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ABSTRACT

For every integer x , construct a stationary continuous-time Markov process $\gamma(x;t)$, with state space $\{-1,+1\}$ (all processes independent, and having the same distributions). Consider a particle moving at unit speed along the real line, with its direction completely determined by the γ 's, as follows: if S_t is its position at time t , and $X_n = S_n - S_{n-1}$, then $S_0 = 0$ and $X_{i+1} = \gamma(S_i;i)$ for $i = 0,1,2,\dots$. The increments X_n are not stationary, nor is S_n Markov, yet we show that this process has much in common with the classical random walk, including: zero-or-one laws; equivalent characterizations of recurrence; a strong law of large numbers; and a central limit theorem. We prove that the process is recurrent if and only if $p = 1/2$, where $p = P\{\gamma(0;0)=+1\}$. We also show that $\lim_{n \rightarrow \infty} (S_n/n)$ is sometimes different from $2p-1$. Coupling and graphical methods are used extensively.

1. Introduction and summary of results. Let $\{y(t), t \geq 0\}$ be a stationary two-state continuous time Markov process on the state space $\{-1, +1\}$, with transition matrix

$$(1.1) \quad \begin{bmatrix} P_{-1,-1}(t) & P_{-1,+1}(t) \\ P_{+1,-1}(t) & P_{+1,+1}(t) \end{bmatrix} = \begin{bmatrix} q + p r^t & p - p r^t \\ q - q r^t & p + q r^t \end{bmatrix}$$

where

$$(1.2) \quad p = \beta/(\beta+\delta), \quad q = \delta/(\beta+\delta), \quad \text{and} \quad r = e^{-(\beta+\delta)}.$$

Also, stationarity requires that $\{y(0)=+1\}$ has probability p . On a probability space $(\Omega, \mathcal{F}, P)$, construct a family of independent copies of $y(t)$, indexed by $x \in \mathbb{Z}$; call these processes $y(x;t)$.

Now introduce a particle moving along the real line at unit speed, starting at time $t=0$, from position $x=0$. Its speed remains constant, but its direction is determined by the $y(x;t)$ processes, as follows: if $y(0;0) = +1$, then the particle begins by moving towards $+1$; if $y(0;0) = -1$, then it moves towards -1 . In general, if the particle is at position x' ($\in \mathbb{Z}$) at time t (t an integer), then it moves towards $x' + y(x';t)$ (changing its direction at time t , if necessary).

Explicitly, let S_t be the position of the particle at time $t \geq 0$; let $X_n = S_n - S_{n-1}$. We define

$$(1.3) \quad \begin{aligned} S_0 &= 0 \\ X_{i+1} &= y(S_i; i), \quad i = 0, 1, 2, \dots \\ S_t &= S_i + (t-i)X_{i+1}, \quad i < t \leq i+1. \end{aligned}$$

We define a walk to be a finite sequence of integers $W = (w_0, w_1, \dots, w_n)$ such that $w_0 = 0$ and $|w_i - w_{i-1}| = 1$ for $i=1, 2, \dots, n$. W will also denote the event $\{S_0 = w_0, \dots, S_n = w_n\}$. If $V = (v_0, \dots, v_m)$ is a second walk, we define a third walk $V*W = (v_0, \dots, v_m, v_m + w_1, \dots, v_m + w_n)$.

Let $\Delta w_i = w_i - w_{i-1}$. The probability of $W = (w_0, \dots, w_n)$ is computed as follows:

$$\begin{aligned} (1.4) \quad P(W) &= \prod_{i=1}^n P(X_i = \Delta w_i \mid X_j = \Delta w_j, \quad j=1, \dots, i-1) \\ &= \prod_{i=1}^n P(\gamma(w_{i-1}; i-1) = \Delta w_i \mid \gamma(w_{i-1}; T(i-1, W)) = \\ &\quad \Delta w_{T(i-1, W)+1}) \end{aligned}$$

where $T(k, W) = \sup\{j: j < k, w_j = w_k\}$ (using the conventions that the sup of the empty set is $-\infty$, and $\gamma(x; -\infty) = +1$). (1.4) may be evaluated using (1.1); for example,

$$\begin{aligned} P((0, 1, 0, 1, 2)) &= P_{+1, +1}^{(+\infty)} P_{+1, -1}^{(+\infty)} P_{+1, +1}^{(2)} P_{-1, +1}^{(2)} \\ &= p \, q \, (p + p \, r^2) (p - q \, r^2) . \end{aligned}$$

The process $\{S_n, n=0, 1, 2, \dots\}$ is obviously not Markovian, nor is it stationary. However, we will show that it has properties similar to those of classical random walks. A key tool is the following lemma:

LEMMA 1.1. There exist constants L and U ($0 < L < 1 < U < \infty$), depending only on β and δ , such that for any walks V and W we have

$$L P(V) P(W) \leq P(V*W) \leq U P(V) P(W) .$$

(Note that in the classical random walk, $L=U=1$.)

We use the notation $\sigma(C)$ to denote the sigma-algebra generated by C , where C is either a collection of sets or a collection of random variables. We define the following sub-sigma-algebras of F :

$$\begin{aligned}
 (1.5) \quad F_n &= \sigma(\{X_1, \dots, X_n\}) \\
 F_\infty &= \sigma(\bigcup_{n=1}^{\infty} F_n) \\
 F_n^+ &= \sigma(\{X_k : k > n\}) \\
 F^+ &= \bigcap_{n=1}^{\infty} F_n^+
 \end{aligned}$$

Define a "shift operator" $\theta: F_\infty \rightarrow F_\infty$ which satisfies $\theta\{X_n = +1\} = \{X_{n+1} = +1\}$ for $n=1, 2, \dots$, and preserves complements and countable intersections. This enables us to formulate a zero-or-one law:

THEOREM 1.2. If $A \in F_\infty$ and $A = \theta A$, then $P(A)$ is either zero or one.

We call the process S_n "recurrent" if $P\{S_n = 0 \text{ for some } n > 0\} = 1$; otherwise, we call it "transient". The borrowing of this Markov chain terminology is justified by the next theorem. (The notation $\{A_n \text{ i.o.}[n]\}$ denotes $\{A_n \text{ occurs infinitely often}\}$.)

THEOREM 1.3. The following are equivalent:

- (a) S_n is recurrent;
- (b) $P\{S_n = x \text{ i.o.}[n]\} > 0$ for at least one x ;
- (c) $P\{\text{for every } x \in \mathbb{Z}: S_n = x \text{ i.o.}[n]\} = 1$;
- (d) $P\{\text{for at least one } x \in \mathbb{Z}: S_n = x \text{ i.o.}[n]\} = 1$;

$$(e) \quad E(\text{cardinality}\{n: S_n = 0\}) = \infty.$$

We remark here that the process may be easily generalized so that S_t takes values in R^v , always moving parallel to one of the coordinate axes, and changing directions only at points $x \in \mathbb{Z}^v$. In this case, each $y(x;t)$ is a Markov process with $2v$ states. Similarly, we could consider a process with bounded increments X_n . The definitions and theorems presented above continue to hold, and the proofs are essentially unchanged from those given in sections 2 and 3. The remainder of the paper, however, uses techniques which apply only to the one-dimensional processes.

COROLLARY 1.4. Let $D+ = \{\lim S_n = +\infty\}$, $D- = \{\lim S_n = -\infty\}$. Then exactly one of (a), (b), or (c) is true:

- (a) S_n is recurrent;
- (b) $P(D+) = 1$;
- (c) $P(D-) = 1$. Moreover, if $p = 1/2$, then (a) holds.

PROOF: Suppose (a) is false. By (d) of Theorem 1.3, the sequence $\{S_n(\omega), n=1,2,\dots\}$ has no finite limit points, with probability 1. Hence $P(D+ \cup D-) = 1$. So by Theorem 1.2, either (b) or (c) is true. The final statement follows from the symmetry of S_n in the case $p = 1/2$, which implies, a priori, that $P(D+) = P(D-)$.///

The remainder of the paper uses graphical methods which make strong use of the nearest-neighbor characteristic: $|X_n| = 1$. We will use the coupling of an infinite number of particles, one starting at each even integer. $\{S_t^z, t \geq 0\}$ (defined as in (1.3), except

$S_0^z = z$) is the path of the particle starting from $z \in \mathbb{Z}$. Then if $x < y$, we have $S_t^x \leq S_t^y$ for all t . This follows from the observation that paths do not cross one another; if two particles meet, they coalesce. In fact, it is proven that any two particles must meet eventually (Lemma 4.2). A corollary of this is

THEOREM 1.5. Let $A \in \mathcal{F}'$. Then $P(A)$ is either 0 or 1.

