DEPARTMENT OF OPERATIONS RESEARCH
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK

TECHNICAL REPORT NO. 227

September 1974

MEASURING POWER IN WEIGHTED
VOTING SYSTEMS*

by

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* Preparation of this paper was supported in part by the Office of Naval Research under contract N00014-67-A-0077-0014, and by the National Science Foundation under grant GP-32314X.

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PREFACE

This paper describes two numerical indices, due to Shapley and Shubik and Banzhaf, which are useful for measuring political power in various weighted voting schemes in which some voters effectively cast more or heavier weighted votes than others. A rather detailed review is given of the many uses of these indices in practical situations, as well as some suggestions for potential new applications, research projects, and future directions. It is intended that this presentation serve both as a survey for the interested reader, and as educational work useful in the college classroom. As a result of this purpose, it will appear as Technical Report No. 227 from the Department of Operations Research at Cornell University (Ithaca, New York 14853; September 1974), and as one of about nine chapters in a forthcoming (1975) publication of the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America.

The purpose of this CUPM publication is to present some educational modules for use in the undergraduate curriculum which illustrate the full cycle, from beginning to end, of the complete applied mathematical methodology; that is, from the real world to the mathematical model and its analytic solution, on through data, computation, interpretation, and criticism. The object is to involve the student in the full research activity on some simple but current and relevant problems. This material is designed to be used both in the traditional lecture format as well as in a more experimental or open-ended manner. This latter approach will likely require a somewhat slower pace, however.

There are only a very few simple mathematical concepts among the few formal prerequisites required of the reader. Most of this paper can be

llowed by one with a familiarity with the most basic concepts from set theory
us a knowledge of permutations and combinations from elementary algebra. Most
the analysis is of a discrete or combinatorial nature. However, some of
e arithmetical or computational techniques are fairly complicated in some
rts of the later sections, and a higher level of mathematical maturity or
phistication will be required at these points. Nevertheless, some parts of
e paper are suitable for use at various levels from secondary school through
llege and beyond. On the other hand, the material about large scale
omputation in Section 6 is somewhat more difficult and presumes some know-
edge of more specific college mathematics; the details of which can be seen
r skimming through this Section. And it is most desirable that anyone who
ll be involved with such calculations have some computer experience before-
nd. The user should be alerted to the fact that some part of Section 6 will
e necessary for anyone who wishes to go into the detailed computational
rojects suggested in Section 7. The instructor however should be easily able
o fill in most necessary details. Sections 8 and 9 are independent of the
arlier sections.

A large number of projects are recommended throughout the paper. These
ary in length from routine class exercises (some with answers and some
ithout), to mini projects which should take a few days to a couple of weeks,
o major projects such as in Section 7 which should take from about four to
ight weeks, and finally to some suggestions which can develop into long term
riginal research topics. The mini projects in Section 5 are often appro-
riate for student teams of two or three persons, and those in Section 7
ould involve up to about six people. The instructor may decide to give none,
ome, or all of this material to the students as best fits his needs. There
ay be some advantages in withholding much of this paper in the early stages

that the students can get more practice in the actual model building aspect research. They may, after examining several examples, be able to derive one of those two power indices, or an alternate one, by themselves. This free undirected approach will of course lead to student doubts and anxieties, and to crises in confidence; and the instructor will continually need to provide guidance and direction, and to rekindle enthusiasm by feeding in bits of new information. It is also recommended that the instructor or students make some contact with colleagues in their school's political science or government department. Such "outsiders" may serve in various capacities from free critic up to members of a team teaching or research endeavor.

The intention was to write this paper so as to be fairly self-contained and to require few, if any, of the works listed in the references. Some of the unpublished reports, RAND memoranda, and papers appearing in law journals might not be readily available to all instructors, and it may take some lead time to acquire them. It will be fairly clear in reading Section 5 as to which references should be available in order to do particular mini projects. The paper by Banzhaf (1966) or Johnson (1969) should prove most useful in the latter parts of Section 5. Banzhaf (1968), Peirce (1968), and perhaps an appropriate reference mentioned in Section 6, would be helpful for doing the projects in Section 7. The books by Farquharson (1969), Papayanopoulos (1973), Tucker (1962a), and Shubik (1964), as well as those containing the papers by Tucker and Shapley (1968) and Shapley (1953) should be available in most school libraries, and can serve as more general reading.

An incomplete first draft of this paper, dated October 1973, was used in one way or another by about a dozen teachers, including the author, in the spring of 1974. A few of these instructors were part of the official CUPM evaluation, but most of them had contact with the author directly. Preliminary

reports indicate a fair amount of success. Incidentally, in at least three of these classes the students arrived at one of these two power indices by themselves - with some hints perhaps. The author is grateful for the comments from these instructors, from the other members of this CUPM panel, as well as to Louis Billera, Pradeep Dubey, Irwin Mann, Joseph Malkevitch, Lloyd Shapley, and Robert Weber who read and made suggestions on the draft version. Hopefully, the number of remaining errors in this revision and extension are rather small; but then again it is not an uncommon part of the model building process to detect mistakes in previous studies.

1. INTRODUCTION

1.1. Weighted Voting. There are a large number of voting situations which some individuals or blocks of voters effectively cast more ballots than others. Such weighted voting systems are found in governmental bodies such as the U.S. Congress, some state legislatures and county boards, in the Electoral College, in voting by stockholders in a corporation, in several university senates, in many other multimember electoral districts in which several representatives are elected at-large from a single district, as well as when strictly disciplined political parties vote as a single bloc.

A prime concern in designing such policy or decision making assemblies is that they are fair and equitable in the sense that they give equal representation to their constituents. There have been many challenges in recent times against existing legislative bodies. There are frequently heard charges of undemocratic inequity, unfairness, disparities, bias, handicaps, gerrymandering, denying, gross variances, impairment, prohibitions, expediency, malapportionment, disenfranchising, ambiguous and archaic systems, diluting and devaluating the vote, and "stuffing" the ballot boxes. The courts have frequently found deliberative and legislative bodies wanting, and have often ruled in favor of impartiality, equal protection, political justice, full participation and representation, the right of suffrage, and the principle "one man - one vote" for all citizens.

1.2. Voting Power and Its Relation to Weights. Some proponents of weighted voting argue that such systems can be used to adjust for or to cancel out inequalities at another level, such as differences in the populations

ch various delegates represent. However, the fraction of the total number votes which a representative possesses, or the number of members in a party, not generally synonymous with any meaningful measure of his or its voting power! The ability to cast more ballots does not in itself necessarily increase one's power nor does it do so in a directly proportional way. There may exist large discrepancies between the weights voters have and a good measure of their political influences. It is fallacious to expect that one's voting power is directly proportional to the number of votes he can deliver. Yet many attempts to correct inequalities merely assign weights to a delegate proportional to the number of inhabitants he represents, and it is felt that this preserves some equality at the level of the individual citizens. Paradoxically, those who advocate that they are the main beneficiaries of weighted systems such as the Electoral College are very often the ones most hurt by it in terms of power indices discussed below. The discrepancies between the ratio of weights and the ratio of power will be illustrated by several examples in the following sections. Power is not a trivial function of one's strength as measured by the number of votes. Simple additive or division arguments are not sufficient, more complicated relations are necessary to understand the real distribution of influence.

1.3. Applications. The power indices discussed below do not constitute an attack on all weighted voting systems per se. Instead, they provide a method for structuring such systems with more understanding, so that certain individual powers can be preserved. Presumably the courts are against indirect sophisticated discrimination as well as the more obvious types. Weighted voting with periodic adjusting of the weights, when done properly, can perhaps serve as an alternative to creating other inequities, caused by events such as

districts of varying size due to shifting populations. It can be used to maintain present advantages or to avoid going through frequent redistricting, reapportionment, realignments, or otherwise disturbing existing constituencies, such as cutting county boundaries to obtain state wide districts; which in turn may give rise to other potential difficulties such as gerrymandering. Proper weighting can possibly compensate or offset for some other problems or defects which arise in attempting to make all districts "equal".

1.4. Measures for Voting Power. Several examples of weighted voting systems will be scrutinized in what follows in order to gain some insight into the complexities and ramifications inherent in the apportionment of unequal voting strengths. Two of the numerical indices which have been used to measure a subject's share of the power to control or influence outcomes will be discussed. This pinpointing of some of the more formal, technical, and quantifiable aspects of bloc voting should clarify many of the implications and consequences which are built into such divisions of power. From a more complete account and accurate evaluation, one should be able to conclude in a more intelligent manner where the relative power to influence and affect aggregate decisions or laws resides.

These analytical techniques should prove useful in reevaluating some of the existing democratic institutions in terms of fairness, concealed biases, and the degree of protection they provide their constituents. Procedures for adequately modifying, altering or revising current assemblies to reflect certain changes should be less ambiguous. These methods can be used to evaluate a newly proposed system, or to design a new legislative body from scratch like a college senate (either a hypothetical or proposed one, or to revise an existing one) so as to provide each with the influence he is entitled to.

course, no scientific theory or mathematical formula will suffice to reveal of the poorly understood and subtle details, nor the behind-the-scene nuances, which can have significant influence on the outcomes reached by voting in a deliberating or legislative council. Additional comments on the possible inadequacy of such theories are contained in Section 9.

1.5. Outline. Some simple examples of weighted voting systems plus the notation and definitions will be presented in Section 2, and some less trivial examples will be mentioned and evaluated for power ratios in Section 4. The power indices introduced by Shapley and Shubik (1954) and by Banzhaf (1965) will be discussed in Section 3. Some mini projects are given in Section 5, and these may be undertaken with or without the suggestions for large size calculations and machine computation described in Section 6. The main project of evaluating the current Electoral College and some of its proposed alternatives is presented in Section 7. Some possible extensions and generalizations for this study and other work on voting appear in Section 8, and some criticisms of this whole approach are mentioned in the final Section.

2. SIMPLE EXAMPLES

2.0. Introduction. Several simple examples of weighted voting schemes will be considered in this section. We will first introduce the notation for a weighted majority game:

$$[q; w_1, w_2, \dots, w_n] .$$

where there are n voters, or players or citizens, denoted by $1, 2, \dots, n$; and w_i is the voting weight of player i and is assumed to be a nonnegative number. Let

$$N = \{1, 2, \dots, n\}$$

be the set of all n voters, and let S be a typical coalition or bloc of voters, i.e., a subset of N . A coalition wins a vote, or is called winning, whenever

$$\sum_{i \in S} w_i \geq q$$

where we will assume that

$$q > \frac{1}{2} \sum_{i \in N} w_i .$$

This q is called the quota for the game. In some cases, but not all, one also assumes that each $w_i < q$, and one often restricts the w_i and q to integer values.

2.1. Stockholders. The percent of a company's stock which an investor owns can serve as a weight in a weighted majority game, but it is often not a good measure of his influence or voting power. A man or coalition with over 50% of the stock has full control or has 100% of the voting power whenever decisions are made by a simple majority, i.e., is a dictator. A group with exactly 50%, or more, of the stock can veto or block any such action. In the game $[51;28,24,24,24]$ the first voter seems to be much stronger than the last three, since he needs only one other to pass an issue whereas the other three must all combine in order to win. In the situation $[51;26,26,26,22]$ the last player seems powerless since any winning coalition containing him can just as well win without him. The two games $[51;40,30,20,10]$ and $[51;30,25,25,20]$ seem identical in terms of voting influence, since the same coalitions are winning in both cases.

2.2. Equal Power. It is clear that in the game $[q;1,1,\dots,1]$ each player has equal power. Such games arise in pure bargaining situations, and in deterrence encounters where each participant has the same amount to lose. However, games such as $[3;2,2,1]$, $[8;7,5,3]$ and $[51;49,48,3]$ are similar to $[2;1,1,1]$ in terms of power, since they give rise to the same collection of winning coalitions. The last of these illustrates the potential value of a small third party such as the Liberal Party in Great Britain. If we add to the game $[3;2,1,1]$ the rule that player 2 (or 3) can cast an additional vote in the case of a 2 to 2 tie, then it is effectively $[3;2,2,1]$. But if player 1 can cast the tie breaker, then it becomes $[3;3,1,1]$ and he is a dictator. More generally, in the game $[10(n-1)+1;100,100,\dots,100,1]$ player n has the same power as the others.

When n is odd, i.e., the game is similar to one in which all of the players have the same weights or number of votes.

2.3. Dummy Players. A player who has no real voting power is called a dummy. Any winning coalition which contains such an impotent voter could win just as well without him. Player 4 in the game $[51; 26, 26, 26, 22]$ is an example. Also player n in the game $[50(n-1)+1; 100, 100, \dots, 100, 1]$ is a dummy when n is even. In the game $[16; 12, 6, 6, 4, 3]$ player 5, with 3 votes, is a dummy; since no subset of the numbers 12, 6, 6, 4 sums to 13, 14, 15, and thus player 5 could never be pivotal in the sense that by adding his vote a coalition would just reach or surpass the quota of 16. On the other hand, one can show that player 4 with 4 votes has the same influence in terms of winning coalitions as either player 2 or 3 with 6 votes apiece. This example can be generalized to games like $[68; 48, 36, 30, \dots, 6, 4, 3]$ where the missing weights are multiples of six. The last three players in the situation $[10; 5, 5, 5, 2, 1, 1]$ are all dummies.

Banzhaf (1965) describes a weighted voting system which was used in Nassau County, New York by the Board of Supervisors in which three of the six municipalities share all of the power equally and three others are dummies in both instances, and this is described in Table 2.3.1. Each municipality has one representative whose vote is weighted according to the number of votes listed in the Table. Of course, the dummies may be able to influence or sway decisions by their right to debate or in some other way, such as by membership in committees. A more recent revision of this Nassau County example which avoids dummies is mentioned in Section 5.6.

