $(\mathcal{F}_t)$ is the filtration generated by an observable process $(X(t))$ where $X(t)$ describes the "state" of the system at time t . We denote the failed system state by $\emptyset$ so that, were replacement not to be made, we would have $X(t) = \emptyset$ if and only if $t \geq \zeta$. The

event that $\{\zeta \leq t\}$ is in $\mathcal{F}_t$ since $(\mathcal{F}_t)$ is generated by $((X(t)))$, and $X(t) = \emptyset$ for $t \geq \zeta$. The failure time and the state process are connected by an assumed known failure rate function r with the property that, for any event A in $\mathcal{F}_t$,

$$P(A \cap \{\zeta > t+x\}) = \int_{A \cap \{\zeta > t\}} \exp\{-\int_t^{t+x} r(X(s)) ds\} dP. \quad (2.1)$$

A more intuitive interpretation is that

$$P(t < \zeta < t + dt \mid t < \zeta, \mathcal{F}_t) = r(X(t))dt. \quad (2.2)$$

It is usual to assume that the system generates income while operating, and that there is a cost associated with its replacement, both varying with the system state. Let $i(x)$ be the rate of income per unit time for a operational system in state x , and let $c(x)$ be the cost to replace a system in state x . Then $c(\emptyset)$ is the cost to replace a failed system and may include a penalty for failure in service. To avoid trivialities, we always assume that $c(X(0)) = c > 0$. The replacement cost associated with the replacement time T is $c(X(T))$, and the cumulative income is

$$I(T) = \int_0^T i(X(s))ds. \quad (2.3)$$

Operation begins with a new system and continues until a planned or failure replacement occurs at which point the original system is replaced with a statistically identical new one, and this process repeats. We focus on the criterion of maximum long run net income per unit time. Using renewal theory, the long run net income rate φ_T that is associated with a replacement time T can be rephrased in terms of expected values over the first replacement cycle ([9],[15]) in the form

$$\varphi_T = \frac{E[I(T) - c(X(T))]}{E[T]}. \quad (2.4)$$

Note that the class of replacement times may need to be limited to a strict subset of the class of all Markov times $T \leq \zeta$ in order that φ_T be well defined. For example, the integral in $I(T)$ must make sense and not both the numerator and denominator in φ_T can be infinite.

An argument that goes back at least to Brender [5], (See Barlow and Proschan [2], p.115), Breiman [4], and Taylor [16], and that has been used most recently by Aven [1], shows that a replacement time T^* maximizes φ_T if and only if it maximizes the linear objective function

$$\Theta(T, \varphi) = E[I(T) - c(X(T)) - \varphi T] \quad (2.5)$$

for the value $\varphi = \varphi^*$ such that

$$\Theta(T^*, \varphi^*) = 0. \quad (2.6)$$

Let us suppose that there exists a replacement cost differential rate function g with the property that

$$E[c(X(T))] = E[\int_0^T g(X(s)) ds] + c \quad (2.7)$$

for all replacement times T . (Recall that $c = c(X(0)) > 0$ is the cost to replace a new system.) Then the linear objective function in (2.5) can be written in the form

$$\Theta(T, \varphi) = E[\int_0^T \{i(X(s)) - g(X(s)) - \varphi\} ds] + c. \quad (2.8)$$

We will always suppose that the integrand $i(X(s)) - g(X(s)) - \varphi$ is monotonically decreasing in s almost surely. Then it is clear that one should continue to operate as long as the integrand is positive, and stop or replace at the first instant that it becomes negative. That is, the replacement time that maximizes $\Theta(T, \varphi)$ is the exit time from the set $\{x: i(x) - g(x) \geq \varphi\}$. Let us call this time $T(\varphi)$:

$$\begin{aligned} T(\varphi) &= \sup\{t \geq 0: i(X(t)) - g(X(t)) \geq \varphi\} \\ &= \inf\{t \geq 0: i(X(t)) - g(X(t)) < \varphi\}. \end{aligned} \quad (2.9)$$

Then the optimal replacement time is $T(\varphi^*)$ where φ^* is such

that

$$\Theta(T(\varphi^*), \varphi^*) = 0. \quad (2.10)$$

The unique φ^* satisfying (2.10) is the maximal long run net income rate per unit time.