Another consequence of Lemma 4.2 is

THEOREM 1.6. Let $D \in \mathcal{F}_\infty$. Then $\lim_{n \rightarrow \infty} P(\theta^n D)$ exists.

Moreover, if we now let $\hat{P}(D) = \lim_{n \rightarrow \infty} P(\theta^n D)$, then $\hat{P}$ is a probability measure on $(\Omega, \mathcal{F}_\infty)$, and θ is measure-preserving and ergodic with respect to $\hat{P}$, so Birkhoff's ergodic theorem is applicable. In terms of P , this yields a strong law of large numbers:

THEOREM 1.7. Let $\hat{p} = \hat{P}\{X_1 = +1\}$. Then $\lim_{n \rightarrow \infty} S_n/n = 2\hat{p}-1$ a.s. [P].

If $\beta = \delta$, then symmetry implies that $P\{X_n = +1\} = 1/2$ for every n , so $\hat{p} = 1/2$. The next theorem proves that S_n is transient when $\beta \neq \delta$.

THEOREM 1.8. If $\beta > \delta$, then $\hat{p} > 1/2$. (Symmetrically, if $\beta < \delta$, then $\hat{p} < 1/2$.)

To describe further how $\hat{p}$ depends on β and δ , let $\lambda = \beta + \delta$; then we can write $\hat{p}$ as a function of p and λ .

THEOREM 1.9. For fixed p ($p \neq 1/2$),

$$(a) \quad \lim_{\lambda \rightarrow 0} \hat{p}(p, \lambda) = 1/2 .$$

$$(b) \quad \lim_{\lambda \rightarrow \infty} \hat{p}(p, \lambda) = p .$$

Intuitively, the case where $\lambda \rightarrow \infty$ (infinite flip rate) gives a process in which the increments X_n are almost independent; and the case $\lambda \rightarrow 0$ (zero flip rate) describes a process in which the environment is frozen, so the particle gets trapped. These ideas are formalized in the proof given in section 6.

The final result that we present is a central limit theorem.

THEOREM 1.10. There exist sequences of constants a_n and b_n such that $(S_n - a_n)/b_n$ converges weakly to the standard normal distribution.

2. Preliminaries and proof of Lemma 1.1. For notational

convenience, we only deal with the case $\beta \geq \delta$. (This is no loss of generality, since $P\{S_n = x | \beta = a, \delta = b\}$ is the same as $P\{S_n = -x | \beta = b, \delta = a\}$.)

For each $x \in Z$, let $\rho^{(x)}$ be a Poisson point process on the line $\{(x, t) : -\infty < t < +\infty\}$ with intensity $\beta + \delta$ (i.e., $\rho^{(x)}$ is a random subset of this line such that $P\{|\rho^{(x)} \cap (\{x\} \times A)| = n\} = e^{-c} c^n / n!$, where A is any Borel subset of R and $c = (\beta + \delta) \times (\text{Lebesgue measure of } A)$, and such that $\{|\rho^{(x)} \cap A_i|, i = 1, \dots, k\}$ are independent if the A_i are disjoint). We also define the random sets $\rho_1^{(x)}$, $\rho_2^{(x)}$, and $\rho_3^{(x)}$ as follows: each point of $\rho^{(x)}$ is independently assigned to exactly one of $\rho_1^{(x)}$, $\rho_2^{(x)}$, or $\rho_3^{(x)}$, with probabilities $\delta/(\beta + \delta)$, $\delta/(\beta + \delta)$, and $(\beta - \delta)/(\beta + \delta)$ respectively. Note that $\rho_1^{(x)}$, $\rho_2^{(x)}$, and $\rho_3^{(x)}$ are independent Poisson processes with intensities δ , δ , and $(\beta - \delta)$ respectively. We construct all of these processes to be independent on a single probability space (Ω, F, P) .

For $t \in R$, $x \in Z$, define

$$y(x; t) = \begin{cases} +1 & \text{if } \sup\{s \leq t : (x, s) \in \rho^{(x)}\} \text{ is a point of } \rho_1^{(x)} \cup \rho_3^{(x)} \\ -1 & \text{otherwise.} \end{cases}$$

Without loss of generality, we may assume:

- (2.1) (i) that this construction has been carried out homogeneously on $Z \times R$, in the sense that for any array $\{a_{ij} : i, j \in Z\}$ with $a_{ij} \in \{-1, +1\}$, there exists an ω in Ω such that $y(i; j) = a_{ij}$ for all $i, j \in Z$;
- (ii) that $\{\omega \in \Omega : |\rho^{(x)} \cap \{(x, t) : -T \leq t \leq T\}| = \infty\} = \emptyset$, for all $T \in R$ and for all $x \in Z$.

It is easy to check that $y(x;t)$ is (for each x) a stationary Markov process with transition probabilities described by (1.1) and (1.2). (Our reasons for introducing three $\rho_i^{(x)}$'s instead of two will be explained in section 5.)

Let $a, b \in \mathbb{R}$ ($a < b$) and $x \in \mathbb{Z}$. Let $A \in \sigma\{y(x;t): t \leq a\}$. Then it is easy to see from the construction that

$$\begin{aligned} (2.2) \quad P\{y(x;b)=+1 \mid A \cap [\rho^{(x)} \cap \{(x;t): a < t < b\} \neq \emptyset]\} &= \beta/(\beta+\delta) \\ &= P\{y(x;b)=+1\}. \end{aligned}$$

To extend (2.2), we consider two functions $f, g: \mathbb{Z} \rightarrow \mathbb{R}^+$ such that $f(x) < g(x)$ for all $x \in \mathbb{Z}$. Define

$$\begin{aligned} H_f &= \sigma\{y(x;t): x \in \mathbb{Z}, t \leq f(x)\} \\ H_g &= \sigma\{y(x;t): x \in \mathbb{Z}, t \geq g(x)\} \\ \mathcal{G}_{g,f} &= \sigma\left\{\bigcup_{x \in \mathbb{Z}} (\rho^{(x)} \cap \{(x,t): f(x) < t < g(x)\})\right\} \\ I(f,g) &= \{\omega \in \Omega: \text{for each } x \in \mathbb{Z}, \rho^{(x)} \cap \{(x,t): f(x) < t < g(x)\} \neq \emptyset\}. \end{aligned}$$

If $C \in \mathcal{G}_{g,f}$ and $C \subset I(f,g)$, then we have, as in (2.2), that the conditional distribution of $\{y(x;g(x)): x \in \mathbb{Z}\}$ given the event AC is simply the distribution of a collection of $\{-1, +1\}$ -valued i.i.d. random variables with $P\{y(x;g(x))=+1\} = \beta/(\beta+\delta)$. But this is just the unconditional distribution of $\{y(x;g(x)): x \in \mathbb{Z}\}$. So, using the Markov property of $y(x; \cdot)$, we have

$$\begin{aligned} (2.3) \quad P\{B \mid AC\} &= P(B), \quad \text{for } A \in H_f, \quad B \in H_g, \text{ and} \\ &\quad C \in \mathcal{G}_{g,f} \text{ with } C \subset I(f,g). \end{aligned}$$

LEMMA 2.1. For $f, g, A,$ and B as above,

$$|P(B \cap A) - P(A)P(B)| \leq 2 P(I(f,g)^C) P(A) .$$

PROOF: $P(B \cap A) = P(B|A \cap I(f,g))P(A \cap I(f,g)) + P(B \cap A \cap I(f,g)^C)$. The lemma now follows from (2.3) (with $C=I(f,g)$), the independence of A and $I(f,g)$, and simple manipulation.///

Let $L = \{(z,t) \in \mathbb{Z} \times \mathbb{N}_0 : z+t \in 2\mathbb{Z}\}$. ($\mathbb{N}_0 = \{0,1,2,\dots\} = \mathbb{N} \cup \{0\}$.) For $(z,T) \in L$, we define $\{S_t^{(z,T)}, t \geq T\}$, the path of a particle starting from position z at time T :

$$\begin{aligned} (2.4) \quad S_T^{(z,T)} &= z \\ X_{i+1}^{(z,T)} &= \gamma(S_i^{(z,T)}; i), \quad i = T, T+1, \dots \\ S_t^{(z,T)} &= S_i^{(z,T)} + (t-i) X_{i+1}^{(z,T)}, \quad i < t \leq i+1 . \end{aligned}$$

