A Trust Region and Affine Scaling Interior Point Method for Nonconvex Minimization with Linear Inequality Constraints

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Abstract. A trust region and affine scaling interior point method (TRAM) is proposed for a general nonlinear minimization with linear inequality constraints [8]. In the proposed approach, a Newton step is derived from the complementarity conditions. Based on this Newton step, a trust region subproblem is formed, and the original objective function is monotonically decreased. Explicit sufficient decrease conditions are proposed for satisfying complementarity, dual feasibility and second order optimality.

The objective of this paper is to establish global and local convergence properties of the proposed trust region and affine scaling interior point method. It is shown that the proposed decrease conditions are sufficient for achieving complementarity, dual feasibility and second order optimality respectively. It is also established that a trust region solution is asymptotically in the interior of the proposed trust region subproblem and a damped trust region step can achieve quadratic convergence.

Key words. trust region, interior point method, Dikin-affine scaling, Newton step

1. Introduction. When solving a nonlinear programming problem, the number of function and derivative evaluations is often regarded as the main cost indicator. This cost often greatly surpasses that of linear algebra work. An efficient algorithm for a nonlinear programming problem should require as few evaluations as possible. For convex programming problems, interior point methods have proven to be an efficient approach; see [17] for a comprehensive bibliography on these methods. Using these methods, a small number of iterations is typically required to solve large problems [1, 4, 26]. The quest for similar successful interior point algorithms for nonconvex programming problems has become increasingly important [13, 7, 9, 10, 18, 15, 3].

Negative curvature in a nonconvex programming problem implies that there can be many local minimizers: a computational method is typically able to compute one of them. Assume that an initial feasible point x_0 is available. An algorithm for which the original objective function is monotonically decreased is desirable in a general nonlinear minimization context. The majority of interior point methods, e.g., a path following (see, e.g., [16]) or a potential function reduction method (see, e.g., [24]), do not have this monotonicity property. Despite lack of polynomial convergence properties, an affine scaling method, e.g., [12, 1, 4, 26], is the only interior point method which approaches a solution by monotonically decreasing the original objective function.

For various structured problems, affine scaling Newton methods have been proposed [6, 5, 19, 7, 20, 18, 2, 10]. Using these methods, a sequence of “interior” points $\{x_k\}$, with the objective function values monotonically decreasing, are generated to converge quadratically to a solution. To extend this approach to a general nonlinear programming problem, we consider the problem of minimizing a nonlinear (nonconvex) function subject to linear inequality constraints,

$$\min_{x \in \mathbb{R}^n} f(x)$$

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$$(1.1) \quad \text{subject to } Ax \geq b,$$

where $A^T = [a_1, \dots, a_m] \in \mathbb{R}^{n \times m}$. In this paper, $\mathcal{F} \stackrel{\text{def}}{=} \{x : Ax \geq b\}$ denotes the feasible region. Moreover, it is assumed that the interior of the feasible region $\text{int}(\mathcal{F}) \stackrel{\text{def}}{=} \{x : Ax > b\}$ is not empty, $f(x)$ is at least continuously differentiable in $\mathcal{F}$ and twice continuously differentiable if second order convergence is considered. Moreover, a strictly feasible initial point $x_0 \in \text{int}(\mathcal{F})$ is assumed to be given.

In [8], a trust region and affine scaling interior point method (TRAM) is proposed for (1.1). Given the current interior point x_k , an improved strictly feasible iterate $x_{k+1} \in \text{int}(\mathcal{F})$ is generated by solving a trust region subproblem with a quadratic objective function and a trust region bound using a 2-norm measurement. Asymptotically, solutions of the trust region subproblems are approximate affine scaling Newton steps for the complementarity conditions.

The main objective of this article is to analyze global and local convergence properties of TRAM. In §2, the proposed TRAM and its explicit decrease conditions are described. The global convergence properties are established in §3 and local convergence properties in §4. Conclusion and future work are discussed in §5. In [8], an interested reader can find a more detailed motivation for TRAM and its preliminary computational results.

2. The Trust Region and Affine Scaling Interior Point Algorithm. The idea behind a trust region method for unconstrained minimization is intuitive and simple. In a neighborhood of the current point x_k , a quadratic objective is minimized over a “region of trust” to yield a sufficient decrease of the original objective function. The size of the trust region neighborhood is updated according to agreement of the objective function and its approximation.

Constraints make it difficult to formulate a similar subproblem for which a global solution can be computed by existing software. The algorithm TRAM proposed in [8] uses a trust region mechanism with a 2-norm trust region bound. In this approach, constraints in the original minimization do not appear in the subproblem explicitly; difficulties of constraints are overcome using affine scaling to formulate an appropriate quadratic function and trust region. We briefly describe TRAM next and refer a reader to [8] for a more detailed discussion.

Let x_k be the current strictly feasible iterate and λ_k be an approximation to the Lagrangian multipliers. Let

$$(2.1) \quad C_k \stackrel{\text{def}}{=} \text{diag}(|\lambda_k|), \quad D(x) \stackrel{\text{def}}{=} \text{diag}(Ax - b), \quad \text{and} \quad D_k \stackrel{\text{def}}{=} D(x_k).$$

Assume $\psi_k(d)$ denotes the quadratic approximation to $f(x)$ at x_k , i.e.,

$$(2.2) \quad \psi_k(d) \stackrel{\text{def}}{=} \nabla f_k^T d + \frac{1}{2} d^T B_k d,$$

where B_k is symmetric and approximates the Hessian $\nabla^2 f_k$ of $f(x)$ at x_k .

In [8], a trust region subproblem is derived from the Newton step for the complementarity conditions of (1.1). The *complementarity conditions* of (1.1), e.g., [14], can be expressed as an $(m+n)$ -by- $(m+n)$ system of nonlinear equations

$$(2.3) \quad \text{diag}(Ax - b)\lambda = 0 \quad \text{and} \quad A^T \lambda - \nabla f = 0.$$

Consider the solution $(p_k^N; \lambda_{k+1}^p - \lambda_k)$ to the following system,

$$(2.4) \quad \begin{bmatrix} \nabla^2 f_k & -A^T \\ C_k A & D_k \end{bmatrix} \begin{bmatrix} p_k^N \\ \lambda_{k+1}^p - \lambda_k \end{bmatrix} = - \begin{bmatrix} \nabla f_k - A^T \lambda_k \\ D_k \lambda_k \end{bmatrix},$$

It is easy to verify that $(p_k^N; \lambda_{k+1}^p - \lambda_k)$ is an approximate Newton step for the complementarity condition (2.3) in a neighborhood of a local minimizer of (1.1).

Let $\hat{p}_k \stackrel{\text{def}}{=} D_k^{-\frac{1}{2}} A p_k$. When $\|(p_k^N; \hat{p}_k^N)\|_2 < \Delta_k$, $(p_k^N; \hat{p}_k^N)$ is the Newton step of the following trust region subproblem

$$(2.5) \quad \begin{aligned} & \min_{d \in \mathbb{R}^n, \hat{d} \in \mathbb{R}^m} \nabla f_k^T d + \frac{1}{2} d^T \nabla^2 f_k d + \frac{1}{2} \hat{d}^T C_k \hat{d} \\ & \text{subject to } Ad - D_k^{\frac{1}{2}} \hat{d} = 0 \\ & \|(d; \hat{d})\|_2 \leq \Delta_k. \end{aligned}$$

Let $\hat{H}_k$ denote the Hessian of (2.5), i.e.,

$$\hat{H}_k \stackrel{\text{def}}{=} \begin{bmatrix} \nabla^2 f_k & 0 \\ 0 & C_k \end{bmatrix}.$$

Let the columns of Z_k denote an orthonormal basis for the null space of $[A, -D_k^{\frac{1}{2}}]$. Then the projected Hessian of the subproblem (2.5) is $Z_k^T \hat{H}_k Z_k$. The optimality conditions of (2.5) imply that there exists a parameter $\nu_k \geq 0$ such that

$$(2.6) \quad (\hat{H}_k + \nu_k I)(p_k; \hat{p}_k) = - \begin{bmatrix} \nabla f_k \\ 0 \end{bmatrix} + \begin{bmatrix} A^T \\ -D_k^{\frac{1}{2}} \end{bmatrix} \lambda_{k+1}^p,$$

with $\nu_k(\Delta_k - \|(p_k; \hat{p}_k)\|_2) = 0$. Then $(p_k; \hat{p}_k) = Z_k Z_k^T (p_k; \hat{p}_k)$ and the projected Hessian $Z_k^T (\hat{H}_k + \nu_k I) Z_k$ is positive semi-definite, i.e.,

$$(2.7) \quad Z_k^T (\hat{H}_k + \nu_k I) Z_k = R_k^T R_k,$$

where R_k is an upper triangular matrix.

