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SYMMETRIC POLYNOMIALS OF RANDOM VARIABLES
ATTRACTED TO AN INFINITELY DIVISIBLE LAW

by

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Abstract

We consider the symmetric polynomials $M_k^n(x_\bullet) = \sum_{i=1}^n x_i^k$ and $S_k^n(x_\bullet) = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}$ and also the stochastic processes $M_k(t) = \sum_{0 \leq t_1 \leq t} x_{i_1}^k$ and $S_k(t) = \sum_{0 \leq t_1 < \dots < t_k \leq t} x_{i_1} x_{i_2} \dots x_{i_k}$ obtained by viewing $\xi = (t_i, x_i)_{i=1}^\infty$ as a Poisson point process in the plane. We outline a unified approach to the study of these polynomials and processes. The process $M_1(t)$ is the pure jump Levy process with associated jump process ξ . The processes $M_k(t)$ and $S_k(t)$ are respectively the k^{th} order weak variation and k^{th} order multiple integral of $M_1(t)$. We also obtain the following extension of the Central Limit Theorem: Suppose that $X_{i,n}$ is a triangular array in the domain of attraction of $M_1(t)$, i.e. $M_1^{[nt]}(X_{\bullet,n}) \xrightarrow{w} M_1(t)$. Then, jointly in k , $M_k^{[nt]}(X_{\bullet,n})$ and $S_k^{[nt]}(X_{\bullet,n})$ converge weakly as $n \rightarrow \infty$. The limits are respectively, the k^{th} order weak variation and the k^{th} order multiple integral of $M_1(t)$.

1. Introduction.

This paper suggests a unified approach to the study of symmetric polynomials in i.i.d. random variables. These polynomials play an important role in the theory of non-linear filtering (Wiener 1958, Kallianpur 1980) and in the theory of symmetric statistics (Mandelbaum and Taqqu 1984). They have been investigated in detail under the assumption that the random variables have finite variance. We make no such assumption here. We develop the notions of weak variation and multiple integrals. We also obtain a limit theorem for symmetric polynomials in random variables that are in the domain of attraction of a pure jump Levy process. The techniques developed by Resnick (1983) will be heavily used.

The symmetric polynomials are best introduced in the following non-probabilistic context.

A. Given a sequence of numbers $x_{\bullet} = (x_i)_{i=1}^{\infty}$, put

$$(1.1) \quad M_k^n(x_{\bullet}) = \sum_{i=1}^n x_i^k,$$

and

$$(1.2) \quad S_k^n(x_{\bullet}) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

By convention, $S_0^n = 1$ and $S_k^n = 0$ for $k > n$. The generating function of S_k^n is

$$(1.3) \quad \phi^n(v) = \sum_{k=0}^n S_k^n v^k = \prod_{i=1}^n (1 + vx_i)$$

and, when $|v|$ is small,

$$\begin{aligned}
 (1.4) \quad \phi^n(v) &= \exp\left\{\sum_{i=1}^n \ln(1+vx_i)\right\} \\
 &= \exp\left\{\sum_{i=1}^n \sum_{k=1}^{\infty} (-1)^{k-1} \frac{v^k}{k} x_i^k\right\} \\
 &= \exp\left\{\sum_{k=1}^{\infty} (-1)^{k-1} \frac{v^k}{k} M_k^n\right\}.
 \end{aligned}$$

$\phi^n(v)$ satisfies the recurrence relation

$$(1.5) \quad \phi^n(v) = 1 + \sum_{i=1}^n v \phi^{i-1}(v) x_i.$$

The symmetric polynomials M_k^n and S_k^n are related by Newton's identities

$$(1.6) \quad k S_k^n = \sum_{i=1}^k (-1)^{i-1} S_{k-i}^n M_i^n,$$

which, iterated, give

$$(1.7) \quad S_k^n = P_k(M_1^n, M_2^n, \dots, M_k^n).$$

The polynomial P_k is given by

$$(1.8) \quad P_k(x_1, x_2, \dots, x_k) = (-1)^k \sum_{i=1}^k \prod_{j=1}^i \frac{(-x_j)^{m_j}}{x_j^{m_j} (m_j!)}.$$

where the summation is over all non-negative integers $m_1, m_2, \dots, m_k$ satisfying $\sum_{j=1}^k j m_j = k$. Thus,

$$S_2^n = \frac{1}{2} (M_1^n)^2 - \frac{1}{2} M_2^n$$

$$S_3^n = \frac{1}{6} (M_1^n)^3 - \frac{1}{2} M_1^n M_2^n + \frac{1}{3} M_3^n.$$

Newton's identities and relation (1.8) are proved in the Appendix. Note that the polynomials S_k^n also satisfy the following recurrence relation:

$$(1.9) \quad S_k^n = \sum_{i=k}^n x_i S_{k-1}^{i-1}.$$

B. We develop the following idea of Feinsilver (1978) (see his Theorem 4 and Propositions 47 and 48): if a triangular array of i.i.d. random variables $X_{i,n}$ with finite variance is in the domain of attraction of a process X_t , then the finite-dimensional distributions of $S_k^{[nt]}(X_{\cdot,n})$ converge to the multiple integrals of the process X_t .

Our paper extends Feinsilver's result in three directions:

- a) by dropping the second moments assumption on $X_{i,n}$;
- b) by strengthening the convergence to weak convergence of processes with respect to the Skorohod topology on the CADLAG space $D(0,\infty)$;
- c) by obtaining a corresponding result about the convergence of $M_k^{[nt]}(X_{\cdot,n})$.

The precise result is stated in Section 3 as Theorem 3.1.

