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**CLOSED SEQUENTIAL PROCEDURES
FOR SELECTING THE MULTINOMIAL EVENTS
WHICH HAVE THE LARGEST PROBABILITIES**

by

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Abstract

Single-stage and closed sequential procedures for selecting the multinomial events which have the largest probabilities are considered. Two goals, Goal I (Selecting the s ($1 \leq s \leq k-1$) best categories without regard to order) and Goal II (Selecting the s ($1 \leq s \leq k-1$) best categories with regard to order) are studied in detail; here $k \geq 2$ is the number of categories in the multinomial distribution. Goal I includes as special cases, the goals of Bechhofer, Elmaghraby and Morse [1959] and Alam and Thompson [1972] which correspond here to the cases $s = 1$ and $s = k-1$, respectively; both foregoing articles gave single-stage procedures which when used with an appropriate single-stage sample size n guarantee a probability requirement which employs the so-called indifference-zone approach.

The sequential procedures that we propose achieve the same probability of a correct selection as do the corresponding single-stage procedures, uniformly in the unknown event probabilities, $(p_1, p_2, \dots, p_k)$. Moreover, this is accomplished with a smaller expected number of vector-observations than that required by the corresponding single-stage procedures. The properties of these sequential procedures are studied analytically, and tables are provided which show the savings in expected sample size when they are used in place of the corresponding single-stage procedures. Comparisons are also made with the expected sample size required by a curtailed sequential procedure proposed by Gibbons, Olkin and Sobel [1977], and it is demonstrated that our sequential procedures are uniformly superior to theirs in terms of expected number of vector-observations.

Key Words

Multinomial selection problem, selection procedures, ranking procedures, sequential procedures, closed procedures, curtailed sampling.

1. Introduction and summary

Let $X_j \equiv (X_{1j}, X_{2j}, \dots, X_{kj})$ ($j = 1, 2, \dots$) denote independent vector-observations from a single multinomial population (Π) having an unknown probability vector $\underline{p} = (p_1, p_2, \dots, p_k)$. Here p_i ($0 \leq p_i \leq 1$, $\sum_{i=1}^k p_i = 1$) denotes the probability of the event E_i ($1 \leq i \leq k$), the events $E_1, E_2, \dots, E_k$ being mutually exclusive, and $X_{ij} = 1$ or 0 according as E_i does or does not occur on the j th observation ($1 \leq i \leq k$, $j = 1, 2, \dots$). Let $p_{[1]} \leq p_{[2]} \leq \dots \leq p_{[k]}$ denote the ordered p -values. It is assumed that the experimenter has no prior knowledge concerning the values of the $p_{[j]}$ or the pairing of the $p_{[j]}$ with the E_i ($1 \leq i, j \leq k$). The category associated with $p_{[k]}$ ($p_{[1]}$) will sometimes be referred to as the "best" ("worst") category. Statistical procedures for selecting the best (worst) category have received considerable attention in recent years. Bechhofer, Elmaghraby and Morse [1959] (hereinafter referred to as B-E-M) proposed a single-stage procedure for the goal of selecting the category associated with $p_{[k]}$. They used the indifference-zone approach of Bechhofer [1954] with the "distance" measure $\theta_{ij} = p_{[i]}/p_{[j]}$ ($i \geq j$). Kesten and Morse [1959] determined the so-called least-favorable (LF) configuration of the p_i ($1 \leq i \leq k$) for the B-E-M procedure, a result that was necessary in order to determine the smallest number (n) of vector-observations necessary to guarantee the associated indifference-zone probability requirement; recently Lam [1981] obtained an alternate proof of the same result. Optimum properties of the procedure were proved by Lehmann [1966] and Eaton [1967]. Alam and Thompson [1972] (A-T) proposed a single-stage procedure for the goal of selecting the category associated with $p_{[1]}$; they also adopted the indifference-zone approach but used the distance measure $\Delta_{ij} = p_{[i]} - p_{[j]}$ ($i \geq j$). For n properly chosen, the B-E-M procedure guarantees that the probability of a

correct selection ($P\{CS\}$) is equal to or greater than P^* ($1/k < P^* < 1$) whenever $P_{[k]}/P_{[k-1]} \geq \theta^*$ ($1 < \theta^* < \infty$) while the A-T procedure guarantees $P\{CS\} \geq P^*$ ($1/k < P^* < 1$) whenever $P_{[2]} - P_{[1]} \geq \Delta^*$ ($0 < \Delta^* < 1$) where $\{\theta^*, P^*\}$ and $\{\Delta^*, P^*\}$ are constants specified prior to the start of experimentation. (A-T had shown that the requirement $P\{CS\} \geq P^*$ whenever $P_{[2]}/P_{[1]} \geq \theta^*$ cannot be guaranteed using a single-stage procedure for $k > 2$ for the goal of selecting the category associated with $p_{[1]}$; later Lee [1975] proved a more general result.) Tables of n necessary to implement the B-E-M procedure for specified $\{\theta^*, P^*\}$, and the A-T procedure for specified $\{\Delta^*, P^*\}$, are given in Gibbons, Olkin and Sobel [1977] (G-O-S) as Table H.1 and H.2, respectively.

Bechhofer, Kiefer and Sobel [1968], Section 5.3.3, p. 121, gave an open sequential procedure which could accommodate both the B-E-M and the A-T goals for $k \geq 2$, as well as more general goals using the indifference-zone approach with the measure of distance θ_{ij} ($i \geq j$); their procedure was a generalization of one described earlier by Bechhofer and Sobel [1956]. Other research workers who developed sequential procedures for the B-E-M goal using the indifference-zone approach with measure of distance θ_{ij} were Cacoullos and Sobel [1966] who used a stopping rule based on inverse-sampling, Alam [1971] and Levin and Robbins [1981] who used the stopping and terminal decision rules of B-K-S, equation (12.9.4), Alam, Seo and Thompson [1971] and Ramey and Alam [1979]. Ramey and Alam [1980] also studied the B-E-M goal from a Bayesian point of view.

Additional relevant references are Bechhofer and Sobel [1958], Dudewicz [1971], Dudewicz and Fan [1973], Gibbons, Olkin and Sobel [1978], and Hsuan, Hwang and Parnes [1982].

The multinomial selection problem has also been formulated and studied from the subset selection (as opposed to the indifference-zone) point of view; Gupta and Panchapakesan [1979], Section 13.6.1, discuss such procedures.

In Section 2 of the present paper we consider two general goals, and single-stage procedures associated with each goal. We refer to these goals as Goal I (Selecting the s ($1 \leq s \leq k-1$) "best" multinomial categories without regard to order) and Goal II (Selecting the s ($1 \leq s \leq k-1$) "best" multinomial categories with regard to order). Thus the goals of B-E-M and A-T are special cases of Goal I with $s = 1$ and $s = k-1$, respectively.

Distance measures play no role in our considerations. The criterion for choosing the single-stage sample sizes (n) for Goals I and II is also of no concern to us. For each goal with given (k,s) our initial interest is in the $P\{CS\}$ achieved by these single-stage procedures for fixed n as a function of $\underline{p}$.

In Section 3 we propose closed sequential procedures for Goals I and II; for prespecified n these sequential procedures, referred to hereinafter as (R^*, S^*, T^*) , achieve the same $P\{CS\}$, uniformly in $\underline{p}$, as do the single-stage procedures which take exactly n vector-observations. Moreover, use of these sequential procedures results in savings in the expected number of vector-observations necessary to terminate sampling ($E\{N\}$) relative to the number of vector-observations (n) required by the corresponding single-stage procedures.

Gibbons, Olkin and Sobel [1977], pp. 178-183, had proposed a sequential procedure, referred to hereinafter as (R^*, S_C, T^*) , which is a curtailed version of the B-E-M single-stage procedure; they provided a table, Table I.1, pp. 453-456, which gives $E\{N\}$ for their sequential procedure. Sobel, Uppuluri and Frankowski [1977], pp. 32-38, proposed an analogous sequential

procedure which is a curtailed version of the A-T single-stage procedure; they discussed computation of $E\{N\}$ for their sequential procedure. In Section 3.4 we consider the G-O-S and S-U-F procedures, and show why our sequential procedures perform at least as well (and sometimes better than) their curtailed versions of the corresponding single-stage procedures.

In Section 4 we consider the $P\{CS\}$ - and $E\{N\}$ -functions of (R^*, S^*, T^*) , and give some exact formulae for these functions, as well as some special properties of the $P\{CS\}$ -function. Section 5 contains tables of the $P\{CS\}$ and $E\{N\}$ of (R^*, S^*, T^*) and (R^*, S_C, T^*) . These were computed using recursive methods analogous to those employed by Sobel, Uppuluri and Frankowski [1977]. The results presented in these tables permit a direct comparison between the $E\{N\}$ -values obtained with (R^*, S^*, T^*) and (R^*, S_C, T^*) as well as with the single-stage n ; the magnitude of the savings in $E\{N\}$ when (R^*, S^*, T^*) is used can then be assessed.

2. Single-stage procedures for Goals I and II

In this section we consider single-stage procedures for the Goal I and Goal II multinomial selection problems. If n vector-observations are taken from the multinomial population Π , let $y_{i,n} = \sum_{j=1}^n x_{ij}$ ($1 \leq i \leq k$) denote the number of occurrences of the event E_i .

2.1 Procedure (R_{SS}, T_{SS}) for Goal I

PROCEDURE FOR GOAL I (Selecting the s ($1 < s < k-1$) best of k

categories without regard to order):

Sampling rule (R_{SS}) : Take n independent vector-observations (2.1a)
from Π .