NASSAU COUNTY

Municipality	Population 1954	No. of Votes 1958	Population 1960	No. of Votes 1964
Hempstead, No. 1	618,065	9	728,625	31
Hempstead, No. 2		9		31
North Hempstead	184,060	7	213,225	21
Oyster Bay	164,716	3	285,545	28
Glen Cove	19,296	1	22,752	2
Long Beach	17,999	1	25,654	2
Total	1,004,136	30	1,275,801	115

TABLE 2.3.1

The Canadian election in the fall of 1972 gave the results in Table 2.3.2, where each of the three listed parties had equal power and those in the 'others' category constitute a dummy (individually or as a group). The Liberals have since won a majority in the elections of 1974.

CANADA

Party	Leader	No. of Seats
Liberals	Pierre E. Trudeau	109
Tories	Robert L. Stanfield	107
New Democrats	David Lewis	31
Others		17

TABLE 2.3.2

2.4. Veto Power and Dictators. A player or coalition is said to have veto power if no coalition is able to win a ballot without his or their consent. A voter is a dictator if he controls any vote by himself, i.e., his weight $w_i \geq q$. There can only be one dictator (recall that we assumed $2q > \sum w_i$), and the other players are then all dummies. Player 1 has veto power in the game $[51; 50, 49, 1]$ which is essentially the same as $[3; 2, 1, 1]$; and if he is a chairman with additional power to break ties then the game becomes $[3; 3, 1, 1]$ and he is a dictator. Note that the ability of an individual to break tie votes in $[[n/2]+1; 1, 1, \dots, 1]$ adds power when n is even and adds nothing when n is odd. This presumes no change in q due to absences or abstentions. Here, $[n/2]$ stands for the greatest integer in $n/2$.

3. MEASURES OF VOTING POWER

3.0. Introduction. Several quantitative measures for evaluating the abstract power of a voter or coalition have been proposed. The Shapley-Shubik index as well as the Banzhaf index will be introduced in Sections 3.3 and 3.5, respectively. These two value concepts have received the most theoretical attention as well as application to real-world political structures. These indices will be computed for some very simple examples throughout Sections 3 and 4, and applied to some more realistic situations in Sections 5 and 7.

3.1. The Notion of Power. It is clear from our examples that voting power, expressed in some formal or functional sense, is not directly proportional to the number of votes one casts. In fact, an exact proportionality between weights and the power indices defined below seems to be rare in cases other than the one man - one vote situation. We are thus concerned with developing indices which give a reasonable measure of voting power in some technical sense. An individual's index should relate, in a preferably simple way, to one's true ability to affect group decisions by means of casting his weighted vote; and it should hopefully capture and crystallize some of our more intuitive concepts about power, such as why some legislators seem to have a better chance on average of being on the winning side than another. It should indicate one's relative influence, in some numerical way, to bring about the passage or defeat of some bill. It should be based somehow upon the importance of the individual in casting the deciding vote which will guarantee that some issue will carry. It should compare

all the opportunities which each voter has to be a sort of critical swing-man in causing a desired outcome. This index should depend upon the number n of players involved, on one's fraction of the total weight, and upon how the remainder of the weight is distributed.

Power depends essentially upon being on the winning side of a division, that is, upon the ability to succeed in joining a winning coalition which support one's views. Furthermore, it is the possibility of membership in the "minimal" winning coalitions which is sufficient or significant; since any additional votes afterwards become unnecessary and irrelevant, whereas deleting a member beforehand fails to insure a victory. It is what one brings to a successful alliance that is crucial, and the one who is "last" to join a minimal winning coalition is particularly influential. The possibility of reversing an outcome by changing one's vote on a question is a most important one. The person whose support is necessary for success has a rather strong bargaining position, and one in this uniquely influential marginal or pivotal position is often rewarded handsomely for casting the deciding vote which carries his coalition "over the top". It is assumed for an abstract theory that all theoretically possible voting alignments, or combinations, are equally likely to occur, and thus the scientific measures which are derived below ignore a large number of less technical concepts such as voting in alphabetical order, political ideologies, and many other aspects as will be mentioned in Section 9.

3.2. Terminology. As we mentioned above, a coalition of players which possess enough votes to guarantee passage of an issue is called winning. A winning coalition is said to be minimal winning if no proper subset of it is winning, and these are important ones in the following Section.

A coalition which is not winning is called losing. A subset S of voters is a blocking coalition or has veto power if its complement $N-S$ is not winning. (Some authors also require that S itself be losing in order to be a blocking coalition.) A player i is a dictator if he forms a winning coalition $\{i\}$ by himself. A voter i is a dummy if every winning coalition which contains him is also winning without him, that is, he is in no minimal winning coalition.

There are many interrelations and properties for the concepts just defined; especially if additional conditions are assumed, such as our quota q consisting of a simple majority of the total voting weights. In fact, a whole algebra or combinatorial structure of simple games can be built up from these ideas as illustrated in Shapley (1962) and by later generalizations, but this will not be done here. The interested reader should see page 87 of Shapley (1973) for references on this and some related theories. A good exercise at this point would be to describe which coalitions in the examples of Section 2 are of the various types mentioned in this Section. It may prove helpful to actually draw the lattice of all subsets of the set of players for these examples.

3.3. The Shapley-Shubik Index. Shapley and Shubik (1954) introduced an index for measuring an individual's voting power, which is a special application of a more general value concept introduced by Shapley (1953) in the context of the general von Neumann-Morgenstern (1944) theory of multiperson cooperative games. A voter's value is the a priori chance that he will be the last member added to turn a losing coalition into a winning one. It thus assigns him the probability of his casting the deciding vote. This expected

frequency with which a man is the pivot, over all possible alignments of the voters, appears to be a good indication of his voting power.

More precisely, one looks at all possible orderings of the n players, and considers this as all of the potential ways of building up toward a winning coalition. There are $n!$ such ordered sequences, or permutations, of the n voters. The "!" stands for factorial; $n! = n \times (n-1) \times \dots \times 2 \times 1$ and $0! = 1$. For each one of these permutations, some unique player joins and thereby turns a losing coalition into a winning one, and this voter is called the pivot. That is, if in the sequence of players $x_1, x_2, \dots, x_{i-1}, x_i, \dots, x_n$, $\{x_1, x_2, \dots, x_{i-1}, x_i\}$ is a winning coalition, but $\{x_1, x_2, \dots, x_{i-1}\}$ is losing, then i is in the pivotal position. Consider the number of permutations of the n voters in which a particular player i is the pivot; and then this number divided by the total number of alignments, which is $n!$, is the Shapley-Shubik power index or voting value, and is denoted by ϕ_i ;

$$\phi_i = \text{Number of sequences in which } i \text{ is a pivot} \div n! .$$

Also, let

$$\phi = (\phi_1, \phi_2, \dots, \phi_n) .$$

We are assuming here that each of the $n!$ alignments is equiprobable. One may consider the random order as a ranking of the voters according to their degree of enthusiasm or intensity of feeling in support of the issue being voted on, and the issue as a random variable. We are not concerned here with the most likely order of voting in some real convention or assembly, such as voting as you enter through some door, or alphabetically, or some other historical or traditional way; nor with developing a realistic

theory of how voting coalitions do grow in practical circumstances, even though the formation of a good theory along these lines would be most desirable.

This power index can be expressed by the formula

$$\phi_i = \sum \frac{(s-1)!(n-s)!}{n!}$$

where

$$s = |S| = \text{Number of voters in } S,$$

and where the summation is taken over all winning coalitions S for which $S - \{i\}$ is losing. This follows from the fact that there are $(s-1)!$ orders in which the given $s-1$ players can enter S before i and there are $(n-s)!$ different orders in which the remaining $n-s$ players can be added to S to form the grand coalition N .

In addition to the probabilistic approach taken above, this same index can be derived by several other approaches: by means of a somewhat dual theory using blocking coalitions instead of winning ones; from a bargaining or fair division scheme suggested by John Harsanyi (1959, 1963); or from a simple set of axioms (symmetry, additivity, and dummies are powerless) given by Shapley (1953) for his more general value concept. Shapley's axioms give a unique value formula in the more general game theory context, but not when restricted to the class of majority voting games. A variation of his axioms due to Dubey (1974) gives a uniqueness result in the case of all "simple" (but not necessarily "monotone simple") games. In addition to these other approaches, this index has many other nice properties, but we only mention here that the ϕ_i are nonnegative,

$$\sum_{i \in N} \phi_i = 1$$

and

$$\sum \frac{(s-1)!(n-s)!}{n!} = \frac{1}{s}$$

where the summation is taken over all coalitions S with a fixed number s of players. The reader can verify these.

3.4. An Example. The 24 permutations of the four players 1, 2, 3, 4 in the weighted majority game $[51;40,30,20,10]$ are listed below. The "*" indicates which player is pivotal in each alignment.

1 2*3 4	2 1*3 4	3 1*2 4	4 1 2*3
1 2*4 3	2 1*4 3	3 1*4 2	4 1 3*2
1 3*2 4	2 3 1*4	3 2 1*4	4 2 1*3
1 3*4 2	2 3 4*1	3 2 4*1	4 2 3*1
1 4 2*3	2 4 1*3	3 4 1*2	4 3 1*2
1 4 3*2	2 4 3*1	3 4 2*1	4 3 2*1

It is clear from this array that

$$\phi = (10,6,6,2)/24 .$$

Or, from the formula we see, for example, that

$$\phi_4 = 3 \cdot \frac{(3-1)!(4-3)!}{4!} + 2 \cdot \frac{(2-1)!(4-2)!}{4!}$$

since player 4 is pivotal in 3 coalitions consisting of three players and in 2 coalitions of two players.

3.5. The Banzhaf Index. Another value concept for measuring voting power was introduced by Banzhaf (1965). He is a lawyer and much of his work has appeared in law journals; and his index, even more so than the one above, has been used in arguments in various legal proceedings. Some of

the mathematical experts in this area have presented affidavits in cases before the courts (see Banzhaf (1968), page 306). Banzhaf's index is also concerned with the fraction of possibilities in which a voter is in the crucial position of being able to change an outcome by switching his vote, that is, being able to alter the whole group's decision by he alone changing. However, he considers all significant combinations of "yes" and "no" votes, rather than permutations of the players as in the Shapley-Shubik case.

Each voter can vote yea or nay on a particular question; so one can imagine all 2^n possible combinations of such votes by these n players. A player is said to be marginal, or a swing or critical, in a given combination if he can change the outcome, resulting from this combination, from passage to defeat or vice versa by changing just his vote on the issue. Power appears to rest in precisely such marginal situations in which a defection from a bare majority produces a different result, or achieves the exactly opposite goal. How often one appears in such a marginal or swing position is taken as the relative index of his influence. The ratio of voting power (in the sense of Banzhaf) for player i to that for player j is the number of combinations in which i is marginal, divided by the number in which j is marginal. Consider the number b_i of voting combinations in which voter i is marginal. The Banzhaf power index or value for player i is defined as b_i divided by the sum of all of the b_j taken over the n players. We will denote player i 's Banzhaf value by β_i and let

$$\beta = (\beta_1, \beta_2, \dots, \beta_n) .$$

We again assume that all voting combinations, i.e., divisions into yeas and nays, are equally probable; since one can hardly assert in advance in the

case of an abstract theory which possibilities are more likely to occur or prove most significant. We assume that everyone votes on each issue, and that no abstentions are allowed.

Note that the Shapley-Shubik index uses permutations of the players, and is concerned with order in which winning coalitions are constructed. It assigns importance to the voter who is the last to join a coalition and thus make it a winning one. Whereas the Banzhaf index employs combinations, and considers just the number of such in which one plays a significant role. It does not look at the chronological order in which the winning coalitions were formed. It is concerned with which subsets voted yea or nay, however, and not with the numerical outcome of the vote. Comparing the values ϕ and β , Riker and Shapley (1968) state on page 204 that "Although the numerical values obtained are slightly different, we know of no significant qualitative differences that would arise from defining the power index in this way". Some more recent work however does suggest some differences, and with the recent axiomatization of β by Dubey in Dubey and Shapley (1974), these differences should become more clear before long. Both value concepts seem to be effective and objective quantitative measures of power, which can be accurately and readily calculated in given cases. They have been accepted by several political scientists and mathematicians as reasonable and believable for understanding the logical structure of many voting problems, and they are not based on any excessive assumption.

Furthermore, both of these value concepts clearly satisfy some of our intuitive feeling towards power expressed earlier; for example, the indices are 1 for dictators and 0 for dummies, and they depend upon winning coalitions and only indirectly upon a voter's numerical weight. By definition the β_i are nonnegative and

$$\sum_{i \in N} \beta_i = 1 .$$

The indices ϕ and β are symmetric and monotone in the sense that $w_i = w_j$ implies $\phi_i = \phi_j$ and $\beta_i = \beta_j$, and $w_i > w_j$ implies $\phi_i \geq \phi_j$ and $\beta_i \geq \beta_j$. One possible approach to deriving a measure of power could be to list desirable conditions like these, and to then search for a function with these properties.

3.6. Computation and Examples. To calculate β for reasonable small examples one can construct a table which lists the n players on top and the 2^n combinations of "yes" and "no" votes below the players, i.e., all the ways a vote could turn out. Next to each combination one can list whether an issue would pass or fail. Ties are not allowed. A second table aside the first one, also headed by the players, can then be used to record for each player next to each combination whether he was marginal or not. One adds up the number of times a player is marginal and the resulting numbers are proportional to the indices β_i . One could eliminate this second table of course by instead underlining or placing a "*" by the marginal voters in the first table.