C. In Section 2, we consider point measures in the plane of the form $m = \sum_{i=1}^{\infty} \delta(t_i, x_i)$ where $\delta(t, x)$ is a point mass at (t, x) and we introduce the following analogues of M_k and S_k :

$$(M_k(m))(t) = \sum_{0 \leq t_i \leq t} x_i^k,$$

and

$$(S_k(m))(t) = \sum_{0 \leq t_{i_1} < t_{i_2} < \dots < t_{i_k} \leq t} x_{i_1} x_{i_2} \dots x_{i_k}.$$

We shall view m as a realization of a Poisson point process ξ in the plane with intensity $dt \times \mu(dx)$ where μ is a Levy measure. Then $M_1(\xi)(t)$ is the pure jump Levy process with jump process ξ . All the $M_k(\xi)(t)$, $k \geq 1$ are in fact pure jump Levy processes (Proposition 2.1).

In Section 4, we prove that $M_k(t) = M_k(\xi)(t)$ can be interpreted as the k^{th} order weak variation of $M_1(\xi)(t)$ and that the process $S_k(t) = S_k(\xi)(t)$ can be represented as a multiple integral of $M_1(\xi)(t)$. Setting $X_t = M_1(\xi)(t)$, we shall write

$$M_k(t) = \int_0^t (dX_s)^k$$

and

$$S_k(t) = \int_{A_t} dX_{s_1} dX_{s_2} \dots dX_{s_k}$$

where

$$(1.10) \quad A_t = \{(s_1, s_2, \dots, s_k) = 0 \leq s_1 < s_2 < \dots < s_k \leq t\}.$$

The processes $M_k(t)$ and $S_k(t)$ satisfy relations (1.6) and (1.7) if one replaces M_k^n by $M_k(t)$ and S_k^n by $S_k(t)$ in these relations.

Probabilistic analogues of relations (1.1) to (1.9) have been considered by several authors (e.g. Itô 1956, Segal 1976). They can be obtained heuristically by replacing x_i by the increments dX_s of a suitably chosen stochastic process X_s . For example, Feinsilver (1978) and Segal (1976) define $S_k(t)$ through an analogue of relation (1.9) namely

$$S_k(t) = \int_0^t S_{k-1}(s-) dX_s$$

where X_s is a finite variance martingale and they obtain several related recurrence relations. Doleans-Dade (1970) studies the following analogues of relation (1.5)

$$\phi_v(t) = 1 + \int_0^t v \phi_v(s-) dX_s.$$

See also Meyer (1976).

The case when X_t is Brownian motion has been treated in the paper of Dynkin and Mandelbaum (1983) and Mandelbaum and Taqqu (1984) in terms of its applications to U-statistics. In that case, $M_k(t) = 0$ for $k \geq 3$ and $M_2(t) = t$. The recurrence relation (1.6) is then of second order, the polynomials P_k in (1.7) become the Hermite polynomials with arguments X_t and t , and finally the generating function in (1.4) becomes $\phi_v(t) = \exp\{vX_t - \frac{v^2}{2} t\}$.

Section 2 discusses the processes $M_k(m)(t)$ and $S_k(m)(t)$. The limit theorem is stated in Section 3. Section 4 concerns the weak variations and multiple integrals of a pure jump Levy process. Section 5 contains the proof of the limit theorem. Additional remarks and extensions are included in Section 6. The Appendix contains the proof of Newton's identities and of relation (1.8).

2. The processes $M_k(t)$ and $S_k(t)$.

A measure μ is a Levy measure if

$$\int_{-\infty}^{\infty} (x^2 \wedge 1) \mu(dx) < \infty$$

and

$$\mu(0) = 0.$$

We shall suppose throughout this paper that μ is a Levy measure which satisfies in addition either one of the following conditions:

$$(A) \quad \int_{-\infty}^{+\infty} (|x| \wedge 1) \mu(dx) < \infty;$$

or

$$(B) \quad \mu \text{ is symmetric.}$$

When (A) holds, we shall say that X_t is a pure jump Levy process with Levy measure μ , if

$$E e^{iuX_t} = \exp\{t \int_{-\infty}^{\infty} (e^{iux} - 1) \mu(dx)\}.$$

In that case X_t has bounded variation.

When (B) holds, we shall say that X_t is a pure jump Levy process with Levy measure μ if we have

$$E e^{iuX_t} = \exp\{t \int_{-\infty}^{\infty} (e^{iux} - 1 - \frac{iux}{1+x^2}) \mu(dx)\}.$$

The processes X_t can also be expressed in terms of point measures.

Let $\delta \geq 0$ and set $R^\delta = R - [-\delta, \delta]$. In particular, $R^0 = R - \{0\}$. Introduce the space $E = (0, \infty) \times R^0$, endowed with the Borel σ -algebra $\mathcal{B}$.

Let $M_p(E)$ denote the set of point measures on E such that

$$M_p(E) = \{m: m = \sum_{i=1}^{\infty} \delta(t_i, x_i) \text{ where only finitely}$$

many of the points $(t_i, x_i), i \geq 1$ fall

in any given compact set of $E\}$.

$M_p(E)$ is endowed with the smallest σ -algebra which makes the functions $m(B)$ measurable, $B \in \mathcal{B}$, and it is also endowed with the topology induced by the vague convergence of measures.

Let $D(0, \infty)$ denote the space of functions continuous to the right and with limits to the left, endowed with the Skorohod topology, as presented, for example, in Whitt (1980). Convergence of functions in $D(0, \infty)$ is denoted by $\xrightarrow{S}$.

