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**The Analysis of Finite Security Markets
Using Martingales**

by

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THE ANALYSIS OF FINITE SECURITY MARKETS
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Abstract: The theory of finite security markets was originally developed by Harrison and Pliska. In this paper, we treat this theory by using discrete martingale techniques. Since the markets are assumed to allow no arbitrage opportunities, we focus on the price increments of the stocks between one time period to the next and convert these increments to martingale differences through an equivalent change of measure. We make precise the connection between diverse expressions of economic equilibrium such as "absence of arbitrage", "martingale property" and "complementary slackness property". Our method has a geometric interpretation which provides a connection with linear programming. It also yields an algorithm for analyzing finite security markets.

Keywords: security markets, arbitrage, martingales, change of measure, convex hull, complete markets, extremal measures, martingale representation property, linear programming, duality, equilibrium.

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1. Introduction

This paper concerns the theory of finite security markets where trading takes place at discrete points in time and the stocks assume only a finite number of values. The model was developed by Harrison and Pliska [1]. We begin by recovering their main results, using martingale techniques instead of a "consistent price system". Martingale techniques seem to be the natural ones in this context because they allow us to focus on the price increments between one time period to the next and convert these to martingale differences through a change of measure. Our approach moreover, extends to markets whose stocks do not necessarily assume a finite number of values. (Details about this extension will appear elsewhere.)

We show that the economically meaningful requirement of "no arbitrage" can be analyzed by looking at the sample paths of a given vector-valued stochastic process representing the different stock prices. Under "no arbitrage", the process can be transformed in a martingale by means of an equivalent change of measure. We can then characterize "complete" markets and also characterize stochastic processes that have the martingale representation property under the new measure.

In addition, the martingale approach allows us to show that finite security markets possess a "duality structure" which can be formulated in terms of a certain primal-dual pair of linear programs. We shall provide an algorithm based on linear programming methods that checks for "completeness" of the market, finds an equivalent martingale measure (if one exists), checks for attainability of contingent claims such as options, determines their prices, and provides a strategy that generates a given claim.

The theory of finite markets is a special case of the more general theory of continuous (frictionless) security markets where trading is continuous and the underlying probability space is arbitrary. In [1], the question of "complete" markets is treated in a way that is specific to the finite market model. Here, we relate the "completeness" problem in finite markets to the fundamental notion of "extremal" probability measures (see Theorem 4.1. below). Such a relationship is known to exist in the continuous market model (see [2]), but, as we will show, it becomes particularly transparent in the finite market model.

The paper is structured as follows. Section 2 contains the formulation of the model. For the convenience of the reader we have included all necessary definitions so as to avoid constant references to the seminal work of Harrison and Pliska [1]. In Section 3, we use a geometric reformulation of the "no arbitrage assumption" to obtain a new proof of the equivalence between the absence of arbitrage opportunities and the existence of an equivalent martingale measure. Section 4 deals with the question of complete markets. We show that completeness is related to the fine structure of the filtration and that it is equivalent to a certain extremality property of the martingale measure constructed in Section 3. We also obtain a geometric characterization of stochastic processes that have the martingale representation property after a change of measure. In Section 5, we point to the connection between some aspects of the theory of finite security markets and linear programming. Section 6 provides an algorithm for analyzing finite market models. In the Appendix, we describe a geometric approach to constructing martingale measures on a finite probability space. The results of the Appendix are used in Sections 3 and 4.

2. Formulation of the finite market model

The model we use is essentially the same as that in Harrison and Pliska [1] from where the following definitions and market interpretations are taken.

Let $(\Omega, \mathcal{F}, P)$ be a given probability space with $|\Omega| =: N < \infty$. The only requirement imposed on the probability measure P is that $P[\{\omega\}] > 0$ for all $\omega \in \Omega$. Thus the various investors may disagree on their choice of P but they all agree on what states of nature are possible.

For a specified time horizon $T < \infty$ (terminal state of all economic activities), we consider a given $(K + 1)$ -dimensional stochastic process $S = (S_t: t = 0, 1, \dots, T)$ defined on $(\Omega, \mathcal{F}, P)$ and with component processes $S^0, S^1, \dots, S^K$. Each component S^k is required to be strictly positive so that S_t^k can be interpreted as the price of security k at time t .

The price process S gives rise to the natural (or minimal) filtration $(\mathcal{F}_t: t = 0, 1, \dots, T)$. By this we mean that $\mathcal{F}_t$ is the σ -algebra generated by $S_0, S_1, \dots, S_t$. Without any real loss of generality, we take $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F} = 2^\Omega$, that is, $\mathcal{F}_T$ is the σ -algebra of all subsets of Ω . The filtration describes how information is revealed to the investors when securities are traded at time $t = 0, 1, \dots, T$: they have access to past and current price information only. To each $\mathcal{F}_t$ there corresponds a minimal partition $\mathcal{P}_t$ of Ω , i.e. at time t the investors only know which cell of this partition contains the true state of nature.

The 0^{th} security S^0 is called the bond: at each time t , S_t^0 determines the relative value of the currency. Without loss of generality,

we set $S_t^0 = 1$ for all t . This is no restriction because we can always consider the ratio S_t^k/S_t^0 , which is the price of security k at time t "corrected for inflation". (For a formal statement, see Harrison and Kreps [3, p. 401].) Harrison and Pliska [1] prefer to think of the 0^{th} security as a riskless asset with interest rate r (constant and positive), i.e. $S_t^0 = (1+r)^t \cdot S_0^0$ with $S_0^0 = 1$.

A trading strategy is defined to be a predictable, $(K + 1)$ -dimensional stochastic process $\phi = (\phi_t: t = 1, \dots, T)$ with components $\phi^0, \phi^1, \dots, \phi^K$. Predictable means that ϕ_t is measurable with respect to the σ -algebra F_{t-1} , i.e. ϕ_t depends only on the information available up to time $t - 1$. ϕ_t^k can be interpreted as the quantity of security k held by the investor between times $t - 1$ and t , namely during the period $[t-1, t)$. The vector ϕ_t is the investor's portfolio at time t and its components may assume negative as well as positive values. The investor is allowed to select his time t portfolio ϕ_t after the prices S_{t-1} are observed because ϕ is assumed to be predictable, but ϕ_t must be established without any knowledge of the prices S_t .

The value process $V(\phi) = \{V_t(\phi): t = 0, 1, \dots, T\}$ for a trading strategy ϕ is defined by

$$V_t(\phi) = \begin{cases} \phi_1 \cdot S_0 = \sum_{k=0}^K \phi_1^k & \text{if } t = 0 \\ \phi_t \cdot S_t = \sum_{k=0}^K \phi_t^k S_t^k & \text{if } t = 1, \dots, T. \end{cases}$$

$V_t(\phi)$ represents the value of the portfolio ϕ_t at time t before any changes are made at that time.

A trading strategy ϕ is called self-financing if

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t \quad \text{for } t = 1, \dots, T-1,$$

that is, if all changes in the value of the portfolio are due to the net gains realized on investments. Let Φ denote the set of all self-financing trading strategies ϕ .

Remarks: 1) Harrison and Pliska [1] impose one more restriction on the set of trading strategies: they only allow self-financing strategies ϕ with the property that the corresponding value process $V(\phi)$ is non-negative. (i.e. $V_t(\phi) \geq 0$ for all t). Such a strategy is then called admissible. This is a modelling assumption which protects the broker from unbounded short sales. It has a minor effect on subsequent results and is not needed from a mathematical point of view. Therefore, we shall not require here that ϕ be admissible.

2) In [1], Harrison and Pliska consider an arbitrary sequence $(F_t: t = 0, 1, \dots, T)$ of σ -algebras with $\{\emptyset, \Omega\} = F_0 \subseteq F_1 \subseteq \dots \subseteq F_T = F$ describing the flow of information in the market model, and they suppose that the price process is adapted to $(F_t: t = 0, 1, \dots, T)$ meaning that

S_t is measurable with respect to F_t for each $t \in \{0,1,\dots,T\}$. In this paper, we take $(F_t: t = 0,1,\dots,T)$ to be the minimal filtration, that is $F_t = \sigma(S_0, S_1, \dots, S_t)$. Whereas the minimal filtration forces investors to base their trading decisions on past and present price information only, an arbitrary filtration allows for the possibility of additional information, including perhaps knowledge of future prices. However, as we will show (see the remarks at the end of Sections 3 and 4), we lose no generality in assuming the minimal filtration.