The example $[51;40,30,20,10]$ considered above gives rise to Table 3.6.1. The resulting indices are $\beta = (10,6,6,2)/24$ which agree in this particular case with ϕ computed above. In the appendix of Banzhaf (1965) there are similar tables for the five-person games $[9;5,5,3,3,1]$ and $[5;4,2,1,1,1]$, and the results are $7\beta = (2,2,1,1,1)$ and $11\beta = (7,1,1,1,1)$, respectively. One should calculate ϕ for these two cases and compare it with β . A similar exercise is to compute β for $[5;4,2,1,1]$ and compare it with $\phi = (3,1,1,1)/6$. In practice, various symmetries allows one to compute β without having to consider the full list of all combinations.

COMPUTATION OF THE BANZHAF INDEX

Players				Pass/Fail		Marginal			
1	2	3	4	P	F	1	2	3	4
Y	Y	Y	Y	P					
Y	Y	Y	N	P		X			
Y	Y	N	Y	P		X	X		
Y	N	Y	Y	P		X		X	
N	Y	Y	Y	P			X	X	X
Y	Y	N	N	P		X	X		
Y	N	Y	N	P		X		X	
N	Y	Y	N		F	X			X
Y	N	N	Y		F		X	X	
N	Y	N	Y		F	X		X	
N	N	Y	Y		F	X	X		
Y	N	N	N		F		X	X	
N	Y	N	N		F	X			
N	N	Y	N		F	X			
N	N	N	Y		F				
N	N	N	N		F				

$$24 \times \beta = (10, 6, 6, 2)$$

TABLE 3.6.1

3.7. Other Value Concepts. Several other value concepts have been suggested for investigating power. The interested reader can consult the references listed on page 329 of Banzhaf (1965) and the discussion in Miller (1964).

4. ADDITIONAL EXAMPLES

4.0. Introduction. Some examples and exercises for voting games which are usually somewhat more complicated than those given in Section 2 are presented in this Section, and some of these are taken from real-world situations.

4.1. The United Nations Security Council. The U.N. Security Council consists of the "big five" countries (France, Great Britain, the People's Republic of China, the Soviet Union, and the United States) who are permanent members, and each individually has veto power; plus ten "small" countries whose membership rotates. It takes nine votes, the "big five" plus at least four others, to carry an issue. This gives the game

$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. Note that a "small" country i can be pivotal in a winning coalition S if and only if S contains exactly nine countries including the five permanent members. There are $\frac{9!}{3!6!}$ such different S which contain i , and the corresponding coefficient in the Shapley-Shubik formula for this 15-person game is $\frac{(9-1)!(15-9)!}{15!}$. The product of these two numbers gives the power index $\phi_s = .001863$ for any nonpermanent member. A member from the "big-five" has index

$\phi_b = (1-10\phi_s)/5 = .1963$. It would be of interest to compute the Banzhaf index for this Council, and to compare these indices with those for the members of the old Security Council before 1963 which was

$[27; 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1]$ (Answer for ϕ : $6\phi_s = 1/77$ and $5\phi_b = 76/77$); or with a potential future Council which adds Japan and/or West Germany as a permanent member, also holding veto power. (The latter two countries now

contribute more financially to the U.N. than do Britain or France.) A much more ambitious project is to compute ϕ or β for the present combined game in the Security Council and the General Assembly, as was done in the case of ϕ by Schwödiauer (1968). (See Section 4.3.)

4.2. Some Other Assemblies. Several other existing assemblies or committees can be found which are weighted voting systems, and it is left as an exercise to express these as weighted majority games and to determine the values for ϕ and β .

The Board of Estimates for New York City has had eight members with voting strength as indicated in Table 4.2.1.

NEW YORK CITY BOARD OF ESTIMATES

Member	No. of Votes	
	Former System	Present System
Mayor	3	4
Controller	3	4
Council President	3	4
Brooklyn Borough President	2	2
Manhattan Borough President	2	2
Bronx Borough President	1	2
Richmond Borough President	1	2
Queens Borough President	1	2

TABLE 4.2.1

The Israeli Knesset in late 1965 had the division indicated in Table 4.2.2, taken from left to right on the political spectrum, as described by Owen (1971).

1965 ISRAELI KNESSET

Party	No. of Seats
New Communists	3
Communists	1
Poaelei Aguda	2
Agudat Israel	4
National Religious	11
Mapam	8
The Alignment (Mapai-Ahdut)	49
Haolan Haze	1
Independent Liberals	5
Rafi	10
Gahal	26

TABLE 4.2.2

The Tokyo Metropolitan Assembly had elections in 1973 with the results as indicated in Table 4.2.3.

TOKYO

Party	No. of Seats
Liberal-Democrats	51
Komeito	26
Communists	24
Socialists	20
Democratic-Socialists	2
Independents	2

TABLE 4.2.3

The 1973 general elections in Sweden ended in a virtual "tie" with a total of 175 seats between the ruling Social Democrats (157) and

Communists (18), as well as for the three opposition parties: Center Party (90), Moderates (51) and Liberals (34).

The State Senate in Hawaii, as reported in Banzhaf (1966), had one district in the State with one legislator, one district with two, two districts with three each, and four districts with four legislators from each of them. He also studied the Arkansas House, the Georgia Senate, the Texas House, and the Wyoming Senate. Consider the power of the various districts under the assumption that all delegates from the same district vote the same way on any question. This multi-member district situation is studied in more depth in Section 5.3.

4.3. Some Large Games. For the 9-person game $[7; 5, 1, 1, 1, 1, 1, 1, 1, 1]$ with one strong player, one can compute that $\phi = (10, 1, 1, 1, 1, 1, 1, 1, 1)/18$. More generally, for the n -person game $[q; w_1, 1, 1, \dots, 1]$, where w_1 is an integer such that $w_1 < q \leq n-1$, $\phi = ((n-1)w_1, n-w_1, n-w_1, \dots, n-w_1)/n(n-1)$. Note that player 1 occupies each position in a random ordering of the n players with equal probability, and so his value is obtained by merely counting the number of times he is in pivotal position and dividing by n . Note that for small n , $\phi_1 = w_1/n$ is somewhat larger than the first player's share of the weights which is $w_1/w = w_1/(w_1+n-1)$, where again $w = \sum_{i \in N} w_i$. But for large n , ϕ_i approaches player i 's share of the weights. Limiting results have been obtained for ϕ for several classes of games of this sort by Shapley and Shapiro (1960), Milnor and Shapley (1961), and Shapley (1961); and a recent book by Aumann and Shapley (1974) describes a whole value theory for various infinite games. Similar types of limiting results have also been obtained by Dubey and Shapley (1974) for the Banzhaf index.

In some decision procedures in Australia, each one of the six states has one vote, and the federal government has two votes plus the ability to break ties, where a simple majority is necessary to win. This can be expressed as a weighted majority game of the type just mentioned, and its ϕ and β can be determined. In the United Nations General Assembly with a required two-thirds majority one has the 130-person game $[87; 3, 1, 1, \dots, 1]$ since the Soviet Union has three votes.

5. MINI PROJECTS

5.0. Introduction. This section includes a collection of relatively small projects, any one of which can be undertaken in a class, or by small subgroups of a class, over a period of two or three weeks. These illustrate applications of the power index concept and are suitable for a class which does not have the prerequisites, time or computer facility in order to pursue the theory or computation required in the next two sections. The calculations for these mini projects can be done by hand if necessary, and can thus be used as a warm-up or for numerical check against any computer programs developed in the following.

5.1. The Proposed Canadian Constitutional Amendment Scheme. A method for amending the Canadian constitution was proposed in 1971 which involved passage by the federal government as well as a ratifying procedure by the ten provinces. Our immediate object is to investigate the voting powers exhibited in only this latter ten-person game between the provinces, and to compare the results with the provincial populations.

The winning coalitions or those with veto power can be described as follows. In order for passage, approval is required of

- a) any province which has (or ever had) 25% of the population,
- b) at least two of the four Atlantic provinces, and
- c) at least two of the four western provinces which currently contain together at least 50% of the total western population.

Using current population figures this means that veto power is held by

- d) Ontario (O) and Quebec (Q),

- (e) any three of the four Atlantic (A) provinces (New Brunswick (NB), Nova Scotia (NS), Prince Edward Island (PEI), and Newfoundland (N)),
- (f) British Columbia (BC) plus any one of the three prairie (P) provinces (Alberta (AL), Saskatchewan (S), and Manitoba (M)), and
- (g) the three prairie provinces taken together.

The various types of winning coalitions are listed in the Table 5.1.1.

Ontario and Quebec are not listed since they are in every winning coalition, but are included in the number s of players in coalition S .

WINNING PROVINCIAL COALITIONS

Type	S	s	No. of such S
1	1P, 2A, BC	6	18
2	2P, 2A, BC	7	18
3	3P, 2A	7	6
4	1P, 3A, BC	7	12
5	3P, 2A, BC	8	6
6	2P, 3A, BC	8	12
7	3P, 3A	8	4
8	1P, 4A, BC	8	3
9	3P, 3A, BC	9	4
10	2P, 4A, BC	9	3
11	3P, 4A	9	1
12	3P, 4A, BC	10	1
Total:			88

TABLE 5.1.1

The numbers in the last column of Table 5.1.1 can be used with the formula for the Shapley value (Section 3.3) to compute ϕ for each of the provinces as well as regional groups, and these are shown in Table 5.1.2. For example,

$$\phi_O = [18(5!4!) + 36(6!3!) + 25(7!2!) + 8(8!1!) + 1(9!0!)] \div (10!) = 53/168 ,$$

and

$$\phi_M = [6(5!4!) + 10(6!3!) + 7(7!2!) + 3(8!1!)] \div (10!) = 1/24 .$$

SHAPLEY-SHUBIK INDEX FOR PROVINCES

Province	ϕ (in %)	% Population	ϕ /Population
BC	12.50	9.38	1.334
AL	4.17	7.33	0.570
S	4.17	4.79	0.872
M	4.17	4.82	0.865
(4 Western)	(25.01)	(26.32)	(0.952)
O	31.55	34.85	0.905
Q	31.55	28.94	1.092
NB	2.98	3.09	0.965
NS	2.98	3.79	0.786
PEI	2.98	0.54	5.53
N	2.98	2.47	1.208
(3 Atlantic)	(11.92)	(9.89)	(1.206)

TABLE 5.1.2

The calculations for ϕ described above were made by D.R. Miller (1973), and he obtained his population figures for the ten-province area from the Canadian Almanac and Directory 1972 (Toronto, 1972). There are several extensions of his work which would also be of interest.

(i) Verify the results for ϕ which are given in Tables 5.1.1 and 5.1.2.

(ii) Determine the Banzhaf value β for this game between the provinces, and compare it to ϕ . Also compute the ratio β (in %) $\div$ % Population.

These answers, as computed by Cornell University students George Kaloroumakis and Eleanor Walthers in May of 1974, are given in Table 5.1.3.

(iii) Use the "square root" result in Sections 6.2 and 7.3 to determine the relative β for the individual voters (assuming all vote) in the voting

game played within their own provinces. Then compute an approximate relative voting power of each citizen in the composed national game by multiplying these latter values by the corresponding ϕ or β for his province as given in (i) or (ii).

BANZHAF INDEX FOR PROVINCES

Province	β (in %)	β /Population
BC	12.44	1.33
AL	5.70	0.78
S	5.70	1.19
M	5.70	1.18
(4 Western)	(29.53)	(1.12)
O	22.80	0.65
Q	22.80	0.79
NB	6.22	2.01
NS	6.22	1.64
PEI	6.22	11.51
N	6.22	2.52
(3 Atlantic)	(24.88)	(2.51)

TABLE 5.1.3

The mathematical results leave room for a good deal of discussion. Some have criticized this scheme (before seeing Miller's results) for giving BC too much power. Are the low figures for AL and especially PEI justifiable? What are some reasonable alternatives to the proposed scheme? One would wish to avoid making PEI a dummy, for example. A brief discussion along these lines is given in Miller's paper.

5.2. Power in a Nine-Man Body. A colleague related to me that he was a member of a nine-man committee on which he and four others formed a coalition. This bloc of five agreed covertly that they would all vote the

ame way on any issue before the full committee (except for a few deviations or diversionary reasons on minor issues or when others not in their coalition also supported their view), and thus a majority of three or more from this coalition effectively controlled the full committee. So each man in this minimal winning coalition increased his voting power from $1/9$ th to $1/5$ th. At the time, however, my friend was wondering whether three of the other four in his coalition had not formed another secret three-man subcoalition without him, which in turn was controlling the whole committee. It would thus become possible to win on a ballot on which there were really only two favorable votes, and six members would be effectively dummies.

In cases like this, as well as others, it is of interest to investigate the power inherent in the cosets in various partitions of a nine-man body. One might also be concerned with changes in power caused by realigning such blocs, and whether the potential gains to counter-organization come from existing blocs or from the nonaligned individuals. The Shapley values for the various sized coalitions in the different types of partitions of a nine-man body have been determined by Krislov (1963) and listed in Table 5.2.1 for the case of a simple majority quota.

Some similar projects along these lines could be undertaken such as the following.

- (i) Compute the Banzhaf value β for all subsets in all partitions of a nine-man body.
- (ii) Determine ϕ and β for subsets of committees of sizes different than nine, e.g., of size 5, 7, 8, or 11; and denote the gains in new power ratios which accrue from coalition building.
- (iii) Consider the previous cases for quotas other than a simple majority, e.g., three-fifths or two-thirds.