We write $S_t^z \equiv S_t^{(z,0)}$ and $S_t \equiv S_t^{(0,0)}$ (recall that S_t was already defined in (1.3)). Define $\Gamma = \{\gamma(x;t) : (x,t) \in L\}$. Then $F_\infty \subset \sigma(\Gamma)$ (F_∞ was defined in (1.5)). Also, since $|S_t^{(z,T)} - z| \leq |T-t|$ for $t \geq T$, we have

$$(2.5) \quad \sigma\{S_t^{(z,T)} : t \leq T\} \subset H^f \cap \sigma(\Gamma), \quad \text{where } f(x) = T + |x-z| .$$

We now define the shift operator θ to be the set transformation $\theta: F_\infty \rightarrow F_\infty$ satisfying

$$(2.6) \quad \theta A = \{\omega \in \Omega : \exists \omega' \in A \text{ such that, for all } k \in \mathbb{N}, X_{k+1}(\omega) = X_k(\omega')\} .$$

We also define the following "translation": for $(x,s) \in L$, $A \in \sigma(\Gamma)$, let

$$(2.7) \quad A[x,s] = \{\omega \in \Omega: \exists \omega' \in A \text{ such that, for all } (z,t) \in L, \\ \gamma(z+x;t+s)(\omega) = \gamma(z;t)(\omega')\}.$$

(2.8) Remarks:

$$(i) \quad \theta(\cap_{i=1}^{\infty} A_i) = \cap_{i=1}^{\infty} \theta A_i, \text{ for } A_1, A_2, \dots \in \mathcal{F}_{\infty}.$$

(PROOF: $\theta(\cap A_i) \subset \cap \theta A_i$ is obvious. For the converse, let $\omega \in \cap \theta A_i$. Then for each i , there is a $\omega_i \in A_i$ such that $X_{k+1}(\omega) = X_k(\omega')$ for all $k \in \mathbb{N}$. So for every i , $X_k(\omega_1) = X_k(\omega_i)$ for all $k \in \mathbb{N}$. Since $A_i \in \mathcal{F}_{\infty}$, we must have $\omega_1 \in A_i$; so $\omega_1 \in \cap A_i$, which implies $\omega \in \theta(\cap A_i)$.)

Similarly, translation preserves intersections.

(ii) For $A \in \sigma(\Gamma)$ and $(x,s) \in L$: $A[x,s] \in \sigma(\Gamma)$ and $P(A) = P(A[x,s])$.

(iii) If $A = \{S_n = a_n\}$ ($a_n \in \mathbb{R}$), then $A[x,t] = \{S_{T+n}^{(x,T)} = x + a_n\}$. Hence, by (i) and (ii), the family $\{S_{T+n}^{(z,T)}: n \in \mathbb{N}_0, z \in \mathbb{Z}\}$ has the same distribution as $\{S_n^Z: n \in \mathbb{N}_0, z \in \mathbb{Z}\}$.

(iv) For a walk W , $\theta W = [(0,+1)*W] \cup [(0,-1)*W]$. (This uses assumption (2.1)(i).)

(v) θ and translation preserve complements. (This also uses assumption (2.1)(i).)

(vi) θ is not measure-preserving with respect to P .

PROOF OF LEMMA 1.1. Let $V = (v_0, \dots, v_m)$ and $W = (w_0, \dots, w_n)$ be two

walks, and let $J = V*W$. Notice that

$$(2.9) \quad (i) \quad T(k, J) = T(k, V) \quad \text{if } k < m; \quad \text{and} \\ (ii) \quad T(k, J) = T(k-m, W) \quad \text{if } k \geq m \text{ and either} \\ T(k, J) = -\infty \text{ or } T(k-m, W) \neq -\infty.$$

Expand $P(V)$, $P(W)$, and $P(J)$ as products of conditional probabilities, as in (1.4). By (2.9)(i),

$$(2.10) \quad \frac{P(J)}{P(V)P(W)} = \prod_{i=1}^n Q_i$$

where Q_i equals

$$\frac{P(\gamma(J_{m+i-1}; m+i-1) = \Delta J_{m+i-1} \mid \gamma(J_{m+i-1}; T(m+i-1, J)) = \Delta J_{T(m+i-1)+1})}{P(\gamma(w_{i-1}; i-1) = \Delta w_i \mid \gamma(w_{i-1}; T(i-1, W)) = \Delta w_{T(i-1, W)+1})}$$

By (2.9)(ii), $Q_i = 1$ unless $T(m+i, J) \neq -\infty$ and $T(i, W) = -\infty$. (This is the case where $i-1$ is the first time that W hits x , but J hits $x+v_m$ at some time $t < m$ (i.e. $J_t = J_{m+i-1} = x+v_m$).)

So if $Q_i \neq 1$, then we can write $Q_i = P_{a,b}(t(i)) / P_{c,d}(+\infty)$, where $t(i) = m+i-1-T(m+i-1, J)$ ($a, b, c, d \in \{-1, +1\}$, and depend on i). Observe that $b = d$ (because $J = V*W$), and $t(i) \geq i$ (because $T(i, W) = -\infty$). Therefore, Q_i equals one of the following expressions: $(q + p r^{t(i)})/q$, $(q - q r^{t(i)})/q$, $(p + q r^{t(i)})/p$, or $(p - p r^{t(i)})/p$. So, since we assumed $p \geq q$, we always have $Q_i \leq 1 + q^{-1} p r^{t(i)} \leq 1 + q^{-1} p r^i$. So by (2.10),

$$\frac{P(J)}{P(V)P(W)} \leq \prod_{i=1}^n (1 + q^{-1} p r^i) < \prod_{i=1}^{\infty} (1 + q^{-1} p r^i) \equiv U < \infty.$$

Similarly, $Q_i \geq 1 - r^{t(i)} \geq 1 - r^i$, so $P(J) / P(V)P(W) >$
 $\prod_{i=1}^{\infty} (1 - r^i) \equiv L > 0$. This proves the lemma.///

3. A zero-or-one law, and a characterization of recurrence.

First, we will prove two extensions of Lemma 1.1.

LEMMA 3.1. Let $A \in F_\infty$, and let $B \in F_n$ for some n (as defined in (1.5)). Then

$$L P(B) P(A) \leq P(B \cap \theta^n A) \leq U P(B) P(A) .$$

PROOF: First suppose $A \in F_m$ for some finite m . Decompose A into a union of (distinct) walks of length m ; i.e., write $A = \bigcup_{i \in I} W_i$ for some finite set I , where W_i is of the form $\{\omega \in \Omega: S_1(\omega)=j_1, \dots, S_m(\omega)=j_m\}$. Similarly, decompose B into a finite union of walks of length n : $B = \bigcup_{j \in J} V_j$. Then

$$\begin{aligned} P(B \cap \theta^n A) &= \sum_{i \in I, j \in J} P(V_j \cap \theta^n W_i) \\ &= \sum_{i \in I, j \in J} P(V_j * W_i) \\ &\leq \sum_{i \in I, j \in J} U P(V_j) P(W_i) \\ &= U P(B) P(A) . \end{aligned}$$

For fixed $B \in F_n$, let $\Lambda(B) = \{A \in F_\infty: P(B \cap \theta^n A) \leq U P(B) P(A)\}$. By the above calculation, $\bigcup_{n=1}^\infty F_n \subset \Lambda(B)$; and $\Lambda(B)$ is a monotone class, so it follows that $\Lambda(B) = F_\infty$ (see, for example, Chung (1974), Theorem 2.1.2). This proves the right-hand inequality of the lemma for general A and B ; the proof of the left-hand inequality is identical.///

LEMMA 3.2. Let τ be a "stopping time" for S_n ; i.e., τ is a random variable taking values in $\{0, 1, 2, \dots, \infty\}$ such that $\{\tau \leq n\} \in F_n$.

Let $F_\tau = \{B \in F_\infty : B \cap \{\tau=n\} \in F_n, \text{ for every } n\}$. If $P(B \cap \{\tau=\infty\}) = 0$, then for $A \in F_\infty$ and $B \in F_\tau$,

$$L P(B) P(A) \leq P(B \cap \theta^\tau A) \leq U P(B) P(A).$$

PROOF: For each n , Lemma 3.1 gives

$$L P(B \cap \{\tau=n\}) P(A) \leq P(B \cap \{\tau=n\} \cap \theta^n A) \leq U P(B \cap \{\tau=n\}) P(A).$$

Summing over all n gives the result.///

PROOF OF THEOREM 1.2: Let $A \in F_\infty$ such that $A = \theta A$. For each n , there is an integer $k(n)$ and a set $B_n \in F_{k(n)}$ such that $P(B_n \Delta A) \leq 2^{-n}$. Using (2.8)(v) and Lemma 3.1,

$$\begin{aligned} P(B_n \Delta A) &= P(B_n \cap \theta^{k(n)}(A^c)) + P(B_n^c \cap \theta^{k(n)}A) \\ &\geq L P(B_n) P(\theta^{k(n)}(A^c)) + L P(B_n^c) P(\theta^{k(n)}A). \end{aligned}$$

Letting $n \rightarrow \infty$, we obtain $0 \geq L P(A) P(A^c) + L P(A^c) P(A)$. This proves the theorem, since $L > 0$.///

The proof of Theorem 1.3 is obtained from a series of lemmas. We begin with some definitions.