Terminal decision rule (T_{SS}) : Compute $y_{i,n}$ ($1 \leq i \leq k$). Let

$A_1, A_2 \subset A = \{1, 2, \dots, k\}$ denote two disjoint sets of order s
and $k-s$, respectively, such that

$$y_{i_1,n} \geq y_{i_2,n} \quad (2.1b)$$

for all $i_1 \in A_1$ and for all $i_2 \in A_2$. If there are r sets $A^{(i)} = \{A_1, A_2\}$ ($1 \leq i \leq r$) which satisfy (2.1b), then select one of them at random and announce for the selected set that A_1, A_2 are associated with $\{p_{[k]}, p_{[k-1]}, \dots, p_{[k-s+1]}\}$ and $\{p_{[k-s]}, \dots, p_{[1]}\}$, respectively.

Example 2.1: ($k = 3, s = 1, n = 2$)

$$\begin{array}{ccc} \underline{E_1} & \underline{E_2} & \underline{E_3} \\ (1 & , & 0 & , & 0) \\ (0 & , & 1 & , & 0) \end{array}$$

Then

$$A^{(1)} = \{\{1\}, \{2,3\}\},$$

$$A^{(2)} = \{\{2\}, \{1,3\}\}$$

satisfy (2.1b). Hence, select one of $A^{(i)}$ ($i = 1,2$) at random.

Example 2.2: ($k = 3, s = 2, n = 1$)

$$\begin{array}{ccc} \underline{E_1} & \underline{E_2} & \underline{E_3} \\ (0 & , & 1 & , & 0) \end{array}$$

Then

$$A^{(1)} = \{\{1,2\}, \{3\}\},$$

$$A^{(2)} = \{\{2,3\}, \{1\}\}$$

satisfy (2.1b). Hence select one of $A^{(i)}$ ($i = 1,2$) at random.

Example 2.3: ($k = 3, s = 2, n = 2$)

$$\begin{array}{ccc} \underline{E_1} & \underline{E_2} & \underline{E_3} \\ (1, 0, 0) \\ (0, 1, 0) \end{array}$$

Then $A^{(1)} = \{\{1,2\}, \{3\}\}$ satisfies (2.1b). Hence, select $A^{(1)}$.

Example 2.4: ($k = 4, s = 2, n = 7$)

$$\begin{array}{cccc} \underline{E_1} & \underline{E_2} & \underline{E_3} & \underline{E_4} \\ (1, 0, 0, 0) \\ (0, 0, 1, 0) \\ (0, 1, 0, 0) \\ (0, 0, 0, 1) \\ (0, 0, 0, 1) \\ (0, 1, 0, 0) \\ (1, 0, 0, 0) \end{array}$$

Then $A^{(1)} = \{\{1,2\}, \{3,4\}\},$

$A^{(2)} = \{\{1,4\}, \{2,3\}\},$

$A^{(3)} = \{\{2,4\}, \{1,3\}\}$

satisfy (2.1b). Hence, select one of $A^{(i)}$ ($1 \leq i \leq 3$) at random.

Remark 2.1: The goal of Example 2.1 was considered by B-E-M while that of Examples 2.2 and 2.3 was considered by A-T (since selecting the two best of three is equivalent to selecting the one worst of three). Neither B-E-M nor A-T considered the goal of Example 2.4.

2.2 Procedure (R_{SS}, T_{SS}) for Goal II

PROCEDURE FOR GOAL II (Selecting the s ($1 \leq s \leq k-1$) best of k categories with regard to order):

Sampling rule (R_{SS}) : Take n independent vector-observations from Π . (2.2a)

Terminal decision rule (T_{SS}) : Compute $y_{i,n}$ ($1 \leq i \leq k$).

Let $A_1, A_2, \dots, A_{s+1} \subset A = \{1, 2, \dots, k\}$ denote $s+1$ disjoint sets with $A_1, A_2, \dots, A_s$ of order one, and A_{s+1} of order $k-s$, such that

$$y_{i_j, n} \geq y_{i_{j+1}, n} \quad (1 \leq j \leq s) \quad (2.2b)$$

for $i_j \in A_j$ ($1 \leq j \leq s$) and for all $i_{s+1} \in A_{s+1}$. If there are r sets $A^{(i)} = \{A_1, A_2, \dots, A_{s+1}\}$ ($1 \leq i \leq r$) which satisfy (2.2b), then select one of them at random and announce for the selected set that

$A_1, A_2, \dots, A_s$ and A_{s+1} are associated with $p_{[k]}, p_{[k-1]}, \dots, p_{[k-s+1]}$ and $\{p_{[k-s]}, \dots, p_{[1]}\}$, respectively.

Example 2.5: ($k = 4, s = 3, n = 4$)

$\underline{E_1}$	$\underline{E_2}$	$\underline{E_3}$	$\underline{E_4}$
(1	, 0	, 0	, 0)
(0	, 0	, 0	, 1)
(0	, 0	, 1	, 0)
(0	, 0	, 0	, 1)

Then

$$A^{(1)} = \{\{4\}, \{1\}, \{3\}, \{2\}\},$$

$$A^{(2)} = \{\{4\}, \{3\}, \{1\}, \{2\}\}$$

satisfy (2.2b). Hence, select one of $A^{(i)}$ ($i = 1, 2$) at random.

3. Sequential procedures for Goals I and II

In this section we consider closed sequential procedures for the Goal I and Goal II multinomial selection problems. By stage m we shall mean that m vector-observations have been taken from the multinomial population Π .

Let $z_{i,m} = \sum_{j=1}^m x_{ij}$ ($1 \leq i \leq k, 1 \leq m \leq n$) denote the number of occurrences of the event E_i through stage m .

3.1 Procedure (R^*, S^*, T^*) for Goal I

PROCEDURE FOR GOAL I (Selecting the s ($1 \leq s \leq k-1$) best of k categories without regard to order):

Sampling rule (R^*): Take independent vector-observations

one-at-a-time from Π .

(3.1a)

Stopping rule (S^*): Stop sampling at the first stage m at which there exist two disjoint sets $A_1, A_2 \subset A = \{1, 2, \dots, k\}$ with A_1 of order s and A_2 of order $k-s$, such that

$$z_{i_1, m} \geq z_{i_2, m} + n - m \quad (3.1b)$$

for all $i_1 \in A_1$ and for all $i_2 \in A_2$.

Terminal decision rule (T^*): Having stopped, if there are r sets

$A^{(i)} = \{A_1, A_2\}$ ($1 \leq i \leq r$) which satisfy (3.1b), then select one of them at random and announce for the selected set that A_1 and A_2 are associated with $\{p[k], p[k-1], \dots, p[k-s+1]\}$ and $\{p[k-s], \dots, p[1]\}$, respectively. (3.1c)

Example 3.1: ($k = 3, s = 1, n = 4$).

Stop if the following sequence of vector-observations is obtained:

$\underline{m}$	$\underline{E_1}$	$\underline{E_2}$	$\underline{E_3}$
1	(0	, 1	, 0)
2	(0	, 1	, 0)

Then $A^{(1)} = \{\{2\}, \{1, 3\}\}$ satisfies (3.1b). Hence, select $A^{(1)}$.

Example 3.2: ($k = 3, s = 2, n = 3$).

Stop if the following sequence of vector-observations is obtained:

$\underline{m}$	$\underline{E_1}$	$\underline{E_2}$	$\underline{E_3}$
1	(1	, 0	, 0)
2	(0	, 0	, 1)

Then $A^{(1)} = \{\{1, 3\}, \{2\}\}$ satisfies (3.1b). Hence, select $A^{(1)}$.

3.2 Procedure (R^*, S^*, T^*) for Goal II

PROCEDURE FOR GOAL II (Selecting the s ($1 \leq s \leq k-1$) best of k categories with regard to order):

Sampling rule (R^*) : Take independent vector-observations

one-at-a-time from Π . (3.2a)

Stopping rule (S^*) : Stop sampling at the first stage m at which there exist $s+1$ disjoint sets, $A_1, A_2, \dots, A_{s+1} \subset \{1, 2, \dots, k\}$ with $A_1, A_2, \dots, A_s$ of order one and A_{s+1} of order $k-s$, such that

$$z_{i_j, m} \geq z_{i_{j+1}, m} + n - m \quad (1 \leq j \leq s) \quad (3.2b)$$

for $i_j \in A_j$ ($1 \leq j \leq s$) and for all $i_{s+1} \in A_{s+1}$.

Terminal decision rule (T^*) : Having stopped, if there are

r sets $A^{(i)} = \{A_1, A_2, \dots, A_{s+1}\}$ ($1 \leq i \leq r$) satisfying (3.2b),

then select one of them at random and announce for the (3.2c)

selected set that $A_1, A_2, \dots, A_{s+1}$ are associated with

$p_{[k]}, p_{[k-1]}, \dots, p_{[k-s+1]}$ and $\{p_{[k-s]}, \dots, p_{[1]}\}$, respectively.

Example 3.3: ($k = 3, s = 2, n = 1$)

Stop if the following sequence of vector-observations is obtained:

$$\begin{array}{c} \underline{m} \\ 1 \end{array} \quad \begin{array}{c} \underline{E_1} \\ (0 \end{array} , \quad \begin{array}{c} \underline{E_2} \\ 0 \end{array} , \quad \begin{array}{c} \underline{E_3} \\ 1) \end{array}$$

Then

$$A^{(1)} = \{\{3\}, \{2\}, \{1\}\},$$

$$A^{(2)} = \{\{3\}, \{1\}, \{2\}\}$$

satisfy (3.2b). Hence, select one of $A^{(i)}$ ($i = 1, 2$) at random.

Example 3.4: ($k = 3, s = 2, n = 4$)

Stop if the following sequence of vector-observations is obtained:

<u>m</u>	<u>E₁</u>	<u>E₂</u>	<u>E₃</u>
1	(0	, 1	, 0)
2	(1	, 0	, 0)
3	(1	, 0	, 0)

Then $A^{(1)} = \{\{1\}, \{2\}, \{3\}\}$ satisfies (3.2b). Hence, select $A^{(1)}$.

