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REPRODUCING KERNEL HILBERT SPACE
FOR SOME NON-GAUSSIAN PROCESSES

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Multiple Wiener - Ito integrals; Self-similar processes;
Hermite processes; Fractional Brownian motion.

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1. Introduction

A sequence $\{f_n(t, \omega), n \geq 1\}$ of $C[0,1]$ -valued random functions obeys a functional law of the iterated logarithm if there exists a subset K of $C[0,1]$ such that

$$\begin{cases} \lim_{n \rightarrow \infty} d(f_n, K) = 0 & \text{a.s.} \\ C\{f_n, n \geq 1\} = K & \text{a.s.} \end{cases}$$

Here d is the sup-norm distance and $C\{X_n, n \geq 1\}$ is the set of limit points (cluster set) of f_n .

If f_n is Gaussian, it is often possible to characterize K

in two equivalent ways. One can specify K directly in terms of the functions that it contains. One can also characterize it as the unit ball of a suitable reproducing kernel Hilbert space (RKHS), denoted $\mathcal{H}(\Gamma)$, with kernel Γ . The kernel Γ is typically the covariance kernel of a Gaussian process to which f_n , adequately normalized, converges weakly in $C[0,1]$.

For example, let $Y_n(t) = 1/\sqrt{n} B(nt)$ where $B(t)$, $t \geq 0$ is standard Brownian motion. The weak limit of $Y_n(t)$ is $B(t)$ whose covariance kernel is $\Gamma(s,t) = \min(s,t)$. If $||\dot{z}||_2^2 = \int_0^1 (dz/dt)^2 dt$, then Strassen's functional law of the iterated logarithm states that the set K of all limit points of

$$f_n(t) = \frac{1}{(2 \log \log n)^{1/2}} \frac{B(nt)}{\sqrt{n}}$$

is a.s.

(1) $K = \{z \in C[0,1]: z(0) = 0, z \text{ absolutely continuous,}$

$$||\dot{z}||_2^2 \leq 1\}$$

or, equivalently,

(2) $K = \text{unit ball of the RKHS } \mathcal{H}(\Gamma) \text{ with } \Gamma(s,t) = \min(s,t).$

(See Jain and Marcus (1978), p. 121).

Consider now a sequence $Y_n(t)$ that converges weakly to a non-Gaussian process with a covariance kernel Γ , and suppose

that a functional law of the iterated logarithm also holds. Is the set of limit points a.s. identical to the unit ball of the RKHS $\mathcal{H}(\Gamma)$? The answer in general is no. We illustrate this by focusing on a class of self-similar processes given by multiple Wiener - Itô integrals. (The Hermite processes, considered in Section 4, are examples of such processes).

Recall that a process $Y(t)$ is self-similar with index $H > 0$ if $Y(at)$ and $a^H Y(t)$ have identical finite-dimensional distributions. The reader may refer to Taqqu and Czado (1985) for a recent survey on functional laws of the iterated logarithm for self-similar processes.

2. RKHS of multiple Wiener - Itô integrals.

Let $Y^{(m)}(t) \in C[0,1]$ be such that

$$Y^{(m)}(t) = \frac{1}{\sqrt{m!}} \int_{\mathbb{R}^m}' k_t^{(m)}(x_1, \dots, x_m) dB(x_1) \cdots dB(x_m). \quad (2.1)$$

Here $k_t^{(m)} \in L^2(\mathbb{R}^m)$ for each t , and $\int_{\mathbb{R}^m}'$ means that this is a multiple Wiener - Itô integral of order m . (For a definition see Itô (1951) or the Appendix of Mandelbaum and Taqqu (1984).)

Heuristically, the notation $\int_{\mathbb{R}^m}'$ indicates that no integration is performed on the hyperdiagonals of $\mathbb{R}^m$.