Now introduce the functions:

$$M_k^\delta: M_p(E) \rightarrow D(0, \infty) \quad \text{and} \quad S_k^\delta: M_p(E) \rightarrow D(0, \infty)$$

defined by

$$(2.1) \quad M_k^\delta(m)(t) = \sum_{\substack{0 \leq t_i \leq t \\ |x_i| > \delta}} x_i^k,$$

and

$$(2.2) \quad S_k^\delta(m)(t) = \sum_{\substack{0 \leq t_{i_1} < t_{i_2} < \dots < t_{i_k} \leq t \\ |x_{i_j}| > \delta}} x_{i_1} x_{i_2} \dots x_{i_k}$$

$$= P_k(M_1^\delta, M_2^\delta, \dots, M_k^\delta)$$

for any m for which the sums converge to functions in $D(0, \infty)$.

Define also the functions:

$$M_k: M_p(E) \rightarrow D(0, \infty) \quad \text{and} \quad S_k: M_p(E) \rightarrow D(0, \infty)$$

by

$$(2.3) \quad M_k(m)(t) = \lim_{\delta \rightarrow 0} M_k^\delta(m)(t),$$

and

$$(2.4) \quad S_k(m)(t) = \lim_{\delta \rightarrow 0} S_k^\delta(m)(t)$$

for any m such that the limits exist and belong to $D(0, \infty)$.

Now view m as a realization of a Poisson point process (P.P.P.) on E , denoted by ξ , with intensity $dt \times \mu(dx)$. Thus for any $B \in \mathcal{B}$, $\xi(B)$ is a Poisson random variable with mean $E\xi(B) = \int_B dt\mu(dx)$, and $\xi(B_1)$ and $\xi(B_2)$ are independent whenever the Borel sets B_1 and B_2 are disjoint. Let N_μ denote the distribution of ξ .

Under these conditions, $M_k^\delta(\xi)(t)$ and $S_k^\delta(\xi)(t)$ are random. They are well-defined because the sums in (2.1) and (2.2) converge a.s. (N_μ) , because a.s. (N_μ) there are only finitely many points in $(0, t) \times R^\delta$. The following proposition concerns the convergence of $M_k^\delta(\xi)(t)$ and $S_k^\delta(\xi)(t)$ as $\delta \rightarrow 0$.

Proposition 2.1. Suppose that μ satisfies either Assumption (A) or (B).

Then

a) a.s. (N_μ) , $M_k(t) = M_k(\xi)(t)$ is well-defined as a stochastic process with paths in $D(0, \infty)$ and the convergence in (2.3) is uniform in t on compact intervals. $M_k(\xi)(t)$ is a pure jump Levy process. Its Levy measure, denoted μ_k , is the measure induced from μ by the transformation

$x \rightarrow x^k$, i.e.

$$\mu_k(B) = \mu(B^{1/k}), \quad B \in \mathcal{B}.$$

b) a.s. (N_μ) , $S_k(t) = S_k(\xi)(t)$ is well-defined as a stochastic process with paths in $D(0, \infty)$ and the convergence in (2.4) is uniform in t on compact intervals. $S_k(t)$ satisfies

$$(2.5) \quad S_k(t) = P_k(M_1(t), M_2(t), \dots, M_k(t))$$

and

$$(2.6) \quad kS_k(t) = \sum_{i=1}^n (-1)^{i-1} S_{k-i}(t) M_i(t).$$

Proof. a) The case $k = 1$ follows from Itô's representation of Levy processes (Itô (1969), 1.7.7); our assumptions eliminate the need for the centering term.

The case $k \geq 2$ is reduced to the case $k = 1$ by the change of variables $x^k = y$. We have then:

$$M_k^\delta(\xi)(t) = M_1^{\delta^k}(\xi_k)(t),$$

where ξ_k is a P.P.P. with intensity $dt \times \mu_k(dx)$. It remains to note that for any $k \geq 2$, μ_k satisfies Assumption (A) since

$$\int_{\mathbb{R}} f(x) \mu_k(dx) = \int_{\mathbb{R}} f(x^k) \mu(dx),$$

yields

$$\begin{aligned} \int_{\mathbb{R}} (|x| \wedge 1) \mu_k(dx) &= \int_{\mathbb{R}} (|x|^k \wedge 1) \mu(dx) \\ &\leq \int_{\mathbb{R}} (x^2 \wedge 1) \mu(dx) \\ &< \infty. \end{aligned}$$

b) The results for S_k follow from the continuity of P_k . Relations (2.5) and (2.6) hold because of (1.7) and (1.6). $\square$

Examples.

a) Suppose that $\mu = \delta(1)$ i.e., μ is a measure with unit point mass at 1. Then for all $k \geq 1$, $\mu_k = \delta(1)$ and

$$\begin{aligned} E e^{iuM_k(t)} &= \exp\left\{t \int_{R^0} (e^{iuy} - 1) \mu_k(dy)\right\} \\ &= \exp\{t(e^{iu} - 1)\}. \end{aligned}$$

Hence $M_k(t) = M_1(t)$ is a Poisson process on R .

As for $S_k(t)$, we have

$$\begin{aligned} S_k(t) &= \sum_{0 \leq t_{i_1} < t_{i_2} < \dots < t_{i_k} \leq 1} x_{i_1} x_{i_2} \dots x_{i_k} \\ &= \sum_{0 \leq t_{i_1} < t_{i_2} < t_{i_k} \leq 1} 1 \\ &= \binom{M_1(t)}{k} \end{aligned}$$

because

$$\text{card } \{t_i: 0 \leq t_i \leq t\} = \sum_{0 \leq t_i \leq t} 1 = M_1(t).$$