Stepsize choice is important for computational and theoretical convergence behaviors of an affine scaling method for linear programming [21, 25]. One would like to take as large a step as possible but this may bring the iterates to the boundary prematurely. Nonlinearity of the problem (1.1) can both alleviate and exacerbate this problem. On the one hand, the iterates may approach a boundary more slowly due to nonlinearity of $f(x)$. On the other hand, once close to a boundary, nonlinearity of $f(x)$ may make it harder to move away. Two strategies are proposed in [8] to overcome this difficulty. Firstly, a simple reflection technique can be used which facilitates departure from a boundary. We refer an interested reader to [8] for details. Secondly, when close to a non-optimal point satisfying complementarity, a perturbed diagonal scaling $\tilde{D}_k \stackrel{\text{def}}{=} \tilde{D}(x_k)$ is used instead, for example,

$$(2.8) \quad (\tilde{D}(x))_{ii} = \begin{cases} (D(x))_{ii} & \text{if } i \neq j_0, \\ 1 & \text{otherwise,} \end{cases}$$

and the iterates are to move away from a constraint j_0 which can be identified as

$$(2.9) \quad (\lambda(x))_{j_0} = \min\{(\lambda(x))_i : |a_i^T x - b_i| < \sigma \text{ and } (\lambda(x))_i < 0\},$$

where $0 < \sigma < 1$ is a small constant. When there is no i with $|a_i^T x - b_i| < \sigma$ and $(\lambda(x))_i < 0$, it is assumed that $j_0 = 0$ and $D(x) = \tilde{D}(x)$. Geometrically, $\tilde{D}_k$ changes the shape of the trust region so that it is elongated along the normal a_{j_0} of the j_0 th constraint. This forces a step away from the nearly binding constraint j_0 .

In general, the trust region subproblem proposed in [8] is,

$$(2.10) \quad \begin{aligned} & \min_{d \in \mathbb{R}^n} \psi_k(d) + \frac{1}{2} d^T A^T S_k^{-1} C_k A d \\ & \text{subject to } \|(d; S_k^{-\frac{1}{2}} d)\|_2 \leq \Delta_k, \end{aligned}$$

where S_k is an affine scaling matrix which typically equals D_k . The diagonal scaling $D_k^{-\frac{1}{2}}$ in the 2-norm trust region bound constraint serves a similar purpose as Dikin affine scaling [12] for a linear programming problem. The classical Dikin affine scaling uses D_k^{-1} rather than $D_k^{-\frac{1}{2}}$. The scaling matrix S_k in the trust region subproblem (2.10) is typically D_k while $\tilde{D}_k$ is occasionally used when an iterate is close to a non-optimal point satisfying complementarity.

An approximate trust region solution p_k is damped for strict feasibility. Assume $0 < \theta_0 < 1$. Let $\theta_k \in [\theta_0, 1)$ be a damping parameter. For example, let p_k be a solution to the trust region subproblem (2.10). A damped step p_k^* can be defined as:

$$(2.11) \quad p_k^* \stackrel{\text{def}}{=} \alpha_k p_k \quad \text{and} \quad \alpha_k = \theta_k \min\{1, \beta_k\},$$

where β_k is the stepsize to the boundary of $\mathcal{F}$ along p_k . The new iterate $x_{k+1} = x_k + p_k^*$ is computed. The trust region size Δ_k is subsequently adjusted based on the agreement between the original objective function and its approximation. This process is then repeated. TRAM is described in FIG. 1.

The quadratic objective in the subproblem (2.10) is a quadratic approximation to the original objective function augmented by a quadratic term adding positive curvature in the space spanned by the constraint normals. This is similar to an augmented Lagrangian function for a constrained minimization. The presence of the affine scaling in both the objective function and the trust region bound constraint ensures that a damped step of the trust region solution is able to yield a sufficient decrease.

Sufficient decrease conditions are useful in determination of approximate trust region solutions. In unconstrained minimization, sufficient decrease is measured against reductions of the quadratic objective along the gradient direction (a first order approximation to the trust region subproblem) and the solution to a trust region subproblem. For the constrained nonlinear minimization problem (1.1), similar explicit decrease conditions are proposed in [8] based on the trust region subproblem (2.10).

A sufficient decrease condition for complementarity (2.3) is derived from the projected gradient direction g_k of the trust region subproblem (2.10) with $S_k = D_k$, i.e.,

$$g_k = -(\nabla f_k - A^T \lambda_k),$$

TRAM Let $0 < \mu < \eta < 1$ and $x_0 \in \text{int}(\mathcal{F})$.

Step 1 Evaluate f_k , ∇f_k and $B_k \approx \nabla^2 f_k$; compute an approximate Lagrangian multiplier λ_k and let $C_k = \text{diag}(|\lambda_k|)$.

Step 2 Compute a step s_k , $x_k + s_k \in \text{int}(\mathcal{F})$, based on the trust region subproblem

$$\begin{aligned} \min_{d \in \mathbb{R}^n} \quad & \psi_k(d) + \frac{1}{2} d^T A^T S_k^{-1} C_k A d \\ \text{subject to} \quad & \|[d; S_k^{-\frac{1}{2}} d]\|_2 \leq \Delta_k. \end{aligned}$$

Step 3 Compute

$$\rho_k = \frac{f(x_k + s_k) - f(x_k)}{\psi(s_k)}.$$

Step 4 If $\rho_k > \mu$ then set $x_{k+1} = x_k + s_k$. Otherwise set $x_{k+1} = x_k$. Update Δ_k as specified.

Updating Trust Region Size Δ_k

Let $0 < \gamma_1 < 1 < \gamma_2$ be given.

1. If $\rho_k \leq \mu$ then set $\Delta_{k+1} \in (0, \gamma_1 \Delta_k]$.
2. If $\rho_k \in (\mu, \eta)$ then set $\Delta_{k+1} \in [\gamma_1 \Delta_k, \Delta_k]$.
3. If $\rho_k \geq \eta$ then set $\Delta_{k+1} \in [\Delta_k, \gamma_2 \Delta_k]$.

FIG. 1. *A Trust Region and Affine Scaling Interior Point Method*

where λ_k is the least squares approximation to the Lagrangian multipliers, i.e.,

$$(2.12) \quad \begin{bmatrix} A^T \\ -D_k^{\frac{1}{2}} \end{bmatrix} \lambda \stackrel{\text{LS}}{=} \begin{bmatrix} \nabla f_k \\ 0 \end{bmatrix}.$$

Let P_k denote the orthogonal projection to the null space of $[A, -D_k^{\frac{1}{2}}]$. Then

$$(2.13) \quad \nabla f_k^T g_k = -\|P_k \begin{bmatrix} \nabla f_k \\ 0 \end{bmatrix}\|_2^2 = -(\|\nabla f_k - A\lambda_k\|_2^2 + \|D_k^{\frac{1}{2}} \lambda_k\|_2^2).$$

Equation (2.13) suggests that a “good” decrease along the projected gradient g_k leads to complementarity: $\lim_{k \rightarrow \infty} \|\nabla f_k - A\lambda_k\|_2 = 0$ and $\lim_{k \rightarrow \infty} \|D_k^{\frac{1}{2}} \lambda_k\|_2 = 0$.