Remark 3.1: It can be seen that if (R^*, S^*, T^*) is used then r is always equal to unity in (3.1c) and (3.2c) except possibly if the stopping stage m equals n (as in Example 3.3).

3.3 The $P\{CS\}$ of (R^*, S^*, T^*) for Goals I and II

Our sequential procedures (R^*, S^*, T^*) for Goals I and II described in Sections 3.1 and 3.2, respectively, possess a very important property relative to the corresponding single-stage procedures (R_{SS}, T_{SS}) for Goals I and II described in Sections 2.1 and 2.2, respectively. This property is summarized in Theorem 3.1, below. We repeat here a basic assumption.

Assumption 3.1: The pairing of the $p_{[j]}$ with the E_i ($1 \leq i, j \leq k$) is assumed to be completely unknown to the experimenter. If two or more events have a common p-value, it is assumed that they are tagged in such a way that the ordering of the p-values is unique.

Under Assumption 3.1 we have the following theorem:

Theorem 3.1:

$$P\{CS|(R^*, S^*, T^*)\} = P\{CS|(R_{SS}, T_{SS})\}$$

uniformly in $\underline{p} = (p_1, p_2, \dots, p_k)$ for both Goal I and Goal II.

The proof of this theorem is analogous to the proof of the corresponding theorem for the Bernoulli selection problem which is given in Bechhofer and Kulkarni [1982a]. It is a special case of a more general theorem proved by Jennison [1983].

As can be seen from Examples 3.1, 3.2, 3.4, sampling for (R^*, S^*, T^*) sometimes terminates before n stages where n is the number of vector-observations taken by the corresponding single-stage procedure. In fact, from (3.1b) and (3.2b) we see that

$$\{N|(R^*, S^*, T^*)\} \leq n \quad \text{for all } (n, s),$$

and

$$E\{N|(R^*, S^*, T^*)\} < n \quad \text{for } n > 1, s = 1.$$

(3.3)

Remark 3.2: Theorem 3.1 does not imply that (R^*, S^*, T^*) and (R_{SS}, T_{SS}) will arrive at the same terminal decision when presented with the same sequence of outcome vectors. Thus, if in Example 2.1 the vectors had been

observed one-at-a-time in the order given, the terminal decision for (R^*, S^*, T^*) would have been to select $A^{(1)}$ after the first vector rather than to randomize between $A^{(1)}$ and $A^{(2)}$ after the second vector. In general, suppose that there are ties among the $y_{i,n}$ ($1 \leq i \leq k$) and that the r sets $A^{(i)}$ ($1 \leq i \leq r$) satisfy (2.1b). Then (R^*, S^*, T^*) will either stop at stage n and randomize over the same set of the $A^{(i)}$ ($1 \leq i \leq r$) as does (R_{SS}, T_{SS}) or it will stop before stage n with exactly one of the $A^{(i)}$ ($1 \leq i \leq r$) satisfying (3.1b).

3.4 Procedure (R^*, S_C, T^*) , the Gibbons-Olkin-Sobel curtailment of the single-stage procedure (R_{SS}, T_{SS}) ,

Gibbons, Olkin and Sobel [1977], pp. 178-183, describe how curtailment can be applied to the Goal I single-stage procedure (R_{SS}, T_{SS}) for $s = 1$; Sobel, Uppuluri and Frankowski [1977], pp. 32-38, do the same for (R_{SS}, T_{SS}) for $s = k-1$. In fact, as we now point out, their method can be used for both Goal I and Goal II for arbitrary s ($1 \leq s \leq k-1$): For Goal I they would use our sampling rule (3.1a) and our terminal decision rule (3.1c) while for Goal II they would use our (3.2a) and (3.2c); however, for Goal I they would replace the weak inequality ($\geq$) in our stopping rule (3.1b) by the strict inequality ($>$), and for Goal II they would do the same in (3.2b). We shall refer to the G-O-S and S-U-F stopping rules which curtail sampling using the strict inequality as P_C , and we shall say that P_C employs weak curtailment whereas P^* employs strong curtailment.

Remark 3.3: Cacoullos and Sobel [1966], p. 445, considered weak curtailment of the B-E-M single-stage procedure for $k = 2$.

If (R^*, S_C, T^*) is used, then it is always the case that (R^*, S_C, T^*) and (R_{SS}, T_{SS}) randomize over the same set of $A^{(i)}$ ($1 \leq i \leq r$), and hence

$$P\{CS | (R^*, S_C, T^*)\} = P\{CS | (R_{SS}, T_{SS})\} \quad (3.4)$$

uniformly in $\underline{p} = (p_1, p_2, \dots, p_k)$ for both Goal I and Goal II.

Also, it is always the case that

$$\{N | (R^*, S^*, T^*)\} \leq \{N | (R^*, S_C, T^*)\} \leq n, \quad (3.5)$$

with certain sequences of outcome vectors yielding smaller N-values for (R^*, S^*, T^*) than for (R^*, S_C, T^*) . This can be seen by the following examples:

Example 3.5: ($k = 3, s = 1, n = 2$)

Suppose that the following sequence of vector-observations is obtained:

<u>m</u>	<u>E₁</u>	<u>E₂</u>	<u>E₃</u>
1	(1 , 0 , 0)		
2	(0 , 1 , 0)		

Then (R^*, S^*, T^*) would stop at stage 1 and select $A^{(1)} = \{\{1\}, \{2,3\}\}$.

However, (R^*, S_C, T^*) would stop at stage 2 and select one of $A^{(1)} = \{\{1\}, \{2,3\}\}$ and $A^{(2)} = \{\{2\}, \{1,3\}\}$ at random (as would (R_{SS}, T_{SS}) if both vector-observations had been taken in one stage).

Example 3.6 ($k = 3, s = 1, n = 3$)

Suppose that the following sequence of vector-observations is obtained:

<u>m</u>	<u>E₁</u>	<u>E₂</u>	<u>E₃</u>
1	(0	, 1	, 0)
2	(0	, 1	, 0)

Then both (R^*, S^*, T^*) and (R^*, S_C, T^*) would stop at stage 2 and select $A^{(1)} = \{\{2\}, \{1, 3\}\}$.

Example 3.7: ($k = 3, s = 1, n = 6$)

Suppose that the following sequence of vector-observations is obtained:

<u>m</u>	<u>E₁</u>	<u>E₂</u>	<u>E₃</u>
1	(1	, 0	, 0)
2	(1	, 0	, 0)
3	(1	, 0	, 0)
4	(0	, 1	, 0)
5	(0	, 1	, 0)
6	(0	, 0	, 1)

Then (R^*, S^*, T^*) would stop at stage 3 and select $A^{(1)} = \{\{1\}, \{2, 3\}\}$ whereas (R^*, S_C, T^*) would stop at stage 6 and also select $A^{(1)} = \{\{1\}, \{2, 3\}\}$.

Remark 3.4: Neither G-O-S nor S-U-F was aware of our procedure (R^*, S^*, T^*) or its properties. In fact, G-O-S state (see their equation (6.6.5), p. 182)

that a necessary and sufficient condition to terminate sampling and reach the same terminal decision as does the single-stage procedure is to employ (R^*, S_C, T^*) ; actually, use of (R^*, S_C, T^*) only guarantees that (R_{SS}, T_{SS}) and (R^*, S_C, T^*) randomize over the same set of terminal decisions as is illustrated by Example 3.5. And, as is illustrated by Example 3.5 and 3.7, (R^*, S^*, T^*) sometimes stops earlier than does (R^*, S_C, T^*) , but both achieve the same $P\{CS\}$ as a consequence of Theorem 3.1 and (3.4).

4. The $P\{CS\}$ - and $E\{N\}$ -functions of (R^*, S^*, T^*)

As a consequence of Theorem 3.1, the $P\{CS\}$ -functions of (R_{SS}, T_{SS}) and (R^*, S^*, T^*) are identical, uniformly in p for both Goal I and Goal II. The calculation of the exact $P\{CS\}$ -function of (R_{SS}, T_{SS}) for Goal I is considered in B-E-M for $s = 1$, and tables of the computed probabilities for p in the so-called LF-configuration ($p_{[k]} = \theta^*/(\theta^*+k-1)$, $p_{[1]} = P_{[k-1]} = 1/(\theta^*+k-1)$) are given there for $k = 2(1)4$, $n = 1(1)30$ and selected θ^* .

We point out below that the $P\{CS\}$ -functions of (R_{SS}, T_{SS}) (and hence of (R^*, S^*, T^*)) for both Goal I and Goal II exhibit a behavior which is quite different for $k \geq 3$, $s > 1$ than for $k \geq 3$, $s = 1$. We summarize these differences in Section 4.1.1.

4.1 Some special properties of the $P\{CS\}$ -functions of (R^*, S^*, T^*)

4.1.1 Limiting values of the $P\{CS\}$

For Goal I with $k \geq 2$, $1 \leq s \leq k-1$, $n \geq 1$ we have

$$\lim_{p_{[k]} \rightarrow 1/k} P\{CS\} = 1/\binom{k}{s} \quad (4.1a)$$

$$\lim_{p_{[k]} \rightarrow 1} P\{CS\} = 1/\binom{k-1}{s-1} \quad (4.1b)$$

For Goal II with $k \geq 2$, $1 \leq s \leq k-1$, $n \geq 1$ we have

$$p_{[k]}^{\lim_{1/k}} P\{CS\} = (k-s)!/k! \quad (4.2a)$$

$$p_{[k]}^{\lim_{1/k}} P\{CS\} = (k-s)!/(k-1)! \quad (4.2b)$$

In particular, for $s = 1$ (in which case Goals I and II coincide) both (4.1b) and (4.2b) equal unity while for $s > 1$ both (4.1b) and (4.2b) are less than unity; also, (4.1b) and (4.2b) obviously cannot be increased by increasing n .