Furthermore,

$$\phi_v(t) = (1+v)^{M_1(t)}.$$

b) Suppose that μ is the Levy measure of a Levy-stable process with index $0 < \alpha < 2$. Then for $x > 0$,

$$(2.7) \quad \mu(x, \infty) = px^{-\alpha}, \quad \mu(-\infty, -x) = qx^{-\alpha},$$

where $p + q = 1$ with $p, q \geq 0$. When $1 \leq \alpha < 2$ we take $p = q = 1/2$ in order to satisfy condition (B), while for $0 < \alpha < 1$, we can take p and q arbitrary since (A) is satisfied. Then, for k odd

$$\mu_k(x, \infty) = px^{-\alpha/k},$$

and

$$\mu_k(-\infty, -x) = qx^{-\alpha/k},$$

and hence $M_k(t)$ is a Levy stable process with index α/k . When k is even,

$$\mu_k(x, \infty) = x^{-\alpha/k},$$

and

$$\mu_k(-\infty, 0) = 0,$$

and in that case $M_k(t)$ is a stable subordinator of index $\alpha/k < 1$.

c) Suppose that μ is the Levy measure of a gamma process $M_1(t)$, i.e.,

$$\mu(-\infty, 0] = 0, \quad d\mu(x) = \theta \frac{e^{-\lambda x}}{x} dx$$

where $\theta, \lambda > 0$. The paths of $M_1(t)$ increase by jumps. The process has stationary independent increments and $M_1(\theta i) - M_1(\theta(i-1))$ are i.i.d. with a distribution that is exponential with parameter λ . The processes $M_k(t)$ are not gamma and the polynomial P_k does not seem to have a particularly simple form.

3. Convergence theorem.

Denote by $\xrightarrow{v}$ the vague convergence of measure on $\mathbb{R}$. Denote by $\xrightarrow{w}$ the weak convergence of measures, whether these are defined on $D(0, \infty)$ or on $M_p(E)$; the convergence has to be understood in terms of the respective topologies introduced in Section 2.

Consider a double array of random variables $X_{i,n}$, $i = 1, \dots, \infty$, $n = 1, \dots, \infty$, with $X_{i,n}$ i.i.d. for every fixed n . $F_{X_{i,n}}$ will denote the distribution of $X_{i,n}$ and ξ_n will denote the point process

$$(3.1) \quad \xi_n = \sum_{i=1}^{\infty} \delta(i/n, X_{i,n}).$$

The Central Limit Theorem below, due to Resnick (1983), states conditions for $X_{i,n}$ to be in the domain of attraction of a pure jump Levy process $M_1(t)$ with Levy measure μ and associated jump process ξ .

Central Limit Theorem (Resnick). If either $X_{i,n}$ and μ are symmetric or if μ satisfies condition (A) of Section 2, then the following three conditions are equivalent:

$$(3.2) \quad nF_{X_{i,n}}(\cdot) \xrightarrow{v} \mu(\cdot)$$

$$(3.3) \quad \xi_n \xrightarrow{w} \xi$$

$$(3.4) \quad M_1^{[nt]}(X_{\cdot, n}) \xrightarrow{w} M_1(t).$$

The equivalence of (3.2) and (3.3) is proved in Proposition 3.1 of Resnick (1983), under the general assumption that μ is a Radon measure. The fact that (3.2) implies (3.4) can be proved by extending Proposition 3.4 of Resnick (1983). It also follows from the results below. The underlying assumptions on $X_{i,n}$ and μ eliminate the need for centerings. The condition $\mu(0) = 0$ forestalls the presence of a Gaussian component in the limit. Condition (3.4), restricted to $t = 1$ is equivalent to (3.2) by the classical central limit theorem. See, for example, Theorem 25.1 of Gnedenko-Kolmogorov (1954).

We establish an extension of this central limit theorem.

Let M_k^n and S_k^n be defined as in (1.1) and (1.2) and let $M_k(\xi)(t)$ and $S_k(\xi)(t)$ be defined as in the Proposition 2.1 of Section 2. Put $\bar{M}_k^n = (M_1^n, M_2^n, \dots, M_k^n)$ and define $\bar{S}_k^n, \bar{M}_k(t)$ and $\bar{S}_k(t)$ in an analogous way.

Theorem 3.1. Let $X_{i,n}, \mu$ and ξ be as in the Central Limit Theorem above. Then (3.2) also implies:

$$(3.5) \quad \bar{M}_k^{[nt]}(X_{\cdot,n}) \xrightarrow{w} \bar{M}_k(\xi)(t)$$

and

$$(3.6) \quad \bar{S}_k^{[nt]}(X_{\cdot,n}) \xrightarrow{w} \bar{S}_k(\xi)(t).$$

This theorem is proved in Section 5.

Remarks: 1) We showed in Section 2 that μ_k always satisfies condition (A) when $k \geq 2$. Relation (3.2) moreover, implies $nF_{X_{i,n}^k}(\cdot) \xrightarrow{v} \mu_k(\cdot)$.

Therefore, by the Central Limit Theorem, we have $M_k^{[nt]}(X_{\cdot,n}) \xrightarrow{w} M_k(t)$.

However, we shall reprove this result by using a unified method that yields both (3.5) and (3.6).

2) Relation (3.6) would follow from (3.5) by an application of the continuous mapping theorem, if we were to weaken (3.6) to convergence of the finite-dimensional distributions. Our result however, cannot be established in such a simple way because the map $P_k: D_{(0,\infty)}^k \rightarrow D(0,\infty)$ is not continuous (Billingsley 1968, Problem 14.3).

4. The Weak Variations and Multiple Integrals of Pure Jump Levy Processes.

Let $X(t)$ be a pure jump Levy process. The k^{th} order weak variation of $X(t)$ is:

$$w - \lim_{n \rightarrow \infty} \sum_{i=1}^{[nt]} [X(\frac{i}{n}) - X(\frac{i-1}{n})]^k$$

and the k^{th} order multiple integral of $X(t)$ is:

$$w - \lim_{n \rightarrow \infty} \sum_{1 \leq i_1 < \dots < i_k \leq [nt]} \prod_{j=1}^k \left[X\left(\frac{i_j}{n}\right) - X\left(\frac{i_{j-1}}{n}\right) \right]$$

The k^{th} order weak variation of $X(t)$ is denoted:

$$\int_0^t (dX_s)^k,$$

and the k^{th} order multiple integral of $X(t)$ is denoted:

$$\int_{A_t} dX_{s_1} dX_{s_2} \dots dX_{s_k}$$

where A_t is as in (1.10).