The above theory has been completely developed in the past literature. Bergman [3] explicitly pointed out that no assumptions concerning the dimensionality or Markov nature of the process $(X(t))$ were necessary as long as $i(X(s)) - g(X(s))$ was monotonic. When $X(t)$ represents some notion of wear or damage accumulated by the system up to time t , this monotonicity assumption is often quite realistic. (Note, however, that there are also realistic situations where the assumption is not appropriate, for instance, where $X(t)$ is a posterior probability on an unobservable system state that is based on some record of observable system output [9]. Then a period of high quality output following a period of low quality output might lead to a reversal in the movement of $X(t)$. See Taylor [15] for an analysis in this case.) In this paper we will always assume that the system state process is monotonic. Within this limitation, however, $X(t)$ could be a quite general multidimensional semimartingale process whose components are monotonic. (See [10], pp. 64-67.) Ito's formula (ibid, pp. 66-67) is used to relate the replacement cost rate differential g to the replace-

ment cost function c as in (2.7). As we do not need this generality in our model, we do not further develop it here.

This model can be extended in many ways. A random nonnegative duration during which replacement takes place may be considered instead of instantaneous replacement [17]. A parallel theory holds when the criterion is mean total discounted cost [1]. The minimization of cost subject to a constraint on the long run mean rate of failures in service is another criterion of importance. A Lagrangian argument can be used to reduce this criterion to an equivalent average cost one.

3. Optimal replacement of fault tolerant systems. Consider a system having N components each of which may fail in some random manner. Assume that the components are distinguishable and that the failure rate of any non-failed or operational component depends on exactly which other components are operational and which have failed.

Components fail, singly or in groups, and once a component has failed, it remains so until the whole system is replaced. System replacement is assumed to occur instantaneously. (The generalization where the time to replace is not negligible is straightforward. See e.g. Taylor [17] for an example.) In this paper we assume that components cannot be replaced individually.

Let $X(t)$ represent the set of nonfailed or operational components at time t . Let $X(0) = 2^N$ denote the initial state, where all components are operational, and let $\emptyset$ denote the state in which all components have failed. We model the system by assuming that $X(t)$ is a Markov process. The state space Λ of the process consists of all possible subsets of $\{1, 2, \dots, N\}$. If A in Λ is a set of operational components, and, say, components i_1 and i_2 in A fail, then the system moves to state $B = A \setminus \{i_1, i_2\} = \{j: j \text{ in } A \text{ and } j \neq i_1, j \neq i_2\}$. Thus the Markov process $\{X(t): t \geq 0\}$ is monotonic in that its probability law allows transitions only to smaller states. We let $\mu(A, B)$ denote the infinitesimal transition rate from state A in Λ to state B in Λ with $B \subsetneq A$. That is,

$$P[X(t+\Delta t) = B \mid X(t) = A] = \mu(A, B)\Delta t + o(\Delta t) \quad (3.1)$$

for all $B \subsetneq A$ with A, B in Λ ;

and

$$P[X(t+\Delta t) = A \mid X(t) = A] = 1 - [\sum_{B \subsetneq A} \mu(A, B)]\Delta t + o(\Delta t) \quad (3.2)$$

for all nonempty sets A in Λ .

The infinitesimal matrix G for this process is given by

$$G(A,B) = \begin{cases} \mu(A,B), & \text{for } B \subsetneq A; \\ -\sum_{D \subsetneq A} \mu(A,D), & \text{for } B = A; \\ 0, & \text{elsewhere.} \end{cases} \quad (3.3)$$

The infinitesimal matrix G is triangular. We assume that the empty set $\emptyset$ is the only absorbing state of the process and then $G(A,A) > 0$ and $G(\emptyset,A) = 0$ for all nonempty A in Λ .

The cost structure. Assume that the system generates income at a rate $i(A)$ while operating in state (set of operational components) A . The cost to replace a system in state A is $c(A)$, with $c(2^N) = c > 0$, and with $c(\emptyset)$ being the cost to replace a failed system. Introduce the differential replacement cost rate function defined by

$$g(A) = \sum_{B \subset A} G(A,B)c(B) \quad \text{for all nonempty } A \text{ in } \Lambda. \quad (3.3)$$

Then, for the finite state process at hand, Dynkin's formula, (Theorem 5.1, Dynkin (1965)) and the convention that $c(2^N) = c > 0$, allow us to write

$$E[c(X(T))] = E\left[\int_0^T g(X(s))ds\right] + c. \quad (3.4)$$

The states in Λ are partially ordered by set inclusion; $B \subset A$ means that B has fewer operational components than A , or "B is worse than A". Thus it is natural to assume that the income function per unit time $i(\cdot)$ is monotonic; that is, for all A, B in Λ ,

$$\text{if } B \subset A, \text{ then } i(B) \leq i(A). \quad (3.5)$$

In words, as the system deteriorates, the income rate decreases. In a similar vein, we assume that the differential replacement cost rate $g(\cdot)$ is increasing, so that

$$\text{if } B \subset A, \text{ then } g(B) \geq g(A). \quad (3.6)$$

This assumption holds in the important special case having a constant planned replacement cost c and a failure penalty K so that

$$c(A) = c \text{ for } A \neq \emptyset, \text{ and } c(\emptyset) = c + K. \quad (3.7)$$