Define $g_k^* \stackrel{\text{def}}{=} \alpha_k g_k$, where $\alpha_k = \theta_k \alpha_k^*$, as a damped minimizer of the augmented quadratic objective along g_k within the *feasible* trust region, i.e., α_k^* solves

$$\min \quad \psi_k(\alpha g_k) + \frac{\alpha^2}{2} g_k^T A^T D_k^{-1} C_k A g_k$$

$$(2.14) \quad \text{subject to} \quad \|\alpha(g_k; D_k^{-\frac{1}{2}} A g_k)\|_2 \leq \Delta_k \\ x_k + \alpha g_k \in \mathcal{F}.$$

Due to the affine scaling matrix $D_k^{-\frac{1}{2}}$, a sufficient decrease of $\psi_k(d)$ measured against the decrease from the damped minimizer g_k^* should lead to satisfaction of complementarity (2.3). This is expressed as condition (AS.1) in FIG. 2.

Assume that strict complementarity holds at $(x; \lambda)$: i.e., $D(x)\lambda = 0$ and $\nabla f(x) - A^T \lambda = 0$ and there is no i such that $\lambda_i = a_i^T x - b_i = 0$. Then dual feasibility $\lambda \geq 0$ is satisfied if and only if the system

$$(2.15) \quad \tilde{D} \tilde{\lambda} = 0 \quad \text{and} \quad \nabla f - A^T \tilde{\lambda} = 0,$$

is satisfied. To achieve dual feasibility the projected gradient $\tilde{g}_k \stackrel{\text{def}}{=} \tilde{g}(x_k)$ is similarly defined,

$$(2.16) \quad \tilde{g}(x) \stackrel{\text{def}}{=} -(\nabla f(x) - A^T \tilde{\lambda}(x)), \quad \begin{bmatrix} A^T \\ -\tilde{D}(x)^{\frac{1}{2}} \end{bmatrix} \tilde{\lambda}(x) \stackrel{\text{LS}}{=} \begin{bmatrix} \nabla f(x) \\ 0 \end{bmatrix}.$$

Under the strict complementarity assumption, dual feasibility is satisfied if and only if $\tilde{g}(x) = 0$. Let $\tilde{P}_k$ denote the orthogonal projection to the null space of $[A, -\tilde{D}_k^{\frac{1}{2}}]$. Then

$$(2.17) \quad \nabla f_k^T \tilde{g}_k = -\|\tilde{P}_k \begin{bmatrix} \nabla f_k \\ 0 \end{bmatrix}\|_2^2 = -(\|\nabla f_k - A \tilde{\lambda}_k\|_2^2 + \|\tilde{D}_k^{\frac{1}{2}} \tilde{\lambda}_k\|_2^2).$$

Equation (2.17) suggests that a “good” decrease along the projected gradient $\tilde{g}_k$ can lead to dual feasibility: $\lim_{k \rightarrow \infty} \|\nabla f_k - A \tilde{\lambda}_k\|_2 = 0$ and $\lim_{k \rightarrow \infty} \|\tilde{D}_k^{\frac{1}{2}} \tilde{\lambda}_k\|_2 = 0$.

The damped minimizer $\tilde{g}_k^* \stackrel{\text{def}}{=} \alpha_k \tilde{g}_k$, $\alpha_k = \theta_k \alpha_k^*$, is along the projected gradient $\tilde{g}_k$ where α_k^* solves

$$(2.18) \quad \min \quad \psi_k(\alpha \tilde{g}_k) + \frac{\alpha^2}{2} \tilde{g}_k^T A^T \tilde{D}_k^{-1} C_k A \tilde{g}_k \\ \text{subject to} \quad \|\alpha(\tilde{g}_k; \tilde{D}_k^{-\frac{1}{2}} A \tilde{g}_k)\|_2 \leq \Delta_k \\ x_k + \alpha \tilde{g}_k \in \mathcal{F}.$$

This leads to condition (AS.2) in FIG. 2 to ensure dual feasibility.

For second order convergence, sufficient decrease can be measured against the decrease generated by the trust region solution. Condition (AS.3) in FIG. 2 is proposed as sufficient decrease for second order optimality and fast local convergence.

Conditions (AS.1), (AS.2) and (AS.3) in FIG. 2 are proposed as sufficient decrease criteria for complementarity, dual feasibility and second order optimality respectively. Using these conditions, desired convergence properties can be achieved. We refer an interested reader to [8] for an example implementation.

3. Global Convergence. We now establish that conditions (AS.1), (AS.2) and (AS.3) in FIG. 2 are sufficient for TRAM to achieve complementarity, dual feasibility and second order optimality respectively. Equation (2.13) suggests that complementarity can be satisfied if decrease is comparable to that of the minimizer along g_k .

Sufficient Decrease Conditions. Let $0 < \beta_{cs}, \beta_{df}, \beta_q < 1$ and $\beta_s > 0$. Let $g_k, \tilde{g}_k$ be defined by (2.12) and (2.16). Let p_k be a solution to (2.10) with $S_k = D_k$ and $B_k = \nabla^2 f_k$. Define $g_k^*, \tilde{g}_k^*$ and p_k^* as in (2.14), (2.18) and (2.11) respectively. Let $\|s_k\|_2 \leq \beta_s \Delta_k$. The sufficient decrease conditions are:

$$(AS.1) \quad \psi_k(s_k) < \beta_{cs}(\psi_k(g_k^*) + \frac{1}{2}g_k^{*T} A^T D_k^{-1} C_k A g_k^*);$$

$$(AS.2) \quad \psi_k(s_k) < \beta_{df}(\psi_k(\tilde{g}_k^*) + \frac{1}{2}\tilde{g}_k^{*T} A^T \tilde{D}_k^{-1} C_k A \tilde{g}_k^*);$$

$$(AS.3) \quad (\psi_k(s_k) + \frac{1}{2}s_k^T A^T D_k^{-1} C_k A s_k) < \beta_q(\psi_k(p_k^*) + \frac{1}{2}p_k^{*T} A^T D_k^{-1} C_k A p_k^*), \text{ where } B_k = \nabla^2 f_k.$$

FIG. 2. *Sufficient Decrease Conditions for the First and Second Order Optimality*

Therefore condition (AS.1) is sufficient if the damped minimizer g_k^* retains a significant portion of this minimizer. Similar remarks are applicable to $\tilde{g}_k^*$ for dual feasibility. An important component of the global convergence analysis is to establish that, due to affine scaling, the *posterior* damping for feasibility does not prohibit sufficient decrease.

To establish convergence properties of TRAM, the following compactness and full row rank assumption is made.

(AS.0) Given an initial point $x_0 \in \text{int}(\mathcal{F})$, it is assumed that the level set $\mathcal{L}$ is compact, where $\mathcal{L} \stackrel{\text{def}}{=} \{x : x \in \mathcal{F} \text{ and } f(x) \leq f(x_0)\}$. Moreover, $[A, -D(x)]$ is assumed to have full row rank for all $x \in \mathcal{L}$.