4.1.2 Maximum values of the $P\{CS\}$

For Goal I (Goal II) with $k \geq 2$, $s = 1$, $n \geq 1$ we have

$$\underset{\sim}{\text{Max}} P\{CS\} = 1. \quad (4.3)$$

This $\underset{\sim}{\text{max}} P\{CS\}$ occurs when $p_{[k]} = 1$.

For Goal I with $k \geq 3$, $k-1 \geq s \geq 2$, $n \geq 1$ we have

$$\underset{\sim}{\text{Max}} P\{CS\} = C_{k,s,n}^I < 1 \quad (4.4)$$

where $C_{k,s,n}^I$ is a constant depending only on (k,s,n) for Goal I.

This $\underset{\sim}{\text{max}} P\{CS\}$ occurs for all n when

$$\begin{aligned} p_{[1]} &= p_{[k-s]} = 0, \\ p_{[k-s+1]} &= p_{[k]} = \frac{1}{s}. \end{aligned} \quad (4.5)$$

We shall refer to (4.5) as the Most Favorable (MF) configuration of the p_i ($1 \leq i \leq k$) for Goal I with $k \geq 2$, $k-1 \geq s \geq 1$ since it is associated with the maximum $P\{CS\}$ that can be achieved for given n . The result (4.5) is proved in Appendix A. As a consequence of (4.5) it can be shown that

i) For $k \geq 3$, $s = 2$, $n \geq 1$ we have

$$C_{k,2,n}^I = 1 - \frac{k-2}{k-1} \cdot \frac{1}{2^{n-1}} \quad (4.6)$$

which occurs when $p_{[k-1]} = p_{[k]} = \frac{1}{2}$.

ii) For $k \geq 4$, $s = 3$, $n \geq 1$ we have

$$C_{k,3,n}^I = 1 - \left[1 - \frac{2}{(k-1)(k-2)} \right] \frac{1}{3^{n-1}} - 3 \left(\frac{k-3}{k-2} \right) \left[\left(\frac{2}{3} \right)^n - 2 \left(\frac{1}{3} \right)^n \right] \quad (4.7)$$

which occurs when $p_{[k-2]} = p_{[k]} = \frac{1}{3}$.

It can be shown that $C_{k,s,n}^I$ of (4.4) is an increasing function of n for fixed (k,s) , and approaches unity as $n \rightarrow \infty$. Exact formulae for the $P\{CS\}$ -function for Goals I and II, $k = 3$, $s = 2$, $n = 1(1)6$ are given in Section 4.2.1. Corresponding formulae for the $E\{N\}$ -function of (R^*, S^*, T^*) are given in Section 4.2.2.

For Goal II with $k \geq 3$, $k-1 \geq s \geq 2$, $n \geq 1$ we have

$$\max_p P\{CS\} = C_{k,s,n}^{II} < 1 \quad (4.8)$$

where $C_{k,s,n}^{II}$ is a constant depending only on (k,s,n) for Goal II.

This $\max P\{CS\}$ occurs at p -values which change with n . Values of $C_{k,s,n}^{II}$ and their associated MF-configuration are given in Table 4.1 for $k = 3, s = 2, n = 1(1)6$. These values indicate that $C_{k,s,n}^{II}$ of (4.8) is a strictly increasing function of n for $n \geq 2$, and approaches unity as $n \rightarrow \infty$.

4.2 Some exact formulae for the $P\{CS\}$ - and $E\{N\}$ -function of (R^*, S^*, T^*) for Goals I and II

The exact formulae in the next subsections are typical of those of (R^*, S^*, T^*) , and indicate their increasing complexity as n increases for fixed (k, s) . For simplicity of notation we assume here that $p_1 \leq p_2 \leq p_3$.

4.2.1 Formulae for the $P\{CS\}$ -function of (R^*, S^*, T^*) for Goals I and II

$P\{CS\}$ for Goal I ($k = 3, s = 2, n = 1(1)6$):

$$n = 1: \frac{1}{2} (p_2 + p_3)$$

$$n = 2: \frac{1}{2} (p_2^2 + p_3^2) + 2p_2p_3$$

$$n = 3: \frac{1}{2} (p_2^3 + p_3^3) + p_2p_3(2 + p_2 + p_3)$$

$$n = 4: \frac{1}{2} (p_2^4 + p_3^4) + 2p_2p_3[3(p_2 + p_3 - p_2p_3) - (p_2^2 + p_3^2)]$$

$$n = 5: \frac{1}{2} (p_2^5 + p_3^5) + 5p_2p_3[6p_2p_3 + 2p_2^2 + 2p_3^2 - 6p_2^2p_3 - 6p_2p_3^2 - p_2^3 - p_3^3]$$

$$n = 6: \frac{1}{2} (p_2^6 + p_3^6) + 3p_2p_3[-3(p_2^4 + p_3^4) + 10(p_2^3p_3 + p_2p_3^3) + 5(p_2^3 + p_3^3) - 20(p_2^2p_3 + p_2p_3^2) + 10p_2p_3] + 80p_2^3p_3^3$$

See (4.6) for the MF-configuration and associated $\text{Max } P\{CS\}$ for Goal I
($k = 3, s = 2, n = 1(1)6$).

$P\{CS\}$ for Goal II ($k = 3, s = 2, n = 1(1)6$):

$$n = 1: \frac{1}{2} p_3$$

$$n = 2: \frac{1}{2} p_3^2 + p_2 p_3$$

$$n = 3: \frac{1}{2} p_3^3 + p_2 p_3 (2p_3 - p_2 + 1)$$

$$n = 4: \frac{1}{2} p_3^4 + p_2 p_3 (6p_3 - 2p_3^2 - 3p_2 p_3)$$

$$n = 5: \frac{1}{2} p_3^5 + 5p_2 p_3^2 [3p_2 + 2p_3 - (3p_2^2 + 3p_2 p_3 + p_3^2)]$$

$$n = 6: \frac{1}{2} p_3^6 + 6p_3^5 p_2 + 15p_3^4 p_2^2 + 15p_3^4 p_2 p_1 \\ + 10 p_3^3 p_2^3 + 60p_3^3 p_2^2 p_1 + 15p_3^2 p_2^2 p_1^2 .$$

See Table 4.1 for the MF-configuration and associated $\text{Max } P\{CS\}$ for
Goal II ($k = 3, s = 2, n = 1(1)6$).

4.2.2 Formulae for the $E\{N\}$ -function of (R^*, S^*, T^*) for Goals I and II

$E\{N\}$ for Goal I ($k = 3, s = 2, n = 1(1)6$):

$$n = 1: 1$$

$$n = 2: 2$$

$$n = 3: 3 - 2(p_1 p_2 + p_1 p_3 + p_2 p_3)$$

$$n = 4: 4 - 3(p_1 p_2^2 + p_1 p_3^2 + p_2 p_1^2 + p_2 p_3^2 \\ + p_3 p_1^2 + p_3 p_2^2)$$

Table 4.1

MF-configuration and associated Max P{CS} for Goal II (k = 3, s = 2)

n	MF-configuration	$C_{3,2,n}^{II}$
1	$p_{[1]} = 0, p_{[2]} = 0, p_{[3]} = 1$	0.5
2	$p_{[1]} = 0, p_{[2]} = 0, p_{[3]} = 1$	0.5
3	$p_{[1]} = 0, p_{[2]} = 0.2, p_{[3]} = 0.8$	0.64
4	$p_{[1]} = 0, p_{[2]} = \frac{\sqrt{21}-3}{2}, p_{[3]} = \frac{5-\sqrt{21}}{2}$ $= 0.2087, \quad = 0.7913$	0.6915
5	$p_{[1]} = 0, p_{[2]} = \frac{2\sqrt{3}-1}{11}, p_{[3]} = \frac{12-2\sqrt{3}}{11}$ $= 0.2240 \quad = 0.7760$	0.7813
6	$p_{[1]} = 0, p_{[2]} = 0.2185, p_{[3]} = 0.7815$	0.8130

$$n = 5: \quad 5-6(p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2) \\ -4(p_1^3 p_2 + p_1^3 p_3 + p_2^3 p_1 + p_2^3 p_3 + p_3^3 p_1 + p_3^3 p_2)$$

$$n = 6: \quad 6-5(p_1^4 p_2 + p_1^4 p_3 + p_2^4 p_1 + p_2^4 p_3 \\ + p_3^4 p_1 + p_3^4 p_2) \\ - 10(p_1^2 p_2^3 + p_1^2 p_3^3 + p_2^2 p_1^3 + p_2^2 p_3^3 \\ + p_3^2 p_1^3 + p_3^2 p_2^3)$$

E{N} for Goal II (k = 3, s = 2, n = 1(1)6)

$$n = 1: \quad 1$$

$$n = 2: \quad 2$$

$$n = 3: \quad 3$$

$$n = 4: \quad 4-3(p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_1^2 + p_2^2 p_3^2 \\ + p_3^2 p_1^2 + p_3^2 p_2^2)$$

$$n = 5: \quad 5-4(p_1^3 p_2 + p_1^3 p_3 + p_2^3 p_1 + p_2^3 p_3 \\ + p_3^3 p_1 + p_3^3 p_2)$$

$$n = 6: \quad 6-5(p_1^4 p_2 + p_1^4 p_3 + p_2^4 p_1 + p_2^4 p_3 \\ + p_3^4 p_1 + p_3^4 p_2) \\ - 10(p_1^2 p_2^3 + p_1^2 p_3^3 + p_2^2 p_1^3 + p_2^2 p_3^3 \\ + p_3^2 p_1^3 + p_3^2 p_2^3)$$

Note: E{N} for Goals I and II (k = 3, s = 2) are the same for
n = 1,2,4 and 6.