3. Probabilistic implications of the "no arbitrage assumption"

"No arbitrage" is an economically meaningful requirement because arbitrage opportunities represent riskless plans for making profit without any investment. Requiring "no arbitrage opportunities" in our market model is equivalent to assuming the existence of an equivalent martingale measure on (Ω, F) . To show this, Harrison and Pliska [1] introduce an extraneous "price system". Here, we shall give a probabilistic proof which points to the fundamental role that martingales play in the theory of market trading. The result is stated in Theorem 3.1 and uses facts obtained in the Appendix. Moreover, Theorem 3.1 provides a geometric interpretation of the "no arbitrage assumption" which points to a connection with linear programming (see Section 5). But first, two definitions.

Definition 3.1 The security market model (introduced in Section 2) is said to contain an arbitrage opportunity if there exists a self-financing strategy ϕ such that $V_0(\phi) = 0$, $V_T(\phi) \geq 0$ and $V_T \not\equiv 0$. It contains no arbitrage opportunities, if for all $\phi \in \Phi$ with $V_0(\phi) = 0$, one has $V_T(\phi) \geq 0 \Rightarrow V_T(\phi) = 0$.

Definition 3.2 A martingale measure is a probability measure Q on $(\Omega, \mathcal{F})$ such that the (discounted) price process $S = (S_t : t = 0, 1, \dots, T)$ is a $(Q, (\mathcal{F}_t))$ - martingale, i.e. $E_Q[S_{t+1} | \mathcal{F}_t] = S_t$ a.s. Q for all $t \in \{0, 1, \dots, T-1\}$, where $E_Q[\cdot]$ denotes the mathematical expectation with respect to Q . (Recall that S_t represents here the discounted price since $S_t^0 = 1$.)

A probability measure Q is equivalent to P if and only if for all $A \in \mathcal{F}$, $Q[A] = 0 \Leftrightarrow P[A] = 0$, thus if and only if $Q[\omega] > 0$ for all $\omega \in \Omega$. Let $\mathcal{P}$ denote the set of all martingale measures Q that are equivalent to P .

We now turn to the probabilistic formulation of the no-arbitrage assumption using discrete martingale theory. The idea is as follows. The absence of arbitrage opportunities enables us, at each time t and based only on the history up to time $t - 1$, i.e. based only on $\mathcal{F}_{t-1}$, to construct a new (conditional) probability such that the increment $S_{t+1} - S_t$ of the price process between times $t - 1$ and t has zero (conditional) expectation under the new probability. By "piecing" the new conditional probabilities together, we obtain a new probability measure on $(\Omega, \mathcal{F})$ that makes the price process S a martingale.

Specifically, note that any probability measure P on $(\Omega, \mathcal{F})$ can be "decomposed" into an initial distribution and successive transition-probabilities. In addition, the transition-probabilities from time $t - 1$ to t ($t = 1, \dots, T$) can be taken to be the conditional distribution of $S_t - S_{t-1}$ given $\mathcal{F}_{t-1}$. Having decomposed P in such a way, our first result is a consequence of the "no-arbitrage requirement" applied to a single time-period.

Lemma 3.1 If the market model contains no arbitrage opportunities, then for all $t \in \{0, 1, \dots, T-1\}$, for all $\phi \in \Phi$ and for any $A \in \mathcal{P}_t$, we have:

$$(i) \quad P[V_{t+1}(\phi) - V_t(\phi) \geq 0 | A] = 1 \Rightarrow P[V_{t+1}(\phi) - V_t(\phi) = 0 | A] = 1.$$

$$(ii) \quad P[V_{t+1}(\phi) - V_t(\phi) \leq 0 | A] = 1 \Rightarrow P[V_{t+1}(\phi) - V_t(\phi) = 0 | A] = 1.$$

Idea of the proof: Suppose for simplicity that there is only one bond and one stock (i.e. $K = 1$). If at some time t , an investor knows for sure that the stock price will not decrease (i.e. $P[S_{t+1} - S_t \geq 0 | A] = 1$ for some $A \in \mathcal{P}_t$), he would buy this stock at time t , sell it at time $t + 1$ and make a profit of $S_{t+1} - S_t$ dollars per unit of stock. He would then invest his profits into riskless bonds ending up with a non-negative return at the terminal time T . To prevent such an arbitrage opportunity one must have $P[S_{t+1} - S_t = 0 | A] = 1$. A similar argument applies when $P[S_{t+1} - S_t \leq 0 | A] = 1$.

Proof of the lemma: Fix $t \in \{0, \dots, T-1\}$ and $\phi \in \Phi$.

(i) To prove the first part, suppose $P[V_{t+1}(\phi) - V_t(\phi) \geq 0 | A] = 1$ for some $A \in \mathcal{P}_t$ and define a new trading strategy ζ for all times $s = 1, \dots, T$ as follows:

For $s \leq t$: $\zeta_s \equiv 0$ ("don't do anything before time t ").

For $s = t + 1$: $\zeta_{t+1}(\omega) = 0$ if $\omega \notin A$, and

$$\zeta_{t+1}^k(\omega) = \begin{cases} \phi_{t+1}^k(\omega) & \text{if } \omega \in A \text{ and } k \in \{1, \dots, K\} \\ \phi_{t+1}^0(\omega) - V_t(\phi)(\omega) & \text{if } \omega \in A \text{ and } k = 0 \end{cases}$$

("If ω happens to be in A at time t , follow strategy ϕ when dealing with the stocks, but modify the bond-holdings appropriately in order to compensate for doing nothing when $\omega \notin A$.)

For $s > t + 1$: $\zeta_s^k \equiv 0$ for $k \in \{1, \dots, K\}$, and

$$\zeta_s^0(\omega) = \begin{cases} V_{t+1}(\zeta)(\omega) & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

("Invest the amount $V_{t+1}(\zeta)$ into riskless bonds if ω happens to be in A otherwise do nothing.")

We first show that $\zeta \in \Phi$. Clearly, ζ is predictable; to see that ζ is self-financing, we compute (for $\omega \in A$)

$$\begin{aligned} \zeta_{t+1} \cdot S_t &= (\phi_{t+1}^0 - V_t(\phi))S_t^0 + \sum_{k=1}^K \phi_{t+1}^k S_t^k \\ &= \sum_{k=0}^K \phi_{t+1}^k S_t^k - V_t(\phi) \\ &= \phi_{t+1} \cdot S_t - V_t(\phi) \\ &= \phi_t \cdot S_t - V_t(\phi) = 0, \end{aligned}$$

using the fact that ϕ is self-financing.

Since $\zeta_s \cdot S_s = 0$ for $s \leq t$, we have $(\zeta_{s+1} \cdot S_s)(\omega) = (\zeta_s \cdot S_s)(\omega)$ for all $s \leq t$ and for all $\omega \in \Omega$. When $s > t + 1$, we hold only bonds (if $\omega \in A$) or we hold nothing at all (if $\omega \notin A$). Therefore, $(\zeta_s \cdot S_s)(\omega) = (\zeta_{s+1} \cdot S_s)(\omega)$ for all $s > t$ and for all $\omega \in \Omega$. The strategy ζ is thus self-financing and hence, $\zeta \in \Phi$.

Next, we note that $V_T(\zeta) \geq 0$; in fact, for all $s \geq t + 1$ and for all $\omega \in A$,

$$\begin{aligned} V_s(\zeta) &= \zeta_s \cdot S_s = \zeta_{t+1} \cdot S_{t+1} \\ &= (\phi_{t+1}^0 - V_t(\phi))S_{t+1}^0 + \sum_{k=1}^K \phi_{t+1}^k S_{t+1}^k \\ &= \sum_{k=0}^K \phi_{t+1}^k S_{t+1}^k - V_t(\phi) \\ &= V_{t+1}(\phi) - V_t(\phi) \geq 0. \end{aligned}$$

Since $V_T(\zeta) = 0$ on A^c , ζ defines a trading strategy with $\zeta \in \Phi$, $V_0(\zeta) = 0$ and $V_T(\zeta) \geq 0$. The absence of arbitrage opportunities now implies $V_T(\zeta) = 0$, or, equivalently,

$$\begin{aligned} 0 &= P[V_T(\zeta) > 0] \\ &= P[\{V_T(\zeta) > 0\} \cap A] \\ &= P[V_{t+1}(\phi) - V_t(\phi) > 0 | A]P[A]. \end{aligned}$$

Hence $P[V_{t+1}(\phi) - V_t(\phi) = 0 | A] = 1$.