The following theorem characterizes the optimal replacement time in the model at hand. Because $\emptyset$ is the only absorbing state, the matrix G , restricted to nonempty states, has an inverse. Let $M = -G^{-1}$ denote the negative of this inverse.

Then $M(A,B)$ evaluates the mean duration spent in state B starting from state A prior to absorption.

Theorem 1: Suppose that the income and differential cost rates are monotonic as in (3.5) and (3.6). Then the optimal replacement time $T^* = T(\varphi^*)$ is the exit time from the class C^* defined by

$$C^* = \{A \text{ in } \Lambda: i(A) - g(A) \geq \varphi^*\} \quad (3.8)$$

where φ^* satisfies

$$\sum_{A \in C^*} M(2^N, A) \{i(A) - g(A) - \varphi^*\} = c. \quad (3.9)$$

More-over, φ^* is the maximal long run net income rate, and C^* is monotonic in the sense: If B is in C^* and $A \supset B$, then A is in C^* . In words, if it pays to continue at state B , then it pays to continue at all states better than B .

Proof. Because i and g are assumed monotonic, it follows easily from (3.7) that C^* is monotonic in the partial order of set inclusion. Because the Markov chain X is monotonic, it then follows that $M(2^N, A)$ gives the mean duration in any state A in C^* prior to exiting C^* . Consequently, (3.8) and (3.9) are the expressions in the model at hand for the general necessary and sufficient conditions given in (2.9) and (2.10).

The algorithm.

STEP 1. Order the states A in Λ according to their values $v(A) = i(A) - g(A)$ in decreasing order. That is,

$$\Lambda = \{A_0 = 2^N, A_1, A_2, \dots, A_{2^N-1} = \emptyset\},$$

such that

$$v(2^N) \geq v(A_1) \geq v(A_2) \geq \dots \geq v(\emptyset).$$

Any states having the same value are listed in order of decreasing size, or listed arbitrarily among states of the same size. Start with the continuation class $C_0 = \{2^N\}$, and set $\varphi_0 = v(2^N) - cG(2^N, 2^N)$. Set $j = 1$.

STEP 2. Consider the state A_j ; if $v(A_j) < \varphi_{j-1}$, go to STEP 3 (STOP). Otherwise, if $v(A_j) \geq \varphi_{j-1}$, set $C_j = C_{j-1} \cup \{A_j\}$, and evaluate

$$\varphi_j = \frac{\sum_{A \in C_j} M(2^N, A) v(A) - c}{\sum_{A \in C_j} M(2^N, A)} \quad (3.10)$$

Add 1 to j . If $j = 2^N$, then STOP. The optimal

continuation class is $C^* = \Lambda \setminus \emptyset$.

Otherwise, return to STEP 2.

STEP 3. STOP. C_{j-1} and φ_{j-1} satisfy conditions (3.7) and (3.8), (see Proposition 1 that follows for proof). Therefore $C^* = C_{j-1}$ is an optimal continuation class, and $\varphi^* = \varphi_{j-1}$ is the maximal long run net income rate.

The algorithm just described enlarges a continuation class state by state until an optimal such continuation class is obtained, as we establish in the next Proposition.

Proposition 1. The $C^* = C_{j-1}$ and $\varphi^* = \varphi_{j-1}$ obtained in the algorithm satisfy (3.8) and (3.9).

Proof: Equation (3.9) holds at every step by the construction in (3.10). By the monotonic order of the $v(A_j)$'s, STEP 2 implies that $v(A) < \varphi^*$ for all states A not in the continuation class C^* . Finally, to see that $v(A) \geq \varphi^*$ for all states A in C^* , note that φ_{j-2} satisfies $\varphi_{j-2} \leq v(A_{j-1})$ by construction, whence

$$\sum_{A \in C_{j-2}} M(2^N, A) v(A) - c \leq v(A_{j-1}) \sum_{A \in C_{j-2}} M(2^N, A);$$

and therefore,

$$\begin{aligned}
\varphi_{j-1} &= \frac{\sum_{A \in C_{j-2}} M(2^N, A) v(A) + M(2^N, A_{j-1}) v(A_{j-1}) - c}{\sum_{A \in C_{j-1}} M(2^N, A)} \\
&\leq \frac{v(A_{j-1}) \sum_{A \in C_{j-2}} M(2^N, A) + M(2^N, A_{j-1}) v(A_{j-1})}{\sum_{A \in C_{j-1}} M(2^N, A)} \\
&= \frac{v(A_{j-1}) \sum_{A \in C_{j-1}} M(2^N, A)}{\sum_{A \in C_{j-1}} M(2^N, A)} = v(A_{j-1}).
\end{aligned}$$