5. Tables of the $P\{CS\}$ and $E\{N\}$ of (R^*, S^*, T^*) and (R^*, S_C, T^*)

In Section 4.2 we gave some representative exact formulae for the $P\{CS\}$ - and $E\{N\}$ -functions of (R^*, S^*, T^*) for Goals I and II with $k = 3, s = 2$. For any (k, s) such formulae are very tedious to derive, and become especially so for large k and n . The same would be true, of course, for the corresponding formulae of (R^*, S_C, T^*) . Therefore, we have used recursion formulae and a computer to calculate numerical values of such quantities. These formulae are valid for $k \geq 2$ for the $P\{CS\}$ -functions and for $k \geq 3$ for the $E\{N\}$ -functions. The derivations of the required algorithms are given for $s = 1$ in Appendix B. To compute $E\{N\}$ for $k = 2$ note the following: If n is odd, then $E\{N\}$ for (R^*, S^*, T^*) is the same as $E\{N\}$ for (R^*, S_C, T^*) ; if n is even, then $E\{N\}$ for (R^*, S^*, T^*) for n is the same as $E\{N\}$ for (R^*, S_C, T^*) for $n + 1$. From equation (6.6.6) of G-0-S we have for odd n that

$$E\{N | (R^*, S_C, T^*)\} = \frac{n+1}{2} \left[\frac{1}{p_{[2]}} I_{p_{[2]}} \left(\frac{n+3}{2}, \frac{n+1}{2} \right) + \frac{1}{p_{[1]}} I_{p_{[1]}} \left(\frac{n+3}{2}, \frac{n+1}{2} \right) \right] \quad (5.1)$$

where $I_x(a, b)$ is the well-known standard incomplete beta function.

Tables 5.1-5.5 illustrate the behavior of (R^*, S^*, T^*) and (R^*, S_C, T^*) for Goal I with $k = 2(1)6$ and $s = 1$ for $n = 2(2)20$ and selected $\underline{p} = (p_{[1]}, p_{[2]}, \dots, p_{[k]})$. We have used $s = 1$ and the parameterization $\theta = p_{[k]}/p_{[k-1]}, p_{[k-1]} = p_{[1]}$ (the so-called slippage configuration) in order to make our numerical values comparable to those given in

Table 5.1

$P\{CS\}$ and $E\{N\}$ of (R^*, S^*, T^*) and (R^*, S_C, T^*)

for $k = 2, s = 1$ when $p_{[2]} = \frac{\theta}{\theta+1}$, $p_{[1]} = \frac{1}{\theta+1}$,

as a function of n and selected θ

n	Key	θ							
		1.0	1.4	1.8	2.2	2.6	3.0	10.0	∞
2	A	0.500	0.583	0.643	0.688	0.722	0.750	0.909	1.000
	B	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	C	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
4	A	0.500	0.624	0.708	0.768	0.811	0.844	0.977	1.000
	B	2.50	2.49	2.46	2.43	2.40	2.38	2.17	2.00
	C	3.75	3.73	3.69	3.64	3.60	3.56	3.25	3.00
6	A	0.500	0.653	0.754	0.820	0.865	0.896	0.993	1.000
	B	4.13	4.08	4.01	3.92	3.84	3.77	3.29	3.00
	C	5.50	5.44	5.34	5.23	5.12	5.03	4.39	4.00
8	A	0.500	0.677	0.788	0.857	0.901	0.929	0.998	1.000
	B	5.81	5.73	5.58	5.43	5.29	5.16	4.40	4.00
	C	7.26	7.17	6.98	6.78	6.61	6.45	5.50	5.00
10	A	0.500	0.698	0.816	0.885	0.926	0.951	0.999	1.000
	B	7.54	7.41	7.17	6.93	6.72	6.54	5.50	5.00
	C	9.05	8.89	8.61	8.32	8.06	7.85	6.60	6.00
12	A	0.500	0.715	0.839	0.907	0.944	0.966		1.000
	B	9.29	9.10	8.77	8.43	8.15	7.91	6.60	6.00
	C	10.84	10.62	10.23	9.84	9.50	9.22	7.70	7.00
14	A	0.500	0.713	0.858	0.924	0.958	0.976		1.000
	B	11.07	10.81	10.36	9.93	9.56	9.26	7.70	7.00
	C	12.65	12.36	11.84	11.35	10.93	10.59	8.80	8.00
16	A	0.500	0.746	0.875	0.937	0.968	0.983		1.000
	B	12.86	12.53	11.96	11.42	10.98	10.62	8.80	8.00
	C	14.47	14.10	13.45	12.85	12.35	11.94	9.90	9.00
18	A	0.500	0.759	0.889	0.948	0.975	0.988		1.000
	B	14.66	14.25	13.55	12.91	12.38	11.96	9.90	9.00
	C	16.29	15.84	15.06	14.34	13.76	13.29	11.00	10.00
20	A	0.500	0.771	0.901	0.957	0.981	0.991		1.000
	B	16.48	15.98	15.15	14.39	13.78	13.31	11.00	10.00
	C	18.12	17.58	16.66	15.83	15.16	14.64	12.10	11.00

Key: $A = P\{CS | (R^*, S^*, T^*)\}$, $B = E\{N | (R^*, S^*, T^*)\}$, $C = E\{N | (R^*, S_C, T^*)\}$

Table 5.2

$P\{CS\}$ and $E\{N\}$ of (R^*, S^*, T^*) and (R^*, S_C, T^*)

for $k = 3, s = 1$ when $P[3] = \frac{\theta}{\theta+2}$, $P[1] = P[2] = \frac{1}{\theta+1}$,

as a function of n and selected θ

n	Key	θ							
		1.0	1.4	1.8	2.2	2.6	3.0	10.0	∞
2	A	0.333	0.412	0.474	0.524	0.565	0.600	0.833	1.000
	B	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	C	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
4	A	0.333	0.457	0.554	0.629	0.688	0.734	0.955	1.000
	B	2.89	2.87	2.83	2.79	2.75	2.70	2.33	2.00
	C	3.89	3.88	3.86	3.83	3.80	3.77	3.42	3.00
6	A	0.333	0.483	0.599	0.687	0.754	0.804	0.986	1.000
	B	4.93	4.88	4.78	4.66	4.55	4.44	3.57	3.00
	C	5.58	5.56	5.51	5.44	5.37	5.31	4.63	4.00
8	A	0.333	0.504	0.636	0.732	0.801	0.851	0.995	1.000
	B	6.72	6.66	6.52	6.36	6.20	6.04	4.78	4.00
	C	7.39	7.34	7.23	7.10	6.97	6.84	5.80	5.00
10	A	0.333	0.527	0.672	0.773	0.842	0.888	0.998	1.000
	B	8.52	8.44	8.26	8.05	7.82	7.61	5.98	5.00
	C	9.31	9.22	9.04	8.82	8.60	8.39	6.95	6.00
12	A	0.333	0.544	0.700	0.804	0.871	0.915	0.999	1.000
	B	10.43	10.31	10.05	9.75	9.45	9.17	7.17	6.00
	C	11.15	11.04	10.78	10.48	10.18	9.91	8.09	7.00
14	A	0.333	0.561	0.725	0.830	0.895	0.934		1.000
	B	12.27	12.12	11.79	11.41	11.03	10.69	8.36	7.00
	C	13.01	12.86	12.53	12.14	11.76	11.42	9.24	8.00
16	A	0.333	0.577	0.749	0.854	0.915	0.950		1.000
	B	14.12	13.94	13.53	13.06	12.60	12.20	9.54	8.00
	C	14.93	12.74	14.31	13.82	13.35	12.93	10.38	9.00
18	A	0.333	0.591	0.769	0.873	0.930	0.961		1.000
	B	16.04	15.80	15.29	14.71	14.17	13.70	10.72	9.00
	C	16.80	16.57	16.06	15.47	14.91	14.42	11.52	10.00
20	A	0.333	0.604	0.786	0.889	0.942	0.970		1.000
	B	17.91	17.63	17.03	16.35	15.73	15.18	11.90	10.00
	C	18.69	18.41	17.80	17.11	16.47	15.91	12.67	11.00

Key: $A = P\{CS | (R^*, S^*, T^*)\}$, $B = E\{N | (R^*, S^*, T^*)\}$, $C = E\{N | (R^*, S_C, T^*)\}$

Table 5.3

$P\{CS\}$ and $E\{N\}$ of (R^*, S^*, T^*) and (R^*, S_C, T^*)

for $k = 4, s = 1$ when $P[4] = \frac{\theta}{\theta+3}$, $P[3] = P[1] = \frac{1}{\theta+3}$,

as a function of n and selected θ

n	Key	θ							
		1.0	1.4	1.8	2.2	2.6	3.0	10.0	∞
2	A	0.250	0.318	0.375	0.423	0.464	0.500	0.769	1.000
	B	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	C	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
4	A	0.250	0.356	0.447	0.523	0.586	0.639	0.930	1.000
	B	3.13	3.11	3.08	3.03	2.99	2.94	2.48	2.00
	C	3.94	3.93	3.92	3.90	3.88	3.86	3.54	3.00
6	A	0.250	0.376	0.485	0.575	0.649	0.708	0.973	1.000
	B	5.09	5.06	5.00	4.93	4.84	4.75	3.82	3.00
	C	5.69	5.67	5.64	5.59	5.53	5.48	4.81	4.00
8	A	0.250	0.398	0.526	0.628	0.708	0.770	0.990	1.000
	B	6.91	6.88	6.79	6.68	6.56	6.43	5.11	4.00
	C	7.63	7.60	7.52	7.41	7.29	7.18	6.03	5.00
10	A	0.250	0.415	0.557	0.669	0.753	0.815	0.996	1.000
	B	8.82	8.77	8.64	8.47	8.29	8.10	6.37	5.00
	C	9.50	9.46	9.34	9.18	9.00	8.83	7.23	6.00
12	A	0.250	0.433	0.588	0.706	0.791	0.851	0.998	1.000
	B	10.69	10.62	10.45	10.22	9.98	9.74	7.62	6.00
	C	11.43	11.36	11.19	10.97	10.72	10.48	8.43	7.00
14	A	0.250	0.448	0.614	0.737	0.822	0.879	0.999	1.000
	B	12.60	12.51	12.28	11.98	11.67	11.36	8.86	7.00
	C	13.33	13.24	13.02	12.73	12.41	12.10	9.62	8.00
16	A	0.250	0.462	0.639	0.765	0.848	0.902		1.000
	B	14.49	14.38	14.10	13.73	13.34	12.97	10.09	8.00
	C	15.25	15.14	14.86	14.49	14.10	13.72	10.82	9.00
18	A	0.250	0.476	0.661	0.789	0.870	0.921		1.000
	B	16.41	16.27	15.92	15.47	15.01	14.57	11.31	9.00
	C	17.17	17.03	16.69	16.24	15.77	15.32	12.01	10.00
20	A	0.250	0.489	0.681	0.810	0.899	0.936		1.000
	B	18.32	18.16	17.74	17.21	16.66	16.15	12.53	10.00
	C	19.10	18.94	18.52	17.99	17.43	16.91	13.21	11.00