(ii) To prove the second part of the lemma, we suppose

$P[V_{t+1}(\phi) - V_t(\phi) \leq 0 | A] = 1$ for some $A \in \mathcal{P}_t$ and define a new trading strategy ζ as before, but with

$$\zeta_{t+1}^k(\omega) = \begin{cases} V_t(\phi)(\omega) - \phi_{t+1}^0(\omega) & \text{if } \omega \in A, k = 0 \\ -\phi_{t+1}^k(\omega) & \text{if } \omega \in A, k \in \{1, \dots, K\}. \end{cases}$$

Then we proceed as before. $\square$

Recall that $\mathbb{P}$ denotes the set of all martingale measures equivalent to the original probability measure P . When is $\mathbb{P}$ non-empty? The following result provides two answers, one of them being geometric in nature. Let $\text{conv}(C)$ denote the convex hull of a finite set $C \subset \mathbb{R}^K$ and let $\text{ri}(C)$ denote the relative interior of C that is, the interior of C when C is viewed as a subset of its affine hull. If $\dim(C) = K$, then $\text{ri}(C)$ is the same as interior of C . When $\dim(C) < K$, the useful notion is that of relative interior because the interior of C is necessarily empty. For $A \in \mathcal{P}_t$ ($0 \leq t < T$), set

$$C(t+1, A) = \text{conv}(\{S_{t+1}(\omega) - S_t(\omega) : \omega \in A\}).$$

Theorem 3.1 The following three conditions are equivalent:

- (1) The market model contains no arbitrage opportunities.
- (2) $0 \in \text{ri}(C(t+1, A))$ for all $A \in P_t$ and all $t \in \{0, 1, \dots, T-1\}$.
- (3) $\mathbb{P} \neq \emptyset$.

Proof: 1) We first show that (1) $\Rightarrow$ (2): Fixing $t \in \{0, 1, \dots, T-1\}$ and $A \in P_t$, we proceed in two steps.

a) Since there are no arbitrage opportunities, we have, by Lemma 3.1.(i)

$$P[V_{t+1}(\phi) - V_t(\phi) \geq 0 \mid A] = 1 \Rightarrow P[V_{t+1}(\phi) - V_t(\phi) = 0 \mid A] = 1$$

for all $\phi \in \Phi$.

In this first step of the proof, we show that this is equivalent to

$$P[\alpha \cdot X \geq 0 \mid A] = 1 \Rightarrow P[\alpha \cdot X = 0 \mid A] = 1$$

for all $\alpha \in R^K$, where

$$X = (S_{t+1}^1 - S_t^1, S_{t+1}^2 - S_t^2, \dots, S_{t+1}^K - S_t^K)^T \in R^K$$

(Note that $V_{t+1}(\phi)$ involves an inner product in R^{K+1} whereas $\alpha \cdot X$ is an inner product in R^K .)

To establish the equivalence, observe first that since $S_t^0 \equiv 1$, we have for all $\phi \in \Phi$,

$$\begin{aligned} V_{t+1}(\phi) - V_t(\phi) &= \phi_{t+1} \cdot S_{t+1} - \phi_t \cdot S_t \\ &= \phi_{t+1} \cdot S_{t+1} - \phi_{t+1} \cdot S_t \\ &= \phi_{t+1} \cdot (S_{t+1} - S_t) = \sum_{k=1}^K \phi_{t+1}^k (S_{t+1}^k - S_t^k) \\ &= \alpha \cdot X \end{aligned}$$

where $\alpha = (\phi_{t+1}^1, \dots, \phi_{t+1}^K)^T \in \mathbb{R}^K$ is predictable, hence α is a constant vector in $\mathbb{R}^K$ when F_t is given. Conversely, given a vector $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^K) \in \mathbb{R}^K$ we can construct the portfolio ϕ_{t+1} at time $t+1$ of a strategy $\phi \in \Phi$ by setting

$$\phi_{t+1}^k = \alpha^k \quad \text{for } k = 1, \dots, K,$$

$$\phi_{t+1}^0 = V_t(\phi) - \sum_{k=1}^K \alpha^k S_t^k.$$

Such a strategy does satisfy the self-financing condition

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t.$$

b) Since, by step a), there exists no $\alpha \in \mathbb{R}^K$ such that, given A , $\alpha \cdot X(\omega) \geq 0$ for all $\omega \in A$ and $\alpha \cdot X \not\equiv 0$, we conclude by Lemma A.1 of the Appendix that $0 \in \text{ri}(C(t+1, A))$.

2) The equivalence (2) $\Leftrightarrow$ (3) follows from Theorem A.4 of the Appendix.

3) To prove (3) $\Rightarrow$ (1) we proceed as in Harrison and Pliska

[1, page 228]: Let $Q \in \mathbb{P}$ and let $\phi \in \Phi$ be such that

$V_0(\phi) = 0$ and $V_T(\phi) \geq 0$. Then $V_T(\phi) = 0$ because

$$\begin{aligned} E_Q[V_T(\phi)] &= E_Q[E_Q[\phi_T \cdot S_T | F_{T-1}]] \\ &= E_Q[\phi_T \cdot E_Q[S_T | F_{T-1}]] \quad (\phi \text{ is predictable}) \\ &= E_Q[\phi_T \cdot S_{T-1}] \quad (S \text{ is a } (Q, (F_t)) - \text{martingale}) \\ &= E_Q[\phi_{T-1} \cdot S_{T-1}] \quad (\phi \text{ is self-financing}) \\ &= \dots \\ &= E_Q[\phi_1 \cdot S_0] \\ &= E_Q[V_0(\phi)] \\ &= 0. \end{aligned}$$

Thus, the market contains no arbitrage opportunities. $\square$

As in Harrison and Pliska [1], we get

Corollary 3.1 For all $Q \in \mathbb{P}$ and any $\phi \in \Phi$, the value process $V(\phi)$ is a real-valued $(Q, (F_t))$ - martingale.

Remark: If the minimal filtration is replaced by an arbitrary filtration $(F_t: t = 0, 1, \dots, T; F_0 = \{\emptyset, \Omega\}, F_T = F)$ with respect to which the price process S is adapted, then there will be in general more partition sets in $\mathcal{P}_t(0 < t < T)$. But this does not affect the validity of Lemma 3.1, of Theorem 3.1 (or of Corollary 3.1).

4. Complete markets

In the market model introduced in Section 2 investors "act" by using "contingent claims", such as call options. Formally, a contingent claim is a non-negative random variable X which is measurable with respect to F . For a less formal definition and an example, see Harrison and Pliska [1]. Let $\mathbb{X}$ denote the set of all contingent claims.

We will assume throughout this section that the market model contains no arbitrage opportunities.

Let Q be a probability measure on (Ω, F) that assigns zero probability to an arbitrary subset of Ω . Thus, we do not insist at this point that Q be equivalent to P . Consider an investor who wants to act only on those ω 's for which $Q[\{\omega\}] > 0$. He would like to determine the price of a given claim $X \in \mathbb{X}$ by following a strategy $\phi \in \Phi$ that would enable him to recover X only on those ω 's for which $Q[\{\omega\}] > 0$. If such a strategy $\phi \in \Phi$ exists, X is said to be attainable under Q , and clearly, the (time-zero) price for the claim X is then the same as the value of the investor's portfolio at time 0.

More precisely, $X \in \mathbb{X}$ is said to be attainable under Q if there exists a trading strategy $\phi \in \Phi$ with associated value process $V(\phi)$ such that $V_T(\phi)(\omega) = X(\omega)$ for all $\omega \in \Omega$ with $Q[\{\omega\}] > 0$. In this case the strategy ϕ is said to generate X and $\pi(X) = V_0(\phi)$ is called the (time-zero) price of the contingent claim X .

But which claims are attainable under Q ? A market is said to be complete under Q if all contingent claims are attainable under Q , or more formally, if for all $X \in \mathbb{X}$, there exists $\phi \in \Phi$ such that $V_T(\phi) = X$ a.s. Q .

The next theorem relates "completeness" of the market to a certain extremality property of a martingale measure. Its proof provides the link to the linear programming approach that is developed in the next section.

Let $M(S)$ denote the set of all probability measures Q on $(\Omega, \mathcal{F})$ (not necessarily equivalent to P) that make the price process S a $(Q, (\mathcal{F}_t))$ - martingale. Note that $\mathbb{P} \in M(S)$ and therefore, the absence of arbitrage opportunities in the market implies that $M(S)$ is non-empty. Since $M(S)$ is a convex set, we can consider the set $M_e(S)$ of its extreme points: a probability measure Q in $M_e(S)$ cannot be expressed as a strictly convex combination of two distinct probability measures in $M(S)$.