The matrix $[A, -D(x)]$ has full row rank if and only if the vectors $\{a_i : a_i^T x - b_i = 0\}$ are linearly independent. In addition $[A, -\tilde{D}(x)]$ has full row rank if $[A, -D(x)]$ has full row rank. Under the full rank assumption, λ_k and $\tilde{\lambda}_k$ can be computed via the normal equations of (2.12) and (2.16), i.e.,

$$(3.1) \quad AA^T \lambda_k + D_k \lambda_k = A \nabla f_k, \quad AA^T \tilde{\lambda}_k + \tilde{D}_k \tilde{\lambda}_k = A \nabla f_k.$$

Equation (3.1) gives the relations below which are frequently used in the subsequent analysis:

$$(3.2) \quad Ag_k = -D_k \lambda_k, \quad A\tilde{g}_k = -\tilde{D}_k \tilde{\lambda}_k.$$

LEMMA 3.1. Assume that (AS.0) holds and $f(x) : \mathcal{F} \rightarrow \Re$ is continuously differentiable. Let $D(x), \tilde{D}(x), g(x), \tilde{g}(x), \lambda(x)$ and $\tilde{\lambda}(x)$ be defined as in (2.1), (2.8), (2.12) and (2.16). Then

- (a) $g(x)$ and $\lambda(x)$ are continuous in $\mathcal{L}$;
- (b) $\tilde{g}(x)$ and $\tilde{\lambda}(x)$ are bounded in $\mathcal{L}$;
- (c) $D(x)^{-\frac{1}{2}} Ag(x)$ and $\tilde{D}(x)^{-\frac{1}{2}} A\tilde{g}(x)$ are bounded in $\mathcal{L}$.

Proof. From (AS.0), $[A, -D(x)]$ has full row rank for any x in $\mathcal{L}$. Then $[A, -\tilde{D}(x)]$ has full row rank for any x in $\mathcal{L}$. Therefore, $(A^T A + D(x))$ and $(A^T A + \tilde{D}(x))$ are nonsingular for any x in $\mathcal{L}$. Hence $g(x), \lambda(x), \tilde{g}(x)$ and $\tilde{\lambda}(x)$ are defined for all x in $\mathcal{L}$.

From definition (2.12) and (2.16), $g(x) = -(\nabla f(x) - A^T \lambda(x))$ and $\tilde{g}(x) = -(\nabla f(x) - A^T \tilde{\lambda}(x))$, where $\lambda(x) = (A^T A + D(x))^{-1} \nabla f(x)$ and $\tilde{\lambda}(x) = (A^T A + \tilde{D}(x))^{-1} \nabla f(x)$.

Since $\nabla f(x)$ and $D(x)$ are continuous in $\mathcal{F}$ and $A^T A + D(x)$ is nonsingular in $\mathcal{L}$, $g(x)$ and $\lambda(x)$ are continuous in $\mathcal{L}$. From compactness of $\mathcal{L}$, $g(x)$ and $\lambda(x)$ are bounded in $\mathcal{L}$. Applying continuity of $\nabla f(x)$ and nonsingularity of $A^T A + \tilde{D}(x)$ in the compact set $\mathcal{L}$, $\tilde{g}(x)$ and $\tilde{\lambda}(x)$ are bounded in $\mathcal{L}$. Using (3.2), $D(x)^{-\frac{1}{2}} A g(x)$ and $\tilde{D}(x)^{-\frac{1}{2}} A \tilde{g}(x)$ are bounded in $\mathcal{L}$. The proof is completed. $\square$

Lemma 3.2 below further describes the relationship between complementarity and the projected gradient g_k , and dual feasibility and $\tilde{g}_k$.

LEMMA 3.2. *Assume that (AS.0) holds, $\{B_k\}$ is bounded, and $f(x) : \mathcal{F} \rightarrow \mathbb{R}$ is continuously differentiable. Moreover assume that $\nabla f_*^T g_* = 0$ at $x_* \in \mathcal{L}$. Then for any $\{x_k\}$ in $\mathcal{L}$ which converges to x_* , $\{\nabla f_k^T g_k\}$ converges to zero. In addition, if there exists a sequence $\{x_k\}$ in $\mathcal{L}$ converging to x_* with $\lim_{k \rightarrow \infty} \nabla f_k^T \tilde{g}_k = 0$ and strict complementarity holds at x_* , then $(\lambda_*)_i > 0$ for any $a_i^T x_* - b_i = 0$.*

Proof. From Lemma 3.1, $g(x)$ is continuous in $\mathcal{L}$. This implies that $\{\nabla f_k^T g_k\}$ converges to zero for any $\{x_k\}$ in $\mathcal{L}$ which converges to x_* .

Assume now that there exists a sequence $\{x_k\}$ in $\mathcal{L}$ converging to x_* with $\{\nabla f_k^T \tilde{g}_k\}$ converging to zero and strict complementarity holds at x_* . If there exists $(\lambda_*)_i < 0$ for some $a_i^T x_* - b_i = 0$, then there exists $j_0 \neq 0$ with $(\tilde{D}_*)_{j_0 j_0} = 1$ and $a_{j_0}^T x_* - b_{j_0} = 0$. Moreover, from $\lim_{k \rightarrow \infty} \nabla f_k^T \tilde{g}_k = 0$, there exists $\tilde{\lambda}_*$ such that $\nabla f_* = A^T \tilde{\lambda}_*$ and $\tilde{D}_* \tilde{\lambda}_* = 0$. If $\tilde{\lambda}_* = \lambda_*$, strict complementarity is violated since $(\tilde{\lambda}_*)_{j_0} = 0$. If $\tilde{\lambda}_* \neq \lambda_*$, the full row rank assumption is violated since $[A, -D_*]^T \tilde{\lambda}_* = (\nabla f_*; 0)$ and $[A, -D_*]^T \lambda_* = (\nabla f_*; 0)$. Hence $(\lambda_*)_i > 0$ for any $a_i^T x_* - b_i = 0$. The proof is completed. $\square$

Lemma 3.3 relates decrease required by (AS.1) and (AS.2) with complementarity and dual feasibility. Its proof is similar to Lemma (4.8) in [22].

LEMMA 3.3. *Assume that $\{x_k\}$ is generated from TRAM, (AS.0) is satisfied, and $f(x) : \mathcal{F} \rightarrow \mathbb{R}$ is continuously differentiable and $\{B_k\}$ is bounded. If (AS.1) is satisfied at the k th iteration, then there exist constants $\chi_\beta, \chi_\nu, \chi_g > 0$ such that*

$$-\psi_k(s_k) \geq \frac{\beta_{cs} \theta_0^2}{2\chi_g} (-\nabla f_k^T g_k) \min\{\Delta_k, \frac{-(\nabla f_k^T g_k)}{\chi_\nu \chi_g}, \chi_\beta \sqrt{\|g_k\|_2^2 + \|D_k^{\frac{1}{2}} \lambda_k\|_2^2}\}.$$

If (AS.2) is satisfied and $(\tilde{\lambda}_k)_{j_0} < 0$, then there exist $\chi_{\tilde{\beta}}, \chi_\nu, \chi_{\tilde{g}} > 0$ such that

$$-\psi_k(s_k) \geq \frac{\beta_{df} \theta_0^2}{2\chi_{\tilde{g}}} (-\nabla f_k^T \tilde{g}_k) \min\{\Delta_k, \frac{-(\nabla f_k^T \tilde{g}_k)}{\chi_\nu \chi_{\tilde{g}}}, \chi_{\tilde{\beta}} \sqrt{\|\tilde{g}_k\|_2^2 + \|\tilde{D}_k^{\frac{1}{2}} \tilde{\lambda}_k\|_2^2}\}.$$

Here the index j_0 is as defined in (2.9).