Theorem 4.1. Let $X(t)$ be a symmetric or a bounded variation pure jump Levy process, with associated jump process ξ , and let $M_k(\xi)(t)$ and $S_k(\xi)(t)$ be defined as in Section 2.

Then the k^{th} order weak variation and k^{th} order multiple integral of $X(t)$ exist and they satisfy

$$(4.1) \quad \int_0^t (dX_s)^k = M_k(\xi)(t)$$

and

$$(4.2) \quad \int_{A_t} dX_{s_1} \dots dX_{s_k} = S_k(\xi)(t)$$

in the sense of equality of probability measures on $D[0, \infty)$.

Proof. Put $X_{i,n} = X(\frac{i}{n}) - X(\frac{i-1}{n})$. Then the $X_{i,n}$ satisfy (3.2), because they are in the domain of attraction of $X(1)$. The conclusion follows from Theorem 3.1 $\square$

In view of Theorem 3.1, we have

Corollary 4.1. Let $X_{i,n}$ and μ be as in the Central Limit Theorem of Section 3.

Then (3.2) implies:

$$M_k^{[nt]}(X_{\cdot,n}) \xrightarrow{w} \int_0^t (dM_1(s))^k$$

and

$$S_k^{[nt]}(X_{\cdot,n}) \xrightarrow{w} \int_{A_t} dM_1(s_1) \dots dM_1(s_k).$$

Thus, when $X_{i,n}$ is in the domain of attraction of $M_1(t)$, the sums of powers and the symmetric polynomials of $X_{i,n}$ converge to the weak variations and multiple integrals of $M_1(t)$.

Corollary 4.1 can be particularized to the case of a sequence X_i that is in the domain of attraction of a stable process.

Corollary 4.2. Consider a sequence of i.i.d. random variables X_i , which satisfy (3.2) with μ as defined in (2.7) and $X_{i,n} = \frac{1}{n^{1/\alpha}} X_i$. In the case $1 \leq \alpha < 2$ assume in addition that the X_i are symmetric. Then

$$\frac{M_k^{[nt]}(X_{\cdot})}{n^{k/\alpha}} \xrightarrow{w} M_k(t)$$

$$\frac{S_k^{[nt]}(X_{\cdot})}{n^{k/\alpha}} \xrightarrow{w} S_k(t)$$

where $M_k(t), S_k(t)$ are respectively the k^{th} weak variation and k^{th} multiple integral of the Levy stable process of index α .

5. Proof of Theorem 3.1.

The proof has the same structure as Resnick's proof of the Central Limit Theorem (1983, Prop. 3.4).

It is convenient to draw the following diagram:

Recall that $M_p(E)$ is endowed with the vague topology and $D(0, \infty)$ is endowed with the Skorokhod topology. Note also that

$$M_k(\xi_n)(t) = M_k^{[nt]}(\chi_{\cdot, n}),$$

and

$$S_k(\xi_n)(t) = S_k^{[nt]}(\chi_{\cdot, n}).$$

By (3.3) we have $\xi_n \xrightarrow{w} \xi$. Since the maps M_k and S_k from $M_p(E)$ to $D(0, \infty)$ are not continuous, we cannot apply the continuous mapping theorem. However,

Lemma 5.1. If $\mu\{-\delta, \delta\} = 0$, then $M_k^\delta(m)$ and $S_k^\delta(m)$ are continuous in a.s. (N_μ) .

The proof can be carried through as in Appendix 3.5 of Resnick (1983). Here is an outline: a.s. (N_μ) , $m \in M_p(E)$ has a finite number of points in $(0, t) \times \mathbb{R}^\delta$. Denote them (t_i, x_i) , $1 \leq i \leq p$. If $m_n \xrightarrow{v} m$, then,

for n big enough, m_n has also p points in $(0, t) \times \mathbb{R}^\delta$. These points, (t_i^n, x_i^n) , $1 \leq i \leq p$, satisfy $(t_i^n, x_i^n) \xrightarrow{n \rightarrow \infty} (t_i, x_i)$. Therefore $M_k^\delta(m)$ and $S_k^\delta(m)$ are step functions with jumps at t_i and $M_k^\delta(m_n)$ and $S_k^\delta(m_n)$ are step functions with jumps at t_{i_n} that are close to t_i . Moreover, since the sizes of the jumps of $M_k^\delta(m_n)$ and $S_k^\delta(m_n)$ are polynomials in x_i^n , they are close to the sizes of the jumps of $M_k^\delta(m)$ and $S_k^\delta(m)$. The preceding facts about jump times and jump sizes imply that $M_k^\delta(m_n)$ and $S_k^\delta(m_n)$ are close to $M_k^\delta(m)$ and $S_k^\delta(m)$ respectively, in the Skorokod topology. Thus, if $m_n \xrightarrow{V} m$, then $M_k^\delta(m_n) \xrightarrow{S} M_k^\delta(m)$ and $S_k^\delta(m_n) \xrightarrow{S} S_k^\delta(m)$. This establishes the continuity of the maps M_k^δ and S_k^δ and concludes the outline of the proof of Lemma 5.1.

Set $\overline{M}_k^\delta = (M_1^\delta, \dots, M_k^\delta)$ and $\overline{S}_k^\delta = (S_1^\delta, \dots, S_k^\delta)$ and note that they are continuous vector maps.