NINE-MAN COMMITTEE

Partition Type	ϕ
5,4 (etc.)	(1,0) (etc.)
4,4,1	(1,1,1)/3
4,3,2	(1,1,1)/3
4,3,1,1	(3,1,1,1)/6
4,2,2,1	(3,1,1,1)/6
4,2,1,1,1	(6,1,1,1,1)/10
4,1,1,1,1,1	(10,1,1,1,1,1)/15
3,3,3	(1,1,1)/3
3,3,2,1	(1,1,1,0)/3
3,3,1,1,1	(9,9,4,4,4)/30
3,2,2,2	(3,1,1,1)/6
3,2,2,1,1	(4,2,2,1,1)/10
3,2,1,1,1,1	(4,2,1,1,1,1)/10
3,1,1,1,1,1,1	(9,2,2,2,2,2,2)/21
2,2,2,2,1	(1,1,1,1,1)/5
2,2,2,1,1,1	(7,7,7,3,3,3)/30
2,2,1,1,1,1,1	(25,25,11,11,11,11,11)/105
2,1,1,1,1,1,1,1	(7,3,3,3,3,3,3,3)/28
1,1,1,1,1,1,1,1,1	(1,1,1,1,1,1,1,1,1)/9

TABLE 5.2.1

Whenever one or more individuals defect from a coalition and migrate into another one (perhaps a new singleton), the realigned coalition structure (i.e., partition) will normally exhibit new power indices for the individuals. If "political man" attempts to maximize his voting power as "financial man" attempts to maximize his profits; then one might expect that such changes, e.g., individuals between political parties, might indicate an increase in the migrator's power or the total power of the party he joins, or that migrations are more common among those with little power to begin with. Riker (1959) studied these hypotheses in the case of the French National Assembly during the 1950's. During the period he considered there were

many parties, some of which exhibited strict party discipline, as well as frequent migrations. For example, a typical Assembly was a majority game such as

$$[313; 105, 100, 88, 85, 75, 55, 46, 32, 23, 13, 1, 1, 1] .$$

Riker developed several formulas to test various power changes before and after migrations. Most of the results were rather ambiguous regarding his hypotheses. It appears that such changes were more for ideological reasons than attempts to gain power, but the situation may have been much too complicated for them to even sense, much less calculate, such indices.

Returning to our nine-man body, we see that several powerful coalitions are quite unstable in the sense that a defector can gain significantly. For example, a partition of type 5,4 seems highly unstable, since a change to type 4,4,1 reduces the power of the five-man coalition from 1 to $1/3$ while increasing the swing-man's own share of the power from $1/5$ to $1/3$.

Some of the notions from Luce's theory of ψ -stability relate to models with changes in partitions. See Chapter 10 in R.D. Luce and H. Raiffa (1957).

Investigations into coalition stability and changes in power would be of interest.

(iv) Use Table 5.2.1, or the results from (i) above, and Riker's paper as an outline to study stability and changes in power to the various individuals and groups in a nine-man committee in which migration is allowed. Similar studies can be done for committees of other sizes, using for example the results in (ii).

(v) Investigate such changes in power for some real-world body, e.g., some local college committee, or the U.S. Supreme Court where death and new appointments or other reasons bring about changes in the coalition structures

in the case of Justice Rutledge. The reference by Schubert (1959) is of me interest here.

5.3. Multi-Member Districts. In many voting assemblies, such as some state legislatures, the Electoral College, a university senate, etc., the voting representatives often come from districts which differ greatly in the number of citizens they have. It is often thought that any inequity created by differences in population can be balanced by electing different numbers of representatives from the various districts. For example, if one district in a state contains a city and has 4,000,000 people whereas all other districts have 1,000,000 inhabitants, then it is viewed as "fair" if the first district elects four times as many representatives at-large from its region as does each other district in the state. This seems to assume that the ability of an individual citizen to affect the election of his representatives varies inversely with population. It is shown in the following examples that this is not a valid assumption in terms of the Banzhaf power index. In fact, a citizen-voter's decreasing influence in electing his representatives varies inversely with the square root of the increasing population, and this will be demonstrated in general in Section 6.2.

Banzhaf (1966) has illustrated how one's influence varies with population for districts with only a few citizens and in the case when the elections are between just two opposing parties, and his example is shown in Table 5.3.1.

SOME MINIATURE DISTRICTS

District Symbol	No. of Voters	No. of Combin.	No. Combin. Marginal	% Combin. Marginal	Individ. % of Pop.		
	n	2^n	b	$100b/2^n$	$100/n$	$100/\sqrt{n}$	$nb/2^n$
A	3	8	4	50.0	33.3	57.7	1.50
B	5	32	12	37.5	20.0	44.7	1.88
C	7	128	40	31.2	14.3	37.7	2.18
D	9	512	140	27.4	11.1	33.3	2.46

TABLE 5.3.1

note there that $b = 2 \times (2m)! / (m!)^2$ where $m = (n-1)/2$. For example, when $n = 9$, the percent of voting combinations in which a citizen is marginal (27.4) is closer to $100/\sqrt{n} = 33.3$ than it is to $100/n = 11.1$. For large n the reciprocal square root approximation, when properly normalized, is extremely accurate (see Section 6.2). So that one could argue in the example mentioned above, that the city with 4,000,000 inhabitants should only have twice the number of representatives as a district with 1,000,000 constituents. This type of argument can be made more forcefully when one considers the influence the representatives from large districts have in turn in the state legislature if they vote as an irrevocable bloc. The effect of so "composing" two such voting games is shown in a simple example below, and for the more realistic Electoral College in Section 7.

Consider a small "state" which is divided into seven districts in which the first district has a voting population of 9 and each of the other six districts has a voting population of 3. Assume that this large district elects 3 representatives at-large by choosing a group of three people from the same party who then vote the same way on any issue before the state

legislature. Each one of the small districts elects one person to the legislature. Furthermore, assume that each representative acts as a true delegate and votes on each question the way in which the majority of his constituents would vote. We see from Table 5.3.1 that a citizen-voter from a small district can influence his or her representative's vote in 50% of the voting combinations, whereas the individual in the large district can influence the vote by the party of three legislators in only 27.4% of the combinations in his district's polls. However, in the legislature itself one has the weighted majority game $[5;3,1,1,1,1,1,1]$. One can show that the Banzhaf power index for this game is $\beta = (5,1,1,1,1,1,1)/11$. This is close to the Shapley-Shubik index of $\phi = (9,2,2,2,2,2,2)/21$ which was computed in Section 4 for all such games with one large player. So a citizen-voter from the large district can influence his group of three representatives 27.4% ($= 140/512$) of the time, and they in turn can sway the legislature 78.1% ($= 100/128$) of the time, for a product of 21.4%. On the other hand, an individual from a small district influences his lone legislator 50% ($= 4/8$) of the time, and his delegate in turn changes the full legislature 15.6% ($= 20/128$) of the time, for a product of 7.8%. These products can approximate an individual's power in the "compound" game.

Previous examples had indicated that in a weighted voting situation the "big" players with the most votes often, but not always, have somewhat larger power indices than their fractions of the total vote would indicate. In this section one observes further that if several delegates are elected at-large from various districts, then the influence of the individual voter in the large districts may be greatly exaggerated relative to the other regions when the number of delegates per district is proportional to its population. Banzhaf (1966) actually applied this sort of an analysis to five state

egislatures. It would make an interesting project to investigate some additional examples which have such multi-member representation, under the assumption of bloc voting behavior.

- i) Investigate some local or state governmental body, board, agency or convention of citizens which has a multi-member representative structure. For example, some school boards have a representation from various constituencies. The school boards in New York City would make an interesting study.
- ii) A college or university has several schools, divisions or departments of various sizes. Investigate how to construct a representative council or some such structure, for example, a mathematics departmental or divisional executive committee which has "fair" representation from each interest or speciality group.

5.4. Some Current Parliaments. A review of recent newspapers and newsmagazines will give interesting data concerning current elections and national assemblies. Especially in the year 1974, when there are several minority governments in the Western World in which a winning coalition of parties must form in order to obtain a ruling government. Some countries also have five to ten different parties in their parliaments, and this requires a reasonable sized computational exercise. Data for four countries are listed below; and one can compute ϕ and β for these assemblies under the assumption of strict party-line votes. This latter assumption has some validity in at least the formative stages of a government, and even at later stages in those countries in which strict party discipline exists.

The following data is from the spring of 1974; and even some of it should be checked further or extended in order to be completely reliable. Such

Figures are frequently in flux, and one is dealing here with rather dynamical situations. For example, some members of the British Parliament have already lost their votes because they have taken leadership roles, etc. The "other" category could also be broken into three singletons. The newspaper reports on Israel were inconsistent and difficult to sort out, and so the following data should be validated. The figures on Denmark do not include a small number of representatives from Greenland and the Faroe Islands. If one wished to do a current and accurate investigation, he should first verify and refine the following data. It is also interesting how most election results in the newspapers are reported as the percent of the popular vote taken by each party (or often just the major parties) rather than the number of seats won; whereas the latter results are the more critical ones, and they are not always easily obtainable from the former percents.

(i) Luxembourg:

Party	No. of Seats	
	Early 1974	Mid 1974
Christian Democrats	21	18
Socialist	12	17
Splinter Socialists	6	5
Liberals	11	14
Communists	6	5
	<hr/>	<hr/>
w	56	59
q	29	30

TABLE 5.4(i)

(ii) Great Britian:

Party	No. of Seats
Labor	301
Conservatives	296
Liberals	14
Irish Unionists	11
Scottish Nationalists	7
Others	3
Welsh Plaid Cymru	2
Irish Catholics	1
w	635

TABLE 5.4(ii)

(iii) Israel:

Party	No. of Seats
Labor	51
Likud	39
National Religious	10
Independent Liberals	4
New Communists	4
Orthodox Agudat Israel	3
Agudat Israel Workers	3
Arab Affiliates	3
Civil Rights	2
Moked	1
w	120

TABLE 5.4(iii)

(iv) Denmark:

Party	No. of Seats	Ordinate*
Communists	6	.00
Socialist Peoples	11	.18
Social Democrats	46	.37
Center Democrats	14	.50
Justice League	5	.54
Radical Left	20	.58
Left (Liberals)	22	.64
Christian Peoples	7	.79
Conservative Peoples	16	.84
Progress	28	1.00
	<hr/>	
w	175	

TABLE 5.4(iv)

Even in countries which have a one-party majority government, one may be able to study the power in various interest groups within this party or the assembly as a whole. See, for example, the study by Lieserson (1968) in Japan.

5.5. Cornell University Senate. A broadly representative University Senate has been in operation at Cornell in recent years, and it has significant powers theoretically over events and budgets in important areas such

These ordinates are an estimate of the parties positions on the political spectrum as measured from left to right, and they will be referred to in Section 8.1.

as student life and activities. The size of this Senate as well as of each of its six major groups has varied, and these numbers are given in Table 5.5.1.

CORNELL SENATE

Constituency	1971	ϕ	β	1972	1974	No.
Students	60	9	4	60	40	15,500
Faculty	60	9	4	60	40	1,550
Employees	5	6	3	13	10	5,000
Alumni	2	2	1	2	1	130,000
Administration	2	2	1	2	2	9
Non-Prof. Acad.	3	2	1	3	2	550
w	132	($\div$)	($\div$)	140	95	
q	67	(30)	(14)	71	48	

TABLE 5.5.1

The number in the Employee's category was increased by eight between 1971 and 1972 for several reasons. Some of these were perhaps more idealistic, such as support for labor in general or because of the large number of members in this class. (At Cornell there are nearly four people in the mostly invisible "supply line" for each professor at the "front".) Other reasons were more practical, such as the fact that the employees were critically involved in the consequences of Senate decisions, and usually it was desirable to have at least one such overworked employee senator on most of the two dozen or so committees and subcommittees. The total size of the Senate was reduced by 1974 in attempts to improve efficiency and because of large absentee problems. The Shapley-Shubik and Banzhaf values for the 1971 Senate are shown in the Table. The correction for the benefit of the Employee group in the two later Senates may have been "overdone" however, since the result

was three groups of equal power and three dummies. (Of course, some students knew right along that the Alumni, Administration (e.g., the Provost and vice presidents) and Non-Professional Academics (e.g., librarians, etc.) were 'dummies'.)

The assumption of bloc voting or "party-line" discipline within the six groups listed in the Table is clearly invalid in practice for this particular Senate, as one would expect in most college situations. Extremes of opinion within some of the large groups could be found on almost any issue. On the other hand, there may be specific local issues on which a partitioning of the Senate into various interest groups might make sense. For example, some groups may have strong feelings in favor or against athletics, whereas different groups may have intense "pro" or "anti" attitudes on Greek living units. Consistent voting patterns might also be discerned on topics such as women or minority students and affirmative action programs. Such data may lead to power analyses and prediction of outcomes on such individual legislation.

There are many interesting projects which one can undertake concerning university, college, school or departmental senates, and a few of these are suggested below. One could use the data in Table 5.5.1, or collect similar figures for his own institution. The numbers for the six constituencies shown in the right hand column in this Table, are for the 1971 Cornell Senate. Except for constitutional amendments and a few procedural matters, a simple majority is used for the quota.

(i) Verify the values of ϕ and β given in Table 5.5.1.

(ii) Do a "two-level" analysis using the ϕ and β for Table 5.5.1 (or your own college data) plus the "square-root" result in Sections 6.2 and 7.3 to estimate the power of an individual person (not senator) in the six groups.