For $x \in Z$, define $\Omega_x = \{\omega \in \Omega : S_n(\omega) = x \text{ i.o. } [n]\}$. $\Omega_x \in F'$, but $\Omega_x \neq \theta \Omega_x$, so we don't yet have a zero-or-one law for $P(\Omega_x)$. Let $D(x, n, t) = \{\omega \in \Omega_x : X_{\tau(x, i, t)+1} \neq +1 \text{ for each } i=1, \dots, n\}$, where $\tau(x, i, t)$ is the i^{th} smallest integer in $\{j > t : S_j = x\}$ (or else $+\infty$, if this set

has fewer than i elements).

For $n \geq 2$,

$$\begin{aligned} P(D(x,n,t) | D(x,n-1,t)) &= E(P_{-1,-1}(\tau(x,n,t) - \tau(x,n-1,t)) | D(x,n-1,t)) \\ &\leq q + p r^2 \end{aligned}$$

because $P_{-1,-1}(s) \leq q + p r^2$ for $s \geq 2$. Therefore, $P(D(x,n,t)) \leq (q + p r^2)^{n-1}$; hence $P(\Omega_x \setminus \Omega_{x+1}) = P(\bigcup_{i=1}^{\infty} [\bigcap_{n=i}^{\infty} D(x,n,t)]) = 0$. By a symmetric argument and induction, $P(\Omega_x \Delta \Omega_y) = 0$ for every $x, y \in \mathbb{Z}$. A direct consequence is the following:

LEMMA 3.3. For any $x \in \mathbb{Z}$:

- (i) $P(\Omega_x \Delta (\bigcap_{y \in \mathbb{Z}} \Omega_y)) = 0$;
- (ii) $P(\Omega_x \Delta (\bigcup_{y \in \mathbb{Z}} \Omega_y)) = 0$.

COROLLARY 3.4. For any given integer x , $P(\Omega_x)$ is either 0 or 1.

PROOF: $\bigcup_{y \in \mathbb{Z}} \Omega_y = \theta \bigcup_{y \in \mathbb{Z}} \Omega_y$. Apply Theorem 1.2 and Lemma 3.3.///

LEMMA 3.5. Let $\tau(x,i) \equiv \tau(x,i,0)$. Assume $P\{\tau(0,1) < \infty\} = 1$. Then $P(\Omega_0) = 1$.

PROOF: We use Lemma 3.2:

$$P\{\tau(0,n) < \infty, \tau(0,n+1) = \infty\} = P(\{\tau(0,n) < \infty\} \cap \theta^{\tau(0,n)} \{\tau(0,1) = \infty\})$$

$$\leq \bigcup P\{\tau(0,n) < \infty\} P\{\tau(0,1) = \infty\} \\ = 0 .$$

Summing over all n gives $P(\Omega_0^c) = 0$.///

LEMMA 3.6. Assume $P\{\tau(0,1) < \infty\} < 1$. Then $E(|\{n: S_n = 0\}|) < \infty$.

PROOF: By Corollary 3.4, $P(\Omega_0) = 0$; hence for some n ,
 $P\{\tau(0,n) < \infty\} < (2U)^{-1}$. As in the proof of Lemma 3.5, we can show

$$(3.1) \quad P\{\tau(0,i+j) < \infty\} \leq \bigcup P\{\tau(0,i) < \infty\} P\{\tau(0,j) < \infty\} .$$

Now, $E(|\{n: S_n = 0\}|) = 1 + \sum_{i=1}^{\infty} P\{\tau(0,i) < \infty\}$. This is a sum of non-increasing terms, so it is finite if and only if $\sum_{k=1}^{\infty} P\{\tau(0,kn) < \infty\}$ is finite. Using induction on (3.1), we find

$$\begin{aligned} \sum_{k=1}^{\infty} P\{\tau(0,kn) < \infty\} &\leq \sum_{k=1}^{\infty} U^{k-1} P\{\tau(0,n) < \infty\}^k \\ &\leq \sum_{k=1}^{\infty} U^{-1} 2^{-k} \\ &< \infty . \end{aligned} \quad ///$$

Let us restate Theorem 1.3 using our new notation:

THEOREM 1.3: The following are equivalent:

- (a) $P\{\tau(0,1) < \infty\} = 1$;
- (b) $P(\Omega_x) > 0$ for at least one $x \in \mathbb{Z}$;
- (c) $P(\bigcap_{y \in \mathbb{Z}} \Omega_y) = 1$;
- (d) $P(\bigcup_{y \in \mathbb{Z}} \Omega_y) = 1$;

$$(e) \quad E(|\{n: S_n = 0\}|) = \infty.$$

PROOF: (c) $\Rightarrow$ (e), (c) $\Rightarrow$ (b), and (b) $\Rightarrow$ (d) are all trivial.

(b) $\Rightarrow$ (c) follows from Corollary 3.4 and Lemma 3.3. (e) $\Rightarrow$ (a) is Lemma 3.6, and Lemma 3.5 gives (a) $\Rightarrow$ (b).///

4. Graphical coupling methods. This section and the next one use methods which rely heavily on the facts that S_n is a one-dimensional process and that $|X_n| = 1$. As noted in the introduction, this gives us a useful coupling of $\{S_t^z, z \in \mathbb{Z}\}$: if $x < y$, then $S_t^x \leq S_t^y$ for all t . This is easily verified (for example, by induction on t), using the observation that if $S_u^x = S_u^y$ for some time u , then $S_t^x = S_t^y$ for all $t \geq u$ (i.e., paths of these processes coalesce). A useful intuitive description of this is the following: when these paths are viewed as curves in the (z,t) -plane, they do not cross over one another. Before we begin to exploit this graphical property, we will prove the following lemma.

LEMMA 4.1. Translation (as defined by (2.7)) is mixing. In particular, if $A \in \mathcal{C}(\Gamma)$ and $P(A) > 0$, then $P(\bigcup_{n=1}^{\infty} A[x,0]) = 1$.

PROOF: If A and B are cylinder sets in $\mathcal{C}(\Gamma)$, then, for sufficiently large x , $A[x,0]$ and B are independent, since the $y(x; \cdot)$ are independent for different x 's. Therefore $\lim_{x \rightarrow \infty} |P(A[x,0] \cap B) - P(A)P(B)| = 0$ for any cylinder sets A and B in $\mathcal{C}(\Gamma)$; hence it also holds for any A and B in $\mathcal{C}(\Gamma)$. This proves that translation is mixing. The second statement of the lemma is a simple consequence of Birkhoff's ergodic theorem applied to the indicator function of A . ///

LEMMA 4.2. Let x and y be even integers. Then $P(S_t^x = S_t^y \text{ for some } t > 0) = 1$.

PROOF: By (2.8)(ii), it suffices to consider the case $x = 0$ and

$y > 0$. First we claim that for any even integer m ,

$$(4.1) \quad P\{S_n^Y - S_n^0 = m, \text{ i.o.}[n]\} = 0.$$

To prove this, set $T(0) = 0$, and for $k = 1, 2, \dots$ define $T(k) = \inf\{n > T(k-1) : S_n^Y - S_n^0 = m\}$. Since $P_{a,+1}(s) > p(1-r)$ and $P_{a,-1}(s) > q(1-r)$ for any $s \geq 2$ and $a \in \{-1, +1\}$, it is easily seen that, for $k \geq 1$,

$$\begin{aligned} P\{S_{T(k)+(m/2)}^Y - S_{T(k)+(m/2)}^0 = 0 \mid T(k) < \infty\} \\ > C \equiv (p(1-r))^{m/2} (q(1-r))^{m/2}. \end{aligned}$$

Therefore, since paths coalesce, $P\{T(k+1) < \infty \mid T(k) < \infty\} < (1-C) < 1$, so $P\{T(k) < \infty \text{ for every } k\} = 0$, which proves (4.1).