Key: $A = P\{CS | (R^*, S^*, T^*)\}$, $B = E\{N | (R^*, S^*, T^*)\}$, $C = E\{N | (R^*, S_C, T^*)\}$

Table 5.4

$P\{CS\}$ and $E\{N\}$ of (R^*, S^*, T^*) and (R^*, S_C, T^*)

for $k = 5, s = 1$ when $P[5] = \frac{\theta}{\theta+4}$, $P[4] = P[1] = \frac{1}{\theta+4}$,

as a function of n and selected θ

n	Key	θ							
		1.0	1.4	1.8	2.2	2.6	3.0	10.0	∞
2	A	0.200	0.259	0.310	0.355	0.394	0.429	0.714	1.000
	B	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	C	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
4	A	0.200	0.289	0.370	0.442	0.504	0.559	0.902	1.000
	B	3.28	3.27	3.24	3.20	3.16	3.12	2.61	2.00
	C	3.96	3.96	3.95	3.94	3.92	3.91	3.63	3.00
6	A	0.200	0.308	0.407	0.493	0.567	0.630	0.957	1.000
	B	5.15	5.14	5.10	5.05	4.99	4.75	4.02	3.00
	C	5.77	5.76	5.73	5.69	5.64	5.60	4.95	4.00
8	A	0.200	0.326	0.443	0.543	0.627	0.695	0.982	1.000
	B	7.10	7.07	7.00	6.91	6.80	6.69	5.37	4.00
	C	7.72	7.70	7.65	7.57	7.48	7.38	6.23	5.00
10	A	0.200	0.342	0.473	0.584	0.674	0.745	0.992	1.000
	B	8.96	8.93	8.84	8.71	8.56	8.41	6.69	5.00
	C	9.62	9.59	9.51	9.40	9.26	9.11	7.49	6.00
12	A	0.200	0.356	0.501	0.621	0.715	0.786	0.997	1.000
	B	10.89	10.85	10.72	10.54	10.34	10.12	7.98	6.00
	C	11.56	11.52	11.40	11.24	11.04	10.84	8.73	7.00
14	A	0.200	0.370	0.527	0.654	0.751	0.821	0.999	1.000
	B	12.79	12.73	12.57	12.34	12.08	11.81	9.26	7.00
	C	13.49	13.44	13.28	13.06	12.81	12.55	9.97	8.00
16	A	0.200	0.383	0.551	0.684	0.781	0.850		1.000
	B	14.72	14.64	14.44	14.15	13.82	13.50	10.52	8.00
	C	15.42	15.35	15.16	14.88	14.57	14.24	11.21	9.00
18	A	0.200	0.395	0.573	0.710	0.808	0.874		1.000
	B	16.64	16.55	16.30	15.94	15.56	15.16	11.79	9.00
	C	17.37	17.28	17.04	16.70	16.31	15.92	12.46	10.00
20	A	0.200	0.406	0.593	0.734	0.831	0.893		1.000
	B	18.57	18.46	18.16	17.74	17.28	16.82	13.04	10.00
	C	19.30	19.20	18.92	18.51	18.05	17.59	13.70	11.00

Key: A = $P\{CS | (R^*, S^*, T^*)\}$, B = $E\{N | (R^*, S^*, T^*)\}$, C = $E\{N | (R^*, S_C, T^*)\}$

Table 5.5

$P\{CS\}$ and $E\{N\}$ of (R^*, S^*, T^*) and (R^*, S_C, T^*)

for $k = 6$, $s = 1$ when $P[6] = \frac{\theta}{\theta+5}$, $P[5] = P[1] = \frac{1}{\theta+5}$,

as a function of n and selected θ

n	Key	θ							
		1.0	1.4	1.8	2.2	2.6	3.0	10.0	∞
2	A	0.167	0.219	0.265	0.306	0.342	0.375	0.667	1.000
	B	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	C	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
4	A	0.167	0.243	0.315	0.380	0.439	0.492	0.871	1.000
	B	3.39	3.38	3.36	3.32	3.29	3.25	2.73	2.00
	C	3.97	3.97	3.97	3.96	3.95	3.94	3.70	3.00
6	A	0.167	0.261	0.350	0.431	0.502	0.565	0.938	1.000
	B	5.20	5.19	5.16	5.13	5.08	5.03	4.20	3.00
	C	5.82	5.81	5.79	5.76	5.73	5.68	5.07	4.00
8	A	0.167	0.275	0.380	0.475	0.558	0.628	0.972	1.000
	B	7.20	7.18	7.13	7.06	6.97	6.87	5.60	4.00
	C	7.76	7.75	7.71	7.66	7.59	7.51	6.41	5.00
10	A	0.167	0.290	0.409	0.516	0.607	0.681	0.987	1.000
	B	9.07	9.04	8.97	8.87	8.75	8.62	6.95	5.00
	C	9.71	9.69	9.63	9.54	9.43	9.31	7.71	6.00
12	A	0.167	0.302	0.434	0.551	0.648	0.725	0.994	1.000
	B	11.00	10.96	10.87	10.73	10.57	10.39	8.28	6.00
	C	11.64	11.61	11.53	11.41	11.25	11.08	9.00	7.00
14	A	0.167	0.314	0.459	0.584	0.685	0.763	0.997	1.000
	B	12.93	12.89	12.76	12.58	12.36	12.13	9.59	7.00
	C	13.60	13.56	13.45	13.28	13.08	12.85	10.29	8.00
16	A	0.167	0.325	0.481	0.613	0.718	0.796	0.999	1.000
	B	14.85	14.80	14.64	14.42	14.15	13.87	10.89	8.00
	C	15.54	15.49	15.35	15.14	14.88	14.60	11.57	9.00
18	A	0.167	0.336	0.502	0.641	0.747	0.824	0.999	1.000
	B	16.79	16.73	16.54	16.26	15.94	15.59	12.19	9.00
	C	17.49	17.43	17.26	17.00	16.68	16.34	12.85	10.00
20	A	0.167	0.346	0.523	0.666	0.773	0.848		1.000
	B	18.73	18.66	18.43	18.10	17.71	17.31	13.48	10.00
	C	19.44	19.37	19.17	18.85	18.47	18.07	14.13	11.00

Key: $A = P\{CS | (R^*, S^*, T^*)\}$, $B = E\{N | (R^*, S^*, T^*)\}$, $C = E\{N | (R^*, S_C, T^*)\}$

Tables A-2, A-3 and A-4 of B-E-M and Table H-1 of G-O-S. Our tables for $k = 2(1)6$; $\theta = 1.0(0.4)3.0, 10.0, \infty$ and $n = 2(2)20$, permit a comparison, over a broad range, of the $E\{N\}$ -values of (R^*, S^*, T^*) with those of (R^*, S_C, T^*) and the single-stage n . It can be seen that $E\{N|(R^*, S^*, T^*)\} < E\{N|(R^*, S_C, T^*)\} < n$ (for $n \geq 2$), and that the absolute savings using either (R^*, S^*, T^*) or (R^*, S_C, T^*) instead of the single-stage procedure of B-E-M increase with n for fixed k and θ , and with θ for fixed k and n . In fact, for fixed k and n we have for $\theta \rightarrow \infty$ that $E\{N|(R^*, S^*, T^*)\} \rightarrow n/2 ((n+1)/2)$ for n even (odd) while $E\{N|(R^*, S_C, T^*)\} \rightarrow (n+2)/2 ((n+1)/2)$ for n even (odd).

6. Concluding remarks

We have demonstrated conclusively that our proposed sequential sampling procedure (R^*, S^*, T^*) yields substantial savings in expected number of vector-observations ($E\{N\}$) over the n required by the single-stage sampling procedure (R_{SS}, T_{SS}) of Bechhofer, Elmaghraby and Morse [1959], when both achieve the same $P\{CS\}$, and that these savings increase dramatically as $P[k]$ approaches unity. Also, procedure (R^*, S^*, T^*) is uniformly (in p, k , and $n \geq 2$) superior in terms of $E\{N\}$ to the sequential sampling procedure (R^*, S_C, T^*) of Gibbons, Olkin and Sobel [1977] which involves curtailment, although the savings here are modest. Finally we point out that (R^*, S^*, T^*) can handle very general goals, is easy to apply, and requires no special tables for its implementation.

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Appendix

A. Derivation of MF-configuration for Goal I

In this appendix we prove (4.5) for Procedure (R_{SS}, T_{SS}) of Section 2.1; as a consequence of Theorem 3.1 the result also holds for (R^*, S^*, T^*) . We use the notation of Section 2 except that we denote $p_{[i]}$ by p_i and $y_{i,n}$ by y_i ($1 \leq i \leq k$). We assume that the s best populations are uniquely defined. Here we let $A_1, A_2 \subset A = \{1, 2, \dots, k\}$ denote two disjoint sets of order s and $k-s$, respectively, such that $p_i > p_j$ for all $i \in A_1$ and for all $j \in A_2$. We summarize our result in the following theorem.