The process $Y^{(m)}(t)$ has mean 0 and covariance kernel

$$\begin{aligned}\Gamma^{(m)}(s, t) &= EY^{(m)}(s)Y^{(m)}(t) \\ &= \int_{\mathbb{R}^m} k_s^{(m)}(x_1, \dots, x_m) k_t^{(m)}(x_1, \dots, x_m) dx_1 \cdots dx_m. \quad (2.2)\end{aligned}$$

No extraneous coefficient appears in (2.2) because of the presence of the coefficient $1/\sqrt{m!}$ in (2.1). Note that $Y^{(m)}(t)$ is Gaussian when $m = 1$ and non-Gaussian when $m > 1$.

Now introduce the Hilbert space

$$\begin{aligned}\mathcal{H}^{(m)} &= \{Y(t) = \int_{\mathbb{R}^m} k_t^{(m)}(x_1, \dots, x_m) g(x_1, \dots, x_m) dx_1 \cdots dx_m : \\ &\quad g \in L^2(\mathbb{R}^m)\} \quad (2.3)\end{aligned}$$

with inner product

$$\langle Y_1, Y_2 \rangle_m = \int_{\mathbb{R}^m} g_1(x_1, \dots, x_m) g_2(x_1, \dots, x_m) dx_1 \cdots dx_m.$$

We say that $Y_1 \in \mathcal{H}^{(m)}$ is represented by $g_1 \in L^2(\mathbb{R}^m)$ and that $Y_2 \in \mathcal{H}^{(m)}$ is represented by $g_2 \in L^2(\mathbb{R}^m)$. Note the following:

- 1) The functions $\Gamma^{(m)}(\cdot, t)$ belong to $\mathcal{H}^{(m)}$ for each $0 \leq t \leq 1$ because, in view of (2.2), $\Gamma^{(m)}(\cdot, t)$ is represented by $k_t^{(m)} \in L^2(\mathbb{R}^m)$.

2) For $y \in \mathcal{H}^{(m)}$,

$$\begin{aligned} \langle y, \Gamma^{(m)}(\cdot, t) \rangle_m &= \int_{\mathbb{R}^m} g(x_1, \dots, x_m) k_t^{(m)}(x_1, \dots, x_m) dx_1 \cdots dx_m \\ &= y(t), \end{aligned}$$

where $g \in L^2(\mathbb{R}^m)$ is the function that represents y .

In view of 1) and 2), we can apply Theorem 2.2 of Marcus and Jain (1978) and obtain

Proposition 1. The Hilbert space $\mathcal{H}^{(m)}$ defined by (2.3) with inner product $\langle \cdot, \cdot \rangle_m$, can be identified with the RKHS $\mathcal{H}(\Gamma^{(m)})$ whose kernel $\Gamma^{(m)}$ is given by (2.2).

The following examples illustrate the case $m = 1$.

(Interesting cases with $m > 1$ are treated in the next section).

Examples. 1) Suppose $m = 1$ and $k_t^{(1)}(x) = 1_{[0, t]}(x)$. Then $Y^{(1)}(t)$ is the Brownian motion $B(t)$ and

$$\mathcal{H}^{(1)} = \{Y(t) = \int_{-\infty}^{\infty} 1_{[0, t]}(x) g(x) dx : g \in L^2(\mathbb{R}^1)\}.$$

Changing parameterization by setting $g(x) = \dot{z}(x) \equiv dz/dx$, one gets

$$\mathcal{H}^{(1)} = \{y(t) = \int_{-\infty}^{\infty} 1_{[0,t]}(x) \dot{z}(x) dx : \dot{z} \in L^2(\mathbb{R}^1)\}$$

$$= \{z(t) : z \text{ absolutely continuous, } z(0) = 0, \dot{z} \in L^2(\mathbb{R}^1)\},$$

which is the RKHS for Brownian motion mentioned in the Introduction.