Proof. Define $\phi(\tau) : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(\tau) = \psi_k(\tau d_k) + \frac{\tau^2}{2} d_k^T A^T D_k^{-1} C_k A d_k \quad \text{if} \quad d_k = \frac{g_k}{\sqrt{\|g_k\|_2^2 + \|D_k^{-\frac{1}{2}} A g_k\|_2^2}},$$

and

$$\phi(\tau) = \psi_k(\tau d_k) + \frac{\tau^2}{2} d_k^T A^T \tilde{D}_k^{-1} C_k A d_k \quad \text{if} \quad d_k = \frac{\tilde{g}_k}{\sqrt{\|\tilde{g}_k\|_2^2 + \|\tilde{D}_k^{-\frac{1}{2}} A \tilde{g}_k\|_2^2}}.$$

Hence $\phi(\tau) = \tau(\nabla f_k^T d_k) + \frac{1}{2}\tau^2\nu_k$, where

$$\nu_k = d_k^T B_k d_k + d_k^T A^T D_k^{-1} C_k A d_k, \quad \text{if} \quad d_k = \frac{g_k}{\sqrt{\|g_k\|_2^2 + \|D_k^{-\frac{1}{2}} A g_k\|_2^2}},$$

and

$$\nu_k = d_k^T B_k d_k + d_k^T A^T \tilde{D}_k^{-1} C_k A d_k, \quad \text{if} \quad d_k = \frac{\tilde{g}_k}{\sqrt{\|\tilde{g}_k\|_2^2 + \|\tilde{D}_k^{-\frac{1}{2}} A \tilde{g}_k\|_2^2}}.$$

Since $\{B_k\}$, $\{\lambda_k\}$ and $\{\tilde{\lambda}_k\}$ are bounded, there exists $\chi_\nu > 0$ such that $|\nu_k| \leq \chi_\nu$. Let τ_k^* be the minimizer of $\phi(\tau)$ on $[0, \min\{\Delta_k, \beta_k\}]$ where β_k is the stepsize along d_k to the boundary of $\mathcal{F}$, i.e.,

$$\beta_k = \min\left\{-\frac{a_i^T x_k - b_i}{a_i^T d_k} : -\frac{a_i^T x_k - b_i}{a_i^T d_k} > 0\right\}.$$

The stepsize $\beta_k \stackrel{\text{def}}{=} +\infty$ if $-\frac{a_i^T x_k - b_i}{a_i^T d_k} \leq 0$ for all i . If $\tau_k^* \in [0, \min\{\Delta_k, \beta_k\})$, then $0 \leq \nu_k \leq \chi_\nu$ and $\tau_k^* = -(\nabla f_k^T d_k)/\nu_k$. From $0 < \theta_0 \leq \theta_k < 1$,

$$(3.3) \quad \phi(\theta_k \tau_k^*) = -\theta_k \frac{(\nabla f_k^T d_k)^2}{\nu_k} + \frac{1}{2} \theta_k^2 \frac{(\nabla f_k^T d_k)^2}{\nu_k} \leq -\frac{\theta_0^2}{2} \frac{(\nabla f_k^T d_k)^2}{\chi_\nu}.$$

Assume $\tau_k^* = \Delta_k$. Since $\nu_k \Delta_k \leq -(\nabla f_k^T d_k)$ when $\nu_k > 0$, and $\phi(\theta_k \tau_k^*) \leq \theta_0^2 \Delta_k (\nabla f_k^T d_k)$ otherwise,

$$(3.4) \quad \phi(\theta_k \tau_k^*) = \phi(\theta_k \Delta_k) \leq \frac{\theta_0^2}{2} \Delta_k (\nabla f_k^T d_k).$$

Assume $\tau_k^* = \beta_k$. Since $\nu_k \beta_k \leq -(\nabla f_k^T d_k)$ when $\nu_k > 0$, and $\phi(\theta_k \tau_k^*) \leq \theta_0^2 \beta_k (\nabla f_k^T d_k)$ otherwise,

$$(3.5) \quad \phi(\theta_k \tau_k^*) = \phi(\theta_k \beta_k) \leq \frac{\theta_0^2}{2} \beta_k (\nabla f_k^T d_k).$$

Considering all the three cases,

$$(3.6) \quad \phi_k(\theta_k \alpha_k^* d_k) \leq \frac{\theta_0^2}{2} (\nabla f_k^T d_k) \min\left\{\Delta_k, \frac{-(\nabla f_k^T d_k)}{\chi_\nu}, \beta_k\right\}.$$

Next we establish lower bounds for β_k for the two definitions of d_k respectively.

First consider $d_k = g_k / \sqrt{\|g_k\|_2^2 + \|D_k^{-\frac{1}{2}} A g_k\|_2^2}$. From (3.2), boundedness of $\{\lambda_k\}$ and $\{x_k\}$, there exists $\chi_\beta > 0$ such that, for some $1 \leq j \leq m$,

$$(3.7) \quad \beta_k \geq \left| \frac{a_j^T x_k - b_j}{a_j^T d_k} \right| = \left| \frac{\sqrt{\|g_k\|_2^2 + \|D_k^{-\frac{1}{2}} A g_k\|_2^2}}{(\lambda_k)_j} \right| \geq \chi_\beta \sqrt{\|g_k\|_2^2 + \|D_k^{-\frac{1}{2}} A g_k\|_2^2}.$$

If $d_k = \tilde{g}_k / \sqrt{\|\tilde{g}_k\|_2^2 + \|\tilde{D}_k^{-\frac{1}{2}} A \tilde{g}_k\|_2^2}$, from $(\tilde{\lambda}_k)_{j_0} < 0$, there exists some $j \neq j_0$ such that

$$\beta_k \geq \left| \frac{a_j^T x_k - b_j}{a_j^T d_k} \right|.$$

From (3.2), $(\tilde{D}_k)_{ii} = (D_k)_{ii}$ when $i \neq j_0$, there exists $\chi_{\tilde{\beta}} > 0$ such that

$$(3.8) \quad \beta_k \geq \chi_{\tilde{\beta}} \sqrt{\|\tilde{g}_k\|_2^2 + \|\tilde{D}_k^{\frac{1}{2}} \tilde{\lambda}_k\|_2^2}.$$

Using Lemma 3.1 and (3.2), there exist $\chi_g, \chi_{\tilde{g}} > 0$ such that

$$(3.9) \quad \sqrt{\|g_k\|_2^2 + \|D_k^{-\frac{1}{2}} A g_k\|_2^2} \leq \chi_g \quad \text{and} \quad \sqrt{\|\tilde{g}_k\|_2^2 + \|\tilde{D}_k^{-\frac{1}{2}} A \tilde{g}_k\|_2^2} \leq \chi_{\tilde{g}}.$$

Since $\psi_k(s_k) \leq \beta_{cs} \phi(\theta_k \alpha_k^*)$ with $d_k = g_k / \sqrt{\|g_k\|_2^2 + \|D_k^{-\frac{1}{2}} A g_k\|_2^2}$ if (AS.1) is satisfied and $\psi_k(s_k) \leq \beta_{df} \phi(\theta_k \alpha_k^*)$ with $d_k = \tilde{g}_k / \sqrt{\|\tilde{g}_k\|_2^2 + \|\tilde{D}_k^{-\frac{1}{2}} A \tilde{g}_k\|_2^2}$, if (AS.2) is satisfied, the results follow from (3.6), (3.7), (3.8), and (3.9). $\square$

LEMMA 3.4. *Assume that (AS.0) holds, $f(x) : \mathcal{F} \rightarrow \Re$ is continuously differentiable, $\{B_k\}$ is bounded and $\{x_k\}$ is generated by TRAM. Then there cannot exist constants $\chi_0, \chi_1 > 0$ such that, for sufficiently large k ,*

$$-\psi_k(s_k) \geq \frac{1}{2} \chi_0 \min\{\Delta_k, \chi_1\}.$$

Proof. The lemma is proved by contradiction. Assume that there exist constants $\chi_0, \chi_1 > 0$ such that, for sufficiently large k ,

$$(3.10) \quad -\psi_k(s_k) \geq \frac{1}{2} \chi_0 \min\{\Delta_k, \chi_1\}.$$

We first prove that

$$(3.11) \quad \sum_{k=1}^{\infty} \Delta_k < +\infty.$$

If there are a finite number of successful iterations then $\Delta_{k+1} \leq \gamma_1 \Delta_k$ for all k sufficiently large. Hence

$$\sum_{k=1}^{\infty} \Delta_k < +\infty.$$

If there is an infinite sequence $\{k_i\}$ of successful iterations, using (3.10), for a successful step and sufficiently large k ,

$$(3.12) \quad f(x_{k_i}) - f(x_{k_i+1}) \geq -\mu \psi_k(s_{k_i}) \geq \frac{1}{2} \mu \chi_0 \min\{\Delta_{k_i}, \chi_1\}.$$

Since $f(x_{k+1}) \leq f(x_k)$ for all k and $\{f(x_k)\}$ is bounded below,

$$\sum_{i=1}^{\infty} \Delta_{k_i} < +\infty.$$

The trust region updating rules of TRAM imply that

$$\sum_{k=1}^{\infty} \Delta_k \leq (1 + \frac{\gamma_2}{1 - \gamma_1}) \sum_{i=1}^{\infty} \Delta_{k_i}.$$

Again (3.11) holds.