Theorem 4.1 Let $Q \in M(S)$. Then $Q \in M_e(S)$ if and only if the market model is complete under Q .

Proof: Consider Q in the "decomposed" form and fix $t \in \{0, 1, \dots, T-1\}$ and $A \in \mathcal{P}_t$ with $Q[A] > 0$. The value of the vector $S_t(\omega) \in \mathbb{R}^{K+1}$ is the same for all $\omega \in A$, but the vector $S_{t+1}(\omega)$ can take n distinct values in $\mathbb{R}^{K+1}$ when $\omega \in A$, each with positive conditional probability

$Q[\cdot|A]$. Here, $1 \leq n \leq N = |\Omega|$. Denote these values

$S_{t+1}(\omega_1), \dots, S_{t+1}(\omega_n)$, and for each security $k \in \{0, 1, \dots, K\}$ let
 $S_{t+1}^k = (S_{t+1}^k(\omega_1), S_{t+1}^k(\omega_2), \dots, S_{t+1}^k(\omega_n))^T \in \mathbb{R}^n$.

To prove the theorem, it is enough to show that

(a) the system

$$\phi^0 S_{t+1}^0 + \phi^1 S_{t+1}^1 + \dots + \phi^K S_{t+1}^K = x$$

has a solution $(\phi^0, \phi^1, \dots, \phi^K)^T \in \mathbb{R}^{K+1}$ for any $x \in \mathbb{R}^n$,

if and only if

(b) the system

$$\sum_{i=1}^n \lambda_i (S_{t+1}(\omega_i) - S_t(\omega_i)) = 0, \quad \sum_{i=1}^n \lambda_i = 1, \quad \lambda_i > 0 \quad \text{for } i = 1, \dots, n$$

has a unique solution $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$.

Indeed, completeness of the market under Q is equivalent to statement

(a) holding for all $A \in \mathcal{P}_t$ with $Q[A] > 0$ ($0 \leq t < T$), and $Q \in M_e(S)$

is equivalent to statement (b) holding for all $A \in \mathcal{P}_t$ with

$Q[A] > 0$ ($0 \leq t < T$) (see Theorem A.3 of the Appendix).

For (a) to hold we need

$$\dim\{\text{span}\{S_{t+1}^0, S_{t+1}^1, \dots, S_{t+1}^K\}\} = n$$

and for (b) to hold we need

$$\dim\{\text{span}\{S_{t+1}(\omega_1) - S_t(\omega_1), \dots, S_{t+1}(\omega_n) - S_t(\omega_n)\}\} = n - 1.$$

But these two conditions are equivalent, since by "going from columns to

rows" and using the fact that $S_{t+1}^0 = S_t^0 = 1$, we get

$$\begin{aligned}
 & \dim\{\text{span}\{S_{t+1}^0, S_{t+1}^1, \dots, S_{t+1}^K\}\} \\
 &= \dim\{\text{span}\{S_{t+1}(\omega_1), S_{t+1}(\omega_2), \dots, S_{t+1}(\omega_n)\}\} \\
 &= \dim\{\text{span}\{S_{t+1}(\omega_1) - S_t(\omega_1), \dots, S_{t+1}(\omega_n) - S_t(\omega_n)\}\} + 1. \quad \square
 \end{aligned}$$

We shall now suppose that $Q \sim P$, that is Q is equivalent to P so that the investor assigns positive probabilities to all states of nature ω .

A contingent claim $X \in \mathbb{X}$ is said to be attainable if it is attainable under P (and therefore under any $Q \sim P$). As before, we call $\pi(X) = V_0(\phi)$ the price of the attainable claim X , when $\phi \in \Phi$ is the strategy that generates X and $V(\phi)$ is the associated value process.

Knowing any $Q \in \mathbb{P}$ allows us to compute the prices of all attainable claims $X \in \mathbb{X}$ via the so-called valuation-formula

$$\pi(X) = E_Q[X] \quad \text{for all } Q \in \mathbb{P}$$

(see Harrison and Pliska [1, p. 228] or use Corollary 3.1). The market model is called complete if all contingent claims are attainable. We can now recover a result of Harrison and Pliska [2] (see also Harrison and Kreps [3]).

Corollary 4.1 The market model is complete if and only if the set $\mathbb{P}$ is a singleton.

Proof: Let $Q \sim P$. If $Q \in M(S)$ and the market model is complete then $Q \in M_e(S)$ by Theorem 4.1 and thus unique by Corollary A.2 of the Appendix. On the other hand, if Q is the unique element in $\mathbb{P}$, then by Theorem A.3 of the Appendix, Q is an extreme point of $M(S)$, and hence the market is complete by Theorem 4.1. $\square$

Remark: Kreps [6] and Harrison and Pliska [1] characterize completeness in a way that is specific to a finite Ω . They introduce a measure for the information that might be received at time $t + 1$ if at time t the event A is known to prevail. Harrison and Pliska [1] call this measure the splitting index of A (denoted by $K(t,A)$) and define it for all $0 \leq t < T$ and all $A \in \mathcal{P}_t$ by

$$K(t,A) \equiv \text{cardinality}(A' \in \mathcal{P}_{t+1} : A' \subseteq A).$$

By using the proof of Theorem 4.1, it is easy to see that a market that contains no arbitrage opportunities is complete if and only if

$$K(t,A) = \dim\{\text{span}\{S_{t+1}(\omega) : \omega \in A\}\}$$

for all $0 \leq t < T$ and all $A \in \mathcal{P}_t$ (see also Kreps [6]). If there are "no redundancies" in the vectors $S_{t+1}(\omega) \in \mathbb{R}^{K+1}$ when $\omega \in A$, in the sense of Harrison and Pliska [1, p. 230], then the condition for completeness is

$$K(t,A) = K + 1 \quad \text{for all } 0 \leq t < T \text{ and all } A \in \mathcal{P}_t.$$

We conclude this section with a geometric characterization of the martingale representation property and we describe the relationship between completeness of the market model and the fine structure of the filtration.

The martingale representation property

In the more general setting of a continuous market, Harrison and Pliska [1,2] proved the equivalence of the following three statements, which we formulate here in terms of the discrete market model:

- (a) The market model is complete under $Q \sim P$
- (b) Every real-valued $(Q, (\mathcal{F}_t))$ - martingale M with $Q \sim P$ is a martingale transform of S with respect to some $\phi \in \Phi$. Formally,

$$M = M_0 + \phi \circ S$$

where for $1 \leq t \leq T$,

$$(\phi \circ S)_t = \phi_1 \cdot S_0 + \sum_{i=1}^t \phi_i \cdot (S_i - S_{i-1})$$

(c) $\mathbb{P}$ is a singleton.

In the "Strasbourg jargon", (b) says that S has the (martingale) representation property (for $(\Omega, (\mathcal{F}_t), Q)$). Harrison and Pliska [1,2] use the expression " S is a complete martingale".

Suppose that we start with a vector-valued stochastic process S on $(\Omega, \mathcal{F}, P)$. Can we perform a change of measure $Q \sim P$, that not only makes S a $(Q, (\mathcal{F}_t))$ - martingale, but also provides S with the representation property? The following theorem characterizes in geometric terms those processes S for which the answer is yes (in which case, we say that $\underline{S}$ can be given the martingale representation property via $Q \sim P$).

Theorem 4.2 The vector-valued stochastic process $S = (S_t : t = 0, 1, \dots, T)$ on $(\Omega, (\mathcal{F}_t), P)$ can be given the martingale representation property via $Q \sim P$ if and only if the following two conditions hold for any $t \in \{0, 1, \dots, T-1\}$ and any $A \in \mathcal{P}_t$:

- (i) $0 \in \text{ri}(\text{conv}(\{S_{t+1}(\omega) - S_t(\omega) : \omega \in A\}))$
- (ii) The conditional distribution (under Q) of $S_{t+1} - S_t$ given A has mean zero and puts positive mass on exactly $n_{t+1,A}$ distinct points, where $n_{t+1,A} = \dim\{\text{span}\{S_{t+1}(\omega) - S_t(\omega) : \omega \in A\}\} + 1$.