2) If $m = 1$ and $k_t^{(1)}(x) = e^{-\lambda(t-x)} 1_{\{x < t\}}$ with $\lambda > 0$, then $y^{(1)}(t)$ is the Ornstein - Uhlenbeck process and its RKHS is

$$\mathcal{H}^{(1)} = \{y(t) = \int_{-\infty}^t e^{-\lambda(t-x)} g(x) dx : g \in L^2(\mathbb{R}^1)\}.$$

3. Self-similar processes

We now introduce integrands $k_t^{(m)}$ in (2.1) such that the corresponding random processes $y^{(m)}(t)$ have covariance kernels that do not depend on the order m of the multiple integral.

For $t \geq 0$, let

$$k_t^{(m)}(x_1, \dots, x_m) = \int_0^t q^{(m)}(v-x_1, \dots, v-x_m) dv \quad (3.1)$$

where $q: \mathbb{R}^m \rightarrow \mathbb{R}$ is such that for all $c > 0$,

$$q^{(m)}(cx_1, \dots, cx_m) = c^{-\lambda} q(x_1, \dots, x_m) \quad (3.2)$$

with $\lambda = (m/2) + 1 - H$ and $1/2 < H < 1$. Moreover, we require that

$$\int_{\mathbb{R}^m} [k_1^{(m)}(x_1, \dots, x_m)]^2 dx_1 \dots dx_m = 1. \quad (3.3)$$

These conditions on $k_t^{(m)}, t \geq 0$ ensure that $Y^{(m)}(t)$ defined by (2.1) has stationary increments, is self-similar with index H , has mean zero and variance t^{2H} . We shall take $Y^{(m)}(t)$ to be the version of the process which belongs to $C[0, \infty)$. Such a version exists because $E|Y^{(m)}(s) - Y^{(m)}(t)|^2 = |s - t|^{2H}$ with $2H > 1$. The covariance kernel of $Y^{(m)}(t)$ is

$$\begin{aligned} \Gamma^{(m)}(s, t) &= EY^{(m)}(s)Y^{(m)}(t) \\ &= \frac{1}{2} \{E[Y^{(m)}(s)]^2 + E[Y^{(m)}(t)]^2 - E[Y^{(m)}(s) - Y^{(m)}(t)]^2\} \\ &= \frac{1}{2} \{s^{2H} + t^{2H} - |s - t|^{2H}\}, \end{aligned}$$

and therefore, the covariance kernel does not depend on m . In view of Proposition 1, we get the following result.

Proposition 2. Suppose that $Y^{(m)}(t)$ is defined as (2.1) with $k^{(m)}$ given by (3.1)-(3.3). Then the Hilbert spaces $(\mathcal{H}^m, \langle \cdot, \cdot \rangle_m)$, $m = 1, 2, \dots$ are all identical to the RKHS $\mathcal{H}(\Gamma)$ with kernel

$$\Gamma(s, t) = \frac{1}{2} \{s^{2H} + t^{2H} - |s - t|^{2H}\}. \quad (3.4)$$

We shall now focus on the process $Y^{(m)}(t)$ defined in (2.1)

with kernel $k^{(m)}$ given by (3.1)-(3.3). From now on we let Γ be the kernel (3.4), $\mathcal{H}(\Gamma)$ be the corresponding RKHS, and U be the unit ball of $\mathcal{H}(\Gamma)$.

The sequence $Y^{(m)}(nt)/n^H$ converges weakly, as $n \rightarrow \infty$, to $Y^{(m)}(t)$ in $C[0,1]$ because the finite-dimensional distributions of $Y^{(m)}(nt)/n^H$ are identical to those of $Y^{(m)}(t)$ and tightness follows from $E|Y^{(m)}(ns)/n^H - Y^{(m)}(nt)/n^H|^2 = E|Y^{(m)}(s) - Y^{(m)}(t)|^2 = |s - t|^{2H}$ with $2H > 1$. Fox (1981) showed that the limit points of

$$f_n^{(m)}(t) = \frac{Y^{(m)}(nt)}{n^H (2 \log \log n)^{m/2}}$$

are a.s. contained in the unit ball U of $\mathcal{H}(\Gamma)$. Mori and Oodaira (1984), extending a result proved by Taqqu (1977) when $m = 1$, show that $f_n^{(m)}(t)$ satisfies a functional law of the iterated logarithm for all $m \geq 1$, and that the set of its limit points as $n \rightarrow \infty$ is a.s.