(iii) If the committee structure of a senate is known, then one can investigate the power of an individual representative in the sequential process of getting a bill through committee and the senate. For example, as was the case in the first Cornell Senate, an alumnus on the powerful Executive Committee would increase his power as well as that of the Alumni group.

(iv) Given the number of student representatives a senate will have, determine a "fair" allocation of this number among the various colleges in a university, or departments in a college.

(v) One can design a senate from scratch by deciding what power the different individuals and/or groups should have and then setting the number of representatives from the various constituencies so as to best achieve this. (See the following Section for additional ideas on (iv) and (v).

(vi) The power of the various committees and/or interest blocs for a university board of trustees or state regents may be of interest, especially in the case of a rather large board as is the case for Cornell University.

5.6. Designing Representative Bodies. In most of the weighted voting problems discussed in this paper, one is given a well structured problem (i.e., the weights) and is asked to determine the corresponding voting power as measured by the Shapley-Shubik and/or Banzhaf indices. On the other hand, one is often faced with the opposite or reverse problem in practice; i.e., if one is designing a legislature, he is given the size of the districts and he must then determine the weights or number of representatives for each in order to achieve a certain given or desirable measure of power. Projects (iv) and (v) in the previous Section are of this type. Most work in this reverse direction uses less elegant or straightforward techniques. One often uses more ad hoc, self correcting, or guess-and-verify methods. Some approximation

algorithms do exist, although they may not always be readily available (without a fee). There are also several substantial and unsolved mathematical problems along these lines, in addition to the practical ones; but the author will not go into further detail here on these more theoretical directions. One should remember, however, that there may be alternate types of solutions to his voting problem, e.g., reapportionment, redistricting, or multi-member districts, in addition to finding the "correct" (unequal) weights for the lone representative from each district.

Let us return to the voting rules in the Nassau County Board of Supervisors which is described in Table 2.3.1 in Section 2.3. Recall that the 1958 and 1964 systems each had three dummy municipalities. In 1971 a new scheme was passed which merely changed the quota from a simple majority of 58 to 63 votes in order to pass a bill. (Actually, the County Charter in 1938 allows one vote per 10,000 residents, and thus Hempstead should currently have a weight of 72. However, since no town or city is allowed by this Charter to have more than one-half of the total vote of the board (i.e., $125 \div 2$), one subtracts ten votes from Hempstead to get 62 and takes w as 115). There are no dummies in the revised scheme, and the new values of ϕ and β , as computed by a Cornell student, Thomas Reahard, with $q = 63$, are shown in Table 5.6.1.

NASSAU COUNTY REVISITED

Municipality	No. of Votes	60 ϕ_i	36 β_i
Hempstead No. 1	31	17	10
Hempstead No. 2	31	17	10
North Hempstead	21	7	5
Oyster Bay	28	13	7
Glen Cove	2	3	2
Long Beach	2	3	2

TABLE 5.6.1

Note that the new power indices are not too extreme in comparison with the proportions of the population. Attempts to invalidate this latter scheme via the courts have failed. Related stories appeared in the New York Times on May 4, 9 and 12, 1971 and February 2, 1974.

- (i) Verify the values of ϕ and β given in Table 5.6.1.
- (ii) Another problem in the Nassau County scheme is that the two representatives from Hempstead are elected at-large from the full district, which gives rise to questions raised in Sections 6.2 and 7.4. Reconsider this scheme in light of this additional problem.

In 1971 a suit was brought against Cortland County, New York which claimed that its one town - one vote voting scheme was unfair, and the state courts agreed that it was unconstitutional. A proportionately-weighted interim scheme was then set up, while the County undertook to devise a new valid apportionment plan. The new system, involving some redistricting, used the 1970 Decennial Census figures and the Banzhaf index for weighted voting. It has since been accepted by the courts, and took effect on January 1, 1974. The new law for the Cortland County Legislature gives one legislator to each of the 19 districts, and weights his vote as indicated in Table 5.6.2. There are different weights necessary for each of three cases, depending upon whether a simple majority, a three-fifths majority, or a two-thirds majority is required to pass an issue. The computation of the desired weights was done by Lee Papayanopoulos, a New York City based consultant, who has a computer program for obtaining approximate weights to satisfy equity in the sense of the Banzhaf index. (The County paid \$2600 for his report, and he later collected \$350 per day for his services in appearing in court.) A study of the Table can point out some "discrepancies" which should give rise to further questions. For example, the nine largest districts control

CORTLAND COUNTY LEGISLATURE

District Number	Popu- lation	% of Pop.	Weights when Majority Quota is			Case $q = 1/2$	
			1/2	3/5	2/3	% of Vote	% of Power
3	2716	5.918	31	40	38	6.739	5.859
7	2560	5.578	27	35	36	5.870	5.612
13	2517	5.484	26	34	34	5.652	5.482
14	2500	5.447	26	33	34	5.652	5.482
2	2478	5.399	25	33	33	5.435	5.358
8	2467	5.375	25	32	33	5.435	5.358
18	2460	5.360	25	33	33	5.435	5.358
15	2444	5.325	25	32	32	5.435	5.358
16	2442	5.321	25	32	32	5.435	5.358
6	2442	5.321	25	32	32	5.435	5.358
5	2440	5.317	25	32	33	5.435	5.358
19	2434	5.304	24	32	33	5.217	5.238
12	2422	5.277	24	32	32	5.217	5.238
1	2406	5.243	24	31	31	5.217	5.238
17	2402	5.234	24	31	31	5.217	5.238
4	2284	4.977	22	28	29	4.783	4.916
9	2187	4.765	20	26	25	4.348	4.735
10	2153	4.691	19	25	24	4.130	4.727
11	2140	4.663	18	25	23	3.913	4.727
Totals	45894	100.000	460	598	598	100.000	100.000

TABLE 5.6.2

35 votes for more than a simple majority of 231, and yet they have only 49.2% of the population. Note also, for example, that in the two-thirds plan, districts 15, 16, and 6 have one less weight than the slightly smaller districts 5 and 19. Incidentally, there was some controversy in designing the current scheme about how to count the students at Cortland State.

iii) Consider a corporation in which four stockholders have 60, 20, 10, and 10 percent of the stock respectively, and for which the company charter calls for all measures to be passed by a simple majority vote of the stock. The directors, knowing that the majority stockholder is a dictator (in our mathematical sense), agree to modify the rules to have "voting power" (say as measured by ϕ and/or β) approximate the relative stock holdings. Two proposals (or is it three or four?) are put forward:

A. Majority rules except when all minority stockholders unify against the majority stockholder. (One may question whether this verbal statement means that the minority have merely veto power or are a winning coalition.)

B. Majority rules except when at least two-thirds of the minority stockholders' votes are opposed to the majority stockholder.

(iv) One might wish to reconsider the projects (iv) and (v) in the previous Section 5.6 in light of the discussion in the present Section.

(v) It would be of interest to undertake a project to redesign some local governmental body. For example, there are still several counties in New York State with unconstitutional systems which are in need of revision. This redesigning has been done for several other counties in New York, and an excellent report on this through 1968 is given in Johnson (1969).

5.7. The U.S. Congress. Several investigations have been made into measures of power for various sorts of legislatures. In a unicameral body in which a winning coalition depends only upon the number of members it contains,

each legislator theoretically has equal power. More interesting problems occur in multicameral bodies or "composed" games, in which various majorities in several houses of different size may be required for passage. An interesting case is the U.S. Congress where a bill requires a majority in the House and the Senate plus the President's signature, or a two-thirds majority of both houses in the case of a veto override. Shapley and Shubik (1954) have determined ϕ as: $5/12$ for the House of Representatives as a whole, $5/12$ for the Senate, and $1/6$ for the President. Several other studies have since been done of the U.S. Congress, and one can refer to Section 12.1 of Luce and Raiffa (1957), to Chapters 14 (especially page 215) and 16 (pages 245-250) of Rapoport (1970), as well as to the references given there for reports on the early work done in this area. Also of interest is Chapter 11 by Luce and Rogow in Shubik (1964), and the paper by Riker and Niemi (1962). Several types of similar studies should be of interest.

- (i) Investigate the various interest blocs in some legislature like the U.S. Congress, at least for the case of some particular interest area, and then determine the voting power of the various blocs.
- (ii) Do a power analysis for some state legislative system.
- (iii) Consider the "committee system" which exists for some legislative body such as the U.S. Congress or a University Senate, and determine power indices under the assumption that a bill must pass in the committee before it reaches the floor.

5.8. Some Mathematical Problems. By now it should be clear to the mathematical reader that there are a great number of mathematical questions and relations which can be pursued, as class projects or basic research, concerning our weighted voting games and power indices. Although our main

val has been to present interesting or relevant applications suitable for mathematical modeling, a few more theoretical sorts of problems are mentioned very briefly in this Section. Although there are a host of such interesting questions, only a couple are suggested. No current survey of the state of this field is intended, and no detailed references are given. Although the suggested problems are often very difficult for general n -person voting situations, suitable class projects result if they are restricted to the cases of $n = 3$ or 4 .

For an arbitrary n and given q , the set of all possible weighted majority games $[q; w_1, w_2, \dots, w_n]$ can be considered as the points in an $(n-1)$ -dimensional simplex W embedded in n -space. One can normalize the weights so that they sum to $w = 1$. To each game there corresponds a particular collection of winning (or minimal winning) coalitions in the lattice of all subsets 2^N of the voter set $N = \{1, 2, \dots, n\}$. Each such collection in turn gives rise to a Shapley-Shubik index ϕ , as well as a Banzhaf index β . Although the set W is infinite, the resulting number of collections of winning coalitions as well as the collections ϕ and β of realizable indices ϕ and β are finite. Since any ϕ_i can be written as an integer divided by $n!$, and any β_i can, for each particular weight vector, be written as an integer over the total number of marginal situations for all n players. One can investigate how W partitions into equivalent games, i.e., games which give the same ϕ (or β). One can pick a particular q , or more generally, partition $W \times [0, 1]$ as q varies over the interval 0 to 1 (or just over $[1/2, 1]$ if one wishes to maintain our earlier assumption). One can see that not all fractions with a denominator of $n!$ can be realized as a ϕ_i .

It is natural to inquire as to which games have $\phi = \beta$, and when is ϕ or β proportional to the weight vector $(w_1, w_2, \dots, w_n)$. A few examples, in the case of ϕ , are given on pages 208-209 of Riker and Shapley (1968).

If an index ϕ , e.g., $\phi = (3,1,1,1)/6$, is considered as a weight vector of a new game, then this game will not in turn have the same index ϕ . Furthermore, not all voting situations (in the sense of minimal winning coalitions) can be represented as weighted majority games. One can consider this question in the case of the Canadian Constitutional Amendment Scheme (Section 5.1); or for the seven-point plane projective geometry discussed on pages 469-470 in von Neumann and Morgenstern (1953), i.e., where the minimal winning coalitions consist of the three points (or voters) which lie on a particular one of the seven lines. One interested in mathematical questions about weights should consult Isbell (1964) and Liapot (1972) and the references they have.

Various methods for composing or combining games can be introduced. The corresponding power indices do not normally compose in any straightforward manner. For example, consider the situation $[3;2,1,1]$ with $\phi = (4,1,1)/6$, in which the third player is really a group of three players whose vote in turn is determined by majority rule. So this "player" really plays the game $[2;1,1,1]$ with $\phi = (1,1,1)/3$ before he votes in the larger situation. We can view this succession of votes as the one game $[6;4,2,1,1,1]$ which has $\phi = (39,9,4,4,4)/60$ and $\beta = (6,2,1,1,1)/11$. So, for example, ϕ_1 has changed from $2/3$ in the original situation to $39/60$ in the composite game.

Some results on the average value of ϕ_i in a given game when the quota q is allowed to vary are discussed in Mann and Shapley (1960), e.g., on page 16.

6. COMPUTATIONAL AIDS

6.0. Introduction. In order to obtain the power indices for games with any players and irregular weights, such as the Electoral College discussed in section 7, it is rather essential to have some computational devices, as well as machine facilities and programs to implement them. Mann and Shapley (1960) used some Monte Carlo techniques to obtain ϕ for the Electoral College, but greater accuracy was desired. Section 6.1 describes some mathematical tricks, suggested by Mann and Shapley (1962), and by David G. Cantor, which simplify the work of calculation; and an actual algorithm for performing this task is referred to. Section 6.2 describes an approximation method using Birtling's formula for computing the number of voting combinations in which an individual voter from a large district is marginal. This technique can be used to compare the relative influences in the sense of Banzhaf of two citizens from different sized districts with equal representation as discussed in Section 5.3, as well as for two individuals from different states in the U.S. Electoral College as is done in Section 7. Section 6.3 mentions an extension of the Shapley value due to G. Owen which gives a good approximation technique for determining ϕ for large voting games.

6.1. Computing the Shapley-Shubik Index. The formula developed above for ϕ was expressed as

$$\phi_i = \sum \frac{(s-1)!(n-s)!}{n!}$$

where the summation is taken over all coalitions S of s players in

which i is pivotal. If one lets

c_{js}^i = Number of ways in which s players, other than i ,
can have a sum of weights equal to j ,

then the formula, in the case of the n -person game $[q; w_1, w_2, \dots, w_n]$, can
be written as

$$\phi_i = \frac{1}{n} \sum_{s=0}^{n-1} \binom{n-1}{s}^{-1} \sum_{j=q-w_i}^{q-1} c_{js}^i,$$

where the numbers

$$\binom{n-1}{s}^{-1} = \frac{s!(n-1-s)!}{(n-1)!}$$

are the binomial coefficients, and can be read off the $(n-1)$ st row of the
Pascal triangle. This latter coefficient, when divided by n , is the number
of ways, in which the s players in a given coalition S which join prior
to i and the remaining $n-1-s$ players in $N-\{i\}-S$ which follow i , can
be ordered. If, and only if, the weights of the s members in S sum to
a j in the range $q - w_i \leq j \leq q - 1$ does player i have enough weight w_i
in order to join S and thereby first reach or exceed the quota q . The
 c_{js}^i then give the number of such distinct coalitions for each j and s .
The number of terms in this double summation for ϕ_i is not excessive for
machine calculations, and thus the major task is to determine the values for
the c_{js}^i .