From (3.11) and $\|x_{k+1} - x_k\|_2 = \|s_k\|_2 \leq \beta_s \Delta_k$, $\{x_k\}$ converges and $\{s_k\}$ converges to zero. Since $\{B_k\}$ is bounded, there exists $\chi_B > 0$ such that $\|B_k\|_2 \leq \chi_B$. Using $f(x_k + s_k) - f(x_k) = \nabla f(x_k + \xi_k s_k)^T s_k$ with $0 \leq \xi_k \leq 1$,

$$\begin{aligned} |f(x_k + s_k) - f(x_k) - \psi_k(s_k)| &\leq \left| \frac{1}{2} s_k^T B_k s_k \right| + |(\nabla f(x_k + \xi_k s_k) - \nabla f_k)^T s_k| \\ &\leq \frac{1}{2} \chi_B \beta_s^2 \Delta_k^2 + \beta_s \Delta_k \|\nabla f(x_k + \xi_k s_k) - \nabla f_k\|_2. \end{aligned}$$

Since $\nabla f(x)$ is continuous in the compact level set $\mathcal{L}$ and $\{x_k\}$ converges, there exists a sequence $\{\epsilon_k\}$ converging to zero such that

$$|f(x_k + s_k) - f(x_k) - \psi_k(s_k)| \leq \epsilon_k \Delta_k.$$

By assumption, for sufficiently large k , $-\psi_k(s_k) \geq \frac{1}{2} \chi_0 \Delta_k$. It is readily obtained that $\{|\rho_k - 1|\}$ converges to zero. Hence $\{\Delta_k\}$ cannot converge to zero, which is a contradiction to (3.11). The proof is completed. $\square$

LEMMA 3.5. *Assume that (AS.0) holds and $f(x) : \mathcal{F} \rightarrow \mathfrak{R}$ is continuously differentiable. Assume further that $\lim_{k \rightarrow \infty} \nabla f_k^T g_k = 0$ and $\liminf_{i \rightarrow \infty} (-\nabla f_{m_i}^T \tilde{g}_{m_i}) > 0$ for a subsequence $\{m_i\}$. Then $(\tilde{\lambda}_{m_i})_{j_0} < 0$ for sufficiently large i , where j_0 is defined by (2.9).*

Proof. Using (3.2),

$$\begin{aligned} \nabla f_k^T \tilde{g}_k &= (A^T \lambda_k)^T \tilde{g}_k + (\nabla f_k - A \lambda_k)^T \tilde{g}_k \\ &= -\lambda_k^T \tilde{D}_k \tilde{\lambda}_k + (\nabla f_k - A \lambda_k)^T \tilde{g}_k. \end{aligned}$$

From $\lim_{k \rightarrow \infty} \nabla f_k^T g_k = 0$ and (2.13), $\lim_{k \rightarrow \infty} (\nabla f_k - A \lambda_k) = 0$ and $\lim_{k \rightarrow \infty} (\lambda_k)_j (\tilde{D}_k)_{jj} = 0$ for all $j \neq j_0$. Since $\liminf_{i \rightarrow \infty} (-\nabla f_{m_i}^T \tilde{g}_{m_i}) > 0$, $(\lambda_{m_i})_{j_0} (\tilde{\lambda}_{m_i})_{j_0} > 0$ for sufficiently large i . From $\lim_{k \rightarrow \infty} \nabla f_k^T g_k = 0$ and $\liminf_{i \rightarrow \infty} (-\nabla f_{m_i}^T \tilde{g}_{m_i}) > 0$, $(\lambda_{m_i})_{j_0} < 0$ for sufficiently large i . This implies that $(\tilde{\lambda}_{m_i})_{j_0} < 0$ for sufficiently large i . $\square$

LEMMA 3.6. *Assume that (AS.0) holds, $f(x) : \mathcal{F} \rightarrow \mathfrak{R}$ is continuously differentiable, and $\{B_k\}$ is bounded.*

- (a) *If (AS.1) holds at every iteration then $\liminf_{k \rightarrow \infty} \nabla f_k^T g_k = 0$.*
- (b) *If (AS.2) holds for sufficiently large k and $\lim_{k \rightarrow \infty} \nabla f_k^T g_k = 0$, then $\liminf_{k \rightarrow \infty} \nabla f_k^T \tilde{g}_k = 0$.*

Proof. We prove each result by contradiction.

- (a) Assume that there is $\epsilon > 0$ such that $-\nabla f_k^T g_k \geq \epsilon$ for all sufficiently large k . From (2.13), there exists $\bar{\epsilon} > 0$ such that, for sufficiently large k ,

$$\|\nabla f_k - A \lambda_k\|_2^2 + \|D_k^{\frac{1}{2}} \lambda_k\|_2^2 \geq \bar{\epsilon}.$$

Since (AS.1) is satisfied, using Lemma 3.3 there exist $\chi_0, \chi_1 > 0$ such that, for sufficiently large k ,

$$-\psi_k(s_k) \geq \frac{1}{2} \chi_0 \min\{\Delta_k, \chi_1\}.$$

But Lemma 3.4 indicates that this is not possible. Hence

$$\liminf_{k \rightarrow \infty} \nabla f_k^T g_k = 0.$$

- (b) Assume now that (AS.2) holds for sufficiently large k and $\lim_{k \rightarrow \infty} \nabla f_k^T g_k = 0$ but $-\nabla f_k^T \tilde{g}_k \geq \epsilon$ for sufficiently large k . From (2.17), there exists $\bar{\epsilon} > 0$ such that, for sufficiently large k ,

$$\|\nabla f_k - A\tilde{\lambda}_k\|_2^2 + \|\tilde{D}_k^{\frac{1}{2}}\tilde{\lambda}_k\|_2^2 \geq \bar{\epsilon}.$$

Lemma 3.5 and 3.3 apply and there exist $\chi_0, \chi_1 > 0$ such that, for sufficiently large k ,

$$-\psi_k(s_k) \geq \frac{1}{2}\chi_0 \min\{\Delta_k, \chi_1\}.$$

Using Lemma 3.4, this is impossible. Hence

$$\liminf_{k \rightarrow \infty} \nabla f_k^T \tilde{g}_k = 0.$$

The proof is completed. $\square$

The next theorem establishes that complementarity and dual feasibility can be satisfied at every limit point.