By the continuous mapping theorem, relation (3.3) and Lemma 5.1, we have:

$$(5.1.a) \quad \overline{M}_k^\delta(\xi_n) \xrightarrow{W} \overline{M}_k^\delta(\xi)$$

and

$$(5.1.b) \quad \overline{S}_k^\delta(\xi_n) \xrightarrow{W} \overline{S}_k^\delta(\xi)$$

as $n \rightarrow \infty$. We also have

$$(5.2.a) \quad \overline{M}_k^\delta(\xi) \xrightarrow{W} \overline{M}_k(\xi)$$

and

$$(5.2.b) \quad \overline{S}_k^\delta(\xi) \xrightarrow{W} \overline{S}_k(\xi)$$

as $\delta \rightarrow 0$, because uniform convergence on compact intervals implies convergence in the Skorohod topology, and obviously, a.s. convergence implies weak convergence.

Then, by Theorem 4.2 of Billingsley (1968), the relations (5.1), (5.2) imply Theorem 3.1 provided that the following holds:

Lemma 5.2.

$$(5.3.a) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P\{\bar{\rho}(\bar{M}_k(\xi_n)(\cdot), \bar{M}_k^\delta(\xi_n)(\cdot)) > \epsilon\} = 0$$

and

$$(5.3.b) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P\{\bar{\rho}(\bar{S}_k(\xi_n)(\cdot), \bar{S}_k^\delta(\xi_n)(\cdot)) > \epsilon\} = 0.$$

Here $\bar{\rho}$ denotes the vector Skorohod metric on $D_{(0,\infty)}^k$ given by $\max_{1 \leq i \leq k} \rho_i$ where ρ_i is the Skorohod metric on the i^{th} coordinate. It is enough to prove that (5.3) holds on finite intervals $[0, T]$, and we shall take $T = 1$ for simplicity.

We shall prove that (5.3) holds even when ρ is replaced by the larger supremum metric. In other words, we show that:

$$(5.4.a) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P\{\max_{1 \leq \ell \leq k} \max_{1 \leq i \leq n} |M_\ell^i(X_{\cdot, n}^\delta)| > \epsilon\} = 0$$

and

$$(5.4.b) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P\{\max_{1 \leq \ell \leq k} \max_{1 \leq i \leq n} |S_\ell^i(X_{\cdot, n}^\delta)| > \epsilon\} = 0,$$

where $X_{i, n}^\delta = X_{i, n} I_{\{|X_{i, n}| \leq \delta\}}.$

Proof of Lemma 5.2. Note that

$$P\{\max_{1 \leq \ell \leq k} \max_{1 \leq i \leq n} |M_\ell^i(X_{\cdot, n}^\delta)| \geq \epsilon\} \leq \sum_{\ell=1}^k P\{\max_{1 \leq i \leq n} |M_\ell^i(X_{\cdot, n}^\delta)| \geq \epsilon\},$$

and that a similar relation holds for (5.4.b), so that it is enough to prove the scalar versions of (5.4), namely:

$$(5.5.a) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P\{ \max_{1 \leq i \leq n} |M_{\ell}^i(X_{\bullet, n}^{\delta})| > \varepsilon \} = 0$$

and

$$(5.5.b) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P\{ \max_{1 \leq i \leq n} |S_{\ell}^i(X_{\bullet, n}^{\delta})| > \varepsilon \} = 0.$$

Suppose first that μ satisfies condition (A). In that case, we shall prove the stronger statements obtained by replacing $X_{\bullet, n}^{\delta}$ by $|X_{\bullet, n}^{\delta}|$, namely

$$(5.6.a) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P\{M_{\ell}^n(|X_{\bullet, n}^{\delta}|) > \varepsilon\} = 0$$

and

$$(5.6.b) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} P\{S_{\ell}^n(|X_{\bullet, n}^{\delta}|) > \varepsilon\} = 0.$$

(Since M_{ℓ}^i and S_{ℓ}^i , when applied to positive numbers, increase in i , $\max_{1 \leq i \leq n}$ is obtained for $i = n$).

By Chebyshev's inequality and relation (3.2),

$$\begin{aligned} \text{L.H.S. of (5.6.a)} &\leq \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \varepsilon^{-1} n \int_{-\delta}^{\delta} |x|^{\ell} dF_{X_{1, n}} \\ &= \lim_{\delta \rightarrow 0} \varepsilon^{-1} \int_{-\delta}^{\delta} |x|^{\ell} \mu(dx) \\ &\leq \varepsilon^{-1} \lim_{\delta \rightarrow 0} \int_{-\delta}^{\delta} |x|^{\ell} \mu(dx) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned}
 \text{L.H.S. of (5.6.b)} &\leq \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \varepsilon^{-1} \binom{n}{\ell} n^{-\ell} \left(n \int_{-\delta}^{\delta} |x| dF_{X_{1,n}} \right)^{\ell} \\
 &\leq \varepsilon^{-1} \lim_{\delta \rightarrow 0} \left(\int_{-\delta}^{\delta} |x| \mu(dx) \right)^{\ell} \\
 &= 0.
 \end{aligned}$$

Suppose next that the $X_{i,n}$ are symmetric.

For (5.5.b), note that $S_{\ell}^i(X_{\bullet,n}^{\delta})$ is a martingale in i , since $X_{j,n}^{\delta}$ has 0 expectation, and $S_{\ell}^i = S_{\ell}^{i-1} + x_i S_{\ell-1}^{i-1}$.