As related by Mann and Shapley (1962), Cantor's contribution was to
point out that the c_{js}^i can be readily obtained from the generating function

$$f_i(x,y) = \prod (1+x^{w_k}y)$$

where the product is taken over all $k \in N - \{i\}$, since the desired c_{js}^i are merely the coefficients of the corresponding $x^j y^s$ terms. The proof of this is left as an exercise. For any i , $F_i(x,y)$ can be obtained from the full n -fold product $\prod (1+x^{w_k}y)$ divided by $(1+x^{w_i}y)$.

The c_{js}^i can also be found as the elements of a matrix C^i of integers, and for each player i this matrix can be generated inductively as follows. Define $C^{(0)}$ so that $c_{00}^{(0)} = 1$ and all other $c_{js}^{(0)} = 0$. Then $C^{(r)}$ is obtained from $C^{(r-1)}$ by the relation

$$c_{js}^{(r)} = c_{js}^{(r-1)} + c_{j-w_p, s-1}^{(r-1)}$$

where the last term is taken as 0 when either of its subscripts is negative, and the successive w_p stand for the weights of the distinct players in $N - \{i\}$. After $n-1$ iterations one gets $C^{(n-1)} = C^i$. One can obtain $C^{(n)}$ by taking all $r \in N$, and then get each C^i by subtracting once, i.e., by reversing our recursive relation above.

Consider the example $[q; w_1, w_2, w_3, w_4] = [6; 4, 3, 2, 1]$, and compute ϕ_1 . Only the nonzero elements of our matrices will be given. Beginning with $C^{(0)}$ with $c_{00}^{(0)} = 1$, and using $c_{js}^{(1)} = c_{js}^{(0)} + c_{j-w_2, s-1}^{(0)}$, gives $C^{(1)}$ with $c_{00}^{(1)} = c_{31}^{(1)} = 1$. Then $c_{js}^{(2)} = c_{js}^{(1)} + c_{j-w_3, s-1}^{(1)}$ gives $C^{(2)}$ with $c_{00}^{(2)} = c_{21}^{(2)} = c_{31}^{(2)} = c_{52}^{(2)} = 1$. And $c_{js}^{(3)} = c_{js}^{(2)} + c_{j-w_4, s-1}^{(2)}$ gives $C^{(3)}$ with $c_{00}^{(3)} = c_{11}^{(3)} = c_{21}^{(3)} = c_{31}^{(3)} = c_{32}^{(3)} = c_{42}^{(3)} = c_{52}^{(3)} = c_{63}^{(3)} = 1$. Our formula restricts j to the range $2 \leq j \leq 5$, and gives

$$\phi_1 = \frac{1121}{4!} \cdot 2 + \frac{2111}{4!} \cdot 3 = \frac{5}{12}.$$

Not all of the above mentioned matrices need be stored in the computer, and only the ranges $0 \leq j \leq q - 1$ and $0 \leq s \leq (n-1)/2$ are necessary since $c^{(n-1)}$ is "symmetric" in the sense that $c_{js} = c_{w-w_p-j, n-1-s}$.

A detailed algorithm suitable for computer programming and designed to calculate ϕ using the approach described above has recently been written by Boyce and Cross (1973).

6.2. The Number of Marginal Combinations. Since the number of voting combinations, 2^n , and the number of marginal combinations, involving factors of the order $n!$, are much too large for even machine calculation, it is necessary to develop simpler approximating devices in order to handle realistic sized voting districts. Assume that a district has n voters where n is taken as odd, and that an election is won by a simple majority, that is, by $(n+1)/2$ votes. A particular voter is marginal whenever the other $n-1$ players divide into exactly one half for an issue and the other half against it. The number of such equal divisions or combinations is given by

$$\frac{2(n-1)!}{\left(\frac{n-1}{2}\right)! \left(\frac{n-1}{2}\right)!}$$

where the 2 in the numerator comes from assigning either side in a division the "yes" vote. One can obtain the relative influence of each player in the sense of Banzhaf by dividing this fraction by 2^n . Using Stirling's formula (which appears in many advanced calculus texts),

$$m! = \sqrt{2\pi m} m^m e^{-m},$$

one can reduce this new fraction to

$$\frac{2(n-1)!}{[(\frac{n-1}{2})!]} \times \frac{1}{2^n} = \frac{1}{\sqrt{2\pi(n-1)}}.$$

If we assume that all voting combinations are equally likely, then this is the fraction of elections in which a particular voter i is decisive, and it varies inversely with the square root of the population. Note that since each voter has equal power, one must get $\beta_i = 1/n$; but β_i does not give the fraction of elections in which player i is in the critical position.

This result now justifies some of the assertions made in Section 5.3, and will prove useful in Section 7. It shows that the discrimination or the deprivation built into voting districts relates to this square root factor, if one accepts the Banzhaf analysis. This approximation is a good one for large districts since Stirling's formula is accurate, for example, to one part in one hundred for $m = 100$. Note that this approximation given by Stirling's formula is in the asymptotic sense, i.e., the ratio of the two sides of the equation approaches one for large m ; the actual gap between $m!$ and the right hand side actually increases with m . Another discussion of this "square root effect" is given in pages 83-85 of Shapley (1973).

In a few footnotes, Banzhaf (1965, page 335; 1968, page 312) mentions some people who assisted him in calculating β for some of his applications. One computer program by M. Sackson was copyrighted, and leads one to wonder

if mathematicians can increase their incomes by copyrighting their important theorems. In Section 5.6, reference was made to Lee Papayanopoulos who assisted Cortland County, New York in designing a weighted voting system using the Banzhaf approach. Johnson (1969) also described some of his computer programs for determining proper weights.

6.3. A Computation Method for Large Games. Owen (1972) introduced a generalization of the Shapley value in the context of general n -person games, and a specific application of his extension is an approximation technique for determining the Shapley-Shubik index. His approach is presented on pages P72-P73 in his paper, and it is quite accurate for games with a large number n of voters if each of the weights w_i is relatively small. It need not be too precise when there is exactly one relatively large weight. Owen's method makes use of the normal probability distribution and numerical integration. However, the actual computations are much easier than the exact values as obtained via Section 6.1 in the case of large games such as the Electoral College given in Section 7. Owen's approach is also discussed in pages 78-83 of Shapley (1973), along with an application to a proposed New Mexico weighted voting scheme.

7. THE ELECTORAL COLLEGE

7.0. Introduction. Previous examples have shown that weighted voting systems occur rather frequently in political situations. This Section will be concerned with the familiar process of electing a president of the United States. This is a three level process in which the citizens in each state vote to determine their choice (the state game), who then receives all of that state's weighted vote in the Electoral College (the national game). If the latter vote is indecisive, then the final choice is made in Congress (the Congressional game). This procedure will be described in more detail in Section 7.1; and the national game, state games, and the combination of these games will be analyzed with respect to voting power in Sections 7.2, 7.3, and 7.4, respectively. Some criticisms of this election process are mentioned in Section 7.5, and some possible alternatives to it are also described there.

The American Electoral College provides a well-known real-world institution which has many nonobvious quantitative aspects despite its apparently transparent structure. It is a topic of at least periodic interest, and major changes in it have frequently been proposed. The mathematical complexity of this election process, and of the proposed alternatives to it, make it appropriate for a classroom project for students with some computing capabilities available.

7.1. Electing a President. Consider the indirect procedure by which an American president is elected. It begins with a national popular vote by all eligible citizens who choose to exercise their right. Each voter picks

a particular candidate representing a certain political party. (We are not considering here the preliminary aspects concerning how the candidates survived state primary elections, party conventions, etc.) The candidate who wins the most votes in a given state, no matter how small his margin, will then receive all of that state's weighted vote in the Electoral College. This is referred to as the "winner take all" or "unit-vote" rule, which is imposed by state laws. The individual electors who actually cast the votes in the Electoral College are usually mere functionaries. This purely mechanical role is contrary to their view in the Constitution and has had a few defectors in recent years, and the legality of some such state laws may be questionable. In short, the popular vote for the president only determines a group of electors for each state who in turn are pledged to the winning candidate in their state.

The number of electors which each state has in the Electoral College is equal to its number of representatives in both houses of the Congress. Each state has two senators plus one to forty-five representatives, where this latter number depends upon its population as determined by the regular decennial census and resulting apportionment (which gives rise to another interesting mathematical problem which will be mentioned again in Section 8). Since there are 100 senators, 435 representatives, plus 3 electors from the District of Columbia, there are a total 538 seats in the Electoral College. Any candidate who reaches or exceeds the simple majority of 270 votes is declared the new president.

In the event that no one receives a majority in the Electoral College, then the process goes to a third level and the deadlock is resolved in the Congress. This occurred in the elections of 1800 and 1824. The House of Representatives chooses the president from among the top three candidates in

the Electoral College. (The Senate picks the vice-president from the top two candidates.) In the House, each state has only a single vote, and a majority of 26 is necessary to win. Note that the small states could have extraordinary power relative to their total population in this Congressional determination. The following mathematical analysis will not be concerned with this third level, or potential "tie-breaking" procedure, in the presidential elections. At this third level some truly bizarre possibilities can occur. For example, at one time before the 1968 election it looked as though the order of finish in the Electoral College could be Nixon-Agnew, Wallace-LeMay, and Humphrey-Muskie with the leading Nixon ticket having less than a majority. This could have resulted in H.H. Humphrey as President and General Curtis LeMay as Vice President. In a fictitious novel, Russell Baker (1968) describes how Lindsay, Johnson and Wallace ran for the Presidency in 1968 with none getting a majority in the Electoral College. As a deadlock developed in the House, the Senate chose R.F. Kennedy as Vice President, who then decided to suspend balloting in the House for President in order to buy time to influence their decision.

Our goal is to investigate some of the mathematical structure of the first two stages in the election of a president, and to analyze how power is distributed between the different states and the individual citizens of these different states. Our previous examples have given reason to believe that such investigations of voting influence can give rise to irregular, surprising and unpredictable comparisons in terms of a voter's ability to affect the outcome of an election. This quantitative approach will confirm the intuitive and historical evidence to the effect that there is a bias in favor of the large states and their citizens.

There is an excellent source book by Peirce (1968) on the Electoral College which discusses its history, past changes, past and present reform efforts, and potential alternatives. The appendix of this book contains a host of data relevant to any study of this institution.

7.2. The National Game. The Electoral College can be viewed as a 51-person weighted majority game between the states. This assumes that each state (plus the District of Columbia) acts as an independent player, i.e., as a free agent without prior commitments and outside influences. For this game, the sum w of the 51 weights is 538 and the simple majority quota q is 270. The weight w_i for each individual state i is equal to its number of electoral votes, and these are listed in Table 7.2.1. We will treat the District of Columbia throughout as though it were a state. These weights can change every ten years as a result of the national census, and they have occasionally been modified slightly by Congress. So the Table lists the weights as they were in the 1964 and 1968 elections under the column "1961", and the way they were for the 1972 election under "1972". These are based upon the 1960 and 1970 census figures, respectively, along with the method used to determine the number of seats each state receives in the Congress.

We are interested in determining the voting power for each state in this national game. These results are given in Table 7.2.2. The first column lists in decending order the number of electoral votes, or weight w_i , which a state may have; and columns (A) and (B) list the number of such states with the corresponding weights for the years 1961 and 1972, respectively. The power indices, in the sense of Shapley-Shubik, are given in columns (C) and (D) for the two periods being considered. The first of these (C), as well as results for some earlier periods, were reported by Mann and

STATE ELECTORAL VOTES

State	1961	1972	State	1961	1972
Alabama	10	9	Montana	4	4
Alaska	3	3	Nebraska	5	5
Arizona	5	6	Nevada	3	3
Arkansas	6	6	New Hampshire	4	4
California	40	45	New Jersey	17	17
Colorado	6	7	New Mexico	4	4
Connecticut	8	9*	New York	43	41
Delaware	3	3	North Carolina	13	13
Dist. of Columbia	3	3	North Dakota	4	3
Florida	14	17	Ohio	26	25
Georgia	12	12	Oklahoma	8	7*
Hawaii	4	4	Oregon	6	6
Idaho	4	4	Pennsylvania	29	27
Illinois	26	26	Rhode Island	4	4
Indiana	13	13	South Carolina	8	8
Iowa	9	8	South Dakota	4	4
Kansas	7	7	Tennessee	11	10
Kentucky	9	9	Texas	25	26
Louisiana	10	10	Utah	4	4
Maine	4	4	Vermont	3	3
Maryland	10	10	Virginia	12	12
Massachusetts	14	14	Washington	9	9
Michigan	21	21	West Virginia	7	6
Minnesota	10	10	Wisconsin	12	11
Mississippi	7	7	Wyoming	3	3
Missouri	12	12	TOTAL	538	538

TABLE 7.2.1

* More recent results give 8 votes to Connecticut and 8 votes to Oklahoma due to a change in how some overseas population is allocated to their home state.