$$K^{(m)} = \{Y(t) = \int_{\mathbb{R}^m} k_t^{(m)}(x_1, \dots, x_m) g(x_1) \cdots g(x_m) dx_1 \cdots dx_m : \|g\|_2^2 \leq 1\}, \quad (3.5)$$

where $\|g\|_2^2 = \int_{-\infty}^{+\infty} g^2(x) dx$. The set $K^{(m)}$ is clearly a subset of

the unit ball of $\mathcal{H}^{(m)}$ since

$$||y||_{\mathcal{H}^{(m)}}^2 = \int_{\mathbb{R}^m} g^2(x_1) \cdots g^2(x_m) dx_1 \cdots dx_m = (||g||_2^2)^m \leq 1.$$

In view of Proposition 2, we get

Proposition 3. When $m \geq 2$, $K^{(m)}$ is a proper subset of the unit ball U of the RKHS $\mathcal{H}(\Gamma)$. The set $K^{(1)}$ coincides with U .

The next proposition states that the set $K^{(m)}$ is determined by a subset of the unit ball of $L^2(\mathbb{R}^1)$.

Proposition 4. Any $y \in K^{(m)}$ can be represented as

$$y(t) = \int_{-\infty}^{\infty} k_t^{(1)}(x) g_m(x) dx \quad (3.6)$$

where g_m is a suitable function in $L^2(\mathbb{R}^1)$ satisfying $||g_m||_2^2 \leq 1$.

Indeed, by Proposition 3, any $y \in K^{(m)}$ can be viewed as a function in $K^{(1)}$. Thus, there must exist a function $g_m \in L^2(\mathbb{R}^1)$ such that

$$\begin{aligned}
y(t) &= \int_{\mathbb{R}^m} k_t^{(m)}(x_1, \dots, x_m) g(x_1) \cdots g(x_m) dx_1 \cdots dx_m \\
&= \int_{-\infty}^{\infty} k_t^{(1)}(x) g_m(x) dx.
\end{aligned} \tag{3.7}$$

Since $\|y(t)\|_{\mathcal{H}(1)}^2 = \|y(t)\|_{\mathcal{H}(m)}^2 \leq 1$, we must have $\|g_m\|_2^2 = \|g\|_2^{2m} \leq 1$.

In principle, the explicit form of g_m can be obtained by solving the functional equation (3.7) in terms of g .

4. The case of the Hermite processes

The functional equation (3.7) simplifies when

$$k_t^{(m)}(x_1, \dots, x_m) = \frac{1}{A_m} \int_0^t \prod_{i=1}^m ((s - x_i)^+)^{-\alpha_m} ds, \tag{4.1}$$

where

$$\frac{1}{2} < \alpha_m < \frac{1}{2} + \frac{1}{2m}$$

and where

$$\begin{aligned}
A_m^2 &= \int_{\mathbb{R}^m} \left\{ \int_0^1 \prod_{i=1}^m ((s - x_i)^+)^{-\alpha_m} ds \right\}^2 dx_1 \cdots dx_m \\
&= \int_0^1 \int_0^1 \left\{ \int_{-\infty}^{s_1 \wedge s_2} (s_1 - x)^{-\alpha_m} (s_2 - x)^{-\alpha_m} dx \right\}^m ds_1 ds_2 \\
&= \left\{ \int_0^\infty (x + x^2)^{-\alpha_m} dx \right\}^m \int_0^1 \int_0^1 |s_1 - s_2|^{-2m\alpha_m + m} ds_1 ds_2 \\
&= \left\{ \frac{\Gamma(1 - \alpha_m) \Gamma(2\alpha_m - 1)}{\Gamma(\alpha_m)} \right\}^m \frac{2}{(1 + m - 2m\alpha_m)(2 + m - 2m\alpha_m)}.
\end{aligned}$$