THEOREM 3.7. *Assume that (AS.0) holds and $f(x): \mathcal{F} \rightarrow \Re$ is continuously differentiable and $\{B_k\}$ is bounded. Let $\{x_k\}$ be generated by TRAM. Assume that (AS.1) holds at every iteration. Then*

- (a) $\lim_{k \rightarrow \infty} \nabla f_k^T g_k = 0$, i.e., the complementarity conditions are satisfied at every limit point of $\{x_k\}$;
- (b) If (AS.2) is satisfied for sufficiently large k , then $\liminf_{k \rightarrow \infty} \nabla f_k^T \tilde{g}_k = 0$. If, in addition, there exists a limit point x_* at which strict complementarity is satisfied and $\nabla f_*^T \tilde{g}_* = 0$, then $\lim_{k \rightarrow \infty} \nabla f_k^T \tilde{g}_k = 0$. Hence, the first order necessary conditions are satisfied at every limit point satisfying strict complementarity.

Proof. The proof is again by contradiction. We consider each result in turn.

- (a) Let ϵ_1 in $(0, 1)$ be given and assume that there is a sequence $\{m_i\}$ such that $\|\nabla f_{m_i}^T g_{m_i}\|_2 \geq \epsilon_1$. Lemma 3.6 guarantees that for any ϵ_2 in $(0, \epsilon_1)$ there is a subsequence of $\{m_i\}$ (without loss of generality we assume that it is the full sequence) and a subsequence $\{l_i\}$ such that

$$(3.13) \quad |\nabla f_k^T g_k| \geq \epsilon_1, \quad m_i \leq k < l_i, \quad |\nabla f_{l_i}^T g_{l_i}| < \epsilon_2.$$

Using (2.13), there exists $\bar{\epsilon}_1 > 0$ such that, for sufficiently large i ,

$$\|\nabla f_k - A\lambda_k\|_2^2 + \|D_k^{\frac{1}{2}}\lambda_k\|_2^2 \geq \bar{\epsilon}_1, \quad m_i \leq k < l_i.$$

If the k -th iteration is successful, using Lemma 3.3,

$$(3.14) \quad f(x_k) - f(x_{k+1}) \geq \frac{\mu\epsilon_1\beta_{cs}\theta_0^2}{2\chi_g} \min\{\Delta_k, \frac{\epsilon_1}{\chi_\nu\chi_g}, \chi_\beta\bar{\epsilon}_1\}, \quad m_i \leq k < l_i,$$

for sufficiently large i . Since $f(x)$ is bounded below on $\mathcal{L}$, $\{f(x_k)\}$ converges and $\{f(x_k) - f(x_{k+1})\}$ converges to zero. From $\|x_{k+1} - x_k\|_2 \leq \beta_s\Delta_k$, it follows that, for sufficiently large i ,

$$(3.15) \quad f(x_k) - f(x_{k+1}) \geq \epsilon_3\|x_{k+1} - x_k\|_2, \quad m_i \leq k < l_i,$$

where $\epsilon_3 = \frac{\mu\epsilon_1\beta_{cs}\theta_0^2}{2\chi_g\beta_s}$. Consider a subsequence of $\{l_i\}$ such that $\{x_{l_i}\}$ converges to x_* and $\liminf_{i \rightarrow \infty} \nabla f_{l_i}^T g_{l_i} = 0$. Without loss of generality, the subsequence is still denoted by $\{l_i\}$. Using (3.15) and the triangle inequality,

$$f(x_{m_i}) - f(x_{k_i}) \geq \epsilon_3 \|x_{k_i} - x_{m_i}\|_2, \quad m_i \leq k_i \leq l_i.$$

Since $\{f(x_k)\}$ converges and $\{x_{l_i}\}$ converges to x_* , $\{x_{m_i}\}$ converges to x_* . From $\{\nabla f_{l_i}^T g_{l_i}\}$ converging to zero, $\nabla f_*^T g_* = 0$. Applying Lemma 3.2, $\{\nabla f_{m_i}^T g_{m_i}\}$ converges to zero. Hence $|\nabla f_{m_i}^T g_{m_i}| < \epsilon_2$ for sufficiently large i . This contradicts (3.13) since $\epsilon_2 < \epsilon_1$. Hence $\{\nabla f_k^T g_k\}$ converges to zero and the complementarity conditions are satisfied at every limit point of $\{x_k\}$;

- (b) From (a), $\lim_{k \rightarrow \infty} \nabla f_k^T g_k = 0$. Assume now that (AS.2) is satisfied for sufficiently large k . Using Lemma 3.6, $\liminf_{k \rightarrow \infty} \nabla f_k^T \tilde{g}_k = 0$. The proof for $\lim_{k \rightarrow \infty} \nabla f_k^T \tilde{g}_k = 0$ is exactly the same as the proof of (a) except that g_k is replaced by $\tilde{g}_k$, and (3.14) follows from Lemma 3.5 and Lemma 3.3 with β_{cs} replaced by β_{df} . Similarly consider a subsequence $\{x_{l_i}\}$ which converges to x_* satisfying strict complementarity and $\liminf_{i \rightarrow \infty} \nabla f_{l_i}^T g_{l_i} = 0$. A contradiction can be derived using Lemma 3.2. Hence $\lim_{k \rightarrow \infty} \nabla f_k^T \tilde{g}_k = 0$. Thus dual feasibility is satisfied at every limit point satisfying strict complementarity.

The proof is completed. $\square$

Before we examine second order convergence, several technical lemmas are required. First, we quote Lemma (4.10) in [23] below.

LEMMA 3.8. *Let x_* be an isolated limit point of a sequence $\{x_k\}$ in $\mathbb{R}^n$. If $\{x_k\}$ does not converge then there is a subsequence $\{x_{l_j}\}$ converging to x_* , and an $\epsilon > 0$ such that $\|x_{l_j+1} - x_{l_j}\|_2 \geq \epsilon$.*

Consequences of (AS.3) are subsequently examined.

LEMMA 3.9. *Assume that $f(x) : \mathcal{F} \rightarrow \mathbb{R}$ is twice continuously differentiable. Let the columns of Z_k form an orthonormal basis for the null space of $[A, -D_k^{\frac{1}{2}}]$ and p_k be a solution to the trust region subproblem (2.10) with $S_k = D_k$ and $B_k = \nabla^2 f_k$. Then*

$$-(\psi_k(s_k) + \frac{1}{2} s_k^T A^T D_k^{-1} C_k A s_k) \geq \frac{\beta_q \theta_0^2 \min\{1, \beta_k^2\}}{2} (\nu_k \Delta_k^2 + \|R_k Z_k^T(p_k; \hat{p}_k)\|_2^2),$$

where β_k is the stepsize along p_k to the boundary of $\mathcal{F}$, $\nu_k \geq 0$ and R_k are as defined in (2.6) and (2.7).

Proof. Let $\phi(\alpha) = \psi_k(\alpha p_k) + \frac{\alpha^2}{2} p_k^T A^T D_k^{-1} C_k A p_k$ and $\alpha \in [0, \min\{1, \beta_k\}]$ where β_k is the stepsize along p_k to the boundary of $\mathcal{F}$. Let α_k^* be the minimizer of $\phi(\alpha)$ in $[0, \min\{1, \beta_k\}]$. Then

$$\begin{aligned} \phi(\alpha) &= \alpha \nabla f_k^T p_k + \frac{1}{2} \alpha^2 (p_k; \hat{p}_k)^T \hat{H}_k(p_k; \hat{p}_k) \\ &= -\alpha (p_k; \hat{p}_k)^T (\hat{H}_k + \nu_k I)(p_k; \hat{p}_k) + \frac{1}{2} \alpha^2 (p_k; \hat{p}_k)^T \hat{H}_k(p_k; \hat{p}_k) \quad (\text{from (2.6)}) \\ &= -\alpha \|R_k Z_k^T(p_k; \hat{p}_k)\|_2^2 + \frac{1}{2} \alpha^2 \|R_k Z_k^T(p_k; \hat{p}_k)\|_2^2 - \frac{1}{2} \alpha^2 \nu_k \Delta_k^2, \quad (\text{from (2.7)}). \end{aligned}$$

But $\alpha_k^* = \min\{1, \beta_k\}$. Hence $\alpha_k^{*2} \leq \alpha_k^*$. From (AS.3), $p_k^* = \theta_k \alpha_k^* p_k$, and $0 < \theta_0 \leq \theta_k < 1$,

$$-(\psi_k(s_k) + \frac{1}{2} s_k^T A^T D_k^{-1} C_k A s_k) \geq \frac{\beta_k \theta_0^2 \min\{1, \beta_k^2\}}{2} (\nu_k \Delta_k^2 + \|R_k Z_k^T(p_k; \hat{p}_k)\|_2^2).$$

The proof is completed. $\square$

The following lemma provides lower bounds for the stepsize sequence $\{\beta_k\}$. The results in Lemma 3.10 hold for any subsequence generated by TRAM (consequently, it holds for the entire sequence as well).