ELECTORAL COLLEGE POWER RATIOS

Column: Number of Electoral Votes w_i	(A) Number of States i		(B)		(C) Power Indices ϕ_i		(D)		(E) Rescaled Indices $538 \phi_i$		(F)		(G)		(H) Power Ratios $538 \phi_i \div w_i$	
	1961	1972	1961	1972	1961	1972	1961	1972	1961	1972	1961	1972	1961	1972	1961	1972
45	0	1	--	.08830938	--	47.510446	--	1.0517806	--	1.0557877						
43	1	0	.08406425	--	45.226568	--	42.893309	--	1.0461782							
41	0	1	--	.07972734	--	41.786804	--	1.0446701	--							
40	1	0	.07767063	--	29.591194	--	27.418153	--	1.0203860							
29	1	0	.05500222	--	--	26.368783	--	1.0154873	--							
27	0	1	--	.05096311	--	25.303807	--	1.0134433	--							
26	2	2	.04901260	.04897682	26.368783	25.285424	21.073062	1.0114169								
25	1	1	.04703309	.04699893	25.303807	16.940359	13.862614	1.0034790								
21	1	1	.03919718	.03916926	21.088085	12.857334	11.838222	.99579468								
17	1	2	.03148765	.03146563	16.940359	10.831753	9.829039	.99018686								
14	2	1	.02578474	.02576694	13.872193	8.830049	8.830049	.9834576								
13	2	2	.02389839	.02388196	12.857334	7.834733	6.843059	.9851844								
12	4	3	.02201920	.02200413	11.846331	6.843059	5.854968	.98470464								
11	1	1	.02014710	.02013337	10.839140	5.854968	4.870519	.98290416								
10	4	4	.01828200	.01826959	9.835718	4.870519	3.889509	.9811659								
9	3	4	.01642383	.01641273	8.836024	4.870519	2.911898	.97934191								
8	3	2	.01457252	.01456270	7.840015	4.870519	2.911898	.97757982								
7	3	4	.01272798	.01271944	6.847652	4.870519	2.911898	.97582844								
6	3	4	.01089014	.01088284	5.858896	4.870519	2.911898	.97410373								
5	2	1	.00905894	.00905301	4.873709	3.889509	2.911898	.97237690								
4	10	9	.00723429	.00722957	3.892051	2.913883	2.913883	.9712958								
3	6	7	.00541615	.00541245	2.913883	2.913883	2.913883	.97063223								
TOTALS: 538	51	51	.99999947	.9999997	537.999790	537.999803	537.999803	50.2102844								

TABLE 7.2.2

Shapley (1962); and the second one (D) was given in Boyce and Cross (1973). Columns (E) and (F) rescale the previous two columns by multiplying by the total weight 538 so that they can be easily compared with the number of electoral votes listed in the first column. Columns (G) and (H) express these comparisons as ratios. It should be noted that the numbers in the table were obtained by decimal calculations on the computer and are rounded off, and are excellent approximations to the exact values which are in fact rational numbers since they have a factor of 511 in the denominator. These results indicate a clear and systematic bias in favor of the large states. However, the total magnitude of the bias is less than ten percent, and is thus rather small in light of the many nonquantifiable aspects in such political institutions. So, to a rather good degree of approximation, a state's voting power is proportional to its influence as expressed by its number of electoral votes.

One can design several interesting major classroom projects which are based upon or motivated by the above results, and a few of these are mentioned in what follows.

- (i) Determine the power, in the sense of ϕ or β , for each of the 13 states in the original Electoral College. Power at other historical periods may also be of interest, e.g., at the time of the Civil War.
- (ii) One can attempt to reproduce the results in Table 7.2.2 for the year 1972. This would make use of some computational techniques such as those described in Section 6.1, plus the design of a computer program to implement these. A detailed algorithm for doing this is described by Boyce and Cross, but one would still have to program this for his local computing facilities.
- (iii) The data in Table 7.2.2 can be analyzed graphically for the 1972 results. One can plot the points (w_i, ϕ_i) in the plane, find the best fitting straight

line, parabola, etc., and test these for goodness of fit. Mann and Shapley (1962) illustrate the data graphically for their 1961 results, and report that it has an excellent parabolic fit. Additional statistical analyses of various deviations in the data may be undertaken.

(iv) One can argue that the states are not independent agents in this national game, but instead that certain states with closely related interests or voting histories combine together and tend to vote alike. One can investigate past voting behavior of states, and aggregate together those who have voted the same way and assume that they are likely to continue to do so in the near future. These new coalitions of states can be treated as the individual players in a new version of the Electoral College game which will now have fewer players but with usually larger weights. This project can be modeled after the recent work by Boyce (1973), and there are a host of statistical hypotheses which can be assumed and investigated. For example, Boyce studies the impact of a third party movement such as the one by George C. Wallace in 1968.

(v) An ambitious project would be to investigate the procedures used by the major parties to nominate their presidential candidates at the national conventions, under say the somewhat unrealistic assumption that all delegates from a given state vote for the same candidate, i.e., the state is bound by the unit rule. This idealized game is somewhat similar to the Electoral College but the weights and number of states are determined in a slightly different fashion. The reference by David (1960) plus some more recent data may prove useful.

7.3. The State Game. Consider the game which is played in a particular state. Here the players are the individual voters in the state and a simple

majority, or often just a plurality, determines the winner, who will then receive all of this state's votes in the national game discussed in the previous section. It is clear that each individual voter in a given state has equal voting power in his state game, and that his power index is $1/N$, where N is the total number of voters in his state. In the following we will take N to be the total population of the state and thus assume that such N are proportional to the numbers of voters which the different states have. The number of electoral votes which a state has is based on such census figures. We also avoid consideration here of the fact that various numbers of voters may not exercise their right to vote. More important than a player's power index however, is the chance that he has of being the marginal voter in his state. Conventional wisdom often holds that this probability varies inversely with the population. On the other hand, we saw from the analysis in Section 6.2 that the possibility of being marginal, as determined by Banzhaf, is instead proportional to $1/\sqrt{N}$. This relationship was also discussed in Section 5.3 in connection with representation from multi-member districts. For example, the 1960 populations of New York State and the District of Columbia were 16,782,304 and 763,956, respectively. The chance that a voter from New York can sway his state's election compared to that for a voter from the District of Columbia is $1/\sqrt{16,782,304}$ to $1/\sqrt{763,956}$, or $(4097)^{-1}$ to $(874)^{-1}$, which gives the ratio 0.2133. In other words a voter from D.C. has 4.687 times the chance of being marginal in his "state" as the voter from N.Y.

(i) In light of this square root factor discussed above, what possible types of errors could be introduced into the various analyses in this paper if one uses a state's population figure rather than the number of its citizens who are actually likely to vote?

7.4. The Combined Game. To obtain some measure of the individual voter's total influence in electing a president, one must take into account both the national game and the state games discussed in the last two sections. To better understand this existing system, one must somehow compose the two previous results. We have seen that it is false to presume that each citizen's relative share of power in this two level game is directly proportional to his state's weight w_i divided by its population N_i . Instead, one can combine the Shapley-Shubik index ϕ for the national game given in Table 7.2.2 with the Banzhaf inverse square root formula for comparing the different state games in order to get some approximate idea of the relative influence of voters from different states. These two measures reflect the power that a given state and an individual within the state have in an election. Consider, for example, a comparison between New York State and the District of Columbia using the 1961 figures in Table 7.2.2. Clearly, N.Y. has more power in the national game, and this can be indicated by the ratio of $\phi_{NY} = .084064$ to $\phi_{DC} = .005416$ which is 15.521, and is a little higher than ratio of weights, i.e., $w_{NY} = 43$ to $w_{DC} = 3$ or 14.333. On the other hand, we saw in the previous section that a voter in N.Y. has only .2133 times as much chance of swaying his larger state as a voter from D.C. has of switching his district. Combining these two effects gives $(15.521) \times (.2133) = 3.311$, which indicates the relative power of an individual in N.Y. to one in D.C. in the two level election process. Note that we have merely multiplied the results of two separate games here, and that we have taken the ϕ measure in one case and used the Banzhaf analysis for the state games. We have not actually solved the single composed game as we did for one simple example in Section 5.3, because this composition would be a huge game in which some 100,000,000 players participate, and because the behavior of power indices (or more

general values in game theory) under various composition is not completely known. We return to this point in (iii) below.

Banzhaf (1968, Table I) has compared the ability of a citizen in one state to affect the outcome of a presidential election with the ability of one from another state, and these are shown in column (I) in Table 7.4.1. Note that N.Y. and D.C. are the extreme cases. Note also that some states have little more power than D.C., and so the latter is not the only one relatively poor in power. The low value for D.C. is expected since its $w_{DC} = 3$ is assigned in a manner different from the states and is not based so much on population. It is clear from the Table that the influence in the various state games greatly exaggerates the small bias in the national game in favor of the larger states. These results do not depend upon some quirk in our definitions of power, but are inherent in the design of the system itself when viewed as an abstract mathematical entity or structure. They surely give rise to some question about whether the "one man - one vote" principle is satisfied in presidential elections.

Some projects parallel to Banzhaf's work can be undertaken.

- (i) One can use the 1972 values for w_i and ϕ_i given in Table 7.2.2 along with the 1970 census figures for N_i to obtain an up-dated version of the results in column (I) of our Table 7.4.1.
- (ii) One can investigate various statistical deviations indicated in column (I), or its up-dated version mentioned in (i); and answer questions about what percent of the states or people benefit or are disadvantaged by the Electoral College system. Some results of this type are given in Banzhaf (1968, Table I).

RELATIVE VOTING POWER - TABLE 7.4.1

Column:		I	II	III
Name Of State	Population 1960 Census	Present Plan	Propor- tional Plan	District Plan
Alabama	3266740.	1.632	1.203	1.302
Alaska	226167.	1.838	5.212	3.075
Arizona	1302161.	1.281	1.509	1.594
Arkansas	1786272.	1.315	1.320	1.459
California	15717204.	3.162	1.000	1.004
Colorado	1753947.	1.327	1.344	1.472
Connecticut	2535234.	1.477	1.240	1.362
Delaware	446292.	1.308	2.641	2.189
Dist. of Columbia	763956.	1.000	1.543	1.673
Florida	4951560.	1.870	1.111	1.197
Georgia	3943116.	1.789	1.196	1.267
Hawaii	632772.	1.468	2.484	2.092
Idaho	667191.	1.429	2.356	2.038
Illinois	10081158.	2.491	1.013	1.059
Indiana	4662498.	1.786	1.096	1.200
Iowa	2757537.	1.596	1.282	1.364
Kansas	2178611.	1.392	1.263	1.399
Kentucky	3038156.	1.521	1.164	1.299
Louisiana	3257022.	1.635	1.206	1.304
Maine	969265.	1.186	1.622	1.691
Maryland	3100689.	1.675	1.267	1.337
Massachusetts	5148578.	1.834	1.068	1.174
Michigan	7823194.	2.262	1.055	1.108
Minnesota	3413864.	1.597	1.151	1.274
Mississippi	2178141.	1.392	1.263	1.399
Missouri	4319813.	1.710	1.092	1.211
Montana	674767.	1.421	2.329	2.026
Nebraska	1411330.	1.231	1.392	1.532
Nevada	285278.	1.636	4.132	2.738
New Hampshire	606921.	1.499	2.590	2.137
New Jersey	6066782.	2.063	1.101	1.162
New Mexico	951023.	1.197	1.653	1.707
New York	16782304.	3.312	1.007	1.000
North Carolina	4556155.	1.807	1.121	1.214
North Dakota	632446.	1.468	2.485	2.903
Ohio	9706397.	2.539	1.053	1.080
Oklahoma	2328284.	1.541	1.350	1.422
Oregon	1768687.	1.321	1.333	1.466
Pennsylvania	11319366.	2.638	1.007	1.043
Rhode Island	859488.	1.259	1.829	1.795
South Carolina	2382594.	1.524	1.319	1.405
South Dakota	680514.	1.415	2.310	2.018
Tennessee	3567089.	1.721	1.212	1.291
Texas	9579677.	2.452	1.025	1.070
Utah	890627.	1.237	1.765	1.764
Vermont	389881.	1.400	3.023	2.342
Virginia	3966949.	1.784	1.189	1.264
Washington	2853214.	1.569	1.239	1.341
West Virginia	1860421.	1.506	1.478	1.514
Wisconsin	3951777.	1.788	1.193	1.266
Wyoming	330066.	1.521	3.571	2.546

(iii) The method of Owen (1972) mentioned in Section 6.3 allows one to construct a nonsymmetric value theory for the quotient game that will yield the correct symmetric indices for the composed game, and it gives good approximations to the 100,000,000-player national game between all the citizens in the U.S., and it does so with simpler computational problems than those discussed in Section 6.1. Apply Owen's approach to compute the relative voting power of the citizens in the different States in the U.S. Presidential Election.

7.5. Electoral College Reform. The unique Electoral College system was created to serve as a compromise between other possibilities. Yet the difficulties and inequities inherent in it have caused continual criticism over the years. Some changes have been made in the past and many unsuccessful reforms have been attempted. Peirce (1968) has an excellent discussion on this. An observation of party campaign strategies or a historical review of close elections confirms, at least empirically, the big state bias in this election process. A good discussion of this is given in an article entitled "The Ox-Cart Way We Pick A Space-Age President" by Hamilton (1968). Some of these close historical results are also mentioned in Banzhaf (1968, footnote on pages 322 and 323) and in Davis (1970, page 135). Interest in reform tends to peak and wane, but it was particularly intense for a few years around 1968 when Wallace's candidacy raised the serious possibility of a deadlock. Some quantitative considerations of the potential power of certain blocs of states in this third party movement are discussed in Boyce (1973). Such reform efforts were ultimately defeated by a successful filibuster carried on by Southern and small state senators led by Sam Ervin, and this is reported in the Congressional Quarterly Almanac (Volume 26, 1970, pages 840-845). Paradoxically, the system was defended by many of those states which fair

rather poorly according to our mathematical analyses of power. So we continue with a system which has given rise to two deadlocks (1800 and 1824) and several presidents who received less than a majority of the popular vote (1824, 1844, 1876, 1884, 1888, 1916, 1948, 1960 and 1968). Some presidents received fewer votes than one of their opponents, and a person can theoretically lose an election in which he obtains a majority of the popular vote. Opinion polls also show that the public is in favor of reform.