Theorem A.1: For (R_{SS}, T_{SS}) applied to Goal I we have that $\max_p P\{CS\}$ occurs for all $k \geq 3$, $k-1 \geq s \geq 2$, $n \geq 1$ when $p_1 = p_{k-s} = 0$, $p_{k-s+1} = p_k = 1/s$.

The proof of Theorem A.1 follows Lemmas 1 and 2, below. Suppose that the y_i ($1 \leq i \leq k$) are known. If $y_i > y_j$ for some $i \in A_2$, $j \in A_1$, then the conditional $P\{CS\} = 0$. Otherwise there exists a partition of $\{1, 2, \dots, k\}$ into B_1, B_2, B_3 where $B_1 \subseteq A_1$, $B_3 \subseteq A_2$, $B_2 \cap A_1 \neq \emptyset$, $B_2 \cap A_2 \neq \emptyset$, $y_i = y^* \forall i \in B_2$, $y_i > y^* > y_j \forall i \in B_1, j \in B_3$. Then with $m_1 = \#(B_2 \cap A_1)$, $m_2 = \#(B_2 \cap A_2)$ we have that the conditional $P\{CS\}$ equals $\{(\frac{m_1+m_2}{m_1})\}^{-1}$.

Lemma 1: Suppose that we fix p_i for $i \neq i_1$ and $i \neq i_2$ leaving $p_{i_1} + p_{i_2} = \lambda$ where $i_1 \in A_1$, $i_2 \in A_2$. Then as p_{i_1}, p_{i_2} vary within this setup, the $P\{CS\}$ is maximized when $p_{i_1} = \lambda$, $p_{i_2} = 0$.

Proof of Lemma 1: Setting $x = y_{i_1} + y_{i_2}$, the joint distribution of $Z = \{y_i (i \neq i_1 \text{ or } i_2), x\}$ is determined. The conditional distribution of $\{y_i\}$ given Z depends on how the x observations on π_{i_1} and π_{i_2} are divided. This conditional distribution for y_{i_1} and y_{i_2} is binomial with

$$P\{y_{i_1} = r\} = \binom{x}{r} \left(\frac{p_{i_1}}{\lambda}\right)^r \left(\frac{p_{i_2}}{\lambda}\right)^{x-r} \quad 0 \leq r \leq x. \quad \text{The conditional}$$

$$P\{CS\} = \left\{ \binom{m_1+m_2}{m_1} \right\}^{-1}, \text{ given all } y_i, \text{ is greatest if } P\{y_{i_1} = x\} = 1 \text{ which}$$

occurs when $p_{i_1} = \lambda, p_{i_2} = 0$. This proves Lemma 1.

By repeated applications of Lemma 1, the $P\{CS\}$ is largest for some configuration where $p_i = 0$ for all $i \in A_2$.

Lemma 2: Suppose that $p_i = 0 \forall i \in A_2$. Fix p_i for $i \neq i_1$ and $i \neq i_2$ leaving $p_{i_1} + p_{i_2} = \lambda$ where now $i_1, i_2 \in A_1$. Then as p_{i_1}, p_{i_2} vary within this setup, the $P\{CS\}$ is maximized when $p_{i_1} = p_{i_2} = \lambda/2$.

Proof of Lemma 2: Setting $x = y_{i_1} + y_{i_2}$, the joint distribution of $Z = \{y_i (i \neq i_1 \text{ or } i_2), x\}$ is determined. The conditional distribution of $\{y_i\}$ given Z depends on how the x observations on π_{i_1} and π_{i_2} are divided. Here again we have a binomial distribution. Given Z there are:

k -s populations in A_2 with 0 occurrences

m (say) populations in A_1 with 0 occurrences

$w = 0, 1$ or 2 out of π_{i_1}, π_{i_2} with 0 occurrences.

The conditional $P\{CS\}$ given w is $\left\{ \binom{m+w}{m} \right\}^{-1}$ which decreases as w increases. If $x = 0$ or 1 , then w is fixed at $2, 1$, respectively. For $x > 1$, we have w equal to 0 or 1 , and we must maximize $P\{W = 1\}$. But

$$P\{\text{One of } y_{i_1}, y_{i_2} \text{ is } 0 \mid x\} = \left(\frac{p_{i_1}}{\lambda}\right)^x + \left(\frac{p_{i_2}}{\lambda}\right)^x$$

which is smallest when $p_{i_1} = p_{i_2} = \lambda/2$. This proves Lemma 2.

By repeated applications of Lemma 2, starting with any $\underline{p}$ with $p_i = 0 \forall i \in A_2$, we can increase the $P\{CS\}$ by replacing both of p_{i_1}, p_{i_2} by $(p_{i_1} + p_{i_2})/2$, $i_1, i_2 \in A_1$, and $\underline{p} \rightarrow \underline{p}^* = (0, 0, \dots, 0, 1/s, \dots, 1/s)$. By continuity, $P\{CS\}$ for $\underline{p}^*$ is greater than or equal to the $P\{CS\}$ for the original $\underline{p}$, and hence $\underline{p}^*$ has the largest $P\{CS\}$ for any $\underline{p}$ with $p_i = 0 \forall i \in A_2$. This result along with Lemma 1 proves Theorem 1.

B. Derivation of an algorithm to compute the $P\{CS\}$ and $E\{N\}$ of (R^*, S^*, T^*) and (R^*, S_C^*, T^*) for Goal I with $s = 1$.

We restrict attention to the so-called slippage configuration of the p_i ($1 \leq i \leq k$), i.e., for arbitrary $\theta \geq 1$ we consider $\underline{p}$ for which

$$p_{[1]} = \dots = p_{[k-1]} = \frac{1}{\theta+k-1}, \quad p_{[k]} = \frac{\theta}{\theta+k-1}. \quad (B.1)$$

For simplicity of notation let $p = 1/(\theta+k-1)$ and $p' = \theta/(\theta+k-1)$. We shall give formulae to compute $P\{CS \mid (R^*, S^*, T^*)\} = P\{CS \mid (R^*, S_C^*, T^*)\}$ when (B.1) holds. In our development we follow the methods of Sobel, Uppuluri and Frankowski [1977], Section 4.4, i.e., let

$$j_{1/k}^{(k,j)}(x,n) = P\{j \text{ specific multinomial cells have frequency } x, \\ \text{and the remaining } k-j \text{ cells have frequency } < x \\ \text{and } \sum_{i=1}^k x_i = n\}$$

In particular we shall denote $J_{1/k}^{(k,0)}(x,n)$ by $J_{1/k}^k(x,n)$. From the definition of $J_{1/k}^k(x,n)$ it is clear that

$$J_{1/k}^k(n+1,n) = 1 \quad (\text{B.2})$$

and

$$J_{1/k}([n/k],n) = 0 \quad (\text{B.3})$$

where $[n/k]$ is the greatest integer less than or equal to n/k .

Computation of $J_{1/k}^k(x,n)$.

For $0 \leq i < x$ and $jx + i = n$ we have

$$J_{1/k}^{(k,j)}(x, jx+i) = \frac{(jx)!}{(x!)^j} \binom{n}{jx} \left(\frac{1}{k}\right)^{jx} \left(1 - \frac{j}{k}\right)^{n-jx} \quad (\text{B.4})$$

$$= \frac{(jx+i)!}{(x!)^j (i)!} \left(\frac{1}{k}\right)^{jx} \left(1 - \frac{j}{k}\right)^i.$$

A recursion relation for J is given by

$$\begin{aligned} J_{1/k}^{(k,j)}(x,n) &= \frac{n(1-j/k)}{n-jx} J_{1/k}^{(k,j)}(x,n-1) \\ &\quad - \frac{(k-j)x}{n-jx} J_{1/k}^{(k,j+1)}(x,n). \end{aligned} \quad (\text{B.5})$$

Equation (B.5) was obtained as in S-U-F. (See their equation (2.9).)

B.1 Computation of $P\{CS | (R^*, S^*, T^*)\} = P\{CS | (R_{SS}, T_{SS})\}$ for $k \geq 2$

Let $X_{(i)}$ denote the number of occurrences of the event associated with $p_{[i]}$ ($1 \leq i \leq k$) in n independent multinomial vector-observations. Then,

$$\begin{aligned} P\{CS\} &= P\{X_{(k)} > X_{(j)} \quad j \neq k\} \\ &+ \frac{1}{2} \sum_{i=1}^{k-1} P\{X_{(k)} = X_{(i)}, X_{(k)} > X_{(j)} \quad j \neq i, k\} \\ &+ \dots \\ &+ \frac{1}{k} P\{X_{(1)} = X_{(2)} = \dots = X_{(k)}\}. \end{aligned}$$

It can be shown that

$$P\{CS\} = \sum_{x=0}^n B_{p'}(n, x) \sum_{t=1}^k \frac{\binom{k-1}{t-1}}{t} J_{1/k-1}^{(k-1, t-1)}(x, n-x)$$

where $B_{p'}(n, x) = \binom{n}{x} p'^x (1-p')^{n-x}$, i.e.,

$$\begin{aligned} P\{CS\} &= \sum_{x=0}^n B_{p'}(n, x) \sum_{t=1}^k \frac{k(k-1)!}{t!(k-t)!k} J_{1/k-1}^{(k-1, t-1)}(x, n-x) \\ &= \sum_{x=0}^n \frac{n!}{x!(n-x)!} p'^x (1-p')^{n-x} \sum_{t=1}^k \frac{\binom{k}{t}}{k} J_{1/k-1}^{(k-1, t-1)}(x, n-x). \end{aligned}$$