Proof: By Theorem A.4 of the Appendix, (i) holds for all $t \in \{0, 1, \dots, T\}$ and all $A \in \mathcal{P}_t$ iff $\mathbb{P} \neq \emptyset$. Also, by Corollary A.2 of the Appendix,

condition (ii) is equivalent to $|\mathbb{P}| = 1$. The above-mentioned equivalences, namely (b) $\Leftrightarrow$ (c), concludes the proof. $\square$

Completeness of the market and the fine structure of the filtration

As indicated at the end of Section 3, the particular structure of the filtration $(F_t: t = 0, 1, \dots, T)$ used for modeling the flow of information in the market is only of minor importance for the existence of an equivalent martingale measure. Since such a measure exists when the results of Section 3 are satisfied by all the sets of the partitions, it is no loss of generality in assuming the minimal filtration from the very beginning. However, the situation is different when dealing with the question of completeness of the market model. In fact, the next result shows that complete markets do not exist when there is more information than that contained in the minimal filtration.

Corollary 4.2 If the market model is complete (or equivalently, if $\mathbb{P}$ is a singleton) then the filtration $(F_t: t = 0, 1, \dots, T)$ is necessarily minimal.

Proof: Let Q denote the unique element in $\mathbb{P}$ and suppose that $(F_t: t = 0, 1, \dots, T)$ is not equal to the minimal filtration $(F_t^* = \sigma(S_0, S_1, \dots, S_t): t = 0, 1, \dots, T)$. Let τ be the first time $t \in \{0, 1, \dots, T-1\}$ such that F_t strictly contains F_t^* . Use P_t and P_t^* to denote the minimal partition of Ω generating F_t and F_t^* respectively ($0 \leq t \leq T$). Then there exists a set $A \in P_\tau^*$ such that

$$A = \bigcup_{i=1}^n A_i, A_i \in P_\tau \text{ for } i = 1, 2, \dots, n \text{ and } n \geq 2.$$

Because $A \in P_\tau^*$, the value of the price process S at time τ is constant ($S_\tau(\omega) = c > 0$, say) for all $\omega \in A$ and therefore equal to c

for each ω in A_i ($i = 1, \dots, n$). Let B denote the unique partition set in $\mathcal{P}_{\tau-1} = \mathcal{P}_{\tau-1}^*$ for which $B \supset A$. The transition from time $\tau - 1$ to time τ partitions B into $m \geq n$ sets $A_1, A_2, \dots, A_n, A_{n+1}, \dots, A_m$ all belonging to $\mathcal{P}_\tau$, where $A_1, \dots, A_n$ are the sets introduced earlier, on which S_τ equals c . By redistributing the probability mass $Q[A|B]$ among $A_1, \dots, A_n$ we can construct many equivalent martingale measures. Indeed, there are different probability measures Q^* on $(\Omega, \mathcal{F})$ satisfying $\sum_{i=1}^n Q^*[A_i|B] = Q[A|B]$, $Q^*[A_i|B] > 0$ ($i = 1, \dots, n$), $Q^*[A_i|B] = Q[A_i|B]$ ($i = n + 1, \dots, m$) and agreeing with Q otherwise. Since $S_\tau - S_{\tau-1}$ is constant on $\bigcup_{i=1}^n A_i$, we have

$$E_{Q^*}[S_\tau - S_{\tau-1} | \mathcal{F}_{\tau-1}](\omega) = E_Q[S_\tau - S_{\tau-1} | \mathcal{F}_{\tau-1}](\omega) = 0$$

for all $\omega \in B$ and hence for all ω .

Thus, $\mathbb{P}$ is not a singleton, which is a contradiction. $\square$

In the splitting index terminology of Harrison and Pliska, Corollary 4.2 can be paraphrased as follows. If the market is complete then the splitting index $K(t, A)$ is determined by the price process $S = (S_t : t = 0, 1, \dots, T)$ only, namely, for each $t \in \{0, 1, \dots, T-1\}$ and each $A \in \mathcal{P}_t$ we have $K(t, A) = \text{card}(\{S_{t+1}(\omega) : \omega \in A\})$.

5. Duality theory for linear programming and the martingale property

In the previous section we introduced the notion of an attainable contingent $X \in \mathbb{X}$ and we investigated market models which contain no arbitrage opportunities and in which each claim is attainable (or attainable under some probability measure Q). We now show that the problem of "attainability" of a claim can be viewed and formulated as the "dual problem" to finding a certain martingale measure for the price process S .

To illustrate this duality, we assume in this section that S is a one-step stochastic process, i.e. $T = 1$ and $S = (S_0, S_1)$ and that the market contains no arbitrage opportunities. Let B denote the matrix with columns formed by the vectors $S_1(\omega_1), S_1(\omega_2), \dots, S_1(\omega_N)$, and let v denote the vector in $\mathbb{R}^N$ with components $X(\omega_i)$ ($i = 1, 2, \dots, N$), where X is a given contingent claim in $\mathbb{X}$. Then we are interested in the following primal-dual pair of linear programs:

$$\begin{array}{ll}
 \text{(P)} \quad \min & v \cdot x \\
 & \text{s.t. } B \cdot x = S_0 \\
 & \quad x \geq 0 \\
 \text{(D)} \quad \max & y \cdot S_0 \\
 & \text{s.t. } y \cdot B \leq v \\
 & \quad y \text{ unrestricted}
 \end{array}$$

Both, (P) and (D) have the following market interpretation.

The primal problem (P) illustrates the behavior of the investor. Interpreting x as the vector of one-step transition probabilities, $v \cdot x$ is the expected price of the contingent claim which the investor attempts to minimize, subject to the requirement that the expected price of the stocks at time 1 is the same as the price at time 0.

The dual problem (D) describes the behavior of the market. Interpret y as the vector denoting the number of stocks bought by an investor at time 0. Thus, $y \cdot S_0$ is the total price paid by the investor at time 0 which the market attempts to maximize, subject to the requirement that the value of the investor's portfolio at time 1 does not exceed the value of the contingent claim in each state of nature ω .

Next note that any (P)-feasible x is automatically a probability measure on $(\Omega, \mathcal{F})$, since the first equation in $B \cdot x = S_0$ is

$\sum_{i=1}^N S_1^0(\omega_i)x_i = S_0^0$ and $S_t^0(\omega) = 1$ for all $\omega \in \Omega$ and $t = 0,1$. Also note that since the market contains no arbitrage opportunities, $M(S) \neq 0$, i.e. (P) is feasible. Therefore, (P) has an optimal feasible solution for any vector v because the feasible region is a bounded convex polyhedron, i.e. a convex polytope.

We now apply the strong duality theorem of linear programming to the primal-dual pair (P) and (D) and get that since (P) has an optimal feasible solution x^* , (D) also has an optimal feasible solution y^* , and moreover, the two optimal objective values are equal, i.e.

$v \cdot x^* = y^* \cdot S_0$. Making use of yet another fundamental result in linear

programming, namely the complementary slackness theorem (also called the price equilibrium theorem), we conclude that if $x_j^* > 0$ for some

$j \in \{1,2,\dots,N\}$, then the dual inequality constraint associated with the primal variable x_j is satisfied as an equation by every (D)-optimal feasible solution y^* , i.e. $x_j^* > 0$ ($j \in \{1,2,\dots,N\}$) implies

$y^* \cdot S_1(\omega_j) = X(\omega_j)$ since $(y^* \cdot B)_j = y^* \cdot S_1(\omega_j)$ and $v_j = X(\omega_j)$.

Thus, by setting $Q[\{\omega_i\}] = x_i^*$ ($i = 1,\dots,N$) and $\phi_1^k = y_k^*$ ($k = 0,1,\dots,K$)

for a pair of primal-dual optimal feasible solutions x^* and y^* , we obtain a martingale measure Q and a trading strategy $\phi = (\phi_1)$ such that the contingent claim X is attainable under Q , where ϕ is the strategy that generates X and where the price $\pi(X)$ is given by the optimal objective value $v \cdot x^* = y^* \cdot S_0$.

In addition to this market interpretation, the equality of the two optimal objective values turn out to have an interesting probabilistic interpretation in terms of the value process $V(\phi) = (V_0(\phi), V_1(\phi))$ for the trading strategy $\phi = (\phi_1)$. On one hand, we have

$$\begin{aligned} v \cdot x^* &= \sum_{j=1}^N x(\omega_j) Q[\{\omega_j\}] = E_Q[X] \\ &= E_Q[\phi_1 \cdot S_1] = E_Q[V_1(\phi) | F_0] \end{aligned}$$

and on the other hand, we also have

$$y^* \cdot S_0 = \phi_1 \cdot S_0 = V_0(\phi).$$

Hence, the equality under optimality between the objective functions in the primal-dual pair of linear programs (P) and (D) is equivalent to the fact that $V(\phi)$ is a $(Q, (F_t))$ - martingale. Either statement can be viewed as an expression of economic equilibrium (or economic stability) resulting from (and implying) the absence of arbitrage opportunities in the market.