By Kolmogorov's inequality,

$$\begin{aligned}
 P\{ \max_{\ell \leq i \leq n} |S_{\ell}^i(X_{j,n}^{\delta})| > \varepsilon \} &\leq \varepsilon^{-2} E[S_{\ell}^n(X_{\bullet,n}^{\delta})]^2 \\
 &= \varepsilon^{-2} E[S_{\ell}^n((X_{\bullet,n}^{\delta})^2)] \\
 &= \varepsilon^{-2} \binom{n}{\ell} (E[X_{1,n}^{\delta}]^2)^{\ell}.
 \end{aligned}$$

Then,

$$\begin{aligned}
 \text{L.H.S. of (5.5.b)} &\leq \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \varepsilon^{-2} \left(n \int_{-\delta}^{\delta} x^2 dF_{X_{1,n}} \right)^{\ell} \\
 &= \varepsilon^{-2} \lim_{\delta \rightarrow 0} \left(\int_{-\delta}^{\delta} x^2 \mu(dx) \right)^{\ell} \\
 &= 0.
 \end{aligned}$$

We now turn to the proof of (5.5.a). Suppose that ℓ is odd. Then, as above, by applying Kolmogorov's inequality to the martingale M_{ℓ}^i , we get:

$$\begin{aligned}
P\{\max_{1 \leq i \leq n} |M_{\ell}^i(X_{\bullet,n}^{\delta})| > \varepsilon\} &\leq \varepsilon^{-2} E[M_{\ell}^n(X_{\bullet,n}^{\delta})]^2 \\
&= \varepsilon^{-2} EM_{\ell}^n((X_{\bullet,n}^{\delta})^2) \\
&= \varepsilon^{-2} n \int_{-\delta}^{\delta} x^{2\ell} dF_{X_{1,n}} \\
&\leq \varepsilon^{-2} n \int_{-\delta}^{\delta} x^{2\ell} dF_{X_{1,n}},
\end{aligned}$$

which implies (5.5.a).

When ℓ is even, M_{ℓ}^i increases with i and the proof is the same as when μ satisfies condition (A). $\square$

6. Extensions.

A. All through the paper

$x_{i,n}$ can be replaced by $f(\frac{i}{n}) x_{i,n}$,

$M_k(t)$ can be replaced by $M_k^f(t) = \sum_{0 \leq t_i \leq t} (f(t_i) x_i)^k$,

$S_k(t)$ can be replaced by $S_k^f = P_k(M_1^f, M_2^f, \dots, M_k^f)$

provided that $f(t)$ is a continuous function. The process $M_k^f(t)$ can be interpreted as the weak k^{th} variation of the process

$$M_1^f(t) = \int_0^t \int_{R^0} f(s) \xi(ds, dx)$$

(6.1)

$$= \int_0^t f(s) dM_1(s)$$

and S_k^f will be the multiple integral

$$(6.2) \quad S_k^f(t) = \int_{A_t} f(s_1) f(s_2) \dots f(s_k) dM_1(s_1) dM_1(s_2) \dots dM_1(s_k).$$

The processes $M_k^f(t)$ and $S_k^f(t)$ satisfy relation (2.6) as well.

There is no underlying ξ process when $M_1(t)$ is Brownian motion. Nevertheless, as it is well known, the weak variations M_k^f and the multiple integrals S_k^f exist as limits of their discrete approximations and they satisfy the same relations as in the cases we have been considering. In

fact, $M_2^f(t) = \int_0^t f^2(s)ds$, and $M_k^f = 0$ for $k \geq 3$.

Itô (1956) and Segal (1976) among others show that the multiple integrals S_k^f provide an orthogonal basis in the space of L^2 functionals of the process $M_1(t)$ when $M_1(t)$ has finite second moments.

B. When $M_1(t)$ is Brownian motion the general multiple integral

$$\int_{A_t} f(s_1, s_2, \dots, s_k) dM_1(s_1) dM_2(s_2) \dots dM_k(s_k)$$

is well-defined provided

$$\int_{A_t} f^2(s_1, s_2, \dots, s_k) ds_1 ds_2 \dots ds_k < \infty.$$

The situation is much more complex when the process M_1 does not process finite second moments. Consider for instance the case where M_1 is a symmetric stable process of index $1 < \alpha < 2$, and suppose for ease of exposition, that in (6.3), f is symmetric, $t_1 = 1$ and that $k = 2$. In the case $f(s_1, s_2) = g(s_1)g(s_2)$, the condition $\int_0^1 |g(s)|^\alpha ds < \infty$ is necessary and sufficient for (6.3) to exist. However, McConnell and Taqqu (1984) show that for general integrands $f(s_1, s_2)$, the condition $\int_0^1 \int_0^1 |f(s_1, s_2)|^\alpha ds_1 ds_2 < \infty$ is not sufficient for (6.3) to exist. They provide a necessary and sufficient condition for the integral to exist based on "summing" properties of the integral operator having integral kernel f .

C. Durrett and Resnick (1978, Theorem 3.1) provide conditions for an array of dependent random variables to be "asymptotically independent", in the

sense that they still satisfy relation (3.3), namely $\xi_n \xrightarrow{w} \xi$ where ξ_n is a point process and ξ is a P.P.P. If these random variables satisfy also condition (5.3.a) for $k = 1$, then the conclusion (3.4) of the Central Limit Theorem also holds.

Our Theorem 3.1 can thus be extended as follows. Suppose that $X_{i,n}$ is a triangular array of dependent random variables defined as in Theorem 3.1 of Durrett and Resnick (1978). If condition (5.3.a) is satisfied then relation (3.5) holds. If condition (5.3.b) is also satisfied then relation (3.6) holds.

D. We can replace $[nt]$ by general time scale $k_n(t)$ which are non-decreasing right continuous functions, with range $\{0,1,2,\dots\}$.