Then, $\varphi_{j-1} \leq v(A_{j-1}) \leq v(A)$ for all A in $C_{j-1} = C^*$, and the verification of (3.10) is complete.

4. An example. Assume that we have a system of N components that is supporting a fixed load. An operating component, having not failed prior to time t and carrying the load $\lambda(t)$, will fail during the interval $(t, t+\Delta t]$ with probability $K[\lambda(t)]\Delta t + o(\Delta t)$, where the function $K[\lambda]$, called the **breakdown rule**, expresses how changes in load affect the failure probability. Under these assumptions, components fail singly. (See e.g. Taylor [18] and Karlin (1984), pp. 264-273.) For the set of operational components A in Λ , let $\lambda(A) = (\lambda_1(A), \dots, \lambda_N(A))$ be an N -vector where $\lambda_i(A)$ is the load on component i . Only components in A share the load so that $\lambda_i(A) = 0$ for i not

in A . Then the infinitesimal transition rates as described in (3.1) for this process are

$$\begin{aligned}\mu(A,B) &= K[\lambda_i(A)], \quad \text{for } B = A \setminus \{i\}, i \text{ in } A \text{ in } \Lambda; \\ &= 0, \quad \text{otherwise.}\end{aligned}$$

Table 1 lists the failure rates for a typical load sharing type system having four components where we use the notation $\mu_i(A) = \mu(A,B)$ for $B = A \setminus \{i\}$. Figure 1 illustrates the possible transitions in the Markov chain that records the set of operational components as a function of time. Table 2 gives the values $v(A) = i(A) - g(A)$ for each state that will be used in a numerical example of the algorithm. The minimal replacement cost is assumed to be $c = c(2^N) = 1$. Since the matrix G is triangular, the mean durations $M(A,B) = -G^{-1}(A,B)$ are easily determined as needed.

We initialize the algorithm with the continuation class $C_0 = \{\{1,2,3,4\}\} = \{2^N\}$. We compute $M(2^N, 2^N) = 0.1$, and then

$$\varphi_0 = v(2^N) - c/M(2^N, 2^N) = 15 - 1/0.1 = 5.$$

Since $v(A_1) = 14 > \varphi_0 = 5$, we continue and enlarge the continuation class, forming $C_1 = \{2^N, A_1\} = \{\{1,2,3,4\}, \{2,3,4\}\}$. Then $M(2^N, A_1) = 0.0056$, and

$$\varphi_1 = \frac{(15)(.1) + (14)(.0056) - 1}{.1 + .0056} = 5.47727.$$

Since $v(A_2) = 11 > \varphi_1 = 5.47727$, again the continuation class is enlarged. We continue in this manner, obtaining the sequence $\varphi_2 = 5.90566$, $\varphi_3 = 6.05578$, $\varphi_4 = 6.267942$, $\varphi_5 = 6.361616$ and $\varphi_6 = 6.40598$. Since $v(A_7) = 6 < \varphi_6 = 6.40598$, the algorithm ends. The optimal continuation class is $C^* = \{A_0, \dots, A_6\}$, as shown in Figure 1.

APPENDIX

The algorithm is described more precisely using the language known as pidgin ALGOL.

We assume that the states in Λ are numbered from zero to $K-1$, where $|\Lambda| = k$, in such a way that if state i is contained in state j , then $i < j$ for $i, j = 0, \dots, k-1$.

Algorithm

INPUT. Array of values $INCOME[j]$, $0 \leq j \leq k-1$.

Array of values $COST[j]$, $0 \leq j \leq k-1$.

Infinitesimal matrix $G[i,j]$, $0 \leq i \leq k-1$, $0 \leq j \leq k-1$.

OUTPUT. A set of states which forms an optimal continuation class C^* .

METHOD. We make use of two procedures: BUBBLESORT, which sorts the elements of an array into nonincreasing order and INVERSE, which obtains the column elements of the inverse of matrix G for the state selected to be in the continuation class.