Finally, note that we can assume that any (P)-optimal feasible solution x^* is actually an extreme point of the set F of all (P)-feasible solutions. Since F can be identified with the set $M(S)$ of all martingale measures by setting $Q[\{\omega_i\}] = x_i$ ($i = 1, \dots, N$), we get that x is an extreme point of F if and only if the corresponding Q is in $M_e(S)$. Therefore, our previous result concerning completeness of the market, namely Theorem 4.1 (or Corollary 4.1), can be viewed as a probabilistic version of the strong duality theorem of linear programming.

When $T > 1$, the various states of nature can be represented as branches of a tree and the above discussion obviously applies to each node of that tree (representing a partition set $A \in P_t$ ($0 \leq t < T$)). Taking Corollary 4.1 into account, we get

Theorem 5.1 Suppose the market contains no arbitrage opportunities. Then the following conditions are equivalent.

- (1) The market is complete.
- (2) $\mathbb{P}$ is a singleton.
- (3) At each node of the tree the corresponding primal problem has an optimal feasible solution with strictly positive components.
- (4) At each node of the tree the corresponding dual problem has an optimal feasible solution which satisfies all the constraints at equality.

Remarks: 1) The tree and the corresponding primal-dual pair of linear programs can be described analytically through the formalism developed in the next section.

2) Condition (3) implies Condition (4) by the Complementary Slackness Theorem. The fact that Condition (4) implies Condition (3) follows from Tucker's Strict Complementarity Theorem: for a primal-dual pair of linear programs (both of which are feasible) there exists a pair of primal and dual feasible optimum solutions satisfying the property that whenever a constraint in one problem is tight, the associated variable in the other problem is nonzero (see Murty [7, page 180]).

3) In the financial literature, one shows that if a market is in equilibrium and contains no arbitrage opportunities, then there is a supporting price system for the securities. It is then also possible to use linear programming duality. See for example Ross [9] and Ingersoll [4].

6. An algorithm for analyzing finite security markets

We shall describe here an algorithm which answers the following questions.

- (A) Is the market complete?
- (B) Does there exist an equivalent martingale measure?

If the answer is yes, obtain one.

- (C) How to check whether a given claim $X \in \mathbb{X}$ is attainable?
- (D) What is the price $\pi(X)$ of a given attainable claim X ?
- (E) If a claim X is attainable at price $\pi(X)$, obtain a trading strategy that generates X .

Note that in certain cases we can answer some of the these questions immediately by using earlier results. If the market is complete then there exists a unique equivalent martingale measure and all claims are attainable (see Corollary 4.1). If one equivalent martingale measure Q is known, then the price $\pi(X)$ of an attainable claim $X \in \mathbb{X}$ can be obtained using the valuation-formula

$$\pi(X) = E_Q[X] = \sum_{i=1}^N X(\omega_i) Q[\{\omega_i\}]$$

($Q \in \mathbb{P}, X \in \mathbb{X}$), stated in Section 4.

In order to obtain answers to questions (A)-(E) in the general case, we prefer to think of the stochastic process $S = (S_0, S_1, \dots, S_T)$ in terms of the associated event-tree corresponding to the information structure provided by the filtration $(\mathcal{F}_t: t = 0, 1, \dots, T)$. Each node of the tree corresponds to a partition set $A \in \mathcal{P}_t$ ($0 \leq t \leq T$) and each edge of the tree represents a possible transition (under P) with an

associated transition probability. Answering each one of the above questions on a microscopic level (i.e. focusing on a specific node $A \in P_t$ ($0 \leq t < T$) of the tree) will provide us with the desired "macroscopic" answers.

To check for market completeness, one merely has to check whether $\dim\{\text{span}\{S_{t+1}(\omega): \omega \in A\}\} = \text{card}\{A' \in P_{t+1}: A' \subseteq A\}$, i.e. whether the rank of the matrix with columns formed by the vectors $S_{t+1}(\omega)$ ($\omega \in A$) equals the splitting index of A .

In order to find out whether there exists an equivalent martingale measure, all one has to do according to Theorem 3.1, is to check whether $0 \in \text{ri}(\text{conv}(\{S_{t+1}(\omega) - S_t(\omega): \omega \in A\}))$. If one is merely interested in the answer to question (B) one has to find a strictly positive feasible solution x to the appropriate primal problem (P) of the preceeding section. By Theorem 3.1, such an x always exists if the market contains no arbitrage opportunities.

But if one is also interested in the answers to (C)-(E) related to a contingent claim X , then one solves the primal problem (P) at hand. The claim is attainable iff (P) has a strictly positive solution. As a by-product, one also obtains the strategy that generates the given claim: it is the solution of the corresponding dual problem (D).

We now introduce a framework that is particularly well suited to answering questions (A)-(E) on the microscopic level in a systematic way. In fact, we shall provide an algorithm for analyzing finite security markets.

Without any loss of generality we can assume that the edges "leaving" a tree-node are enumerated from 1 (for the "uppermost" edge)

to n (for the "lowermost" edge), where n denotes the number of edges originating from that node. For convenience, we also assume that the value of n is the same for every node in the tree. (The general case only requires a more elaborate notation!)

For $0 \leq t \leq T$, let $D(t)$ denote the set of all words of length t over the alphabet $I = \{1, 2, \dots, n\}$, i.e.

$$D(t) = \{m: m = a_1 a_2 \dots a_{t-1} a_t, a_i \in I \text{ for } 1 \leq i \leq t\}$$

with $D(0) = \{\emptyset\}$. Note that with this notation there is a 1-1 correspondence between the partition sets $A \in P_t$ and the elements in $D(t)$ ($0 \leq t \leq T$). We then obtain an enumeration-scheme of all partition sets which can be characterized recursively as follows (see figure 6.1)

- For $t = T$ and $m \in D(T)$, let $A(m) = \{\omega\}$ where ω is the unique branch in the event-tree corresponding to m .
- For $0 < t < T$, let $A(m)$ be defined for $m \in D(t+1)$. Define $A(m)$ for $m \in D(t)$ by $A(m) = \bigcup_{i=1}^n A(mi)$ where mi is a word of length $t+1$ (we write $|m|$ to denote the length of m).
- For $t = 0$, let $A(\emptyset) = \Omega$.

For $0 \leq t \leq T$ and $m \in D(t)$, we set $S(A(m)) = S_t(\omega)$ when $\omega \in A(m)$, so that, for example, the set $\{S(A(mi)): i = 1, 2, \dots, n\}$ denotes all

Figure 6.1: $n = 3$

possible values that S_{t+1} assumes when ω happens to be in $A(m)$ at time t . Similarly, for a given trading strategy $\phi \in \Phi$, $V(\phi)(A(m))$ denotes the value of the value process $V(\phi)$ evaluated at $A(m)$. In particular, $V(\phi)(A(\emptyset))$ equals the price $\pi(X)$ of a claim X that is generated by the strategy ϕ . With this convenient way of denoting all possible values of the processes S and $V(\phi)$ at any point in the event-tree, we can now give a short description of the algorithm that analyzes finite security markets.

The main part of the algorithm consists of the recursive procedure ANALYZE which goes through the event-tree in a depth-first order. (In the case illustrated in Figure 6.1, it processes in succession nodes $A(1)$, $A(2)$, $A(3)$ and then $A(\emptyset)$. At a given node $A(m)$ ($m \in D(t)$, $0 \leq t \leq T-1$) it checks whether $0 \in \text{ri}(\text{conv}(\{S(A(m1)) - S(A(m)), \dots, S(A(mn)) - S(A(m))\}))$. This is done by checking whether the primal problem of the following primal-dual pair of linear programs has a strictly positive feasible solution:

$$\begin{array}{ll} (P(m)) \quad \min & v \cdot x \\ & \text{s.t. } B \cdot x = S(A(m)) \\ & \quad x \geq 0 \\ (D(m)) \quad \max & y \cdot S(A(m)) \\ & \text{s.t. } y \cdot B \leq v \\ & \quad y \text{ unrestricted} \end{array}$$

where B is the $(K+1) \times n$ matrix whose columns are $S(A(m1))$, $S(A(m2))$, $\dots$, $S(A(mn))$, and where v is the vector in R^n with components $V(\phi)(A(mi))$ ($i = 1, \dots, n$). A practical way of attacking this problem, is to transform it into a standard problem in linear programming by replacing the non-negativity constraints in $(P(m))$ by new constraints of the form $x \geq \epsilon$ where $\epsilon > 0$. If $(P(m))$ has a

strictly positive feasible solution for each $m \in D(t)$ ($0 \leq t < T$), one will then be able to construct an equivalent martingale measure. To check a given claim $X \in \mathbb{X}$ for attainability, one has to find a strictly positive optimal feasible solution of $(P(m))$ for each $m \in D(t)$ ($0 \leq t < T$). Note that there is no need in this problem to introduce constraints of the form $x \geq \epsilon$ with $\epsilon > 0$.