By Fatou's Lemma,

$$(4.2) \quad E(\liminf_{n \rightarrow \infty} (S_n^Y - S_n^0)) \leq \liminf_{n \rightarrow \infty} E(S_n^Y - S_n^0).$$

But $E(S_n^Y - S_n^0) = E(S_n^Y) - E(S_n^0) = y$ for every n , so we conclude from (4.2) that $P\{\liminf_{n \rightarrow \infty} (S_n^Y - S_n^0) = \infty\} = 0$. Combining this with (4.1) tells us that $P\{\liminf_{n \rightarrow \infty} (S_n^Y - S_n^0) = 0\} = 1$, which proves the lemma.///

PROOF OF THEOREM 1.5: Assume $P(A) \neq 0$. By Lemma 4.1, we can define (a.s.) a $2N$ -valued random variable $U(\omega)$ such that $\omega \in A \mid U(\omega), 0]$ a.s.; therefore, $P(A \mid U(\omega), 0] = 1$. (Here, $A \mid U, 0] = U_n(A \cap \{U=n\})$.) By Lemma 4.2, it follows that $P\{\text{for some } n, S_n^U = S_n^0\} = 1$; this implies that

$P\{X_n^U = X_n^O \text{ for all but finitely many } n\} = 1$. So since $A \in F'$,
 $P\{A|U, 0] \setminus A[0, 0]\} = 0$. Hence $P(A) = P(A|0, 0] = 1$. ///

LEMMA 4.3. Let $\epsilon > 0$. Then there exist positive integers M and N (depending on ϵ) such that:

$$(4.3) \quad |P(A \cap \theta^{k+M+n}D) - P(\theta^n D)P(A)| \leq 3\epsilon P(A)$$

for all integers $k \geq 1$ and $n \geq N$, and for all events $A \in F_k$ and $D \in F_\infty$.

PROOF: Fix arbitrary ϵ , k , A , and D as in the statement of the lemma. Choose a positive integer M satisfying

$$(4.4) \quad 4r^M / (1-r) < \epsilon.$$

By Lemma 4.2, there exists an integer N such that

$$(4.5) \quad P\{S_n^{-M} \neq S_n^{+M}\} < \epsilon \quad \text{for all } n \geq N.$$

Now fix an integer $n \geq N$.

Fix $a \in \mathbb{Z}$ such that $|a| \leq k$. Let $c = a-M$ and $d = a+M$.

Let

$$(4.6) \quad B_a = \{S_{k+M+n}^{(c, k+M)} = S_{k+M+n}^{(d, k+M)}\} \cap ((\theta^n D)[c, k+M]).$$

Define the functions f and g as follows:

$$f(x) = k \quad \text{for all } x \in \mathbb{Z};$$

$$g(x) \equiv g_{a,M}(x) = k + M + \min\{|x-c|, |x-d|\}.$$

(See Figure 1). Then, as in (2.5), $B_a \in H^g \cap \sigma(\Gamma)$. Let $A_a = A \cap \{S_k^{(0,0)} = a\}$.

Since $A_a \subset \{c \leq S_{k+M}^{(0,0)} \leq d\}$, we can use the fact that S paths do not jump over one another to conclude

$$\begin{aligned} A_a \cap B_a &\subset A_a \cap \{S_{k+M+n}^{(c,k+M)} = S_{k+M+n}^{(0,0)}\} \cap ((\theta^n D)[c, k+M]) \\ &\subset A_a \cap \theta^{k+M+n} D. \end{aligned}$$

Hence

$$(4.7) \quad P(A_a \cap B_a) \leq P(A_a \cap \theta^{k+M+n} D).$$

By (4.5) and (2.8)(iii), $P\{S_{k+M+n}^{(c,k+M)} \neq S_{k+M+n}^{(d,k+M)}\} < \epsilon$; so by (4.6) and (2.8)(ii),

$$(4.8) \quad P(\theta^n D) \leq P(B_a) + \epsilon.$$

Now, by Lemma 2.1 and (4.8),

$$(4.9) \quad P(A_a \cap B_a) \geq P(\theta^n D)P(A_a) - (2P\{I(f, g_{a,M})^c\} + \epsilon)P(A_a).$$

Notice that $P\{I(f, g_{a,M})^c\}$ does not depend on a . In fact,

$$\begin{aligned} (4.10) \quad P\{I(f, g)^c\} &\leq \sum_{x \in \mathbb{Z}} P\{(P^{(x)} \cap \{(x, t) : f(x) < t < g(x)\}) = \emptyset\} \\ &= \sum_{x \in \mathbb{Z}} \exp[-(\beta + \delta)(g(x) - f(x))] \end{aligned}$$

$$\begin{aligned}
&= r^M \sum_{x \in \mathbb{Z}} \max\{r^{|c-x|}, r^{|d-x|}\} \\
&< r^M + \sum_{j=0}^{\infty} r^j .
\end{aligned}$$

So by (4.4),

$$(4.11) \quad P\{I(f,g)^C\} < \epsilon .$$

Now substitute (4.11) and (4.7) into (4.9), and sum over a from $-k$ to $+k$:

$$(4.12) \quad P(A \cap \theta^{k+M+n}D) \geq P(\theta^n D) P(A) - 3\epsilon P(A) .$$

If we replace D by D^C in (4.12), then we can use (2.8)(v) to reverse the inequality and obtain (4.3).///

PROOF OF THEOREM 1.6: In Lemma 4.3, let $A = \Omega$. Fix $\epsilon > 0$. Then there exist integers N and M such that: for any $D \in \mathcal{F}_\infty$, any $n \geq N$, and any $m \geq M+n$,

$$|P(\theta^m D) - P(\theta^n D)| \leq 3\epsilon .$$

Therefore $\{P(\theta^n D), n \geq 1\}$ is a Cauchy sequence, because

$$|P(\theta^j D) - P(\theta^n D)| \leq 6\epsilon \text{ for any } n \geq N \text{ and } j \geq N .///$$

COROLLARY 4.4. Let $\hat{P}(A) = \lim_{n \rightarrow \infty} P(\theta^n A)$ for $A \in \mathcal{F}_\infty$. Then $\hat{P}$ is a probability measure on $(\Omega, \mathcal{F}_\infty)$ satisfying $L P \leq \hat{P} \leq U P$.

PROOF: The inequalities follow from Lemma 3.1 (with $B = \Omega$) . $\hat{P} \geq 0$

and $\hat{P}(\Omega) = 1$, so it only remains to check countable additivity.

Let $A_1, A_2, \dots$ be disjoint members of F_∞ . Then

$$\begin{aligned}\hat{P}(\cup_{k=1}^{\infty} A_k) &= \lim_{n \rightarrow \infty} P(\theta^n(\cup_{k=1}^{\infty} A_k)) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} P(\theta^n A_k) \quad (\text{see (2.8)(i)}) \\ &= \sum_{k=1}^{\infty} \lim_{n \rightarrow \infty} P(\theta^n A_k) .\end{aligned}$$

The last equality follows from dominated convergence: for every n , $P(\theta^n A_k) \leq U P(A_k)$ by Lemma 3.1, and $\sum_k U P(A_k) \leq U < \infty$. ///

PROOF OF THEOREM 1.7: By definition of $\hat{P}$, we have $\hat{P}(\theta A) = P(A)$ for every $A \in F_\infty$. Also, it follows from Theorem 1.2 that if $A \in F_\infty$ and $A = \theta A$, then $\hat{P}(A)$ is either 0 or 1. Therefore θ is a measure-preserving ergodic set transformation on $(\Omega, F_\infty, \hat{P})$. So Birkhoff's ergodic theorem is applicable (see Doob (1953), X, 1-2): $\hat{P}(B) = 1$, where

$$B = \{ \omega : \lim_{n \rightarrow \infty} n^{-1} \sum_{j=1}^n X_j(\omega) = \int_{\Omega} X_1(\omega) \hat{P}(d\omega) \}$$

(we have used the relation $\theta^n \{X_1(\omega) = +1\} = \{X_{n+1}(\omega) = +1\}$). $\theta B = B$, so $P(B) = \lim_{n \rightarrow \infty} P(\theta^n B) = \hat{P}(B) = 1$. This completes the proof, since $\int_{\Omega} X_1 \hat{P}(d\omega) = 2 \hat{p} - 1$. ///

5. Proof of Theorem 1.8. Throughout this section, we will assume $\beta > \delta$. We begin by supplementing the definitions of section 2 by defining

$$y'(x;t) = \begin{cases} +1 & \text{if } \sup\{s \leq t: (x,s) \in (\rho_1^{(x)} \cup \rho_2^{(x)})\} \text{ is a point of } \rho_1^{(x)} \\ -1 & \text{otherwise.} \end{cases}$$