This choice of A_m ensures that $\|k_1^{(m)}\|^2 = 1$. Hence $k_t^{(m)}$ satisfies the Assumptions (3.1)-(3.3). The corresponding process $Y^{(m)}(t)$, defined by (2.1), is

$$\begin{aligned}
Y^{(m)}(t) &= \frac{1}{\sqrt{m!} A_m} \int_{\mathbb{R}^m} \left\{ \int_0^t \prod_{i=1}^m ((s - x_i)^+)^{-\alpha_m} ds \right\} dB(x_1) \cdots dB(x_m) \\
&= \frac{\sqrt{m!}}{A_m} \int_{-\infty}^t dB(x_1) \int_{-\infty}^{x_1} dB(x_2) \cdots \int_{-\infty}^{x_{m-1}} dB(x_m) \\
&\quad \int_0^t \prod_{i=1}^m ((s - x_i)^+)^{-\alpha_m} ds.
\end{aligned}$$

It is known as the normalized Hermite process of order m . The process $Y^{(1)}(t)$ is the Gaussian fractional Brownian motion and the process $Y^{(2)}(t)$ is called the Rosenblatt process (see Taqqu (1985)).

When $k_t^{(m)}$ is given by (4.1), the functional equation (3.7)

becomes

$$\begin{aligned} & \frac{1}{A_m} \int_{\mathbb{R}^m} \left[\int_0^t \prod_{i=1}^m ((s - x_i)^+)^{-\alpha_m} ds \right] g(x_1) \cdots g(x_m) dx_1 \cdots dx_m \\ &= \frac{1}{A_1} \int_{\mathbb{R}} \left[\int_0^t ((s - x)^+)^{-\alpha_1} ds \right] g_m(x) dx. \end{aligned} \quad (4.2)$$

Note that the integral $\int_{-\infty}^s (s - x)^{-\alpha_m} g(x) dx$ converges for $g \in L^2(\mathbb{R}^1)$ for almost every s . Indeed, fix $T < s < 2T$, and express the integral as $I_1(s) + I_2(s)$ where

$$I_1(s) = \int_{-\infty}^T (s - x)^{-\alpha_m} g(x) dx \leq \left(\int_{-\infty}^T (s - x)^{-2\alpha_m} dx \right)^{1/2} \|g\|_2 < \infty$$

and where $I_2(s) = \int_T^s (s - x)^{-\alpha_m} g(x) dx$. Then

$$\int_T^{2T} |I_2(s)| ds \leq \int_T^{2T} |g(x)| \left\{ \int_x^{2T} (s - x)^{-\alpha_m} ds \right\} dx < \infty.$$

Therefore (4.2) becomes

$$\frac{1}{A_m} \int_0^t \left[\int_{-\infty}^s (s - x)^{-\alpha_m} g(x) dx \right]^m ds = \frac{1}{A_1} \int_0^t \left[\int_{-\infty}^s (s - x)^{-\alpha_1} g_m(x) dx \right] ds$$

or simply,

$$\left[\int_{-\infty}^s (s - x)^{-\alpha_m} g(x) dx \right]^m = \frac{A_m}{A_1} \int_{-\infty}^s (s - x)^{-\alpha_1} g_m(x) dx.$$

This is a functional equation involving fractional integration.

Formally, if

$$(I^{1-\alpha}g)(s) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^s (s-x)^{-\alpha} g(x) dx,$$

and if, $I^{\alpha}(I^{-\alpha}g) = g$, then

$$g_m = \frac{A_1}{A_m} \frac{(\Gamma(1-\alpha_m))^m}{\Gamma(1-\alpha_1)} I^{\alpha_1-1} [I^{1-\alpha_m}g]^m.$$

Observe that by setting this expression for $g_m(x)$ in (3.6) one obviously recovers the set of limit points

$$K_m = \{y(t) = \frac{1}{A_m} \int_0^t ds [\int_{-\infty}^s (s-x)^{-\alpha_m} g(x) dx]^m : ||g||_2^2 \leq 1\}$$

given in (3.5).

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