Algorithm

1. Initialize:

set $a = c = e = 1$;

for all $m \in D(T)$, set $V(\phi)(A(m)) = X(A(m))$;

2. Call the procedure $\text{ANALYZE}(A(m))$ described below with $m = \emptyset$.

Calling this procedure will either lead to step 4 or terminate with the following output:

- $e = 1$: there exists an equivalent martingale measure

- a set of transition probabilities

$\{Q(A(m), A(m_i)) : i = 1, 2, \dots, n; m \in D(t); 0 \leq t < T\}$

- the values of a and c :

$a = 1$ or 0 according to whether the claim is attainable or not

$c = 1$ or 0 according to whether the market is complete or not

- a set $\{\phi(A(m)) : m \in D(t), 0 \leq t < T\}$ of vectors in $\mathbb{R}^{K+1}$ characterizing completely a strategy $\phi = (\phi_1, \phi_2, \dots, \phi_T)$ that generates the claim X (when $a = 1$)

- the price $\pi(X) = V(\phi)(A(\emptyset))$ of the claim X (when $a = 1$)

3. Obtain $Q \in \mathbb{P}$ as follows: for each $\omega \in \Omega$ set

$Q[\{\omega\}] = Q(A(\emptyset), A(a_1)) \cdot Q(A(a_1), A(a_1 a_2)) \cdot \dots \cdot Q(A(a_1 a_2 \dots a_{T-1}),$

$A(a_1 a_2 \dots a_T))$ where $m = a_1 a_2 \dots a_{T-1} a_T$ ($a_i \in I, 0 < i \leq T$) is the unique element in $D(T)$ corresponding to ω .

4. $e = 0$: there exists no equivalent martingale measure.

The procedure stopped while processing the node $A(m)$, which is a partition set in $\mathcal{P}_t$ with $t = |m|$. That partition set is uniquely determined by m .

Description of the recursive procedure ANALYZE

procedure ANALYZE($A(m)$);

begin

if $|m| < T - 1$ then for $i = 1$ to n do ANALYZE($A(m_i)$);

if $0 \notin \text{ri}(\text{conv}(\{S(A(m_1)) - S(A(m)), S(A(m_2)) - S(A(m)), \dots, S(A(m_n)) - S(A(m))\}))$ then

begin

set $e = 0$; goto step 4;

end

else

begin

if $c = 1$ then

begin

if $\dim\{\text{span}\{S(A(m_1)), S(A(m_2)), \dots, S(A(m_n))\}\} \neq n$ then set

$c = 0$;

end;

if $a = 1$ then

begin

if $(P(m))$ has an optimal feasible solution x^* with $x^* > 0$

then

```
begin
  for j = 1 to n set Q(A(m),A(mj)) =  $x_j^*$ ;
  for k = 0 to K set  $\phi^k(A(m)) = y_k^*$  (where  $y^*$  is (D(m)) -
    optimal);
  set  $V(\phi)(A(m)) = y^* \cdot S(A(m))$ ;
end
else
begin
  set a = 0;
  for j = 1 to n set Q(A(m),A(mj)) =  $x_j$  (where x is
    (P(m)) - feasible,  $x > 0$ );
  end
end
end
end;
```

This procedure is written in a general purpose language. Being recursive, it calls itself many times. For example, in the case illustrated in Figure 6.1, we have $T = 2$. Calling the procedure ANALYZE(A($\emptyset$)) immediately leads to the call of ANALYZE(A(1)), which in turn results in processing the node A(1) since the condition $|m| < T-1$ is not satisfied, then calls ANALYZE(A(2)) and processes node A(2), calls ANALYZE(A(3)) and processes node A(3), and then drops automatically back to the node A($\emptyset$) and processes it.

Appendix

A Geometric Approach to Constructing Martingale Measures: The Finite Case

Consider a given d -dimensional stochastic process $Z = (Z_t: t = 0, 1, \dots, T < \infty)$, with $d \geq 1$, defined on a finite probability space $(\Omega, \mathcal{F}, P)$, where P gives positive probability to all N points of Ω . For $0 \leq t \leq T$, let $\mathcal{P}_t$ denote the smallest partition of Ω corresponding to the σ -algebra $\mathcal{F}_t = \sigma(Z_0, Z_1, \dots, Z_t)$. Without loss of generality we let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$ be the σ -algebra containing all subsets of Ω . Furthermore, let $M(Z)$ denote the set of all probability measures Q on $(\Omega, \mathcal{F})$ that make the given stochastic process Z a $(Q, \mathcal{F}_t)$ -martingale, and let $M_e(Z)$ denote the set consisting of all extreme points of $M(Z)$.

We first deal with the case of a single period stochastic process $Z = (Z_0, Z_1)$ and focus on the increment $Y = Z_1 - Z_0$. The d -dimensional random variable Y then satisfies $\sigma(Y) = \mathcal{F}$. We shall find a necessary and sufficient condition for the existence of a probability measure Q on $(\Omega, \mathcal{F})$ which is equivalent to P and for which $E_Q[Y] = 0$, where $E_Q[\cdot]$ denotes the mathematical expectation with respect to Q . Note that the equivalence between P and Q implies that Q must also give positive probability to all N points of Ω .

To gain insight, let us first consider the one-dimensional and two-dimensional cases. In the case $d = 1$, it is very easy to prove the following result:

$$\exists Q \sim P \text{ such that } E_Q[Y] = 0 \iff \begin{cases} Y(\omega) \geq 0 \ \forall \omega \in \Omega \Rightarrow Y \equiv 0 \\ Y(\omega) \leq 0 \ \forall \omega \in \Omega \Rightarrow Y \equiv 0. \end{cases} \quad (\text{A.1})$$

This simply means that in order for such a Q to exist, the random variable Y has to assume positive and negative values (except when $Y \equiv 0$), so that probability mass can be "redistributed" in such a way that the "center of gravity" coincides with the origin. In the two-dimensional case, one can easily construct examples that show that condition (A.1) does not guarantee anymore the existence of a Q with the desired properties. However, these examples turn out not to satisfy either condition (I) or condition (II) of Lemma A.1 below.

The convex hull of the set $\{Y(\omega): \omega \in \Omega\} \subset \mathbb{R}^d$ is $C = \text{conv}(\{Y(\omega): \omega \in \Omega\}) = \{x \in \mathbb{R}^d: x = \sum_{\omega \in \Omega} \lambda_{\omega} Y(\omega), \sum_{\omega \in \Omega} \lambda_{\omega} = 1, \lambda_{\omega} \geq 0\}$ and its affine hull is $\text{aff}(C) = \{x \in \mathbb{R}^d: x = \sum_{\omega \in \Omega} \lambda_{\omega} Y(\omega), \sum_{\omega \in \Omega} \lambda_{\omega} = 1\}$. When dealing with convex sets C , the concept of interior is replaced by the more convenient concept of relative interior, reflecting the fact that C , regarded as a subset of $\mathbb{R}^d$, does not have an interior when $\dim(C) < d$. The relative interior of C , denoted $\text{ri}(C)$, is the interior of C when C is viewed as a subset of $\text{aff}(C)$. Formally, $\text{ri}(C) = \{x \in \text{aff}(C): \exists \epsilon > 0 \text{ such that } (x + \epsilon B) \cap (\text{aff}(C)) \subset C\}$ where $B = \{x \in \mathbb{R}^d: |x| \leq 1\}$ is the Euclidean unit ball in $\mathbb{R}^d$. Let $\text{cl}(C)$ denote the closure of C and note that $\text{cl}(C) = \text{cl}(\text{ri}(C))$.

Lemma A.1. The following two statements are equivalent.

(I) $0 \in \text{ri}(C)$.