Many alternatives to the present Electoral College system have been proposed. One frequently heard suggestion is to abolish this set-up completely, and to determine the president directly from the national popular vote. This approach could give rise to some technical difficulties if, for example, we moved towards a system of many parties. But the vote of each individual voter clearly counts the same in this case. However, the plans given serious consideration by Congress recently have usually been variations in the present Electoral College. Two types of these are referred to as the proportional plans and the district plans.

In the proportional plans, a state i with population N_i maintains its electoral vote w_i , but this weight is divided among the candidates in proportion to their statewide popular vote. These numbers need not be restricted to integers. So a citizen who changes his vote causes a shift of w_i/N_i electoral votes in the Electoral College. But voters from different states still have different influences under this plan. The relative ability of individuals from different states to affect an election can be easily computed. This was done by Banzhaf (1968) using the 1960 population figures and is given in column (II) of Table 7.4.1. Since smaller states have more electoral votes per resident than the larger ones, they have more voting power under this plan.

- (i) A simple project is to recompute the numbers in column (II) using the 1972 electoral vote distribution (Table 7.2.1) and the 1970 census figures.
- (ii) A more ambitious project would be to investigate a variation of the proportional plan in which votes are distributed to the candidates but not in a directly proportional plan. For example, a candidate who receives 55% of a state's popular vote might be assigned 65% of its electoral vote, or one who receives as much as 65% of the popular vote may get 95% of the electoral vote. In such a system, what sort of curves (popular vote versus electoral votes) would seem appropriate? It would be of interest to pick some trial curves and study what results they would have produced in previous elections. One may wish to limit considerations to only two major candidates, or to design a scheme for multiple parties.

In the district plans for electing the president, the voters in each state elect two electors at-large plus one elector from each congressional district in the state. Each elector then votes for the candidate who obtained the most votes in his district. The inverse square root formula can be used to determine an individual's relative chance of being marginal in the vote for the two at-large statewide delegates as well as for his own congressional district elector. His power is determined by the sum of these two possibilities. This case was also calculated using 1960 figures by Banzhaf (1968) and is given in column (III) of Table 7.4.1. Again, the more populous states are at the disadvantage. On the other hand, the large states have, contrary to their own interests, tended to support such plans as the proportional and district one, while the smaller ones have resisted reform in this direction.

- (iii) Another project is to calculate the numbers in column (III) using the 1972 data in Table 7.2.1 and the 1970 population figures.

(iv) One can investigate certain statistical deviations in columns (II) and (III), or the new results in (i) or (iii), and discuss questions about the numbers of states or individuals who benefit or are disadvantaged by these systems. A few results along these lines are mentioned in Banzhaf (1968, Tables II and III).

(v) The Behavioral Research Council of Great Barrington, Massachusetts 01230, in their Bulletin of January, 1974, has proposed a "second revised draft" (a "third" is also planned) of the Constitution of the U.S.A., which they suggest be used for study purposes in high schools and colleges. Their proposed election procedures have a weighted-voting aspect to them, and it may be of interest to investigate these in light of the power indices discussed above.

(vi) In a recent book by geographer Pearcy (1973), he suggests that the U.S.A. should be repartitioned into a country with 38 states. He suggests that river valleys, large metropolitan areas, etc., should be part of an individual state rather than serve as boundaries for different states. His states would be more equal in size, and his plan would in theory give significant financial savings. Consider population figures, and design an Electoral College for this new U.S.A. of 38 states. His plan would also change the apportionment of Congress, another type of voting problem mentioned in Section 8.2.

(vii) In an editorial in the New York Times by Joseph Farkas on March 29, 1974, he suggests: "The principle of one man - one vote should therefore be replaced by a system of proportional representation that would weigh each man's vote in proportion to his demonstrated capability to make intelligent choices". Related letters to the editor appeared on April 11. Consider how one might implement this elitist form of democracy in light of the power indices for weighted voting.

8. ADDITIONAL VOTING PROBLEMS

8.0. Introduction. Many types of voting situations give rise to problems with a significant mathematical structure. There are several directions in which the weighted voting discussed in this paper can be extended, as well as many other sorts of voting problems which make use of other mathematical methods. A few of these extensions and alternate approaches will be mentioned in this Section. These are included in order to point the interested reader in a few of the possible directions, and no attempt is made to give any details or to cover all of the possible voting areas. For example, problems involving probabilistic, statistical, clustering, or simulation techniques are not discussed here.

8.1. Affinity to Coalitions. Experience from the real-world indicates that some voting combinations or coalitions of voters are more likely to form than others. This may be caused by personal relations between the players, by closeness of political philosophies, by other ideological grounds, or by common goals or geographical proximity. Owen (1971) has developed a modification of the Shapley value which considers this affinity of certain coalitions. The relative position of the voters is described by positioning them on a hypersphere, and then the likelihood of coalitions depends upon these locations. Owen applied this value to the eleven-party Israeli Knesset in 1965, where the parties were spread from left to right on the political spectrum. It would be an interesting project to apply this to the Israeli Knesset as it exists in 1974 (see Section 5.4 (iii)), where one would perhaps wish to collect opinions on the orthodox-nonorthodox spread as well as the liberal-conservative

spectrum; or to the Danish Parliament in 1974 where some suggested data, taken from four Danish professors, is given in Section 5.4 (iv). The computation of Owen's index is usually not extremely difficult. Another view for incorporating "prior position" into the value or power index is given on pages 76-78 in Shapley (1973).

8.2. Apportionment of Congress. An important problem in the U.S.A., and one with a very interesting history, is how to determine the number of seats which any state will have in the U.S. Congress. It is desirable that any such scheme be fair in the sense that it gives each state its "quota" and that the method has a certain "monotonicity property" (i.e., avoids the "Alabama paradox"). A new Quota Method, which is the unique method satisfying such desirable properties, has just been proposed in two very interesting papers by Balinski and Young (1974, 1975). Many excellent class projects can be designed around this type of problem.

8.3. Redistricting. Another important political problem is how to partition a region into a certain number of "similar" districts, e.g., dividing a state into Congressional districts. It is desirable, or a legal necessity, that the resulting districts have certain properties, e.g., roughly equal population, contiguity (say convex), compactness (say fat or circular or square-like rather than thin), etc. One wishes to avoid gerrymandering. Several mathematical methods have been applied to this problem. Garfinkel and Nemhauser (1970) developed an algorithm by considering techniques from integer programming. They applied it to 26 census tracts in Sussex County, Delaware and to 7 districts and 39 counties in the State of Washington. They had difficulty with 55 counties in the State of West Virginia. Other

approaches, such as simulation, have also been used on this problem. Several class projects along these lines are possible, e.g., consider how to construct a university senate with constituencies of about equal size where each senator represents a rather "common" or "homogeneous" group.

8.4. Sequential Voting Strategies. Many common voting procedures have an important strategic aspect to them. In many sequential situations, such as binary voting as is frequently done on a series of proposed amendments, the final outcome may depend upon the order in which the motions are considered. The result may depend upon the information about another's intentions, and thus one may benefit by deception, misleading or diversionary tactics in the earlier stages when there are multiple votes in a sequence. Optimal voting strategies and various equilibria concepts have been studied, e.g., by using the theory of games in extensive (tree) or normal (matrix) forms. Good introductions to some of these situations appear in the excellent little book by Farquharson (1969) (including historical examples in his Appendix), and in Chapter 14 (especially Section 14.8) in Luce and Raiffa (1957).

8.5. Maps. There has been recent interest in various types of geographical or political maps, called cartograms, which are not drawn in accord with standard map projections. For example, pilots use maps which have a magnification of airport areas, and one could produce highway maps in which cities or complicated interchanges are embedded "continuously" but in a much larger areal scale. A related theorem in differential geometry appears in Sen (1974); and there are potential applications to areas in economics and political science, e.g., see the paper by Tobler (pages 215-220) in Papayanopoulos (1973).

9. LIMITS OF THE MATHEMATICAL MODELS

9.1. General Remarks. An important step in the complete model building process is to validate or verify the accuracy and usefulness of the model to the real-world situation. It is not sufficient to work in a mathematical vacuum, for one is required to check whether his conclusions are realistic or whether they must be carefully limited. Awareness of the theoretical or hypothetical nature of the results must be clearly understood when one makes use of his results in practice. This is particularly true in building mathematical models in political science, where it is so easy to be excessively isolated, to discard too many essentials, or to overlook many less quantifiable customs or procedures. In the social sciences in general it is most difficult to form complete models which contain all relevant aspects of current reality. After all, the social sciences make a much greater use of adjectives than do the physical sciences or mathematics. In this realm it is usually foolish to claim that one has obtained complete understanding, proofs, full explanation, true meaning, etc. Before employing one's abstract analysis as a true imitation of reality, one normally has to become heavily involved with complicated data and other more empirical activities.

9.2. Additional Considerations. It is clear that our mathematical analyses of certain voting situations is in many ways superficial. It does not include much of the political actuality in the global realm of power, influence, legislation, or elections in our governmental institutions. In the realities of political life there are many alliances, differences, cohesive blocs, discriminations, partisan actions, and additional sources which influence

outcomes. One must consider the effects of working behind-the-scene, capabilities in caucus or in the cloakrooms, or other "secondary" considerations; as well as the superimposed or nonelected operating structures or individuals, such as a committee system with a seniority rule of succession to a strong chairman's position. One must also consider the influences of party loyalty, prestige, publicity, bossism, solidarity, persuasion, personal gains, lobbying, fractionalism, bribes, gratuities, patronage, pork barrels, horse trading, campaign financing, and so on. Hopefully, these factors are counter balanced by or at least not completely divorced from such elements as honest debate, sincere deliberations, morality, conscience, ethics, reason, fairness, equity or other rules or "laws" in the game of politics. In extreme cases the whole political structure itself is altered more drastically under coercion, threats, force, wars, coups or assassinations.

In particular, in many of our examples and suggested projects the assumption that there exists irrevocable blocs of voters is frequently less than realistic. Our applications often discounted effects due to absences, abstentions, tie breaking rules, population figures for numbers of voters, and ignored other details of voting procedures. The analysis of the Electoral College in Section 7 clearly overlooks many free-for-all and less-readily-apparent aspects of an actual national convention or political campaign. Some detailed criticisms of Banzhaf's (1968) study by a few U.S. Senators and other experts follow his paper, and also appear in the following volume of this same journal (Volume 14, pages 86-96). However, there is some historical and empirical evidence to support the big-state bias in the Electoral College as is discussed in Peirce (1968), Hamilton (1968), and Uslander (pages 61-76 in Papayanopoulos (1973)). There is clearly a great deal of material which can be used for classroom discussion or criticism of the previous projects.

At another level of criticism there is the major question of what is the role of an elected official. Should he serve in the more routine, mechanical, or plebiscitarian role of the true delegate or representative, who polls, surveys or in some other way has his pulse on the will of his citizens, and who merely funnels or transmits this popular opinion to the legislature? Or should he act in a more patrician or Burkean mode where he is more of a free agent or trustee, to whom the voters abdicate power or whose judgment they rely upon? For example, U.S. Congressmen tend to act in the latter mode on questions about foreign policy or impeachment, whereas they may follow the former role in more local or domestic issues, since their reelection may so depend upon it. Additional moral and philosophical concerns about the ends, means, causes, goals and consequences of power are discussed in some of the writings by Riker.

9.3. Usefulness. It is clear that our weighted voting analysis is only a rather singular, perhaps even trivial, part of the political systems being analyzed. In many cases the weights are not the critical factors, and other practical considerations prove more important. Our conclusions admittedly fall short to the extent that the models were not general enough. However, any model must compromise between excessive complication and incorporation of the total picture. However, one should distinguish between the faults of the analyses, such as erroneous assumptions, incompleteness or incorrect conclusions, on the one hand, as against faults inherent in the actual design or very fabric of the system being modeled, on the other hand, e.g., irrational or illogical behavior, false intuition, or nonlogical structures.

One could nevertheless argue that the power indices are a worthwhile first step in a quantitative investigation of such voting situations. Although limited in scope, such analysis is hardly entirely wanting or completely

irrelevant, and there should be little objection to utilizing the results in the limited or spotty context to which they apply. Such studies should be useful in setting up norms, standards, ground rules, or minimal requirements for a voting situation; and they have recently found some implementation and acceptance by the courts in practice. Although fairness or justice are hardly to be found entirely in some simple mathematical formula or symmetry, it would appear that one should attempt to avoid numerical bias, as well as other types, whenever possible. Hopefully, the power indices surveyed in this paper, will have some value as a quantitative aid in judging equality in certain political structures, and they will become part of the conventional wisdom on the subject. Knowledge of power indices might be compared to knowing the odds of obtaining various poker hands; it is rather helpful, but not sufficient in itself to being a good poker player. It is also most likely that once mathematical techniques have successfully entered into political science, they are unlikely to be completely discarded in the future.

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