(We assume that (2.1)(i) holds for y' as well as for y .) $y'(x;t)$ is a stationary Markov process with transition probabilities as in (1.1), with $p = q = 1/2$ and $r = e^{-2\delta}$. Moreover, y and y' are "coupled" in the sense that

$$(5.1) \quad y'(x;t) \leq y(x;t) \quad \text{for all } x \text{ and } t.$$

Let $E = \{(-1,-1), (-1,+1), (+1,+1)\}$ and let $\eta(x;t) = (y'(x;t), y(x;t))$. Then for each fixed x , $\eta(x;\cdot)$ is a stationary E -valued Markov process (by construction and by (5.1)). Let $\pi(\cdot) \equiv P\{\eta(0;0) = \cdot\}$ be the stationary distribution of η :

$$\begin{aligned} (5.2) \quad \pi((+1,+1)) &= P\{y'(0;0)=+1\} = 1/2 \\ \pi((-1,-1)) &= P\{y(0;0)=-1\} = \delta/(\beta+\delta) \\ \pi((-1,+1)) &= (\beta-\delta) / 2(\beta+\delta). \end{aligned}$$

Now define X_i^n and S_n^n (respectively, $X_i^{n'}(z,T)$ and $S_n^{n'}(z,T)$) as in (1.3) (respectively, (2.4)), except with y replaced by y' . It follows from (5.1) that for any $(z,T) \in L$,

$$(5.3) \quad S_n^{n'}(z,T) \leq S_n^n(z,T) \quad \text{for every } n.$$

S_n^* is symmetric, so $E(S_n^*) = 0$ for every n ; so from section 4 we know that $\lim_{n \rightarrow \infty} S_n^*/n = 0$ a.s.[P]. Our goal in this section is to prove $\hat{p} > 1/2$ (we emphasize that $\hat{p}$ was defined from the S process, and not from S^*). Observe that (5.3) and Theorem 1.7 together imply that $\hat{p} \geq 1/2$.

Let f , g , and h be functions from Z into R such that, for every $x \in Z$, $f(x) < g(x) < h(x)$. Define

$$\overline{H}_f = \sigma\{\gamma(z;t), \gamma'(z;t) : z \in Z, t \leq f(z)\}$$

$$\overline{H}^h = \sigma\{\gamma(z;t), \gamma'(z;t) : z \in Z, t \geq h(z)\}$$

$$I^*(f,g,h) = I(g,h) \cap$$

$$\{\omega \in \Omega : \forall x \in Z, [(\rho_1^{(x)} \cup \rho_2^{(x)}) \cap \{(x,t) : f(x) < t < g(x)\}] \neq \emptyset\}$$

Let $A \in \overline{H}_f$, and let $x \in Z$. Then, using (2.2), we have

$$P\{\gamma(x;h(x)) = -1 \mid I^*(f,g,h) \cap A\} = \beta/(\beta+8)$$

$$P\{\gamma'(x;h(x)) = +1 \mid I^*(f,g,h) \cap A\} = 1/2;$$

so from (5.2) we obtain the following analogue of (2.2):

$$P\{\eta(x;h(x)) = \cdot \mid I^*(f,g,h) \cap A\} = \pi(\cdot).$$

Now, the Markov property of η gives a useful analogue of (2.3):

$$(5.4) \quad P\{B \mid I^*(f,g,h) \cap A\} = P(B) \quad \text{for any } B \in \overline{H}^h.$$

The proof of Theorem 1.8 relies most heavily upon the following lemma.

LEMMA 5.1. $\lim_{n \rightarrow \infty} E(S_n) = +\infty$.

PROOF: Choose $M > 0$; we will show that there exists an N such that $E(S_n) > M$ for all $n \geq N$. Let $J = \prod_{i=1}^{\infty} (1 - e^{-i\delta})^B$; then $J > 0$. Choose an integer $V > M/J$. Define the "random time"

$$\sigma \equiv \sigma_V = \min \{n: S_n - S'_n = 2V\}.$$

Let $D(n)$ be the event $\{S'_{n+V} = S'_n - V, S_{n+V} = S_n + V\}$. Note that $P\{D(n) | \bar{H}_{n-1}\} \geq K^{2V}$ (where the " $n-1$ " subscript is the constant function $n-1$), and

$$K = \inf\{P\{\eta(x;t)=a | \eta(x;s)=b\} : t-s \geq 1; a, b \in E\} > 0.$$

$P\{\sigma > mV\} \leq P(\cap_{k=0}^{m-1} [D(kV)]^C) \leq (1 - K^{2V})^m$, for any $m \in \mathbb{N}$; so we can choose an N such that $P\{\sigma \geq N\} > 1/2$. Since $S_n - S'_n \geq 0$ (by (5.3)), we have

$$(5.5) \quad E(S_n - S'_n) \geq \sum_{k=V}^n \sum_{j=-k}^k E((S_n - S'_n) 1_{\{\sigma=k, S'_k=j\}} 1_{I_{k,j}})$$

where 1_C is the indicator function of the event C ,

$$I_{k,j} = I^*(f_k, g_{kj}, h_{kj}), \text{ and}$$

$$f_k(z) = k-1$$

$$h_{kj}(z) = k + \min\{|z-j|, |z-(j+2V)|\}$$

$$g_{kj}(z) = (f_k(z) + h_{kj}(z))/2.$$

(See Figure 2.) Now, for any j and k ,

$$\begin{aligned} P(I_{k,j}) &= \prod_{x \in \mathbb{Z}} [1 - \exp\{-(\beta + \delta)(h_{kj}(x) - g_{kj}(x))\}] [1 - \exp\{-2\delta(g_{kj}(x) - f_k(x))\}] \\ &> \prod_{x \in \mathbb{Z}} [1 - \exp\{-\delta(\min\{|x-j|, |x-j-2V|+1\})\}]^2 \\ &> \prod_{x \in \mathbb{Z}} [1 - \exp\{-\delta(|x-j|+1)\}]^2 [1 - \exp\{-\delta(|x-j-2V|+1)\}]^2; \end{aligned}$$

therefore,

$$(5.6) \quad P(I_{k,j}) > J.$$

$S_n^{*}(j,k)$ and $S_n^{*}(j+2V,k)$ are $\bar{H}_{f_k}^{h_{kj}}$ -measurable, and $\{\sigma=k, S_k^{*}=j\} \in \bar{H}_{f_k}$, so from (5.4) it follows that

$$\begin{aligned} (5.7) \quad E(S_n^{(j+2V,k)} - S_n^{*(j,k)}) 1_{\{\sigma=k, S_k^{*}=j\}} 1_{I_{k,j}} \\ = E(S_n^{(j+2V,k)} - S_n^{*(j,k)}) P\{\sigma=k, S_k^{*}=j\} P(I_{k,j}) \end{aligned}$$

Using (2.8)(iii) and (5.3), we easily obtain

$$\begin{aligned} (5.8) \quad E(S_n^{(j+2V,k)} - S_n^{*(j,k)}) &= E(2V + S_{n-k}) - E(S_{n-k}^{*}) \\ &\geq 2V. \end{aligned}$$

Now, combine (5.5), (5.7), (5.6), and (5.8) to get

$$(5.9) \quad E(S_n - S_n^{*}) \geq \sum_{k=V}^n \sum_{j=-k}^k 2V P\{\sigma=k, S_k^{*}=j\} J$$

$$= 2V \int P(\phi \leq n) .$$

Since $E(S'_n) = 0$, we have, from (5.9) and the definition of N and V , that $E(S_n) > M$, for $n \geq N$. ///

Using Lemma 5.1, fix an integer N satisfying

$$(5.10) \quad E(S_N) > 2(1-r)^{-2} \quad (\text{recall } r = e^{-(\beta+\delta)}) .$$

Define the indicator function

$$K(z, t, t'; \omega) = \begin{cases} 1 & \text{if } (\rho^{(x)} \cap \{s: t - |z-x| < s < t'\}) \neq \emptyset \\ & \text{for each } x = z-N, \dots, z+N \\ 0 & \text{otherwise .} \end{cases}$$

Define $\phi[0] = 0$, $S_0^* = 0$, and recursively define

$$\begin{aligned} S_n^* &= S_{\phi[k], \phi[k]}^{(S_{\phi[k]}^*, \phi[k])}, \quad \text{for } \phi[k] < n \leq \phi[k+1] ; \\ \phi[k+1] &= \min\{t \in \mathbb{N}: t > \phi[k] + N, K(S_{\phi[k]+N}^*, \phi[k] + N, t; \omega) = 1\} ; \\ S_n^* &= S_{\phi[k]+N}^* - (n - \phi[k] - N), \quad \text{for } \phi[k] + N < n \leq \phi[k+1] . \end{aligned}$$

(Observe that the last line is equivalent to: $S_n^* = S_{n-1}^* - 1$, for $\phi[k] + N < n \leq \phi[k+1]$.) For arbitrary $z \in \mathbb{Z}$, $t < t'$:

$$\begin{aligned} P(K(z, t, t') = 0) &\leq \sum_{x=z-N}^{z+N} P((\rho^{(x)} \cap \{s \in \mathbb{R}: t - |z-x| < s < t'\}) = \emptyset) \\ &= \sum_{y=-N}^N \exp[-(\beta+\delta)(t'-t+|y|)] \\ &< 2r^{|t'-t|} (1-r)^{-1} . \end{aligned}$$

Therefore

$$(5.11) \quad P\{\phi[1]-N > a\} < 2 r^a (1-r)^{-1} \quad (a \in \mathbb{N}) .$$

The left hand side equals $P\{\phi[k+1]-(\phi[k]+N) > a \mid \phi[k] < \infty\}$, so this enables us to conclude that $\phi[k]$ is finite for every k with probability one.