(II) There is no hyperplane that separates C and the origin 0 properly, i.e. there exists no $\alpha \in \mathbb{R}^d$ such that $\alpha \cdot Y(\omega) \geq 0 \forall \omega \in \Omega$ but $\alpha \cdot Y \not\equiv 0$.

Proof. Let C_1 and C_2 be two non-empty convex sets in $\mathbb{R}^d$. Then $\text{ri}(C_1) \cap \text{ri}(C_2) = \emptyset$ iff there exists a hyperplane separating C_1 and C_2 properly (see Rockafellar [8, p. 96]). Setting $C_1 = C$ and $C_2 = \{0\}$ concludes the proof. $\square$

We also need

Lemma A.2. Let A be a given $d \times N$ matrix. Then exactly one of the following two alternatives hold:

either (a) $A \cdot x = 0$ has a solution $x \in \mathbb{R}^N$ with $x > 0$
or (b) there is a vector $y \in \mathbb{R}^d$ with $y \cdot A \geq 0$, $y \cdot A \neq 0$.

Proof. Let $e = (1, 1, \dots, 1) \in \mathbb{R}^N$. Then we have (after scaling, if necessary),

(a) $\Leftrightarrow \exists u \in \mathbb{R}^N$ such that $A \cdot u = A \cdot (-e)$, $u \geq 0$ (set $u = x - e$)
and

(b) $\Leftrightarrow \exists v \in \mathbb{R}^d$ such that $v \cdot A \leq 0$ and $v \cdot (A \cdot (-e)) > 0$
(set $v = -y$).

Setting $b = A \cdot (-e)$, $z = u$ and $w = v$ we get

(a) $\Leftrightarrow (A): \exists z \in \mathbb{R}^N$, $z \geq 0$ such that $A \cdot z = b$

(b) $\Leftrightarrow (B): \exists w \in \mathbb{R}^d$ such that $w \cdot A \leq 0$ and $w \cdot b > 0$.

But Farkas Lemma (see Murty [7, p. 175]) states that exactly one of the two alternatives (A) or (B) hold. $\square$

Theorem A.1. There exists a probability measure Q on $(\Omega, \mathcal{F})$ that is equivalent to P and for which $E_Q[Y] = 0$ if and only if statement (I) (or equivalently (II)) of Lemma A.1 is satisfied.

Proof. The sufficiency is obvious. To prove necessity, define the $d \times N$ matrix A by

$$A = \begin{bmatrix} Y_1(\omega_1) & Y_1(\omega_2) & \dots & Y_1(\omega_N) \\ Y_2(\omega_1) & Y_2(\omega_2) & \dots & Y_2(\omega_N) \\ \vdots & \vdots & & \vdots \\ Y_d(\omega_1) & Y_d(\omega_2) & \dots & Y_d(\omega_N) \end{bmatrix}$$

and suppose that statement (II) of Lemma A.1 holds. Then alternative (b) of Lemma A.2 cannot hold which implies that alternative (a) must hold. Thus, the equation $A \cdot x = 0$ has a strictly positive solution $x^* \in \mathbb{R}^N$. Now set $Q[\{\omega_i\}] = (1/\sum_{j=1}^N x_j^*)x_i^*$ ($i = 1, \dots, N$). Clearly, Q is a probability measure on $(\Omega, \mathcal{F})$, $Q[\{\omega_i\}] > 0$ for $i = 1, \dots, N$, and

$$\begin{aligned} E_Q[Y] &= \sum_{i=1}^N Y(\omega_i)Q[\{\omega_i\}] \\ &= \sum_{i=1}^N Y(\omega_i)(1/\sum_{j=1}^N x_j^*)x_i^* \\ &= (1/\sum_{j=1}^N x_j^*)A \cdot x^* \\ &= 0. \end{aligned}$$

Thus Q has the desired properties. $\square$

The following result is a straightforward modification of Theorem A.1, where we now allow a change of measure that results in a probability measure Q on $(\Omega, \mathcal{F})$ which is absolutely continuous with respect to P (i.e. Q may assign zero mass to some $\omega \in \Omega$) and for which $E_Q[Y] = 0$.

Theorem A.2. There exists a probability measure Q on $(\Omega, \mathcal{F})$ that is absolutely continuous with respect to P and for which $E_Q[Y] = 0$ if and only if (I') (or equivalently (II')) is satisfied, where

(I') $0 \in \text{cl}(C)$.

(II') There is no hyperplane that separates C and the origin 0 strongly, i.e. there exists no $\alpha \in \mathbb{R}^d$ such that $\alpha \cdot Y(\omega) > 0 \ \forall \ \omega \in \Omega$.

Note that in our context, all probability measures on $(\Omega, \mathcal{F})$ are absolutely continuous with respect to P . Therefore, Theorem A.2 follows also from Karr [5, Prop. 2.1] and by applying his Theorem 2.1, we obtain

Theorem A.3. Fix $1 \leq n \leq N$ and let Q be a probability measure on $(\Omega, \mathcal{F})$ that puts positive mass on n points of Ω , say $\omega_1, \omega_2, \dots, \omega_n$, and puts zero mass on the remaining $N - n$ points of Ω . Suppose Q is such that $E_Q[Y] = 0$. Then the following three statements are equivalent:

- (a) Q is an extreme point of the set $\{Q: E_Q[Y] = 0\}$.
- (b) The system

$$(S) \quad \sum_{i=1}^n Y(\omega_i) \cdot q_i = 0, \quad \sum_{i=1}^n q_i = 1, \quad q_i > 0 \quad \text{for } i = 1, \dots, n$$

has the unique solution $q_i = Q[\{\omega_i\}]$ ($i = 1, \dots, n$).

- (c) $\dim\{\text{span}\{Y(\omega_1), \dots, Y(\omega_n)\}\} = n - 1$.

Remark. Note that different Q 's satisfying (a) may put positive mass on different sets of n points of Ω and zero mass on the remaining ones. However, a Q which satisfies (a) is unique if it puts positive mass on all the N points of the sample space Ω , i.e. when $Q \sim P$, because in this case the system (S) has the unique solution $q_i = Q[\{\omega_i\}]$ ($i = 1, \dots, N$). Therefore, we get

Corollary A.1. If $Q \sim P$ and Q is an extreme point of the set $\{Q: E_Q[Y] = 0\}$, then Q is unique.

Returning to the general case of a multi-period stochastic process $Z = (Z_t: t=0,1,\dots,T)$ we note that the underlying probability measure P can be "decomposed" into an initial distribution (which is a degenerate distribution since F_0 is trivial) and successive transition probabilities. The transition probability from time t to $t+1$ is the conditional distribution of $Z_{t+1} - Z_t$ given F_t ($0 \leq t < T$). By viewing the probability measures of the single period case as transition probabilities and by piecing them together, we obtain the following results for the stochastic process Z .

Theorem A.4. The following three statements are equivalent:

- (i) There exists $Q \in M(Z)$ with $Q \sim P$.
- (ii) For all $t \in \{0,1,\dots,T-1\}$ and all $A \in \mathcal{P}_t$,
 $0 \in \text{ri}(\text{conv}(\{Z_{t+1}(\omega) - Z_t(\omega): \omega \in A\}))$.
- (iii) For all $t \in \{0,1,\dots,T-1\}$ and all $A \in \mathcal{P}_t$ there exists no hyperplane that separates $\text{conv}(\{Z_{t+1}(\omega) - Z_t(\omega): \omega \in A\})$ and 0 properly, i.e. there exists no $\alpha \in R^d$ such that
 $\alpha \cdot (Z_{t+1}(\omega) - Z_t(\omega)) \geq 0 \forall \omega \in A$ but $\alpha \cdot (Z_{t+1} - Z_t) \not\equiv 0$.

Theorem A.5. Suppose $Q \in M(Z)$. Then $Q \in M_e(Z)$ if and only if the (Q, F_t) -martingale Z satisfies the following condition:

For each $0 \leq t < T$ and each set $A \in \mathcal{P}_t$ with $Q[A] > 0$, the conditional distribution of the increment $Z_{t+1} - Z_t$ given A has mean zero, and puts positive mass on exactly

$n_{t+1,A}$ points, where

$$n_{t+1,A} = \dim\{\text{span}\{Z_{t+1}(\omega) - Z_t(\omega): \omega \in A, Q[\{\omega\}] > 0\}\} + 1.$$

Corollary A.1. If $Q \sim P$ and $Q \in M_e(Z)$ then Q is unique.

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