But

$$\begin{aligned}
 J_{1/k-1}^{(k-1,t-1)}(x,n-x) &= \sum_{\substack{x_{t+1}, \dots, x_k: \\ x_{t+1} + \dots + x_k + (t-1)x = n-x}} \frac{(n-x)!}{(x!)^{t-1} x_{t+1}! \dots x_k!} \left(\frac{1}{k-1}\right)^{n-x} \\
 &= \frac{(n-x)! x!}{n!} \sum \frac{n!}{(x!)^t x_{t+1}! \dots x_k!} \left(\frac{1}{k}\right)^n \frac{k^n}{(k-1)^{n-x}} \\
 &= \frac{(n-x)! x!}{n!} \frac{k^n}{(k-1)^{n-x}} J_{1/k}^{(k,t)}(x,n) .
 \end{aligned}$$

Thus

$$\begin{aligned}
 P\{CS\} &= \sum_{x=0}^n \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \sum_{t=1}^k \frac{\binom{k}{t}}{k} \frac{(n-x)! x!}{n!} \frac{k^n}{(k-1)^{n-x}} J_{1/k}^{(k,t)}(x,n) \\
 &= \sum_{x=0}^n p^x (p)^{n-x} (k-1)^{n-x} \frac{k^{n-1}}{(k-1)^{n-x}} \sum_{t=1}^k \binom{k}{t} J_{1/k}^{(k,t)}(x,n) \\
 &= \sum_{x=0}^n p^x p^{n-x} k^{n-1} \sum_{t=1}^k \binom{k}{t} J_{1/k}^{(k,t)}(x,n) .
 \end{aligned}$$

But

$$\sum_{t=1}^k \binom{k}{t} J_{1/k}^{(k,t)}(x,n) = P\{\text{maximum cell frequency} = x\} .$$

Hence

$$\begin{aligned}
 P\{CS\} &= \sum_{x=0}^n p^x p^{n-x} k^{n-1} [J_{1/k}^k(x+1,n) - J_{1/k}^k(x,n)] \\
 &= \sum_{x=\lfloor \frac{n}{k} \rfloor}^n p^x p^{n-x} k^{n-1} [J_{1/k}^k(x+1,n) - J_{1/k}^k(x,n)] ,
 \end{aligned}$$

and thus

$$P\{CS\} = \frac{k^{n-1}}{(\theta+k-1)^n} \sum_{x=\lfloor \frac{n}{k} \rfloor}^n \theta^x [J_{1/k}^k(x+1, n) - J_{1/k}^k(x, n)]. \quad (B.6)$$

Equation (B.6) can be used to compute the $P\{CS\}$ for $s = 1$ in the slippage configuration.

B.2 Computation of $E\{N | (R^*, S^*, T^*)\}$ for $k \geq 3$

Let N be the random number of vector-observations required until termination. Then for (R^*, S^*, T^*) we have

$$P\{N \leq m\} = P\{Z_{im} \geq Z_{jm} + (n-m) \text{ for all } j \neq i, \text{ for some } i\}.$$

We can write

$$P\{N \leq m\} = P\{N \leq m, CS\} + P\{N \leq m, ICS\}$$

where ICS denotes incorrect selection. Let $Z_{(i)m}$ denote the number of occurrences of the event associated with $p_{[i]}$ ($1 \leq i \leq k$) at stage m . We first consider

$$\begin{aligned} P\{N \leq m, CS\} &= P\{Z_{(k)m} \geq Z_{(i)m} + (n-m) \forall i \neq k\} \\ &= \sum_y P\{Z_{(k)m} = y, Z_{(i)m} \leq y + m - n \forall i \neq k\} \\ &= \sum_y \sum_{\substack{y_i < y + m - n + 1, \\ \sum_{i \neq k} y_i = m - y}} \frac{m!}{y! \prod_{i \neq k} y_i!} p^y p^{m-y} \\ &= \sum_y \frac{m!}{y! (m-y)!} p^y (1-p')^{m-y} \sum_{\substack{y_i: y_i < y + m - n + 1, \\ \sum_{i \neq k} y_i = m - y}} \frac{(m-y)!}{\prod_{i \neq k} y_i!} \left(\frac{p}{1-p'}\right)^{m-y} \\ &= \sum_y \binom{m}{y} p^y (1-p')^{m-y} J_{1/k-1}^{k-1}(y + m - n + 1, m - y) \text{ since } \frac{p}{1-p'} = \frac{1}{k-1}. \end{aligned}$$

Thus

$$\begin{aligned}
 P\{N \leq m, CS\} &= \sum_y B_{p'}(m, y) J_{1/k-1}^{k-1}(y+m-n+1, m-y) \\
 &= \sum_{y=n-m}^m B_{p'}(m, y) J_{1/k-1}^{k-1}(y+m-n+1, m-y) .
 \end{aligned} \tag{B.7}$$

$$\begin{aligned}
 P\{N \leq m, \text{select event } E_{(i)}\} &= \sum_y P\{Z_{(i)m} = y, Z_{(j)m} < y+m-n+1 \forall j \neq i\} \\
 &= \sum_y \sum_{\substack{y_j < y+m-n+1 \\ \sum_{j \neq i} y_j = m-y}} \frac{m!}{y! \prod y_j!} p^{y_{p'}} p_k^{y_k} p^{m-y-y_k} \\
 &= \sum_y B_p(m, y) \sum_{\substack{y_j < y+m-n+1 \\ \sum_{j \neq i} y_j = m-y}} \frac{(m-y)!}{\prod y_j!} p_1^{y_k} p_2^{m-y-y_k}
 \end{aligned} \tag{B.8}$$

where $p_1 = \frac{p'}{1-p}$ and $p_2 = \frac{p}{1-p}$.

Note that $(k-2)p_2 + p_1 = 1$. Now,

$$\sum_{\substack{y_j < y+m-n+1 \\ \sum_{j \neq i} y_j = m-y}} \frac{(m-y)!}{\prod y_j!} p_1^{y_k} p_2^{m-y-y_k} = J_{(p_1, \underline{p_2})}^{(k-1)}(y+m-n+1, m-y)$$

where $(p_1, \underline{p_2})$ represents the fact that the multinomial probabilities are $(p_1, p_2, \dots, p_2)$ where in the $(k-1)$ -vector, p_2 is repeated $(k-2)$ times. For any (r, s) ,

$$\begin{aligned}
J_{(p_1, \underline{p_2})}^{(k-1)}(r, s) &= \sum_{\substack{y_i < r \\ \sum y_i = s}} \frac{s!}{y_1! \dots y_{k-1}!} p_1^{y_1} p_2^{s-y_1} \\
&= \sum_{y_1=0}^{\min(r-1, s)} \frac{s!}{y_1! (s-y_1)!} p_1^{y_1} (1-p_1)^{s-y_1} \sum_{\substack{y_i < r \\ y_2 + \dots + y_{k-1} = s-y_1}} \frac{(s-y_1)!}{y_2! \dots y_{k-1}!} \left(\frac{p_2}{1-p_1}\right)^{s-y_1} \\
&= \sum_{y_1=0}^{\min(r-1, s)} B_{p_1}(s, y_1) J_{1/k-2}^{k-2}(r, s-y)
\end{aligned}$$

since $\frac{p_2}{1-p_1} = \frac{1}{k-2}$. Thus,

$$J_{(p_1, \underline{p_2})}^{(k-1)}(y+m-n+1, m-y) = \sum_{y_1=0}^{\min(y+m-n, m-y)} B_{p_1}(m-y, y_1) J_{1/k-2}^{k-2}(y+m-n+1, m-y-y_1) .$$

Hence

$$\begin{aligned}
P\{N \leq m, \text{ICS}\} &= (k-1) \sum_{y=n-m}^m B_p(m, y) J_{(p_1, \underline{p_2})}^{(k-1)}(y+m-n+1, m-y) \\
&= (k-1) \sum_{y=n-m}^m B_p(m, y) \sum_{y_1=0}^{\min(y+m-n, m-y)} B_{p_1}(m-y, y_1) J_{1/k-2}^{k-2}(y+m-n+1, m-y-y_1) .
\end{aligned} \tag{B.9}$$

Using (B.7) and (B.9) we have

$$\begin{aligned}
 E\{N\} &= n - \sum_{m=0}^{n-1} P\{N \leq m\} \\
 &= n - \sum_{m=\lceil n/2 \rceil}^{n-1} P\{N \leq m\} \quad \text{where } \lceil x \rceil = \text{smallest integer } \geq x. \\
 &= n - \sum_{m=\lceil n/2 \rceil}^{n-1} [P\{N \leq m, CS\} + P\{N \leq m, ICS\}] \tag{B.10} \\
 &= n - \sum_{m=\lceil n/2 \rceil}^{n-1} \left[\sum_{y=n-m}^m B_p(m, y) J_{1/k-1}^{k-1}(y+m-n+1, m-y) \right. \\
 &\quad \left. + (k-1) \sum_{y=n-m}^m B_p(m, y) \sum_{y_1=0}^{\min(y+m-n, m-y)} B_{p_1}(m-y, y_1) J_{1/k-2}^{k-2}(y+m-n+1, m-y-y_1) \right].
 \end{aligned}$$

Using the recursion equations for J , one can compute $E\{N\}$ in the configuration (B.1).

B.3 Computation of $E\{N|(R^*, S_C, T^*)\}$ for $k \geq 3$

By a similar development we can show that the expected value of N using (R^*, S_C, T^*) is given by

$$\begin{aligned}
 E\{N\} = n - \sum_{m=\lceil (n+1)/2 \rceil}^{n-1} \left[\sum_{y=n-m+1}^n B_p(m, y) J_{1/k-1}^{k-1}(y+m-n, m-y) \right. \\
 \left. + (k-1) \sum_{y=n-m+1}^m B_p(m, y) \sum_{y_1=0}^{\min(y+m-n-1, m-y)} B_p(m-y, y_1) J_{1/k-2}^{k-2}(y+m-n, m-y-y_1) \right].
 \end{aligned}
 \tag{B.11}$$

We have used equations (B.6), (B.10) and (B.11) to compute the entries in Tables 5.1-5.5.

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