Since paths don't jump over one another, we see that $S_n^* \leq S_n$ ($= S_n^{(0,0)}$) for every n . We can describe the S^* process informally as follows. Let a particle move according to $\gamma(\cdot, \cdot)$ as usual, for N steps; then take one step to the left for each unit of time until K occurs. This brings us to time $\phi[1]$. Repeat this procedure, bringing us to time $\phi[2]$, and so on. Based on this intuitive description, and an understanding of the role of K as an "indicator of forgetfulness", the next lemma is almost obvious.

LEMMA 5.2. Each of the following is an i.i.d. sequence:

- (i) $\{Z_k : k \geq 1\}$, where $Z_k = \phi[k] - \phi[k-1]$;
- (ii) $\{W_k : k \geq 1\}$, where $W_k = S_{\phi[k]}^* - S_{\phi[k-1]}^*$.

PROOF: (i) Fix positive integers n and k . For given integers $\{a_i, b_i : i \geq 1\}$, let

$$A = \{\phi[i] = a_i \text{ for } 1 \leq i \leq k; \text{ and } S_j^* = b_j \text{ for } 1 \leq j \leq a_k + N\} .$$

Let $c = b_{a_k + N}$, and let $f(x) = a_k + N - |x - c|$. Then $A \in H_f$ (c.f. (2.5)). Let

$$B = \{(x, t) \in \mathbb{Z} \times \mathbb{R} : 0 \leq N - |x| < t\}.$$

It is immediate from the definitions that we can write $\phi[k+1] - \phi[k]$ as a non-random functional F of a random set, as follows:

$$\phi[k+1] = \phi[k] + F\left(\left(\bigcup_{x \in \mathbb{Z}} \rho^{(x)}\right) \cap \{B + (S_{\phi[k]+N}^*, \phi[k])\}\right),$$

where the expression in brace brackets is a translation of B in the plane by a (random) element of L . So

$$\begin{aligned} (5.12) \quad P\{Z_{k+1}=n \mid A\} &= P\{F\left(\left(\bigcup_{x \in \mathbb{Z}} \rho^{(x)}\right) \cap \{B+(c, a_k)\}\right) = n \mid A\} \\ &= P\{F\left(\left(\bigcup_{x \in \mathbb{Z}} \rho^{(x)}\right) \cap \{B+(c, a_k)\}\right) = n\}. \end{aligned}$$

The second equality follows from the independence property of Poisson point processes: $\{B+(c, a_k)\} \cap \{(x, t) : t \leq f(x)\} = \emptyset$, so $F\left(\left(\bigcup_{x \in \mathbb{Z}} \rho^{(x)}\right) \cap \{B+(c, a_k)\}\right)$ is independent of A . Moreover, the final expression in (5.12) does not depend on k , c , or a_k , by the spatial homogeneity of $\rho^{(x)}$ (with respect to translations by elements of L). Hence, $P\{Z_{k+1}=n \mid A\}$ does not depend on A at all, so $\{Z_k\}$ is an i.i.d. sequence.

(ii) Now let $A' = \{\phi[i]=a_i, S_{\phi[i]}^*=c_i, \text{ for } i=1, \dots, k\}$. Letting $a_0 = 0$ and $d_i = c_i + (a_i - a_{i-1} - N)$, we use the identity $S_{\phi[i]}^* = S_{\phi[i-1]+N}^* - (\phi[i] - \phi[i-1] - N)$ to obtain

$$A' = \{\phi[i]=a_i, S_{\phi[i-1]+N}^*=d_i, \text{ for } i=1, \dots, k\}.$$

Now define

$$\begin{aligned}
D_0 &= Z, & D_i &= \{d_i - N, \dots, d_i + N\} & (i \geq 1); \\
f_0(x) &= -\infty, & f_i(x) &= a_{i-1} + N - |x - d_i| & (i \geq 1); \\
g_0(x) &= 0, & g_i(x) &= a_i & (i \geq 1); \\
f(x) &= \max\{f_i(x) : 0 \leq i \leq k \text{ and } x \in D_i\}, \\
g(x) &= \max\{g_i(x) : 0 \leq i \leq k \text{ and } x \in D_i\}.
\end{aligned}$$

So $P\{A' \setminus I(f, g)\} = 0$. We can write A' as $A^* \cap C^*$, where $A^* \in \mathcal{H}_f$ and $C^* \in \mathcal{G}_{g, f}$. Let $B' = \{S_{a_k + N}^{(c_k, a_k)} = m + c_k\}$. Then, using (2.3),

$$\begin{aligned}
P\{W_{k+1} = m \mid A'\} &= P\{B' \mid A'\} \\
&= P\{B' \mid A' \cap I(f, g)\} \\
&= P\{B' \mid A^* \cap C^* \cap I(f, g)\} \\
&= P\{B'\} \\
&= P\{S_N = m\}.
\end{aligned}$$

The last term has no dependence on k , a_k , or c_k , so it follows that $\{W_k\}$ is an i.i.d. sequence. ///

By (5.11),

$$(5.13) \quad E(\varphi[1] - N) < \sum_{a=0}^{\infty} 2 r^a (1-r)^{-1} = 2 (1-r)^{-2}.$$

So, by (5.13) and (5.10),

$$(5.14) \quad E(W_1) = E(S_N - (\varphi[1] - N)) > 0.$$

The proof of Theorem 1.8 is now a simple consequence of the strong law

of large numbers and Lemma 5.2. $\lim_{k \rightarrow \infty} (S_{\phi[k]}^* / \phi[k]) = E(W_1)$

a.s.[P], and $\lim_{k \rightarrow \infty} \phi[k]/k = E(\phi[1])$ a.s., so $\lim_{k \rightarrow \infty} (S_{\phi[k]}^* / \phi[k]) =$

$E(W_1)/E(\phi[1])$ a.s. By Theorem 1.7, $\hat{2p-1} = \lim_{k \rightarrow \infty} (S_{\phi[k]} / \phi[k])$.

Since $S_n \geq S_n^*$, we conclude $\hat{2p-1} \geq E(W_1)/E(\phi[1]) > 0$ (using

(5.13) and (5.14)). Thus the theorem is proven.

6. Further results. In this section, we prove Theorems 1.9 and 1.10.

PROOF OF THEOREM 1.9. (a) By symmetry, it suffices to consider $p > 1/2$. Define a non-decreasing process R_t as follows:

$$W[0] = \min\{i \in \{0, 2, 4, \dots\} : \gamma(0; i) = +1\}$$

$$R_t = \begin{cases} 0 & \text{for } 0 \leq t \leq W[0] \\ t - W[0] & \text{for } W[0] \leq t \leq W[0] + 1 \end{cases}$$

$$W[k+1] = \min\{i \in \{W[k]+1, W[k]+3, W[k]+5, \dots\} : \gamma(k+1; i) = +1\}$$

$$R_t = \begin{cases} k+1 & \text{for } W[k]+1 \leq t \leq W[k+1] \\ k+1 + (t - W[k+1]) & \text{for } W[k+1] \leq t \leq W[k+1] + 1 \end{cases}$$

Let $V_0 = W[0]$, and let $V_i = W[i] - (W[i-1] + 1)$ for $i \geq 1$. Then V_k is the length of time that R_t spends at position k . Since this length of time is always an even integer (possibly 0), it follows that the path of R_t (when graphed in the (z, t) -plane) can only change direction at points of L . From this, it is not difficult to see that $S_t \leq R_t$ for all $t > 0$.