The algorithm itself is:

begin

for $i \leftarrow 0$ **until** $k-1$ **do**

begin

for $j \leftarrow 0$ **until** $k-1$ **do**

$GC[i] \leftarrow GC[i] + G[i,j]*COST[j];$

$V[i] \leftarrow INCOME[i] - COST[i],$

$P[i] \leftarrow i$

end

 BUBBLESORT(V,P);

 Let the starting continuation class be the state 0;

$NPHI \leftarrow (1/-G[0,0])*V[0] - COST[0];$

$DPHI \leftarrow 1/-G[0,0];$

$PHI \leftarrow NPHI/DPHI;$

$j \leftarrow 1$

while $V[j] > PHI$ **do**

begin

 Enlarge continuation class by the state $P[j],$

 INVERSE(j,p);

$NPHI \leftarrow NPHI + GINV[0,P[j]]*V[P[j]];$

$DPHI \leftarrow DPHI + GINV[0,P[j]];$

$PHI \leftarrow NPHI/DPHI;$

$j \leftarrow j+1$

end

write the actual continuation class and PHI.

end

procedure BUBBLESORT(A,B):

for j ← k-2 **step** -1 **until** 0 **do**

for i ← 0 **step** 1 **until** j **do**

if A[i+1] > A[i] **then**

begin

 interchange A[i] and A[i+1];

 interchange B[i] and B[i+1];

end

Note: The array B is used as a pointer to recognize the states in V after it has been sorted.

procedure INVERSE(i,P):

 GINV[P[j],P[j]] ← 1/-G[P[j],P[j]];

 GAUX ← 0

for i ← 1 **until** j **do**

begin

for l ← 1 **until** i **do**

 GAUX ← GAUX + G[P[j-i],P[j-i+l]]*GINV[P[j-i+l],P[j]];

 GINV[P[j-i],P[j]] ← GAUX*1/-G[P[j-i],P[j]];

 GAUX ← 0

end

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System State A FROM	Failure of Component i	System State A \ {i} TO	Failure Rate $\mu_i(A)$
{1,2,3,4}	1 2 3 4	$\begin{Bmatrix} 2,3,4 \\ 1,3,4 \\ 1,2,4 \\ 1,2,3 \end{Bmatrix}$	1 2 3 4
{2,3,4}	2 3 4	$\begin{Bmatrix} 3,4 \\ 2,4 \\ 2,3 \end{Bmatrix}$	5 6 7
{1,3,4}	1 3 4	$\begin{Bmatrix} 3,4 \\ 2,4 \\ 1,3 \end{Bmatrix}$	8 9 10
{1,2,4}	1 2 4	$\begin{Bmatrix} 2,4 \\ 1,4 \\ 1,2 \end{Bmatrix}$	11 12 13
{1,2,3}	1 2 3	$\begin{Bmatrix} 2,3 \\ 1,3 \\ 1,2 \end{Bmatrix}$	14 15 16
{1,2}	1 2	$\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$	17 18
{1,3}	1 3	$\begin{Bmatrix} 3 \\ 1 \end{Bmatrix}$	19 20
{1,4}	1 4	$\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$	21 22
{2,3}	2 3	$\begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$	23 24
{2,4}	2 4	$\begin{Bmatrix} 4 \\ 2 \end{Bmatrix}$	25 26
{3,4}	3 4	$\begin{Bmatrix} 4 \\ 3 \end{Bmatrix}$	27 28
{1}	1	$\emptyset$	29
{2}	2	$\emptyset$	30
{3}	3	$\emptyset$	31
{4}	4	$\emptyset$	32
$\emptyset$	absorbing state		

Table 1 Failure rates.

System State A	$v(A) = i(A)$	Ordered States
$\{1,2,3,4\}$	15	A_0
$\{2,3,4\}$	14	A_1
$\{1,3,4\}$	6	A_7
$\{1,2,4\}$	9	A_4
$\{1,2,3\}$	11	A_2
$\{1,2\}$	8	A_5
$\{1,3\}$	5	A_8
$\{1,4\}$	4	A_9
$\{2,3\}$	11	A_3
$\{2,4\}$	8	A_6
$\{3,4\}$	2	A_{10}
$\{1\}$	1	A_{11}
$\{2\}$	1	A_{12}
$\{3\}$	1	A_{13}
$\{4\}$	1	A_{14}
$\emptyset$	0	A_{15}

Table 2 The values associated with the states, listed in decreasing order.

Figure 1 Sets of operational components and the transition rates between them.