Condition (3.2) becomes then

$$k_n(t)F_{X_{i,n}}(\cdot) \xrightarrow{v} t_\mu(\cdot)$$

and ξ_n is to be replaced by $\sum_{i=1}^{\infty} \delta(t_{i,n}, X_{i,n})$, where $t_{i,n}$ are the points of increase of $k_n(t)$. Our results apply because one still has $\xi_n \xrightarrow{w} \xi$ (see Theorem 3.1 of Durrett and Resnick 1978).

Appendix

A. We first give a proof of Newton's identities (1.6), starting from the expression (1.4) of the generating function.

Lemma A.1. The coefficients S_k of the formal generating function

$$\phi(v) = \exp\left\{\sum_{k=1}^{\infty} (-1)^{k-1} \frac{M_k}{k} v^k\right\} = \sum_{k=0}^{\infty} S_k v^k$$

satisfy relation (1.6).

Proof. Let $f(v) = \sum_{k=0}^{\infty} (-1)^k M_{k+1} v^k$. Then $\phi'(v) = \phi(v)f(v)$. Differentiating $(k-1)$ times and applying Leibniz's formula, we get

$$(A.1) \quad \phi^{(k)}(v) = \sum_{i=0}^{k-1} \binom{k-1}{i} \phi^{(k-1-i)}(v) f^{(i)}(v).$$

Set $v = 0$ and note that $\phi^{(k)}(0) = k! S_k$ and $f^{(i)}(0) = i! (-1)^i M_{i+1}$.

Relation (A.1) then becomes

$$\begin{aligned} k! S_k &= \sum_{i=1}^k \binom{k-1}{i-1} ((k-i)!) S_{k-i} ((i-1)!) (-1)^{i-1} M_i \\ &= \sum_{i=1}^k (k-1)! (-1)^{i-1} S_{k-i} M_i, \end{aligned}$$

which yields (1.6). $\square$

Lemma A.2: The solution of the recurrence relation (1.6) is given by formula (1.8).

Proof. Let $\underline{n} = (n_1, n_2, \dots)$ be an infinite sequence of nonnegative integers with finitely many nonzero elements and put:

$$A_k = \{\underline{n}: \sum_{i=1}^{\infty} i n_i = k\}.$$

Let $f(\underline{n})$ denote:

$$f(\underline{n}) = \prod_{i=1}^{\infty} \frac{(-M_i)^{n_i}}{n_i! i^{n_i}}.$$

Then, (1.8) becomes

$$P_k(M_1, \dots, M_2) = (-1)^k \sum_{\underline{n} \in A_k} f(\underline{n}).$$

Also, put $e_i = (\dots, 0, 1, 0, 0, \dots)$ with 1 on the i^{th} place. Note that if $\underline{m} = \underline{n} + e_i$, then

$$f(\underline{m}) = \frac{-M_i}{i(n_i+1)} f(\underline{n}) = -\frac{M_i}{i m_i} f(\underline{n}),$$

and that if $\underline{n} \in A_{k-i}$, then $\underline{m} \in A_k$.

Suppose by induction that the Lemma holds up to $k-1$. Then

$$\begin{aligned} kS_k &= \sum_{i=1}^k (-1)^{i-1} M_i S_{k-i} \\ &= \sum_{i=1}^k (-1)^{i-1} M_i (-1)^{k-i} \sum_{\underline{n} \in A_{k-i}} f(\underline{n}) \\ &= (-1)^k \sum_{i=1}^k \sum_{\underline{n} \in A_{k-i}} f(\underline{n} + e_i) \cdot i(n_i+1). \end{aligned}$$

Putting $\underline{n} + \underline{e}_i = \underline{m}$, and changing the order of summation yields

$$kS_k = (-1)^k \sum_{\underline{m} \in A_k} f(\underline{m}) \sum_{i=1}^k im_i.$$

Since $\sum_{i=1}^k im_i = k$, we conclude that (1.8) holds for k .

References

1. Dynkin, E.B. and Mandelbaum, A. (1983). Symmetric statistics, Poisson point processes and multiple Wiener integrals. Ann. of Statistics 11, 739-745.
2. Doleans-Dade, C. (1970). Quelques applications de la formule de changement de variables pour les semimartingales. Z. Wahrsch. verw. Gebiete 16, 181-194.
3. Durrett, R. and Resnick, I.S. (1978). Functional limit theorems for dependent variables. Ann. of Prob. 6, 829-846.
4. Feinsilver, P.J. (1978). Special functions, probability semigroups and Hamiltonian flows. Lect. Notes in Math., 696. Springer Verlag: New York.
5. Gnedenko, B.V. and Kolmogorov, A.N. (1954). Limit Distributions for Sums of Independent Random Variables. Addison-Wesley: Reading.
6. Itô, K. (1956). Spectral type of the shift transformation of differential processes with stationary increments. Trans. Amer. Math. Soc. 81, 106-134.
7. Itô, K. (1969). Stochastic Processes. Lecture Notes Series, 16, Matematisk Institut, Aarhus Universitet.
8. Kallianpur, G. (1980). Stochastic Filtering Theory. Springer Verlag: New York.
9. Mandelbaum, A. and Taqqu, M.S. (1983). Invariance principle for symmetric statistics. Preprint. To appear in Ann. of Statistics.
10. McConnell, T. and Taqqu, M.S. (1984). Double integration with respect to stable processes. Preprint.
11. Meyer, P.A. (1976). Un cours sur les intégrales stochastiques. Lect. Notes in Math., 511, Springer Verlag: New York.
12. Resnick, S.I. (1983). Point processes, regular variation and weak convergence. Preprint. To appear in Advances in Appl. Prob.
13. Segal, A. and Kailath, T. (1976). Orthogonal functionals of independent-increment processes. IEEE Inf. Theory. IT-22, 287-298.
14. Whitt, W. (1980). Some useful functions for functional limit theorems. Math. of Op. Research 5, 67-85.