Moreover, $V_0, V_1, \dots$ are i.i.d., with $P(V_0 > 2m) = P(V_0 > 2m+1) = q(q+pr^2)^m$, so

$$(6.1) \quad E(V_0) = 2 \sum_{m=0}^{\infty} q(q+pr^2)^m = 2q(p-pr^2)^{-1}.$$

$R_{W[n]} / W[n] = (n+1) / (n+V_0+\dots+V_n)$, so by the strong law of large

numbers, $R_{W[n]}/W[n]$ converges to $(1+E(V_0))^{-1}$ a.s. Since $S_{W[n]}/W[n]$ converges to $2p-1$ a.s., we use Theorem 1.8 and the fact that $S_t \leq R_t$ to conclude that

$$1/2 < \hat{p}(p, \lambda) \leq [(1+E(V_0))^{-1} + 1]/2.$$

Finally, since $r = e^{-\lambda}$, this and (6.1) imply $\lim_{\lambda \rightarrow 0} \hat{p}(p, \lambda) = 1/2$.

(b) For any walk $(w_0, \dots, w_{n-1})$,

$$\begin{aligned} p - p r^2 &\leq P\{X_n = +1 \mid S_0 = w_0, \dots, S_{n-1} = w_{n-1}\} \\ &\leq p + q r^2. \end{aligned}$$

Since $\hat{p} = \lim_{n \rightarrow \infty} P\{X_n = +1\}$, we have

$$p - p e^{-2\lambda} \leq \hat{p}(p, \lambda) \leq p + q e^{-2\lambda},$$

from which (b) is immediate.///

LEMMA 6.2. Let

$$\varphi(m) = \sup \left\{ \frac{|\hat{P}(B \cap C) - \hat{P}(B)\hat{P}(C)|}{\hat{P}(B)} : n \geq 0, \hat{P}(B) > 0, B \in \mathcal{F}_n, C \in \mathcal{F}_{n+m} \right\}$$

Then $\lim_{m \rightarrow \infty} \varphi(m) = 0$; that is, $\{X_n\}$ is uniformly mixing with respect to $\hat{P}$, in the terminology of Ibragimov and Linnik (1971) (other authors refer to this as " φ -mixing").

PROOF: Let $\epsilon > 0$. In Lemma 4.3, replace k by $s+j$ and A by $\theta^s B$ to obtain: there exist integers M and N such that

$$|P\{(\theta^s B) \cap (\theta^{j+s+M+n} D)\} - P\{\theta^n D\}P\{\theta^s B\}| \leq 3\epsilon P\{\theta^s B\}$$

for all $j \geq 1$, $s \geq 1$, $n \geq N$, $B \in F_j$, and $D \in F_\infty$. Now let $s \rightarrow \infty$; then, for the same possible j , n , B , and D , we obtain (with the help of (2.8)(i))

$$(6.2) \quad |\hat{P}\{B \cap \theta^{j+M+n} D\} - P\{\theta^n D\}\hat{P}\{B\}| \leq 3\epsilon \hat{P}\{B\}.$$

Taking $B = \Omega$ in (6.2) and using $\hat{P}(\theta D) = \hat{P}(D)$ gives

$$(6.3) \quad |P(\theta^n D) - \hat{P}(D)| \leq 3\epsilon \quad \text{for all } D \in F_\infty \text{ and all } n \geq N;$$

putting (6.3) back into (6.2) yields

$$|\hat{P}\{B \cap \theta^{j+M+n} D\} - \hat{P}\{D\}\hat{P}\{B\}| \leq 6\epsilon \hat{P}\{B\},$$

for the same possible j , n , B , and D . Since any $C \in F_k^*$ can be written $C = \theta^k D$ for some $D \in F_\infty$, this becomes:

$$|\hat{P}\{B \cap C\} - \hat{P}\{C\}\hat{P}\{B\}| \leq 6\epsilon \hat{P}\{B\}$$

for all $i \geq M+N$, $j \geq 1$, $B \in F_j$, and $C \in F_{k+1}^*$. Since $\epsilon > 0$ is arbitrary, the lemma is proven.///

LEMMA 6.3. $\hat{\text{Var}}(S_n)$, the variance of S_n with respect to $\hat{P}$,

diverges to $+\infty$ as $n \rightarrow \infty$.

PROOF: The lemma follows from the following claim: for any sequence of integers $c_1, c_2, \dots$, $\lim_{n \rightarrow \infty} \hat{P}\{S_n = c_n\} = 0$. This in turn will be a consequence of Corollary 4.4 once we prove that

$$\lim_{n \rightarrow \infty} P\{S_n = c_n\} = 0.$$

$$\begin{aligned} 1 + P\{S_n = c_n\} &= P\{S_n \leq c_n\} + P\{S_n \geq c_n\} \\ &= P\{S_n^0 \leq c_n\} + P\{S_n^2 \geq c_n + 2\} \\ &\leq P\{S_n^0 \leq c_n \text{ and } c_n + 2 \leq S_n^2\} + 1 \\ &\leq P\{S_n^0 < S_n^2\} + 1. \end{aligned}$$

By Lemma 4.2, the final expression converges to 1 as $n \rightarrow \infty$; hence,

$$\lim_{n \rightarrow \infty} P\{S_n = c_n\} = 0. \quad ///$$

PROOF OF THEOREM 1.10. We use the following theorem (Ibragimov and Linnik (1971), page 340): If $\{Y_i\}$ is a stationary, uniformly mixing sequence of random variables with mean 0, such that $E(Y_i^{2+\delta}) < \infty$ for some $\delta > 0$, and $\sigma_n^2 \equiv E(\sum_{i=1}^n Y_i)^2 \rightarrow \infty$ as $n \rightarrow \infty$, then $(\sum_{i=1}^n Y_i)/\sigma_n$ converges weakly to the standard normal distribution. In our case, let $Y_i = X_i - (2\hat{p}-1)$; then Lemmas 6.2 and 6.3 (with $\sigma_n = \hat{\text{Var}}(S_n)$) allow us to apply this theorem to $T_n \equiv (S_n - n(2\hat{p}-1))/\sigma_n$ under $\hat{P}$:

$$(6.4) \quad \lim_{n \rightarrow \infty} \hat{P}\{T_n \leq x\} = \Phi(x) \equiv (2\pi)^{-1/2} \int_{-\infty}^x e^{-t^2} dt, \quad \text{uniformly in } x.$$

Our aim now is to replace $\hat{P}$ by P in (6.4).

Fix $\epsilon > 0$. Let $A_{n,x} = \{T_n \leq x\}$. By (6.3), there exists an M such that $|P(\theta^M A_{n,x}) - \hat{P}(A_{n,x})| < \epsilon$ for all n and x . Hence, by (6.4), there exists an N such that

$$(6.5) \quad |P(\theta^M A_{n,x}) - \Phi(x)| < 2\epsilon \quad \text{for all } n \geq N \text{ and all } x.$$

Let $G_{n,M}$ be the distribution function of $R_{n,M} \equiv (S_{n+M} - S_n - n(2p-1))/\sigma_n$ with respect to P ; then $G_{n,M}(x) = P(\theta^M A_{n,x})$, so (6.5) implies

$$(6.6) \quad d(G_{n,M}, \Phi) < 2\epsilon \quad \text{for all } n \geq N,$$

where d is the Levy distance (see, for example, Chung (1974), page 94). Now, by Corollary 4.4 and Chebychev's inequality,

$$P(|T_n| > t) \leq U \hat{P}(|T_n| > t) \leq U t^{-2} \quad \text{for any } t > 0.$$

This shows that $\{T_n, n \geq 1\}$ is a tight sequence with respect to P .

Using this, Lemma 6.3, and the fact that $|\sigma_{n+M} - \sigma_n| < 2M$ (because $|S_{n+M} - S_n - M(2p-1)| < 2M$), it follows that, for all $\epsilon' > 0$,

$$\lim_{n \rightarrow \infty} P \left\{ \left| T_{n+M} \left[1 - \frac{\sigma_{n+M}}{\sigma_n} \right] + \left[\frac{S_n - M(2p-1)}{\sigma_n} \right] \right| > \epsilon' \right\} = 0.$$

This is the same thing as $\lim_{n \rightarrow \infty} P(|T_{n+M} - R_{n,M}| > \epsilon') = 0$; this, in turn, implies that $\lim_{n \rightarrow \infty} d(F_{n+M}, G_{n,M}) = 0$, where F_k is the distribution function of T_k with respect to P . Combining this

with (6.6) tells us that

$$d(F_{n+M}, \Phi) < 3\epsilon \quad \text{for all sufficiently large } n.$$

Since ϵ is arbitrary, F_k converges to Φ in the Levy distance; hence, T_n converges weakly to the standard normal distribution.///

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Fig. 1. (Lemma 4.3)

Fig. 2 (Lemma 5.1)

The event $\{\sigma=k, S'_k=j\}$, and the functions f_k , g_{kj} , and h_{kj} .

REFERENCES

- CHUNG, K. L. (1974). *A Course in Probability Theory*. Academic, New York.
- DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.
- IBRAGIMOV, I. A. and LINNIK, Yu. V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff, Groningen.