LEMMA 3.10. *Assume that (AS.0) holds, $f(x) : \mathcal{F} \rightarrow \Re$ is twice continuously differentiable. Let $\{x_k\}$ be any subsequence generated by TRAM and β_k be the stepsize along p_k to the boundary of $\mathcal{F}$. Then there exist $\chi_0, \chi_1, \chi_2 > 0$ such that, for sufficiently large k ,*

$$(3.16) \quad \beta_k \geq \frac{\nu_k}{\chi_0(\chi_1 + (\chi_2 + \nu_k)\Delta_k)},$$

where $\nu_k \geq 0$ is defined in (2.6). Assume further that $\{x_k\}$ converges to x_* at which strict complementarity and dual feasibility are satisfied, and $\{(p_k; \hat{p}_k)\}$ converges to zero. Then $\liminf_{k \rightarrow \infty} \beta_k > 0$.

Proof. By definition,

$$\beta_k \geq \min\left\{-\frac{a_i^T x_k - b_i}{a_i^T p_k} : -\frac{a_i^T x_k - b_i}{a_i^T p_k} > 0\right\}.$$

From $\hat{p}_k = (Ax_k - b)^{-\frac{1}{2}} Ap_k$ and (2.6),

$$a_i^T p_k = (a_i^T x_k - b_i)^{\frac{1}{2}} (\hat{p}_k)_i = -\frac{(a_i^T x_k - b_i)(\lambda_{k+1}^p)_i}{\nu_k + |(\lambda_k)_i|}.$$

Hence, there exists $1 \leq j \leq m$ such that,

$$(3.17) \quad \beta_k \geq \frac{\nu_k + |(\lambda_k)_j|}{(\lambda_{k+1}^p)_j} \geq \frac{\nu_k}{\|\lambda_{k+1}^p\|_\infty}.$$

From (2.6),

$$(3.18) \quad \begin{bmatrix} A^T \\ -D_k^{\frac{1}{2}} \end{bmatrix} \lambda_{k+1}^p = \begin{bmatrix} \nabla f_k \\ 0 \end{bmatrix} + (\hat{H}_k + \nu_k I)(p_k; \hat{p}_k).$$

Since $[A, -D_k^{\frac{1}{2}}]$ has full row rank in the compact set $\mathcal{L}$, $\{\lambda_k\}$ is bounded (using Lemma 3.1), and $f(x)$ is twice continuously differentiable, there exist $\chi_0, \chi_1 > 0$ such that

$$(3.19) \quad \|\lambda_{k+1}^p\|_\infty \leq \chi_0(\chi_1 + (\chi_2 + \nu_k)\Delta_k).$$

Substitute (3.19) into (3.17), (3.16) holds.

Assume now that $\{x_k\}$ converges to x_* satisfying strict complementarity, dual feasibility, and $\{(p_k; \hat{p}_k)\}$ converges to zero. Since (AS.0) holds, $\{\lambda_k\}$ converges to λ_* .

From strict complementarity and dual feasibility, $(\lambda_*)_i > 0$ for any $a_i^T x_* - b_i = 0$. Since $\{p_k\}$ converges to zero, the index j defining β_k in (3.17) satisfies $a_j^T x_* - b_j = 0$ and $(\lambda_*)_j > 0$ for sufficiently large k .

Let the sequence $\{\Delta_k\}$ be partitioned into two subsequences: $\{k : \Delta_k \leq 1\}$ and $\{k : \Delta_k > 1\}$. For the subsequence $\{k : \Delta_k \leq 1\}$, using (3.17) and (3.19),

$$\beta_k \geq \frac{\nu_k + |(\lambda_k)_j|}{\chi_0(\chi_1 + (\chi_2 + \nu_k))}, \quad \text{for some } a_j^T x_* - b_j = 0.$$

Since $\lim_{k \rightarrow \infty} (\lambda_k)_i > 0$ for all $a_i^T x_* - b_i = 0$, $\liminf_{k \rightarrow \infty} \beta_k > 0$.

Consider the subsequence $\{k : \Delta_k > 1\}$. Using Lemma 3.9,

$$-(\psi_k(p_k^*) + \frac{1}{2} p_k^{*T} A^T D_k^{-1} C_k A p_k^*) \geq \frac{\beta_q \theta_0^2 \min\{1, \beta_k^2\}}{2} \nu_k \Delta_k^2.$$

Since $\{(p_k, \hat{p}_k)\}$ converges to zero, $\lim_{k \rightarrow \infty} (\psi_k(p_k^*) + \frac{1}{2} p_k^{*T} A^T D_k^{-1} C_k A p_k^*) = 0$. Therefore $\lim_{k \rightarrow \infty} \nu_k = 0$ for the subsequence $\{k : \Delta_k > 1\}$. Using (3.18) and $\lim_{k \rightarrow \infty} \lambda_k = \lambda_*$, $\lim_{k \rightarrow \infty} \lambda_k^p = \lambda_*$. Using (3.17), $\liminf_{k \rightarrow \infty} \beta_k > 0$. The proof is completed. $\square$

LEMMA 3.11. Assume that $[A, D_*]$ has full row rank at $x_* \in \mathcal{F}$ and the complementarity conditions are satisfied at $(x_*; \lambda_*)$. Let the columns of Z_* denote an orthonormal basis for the null space of $[A, D_*]$. If $Z_*^T \hat{H}_* Z_*$ is positive semidefinite, then $d^T \nabla^2 f_* d \geq 0$ for any d satisfying $a_i^T d = 0$ for all i with $a_i^T x_* - b_i = 0$.

Proof. Let d satisfy $a_i^T d = 0$ for all i with $a_i^T x_* - b_i = 0$. Define $\hat{d}_i = (a_i^T x_* - b_i)^{-\frac{1}{2}} a_i^T d$ if $a_i^T x_* - b_i > 0$ and $\hat{d}_i = 0$ otherwise. Then $Ad - D_* \hat{d} = 0$. Hence $(d; \hat{d}) = Z_* w$ and $\hat{d}^T C_* \hat{d} = 0$. Since $Z_*^T \hat{H}_* Z_*$ is positive semidefinite, $d^T \nabla^2 f_* d = w^T Z_*^T \hat{H}_* Z_* w \geq 0$. $\square$

The next theorem establishes that the second order necessary conditions can be satisfied.

THEOREM 3.12. Assume (AS.0) holds and $f(x) : \mathcal{F} \rightarrow \Re$ is twice continuously differentiable. Let $\{x_k\}$ be the sequence generated by TRAM. If (AS.1) is satisfied at every iteration and (AS.3) holds for sufficiently large k , then

- (a) There is a limit point x_* such that $d^T \nabla^2 f_* d \geq 0$ for any d satisfying $a_i^T d = 0$ for all i with $a_i^T x_* - b_i = 0$;
- (b) If x_* is an isolated limit point, then $d^T \nabla^2 f_* d \geq 0$ for any d satisfying $a_i^T d = 0$ for all i with $a_i^T x_* - b_i = 0$.

Proof. Using Lemma 3.11, in order to prove $d^T \nabla^2 f_* d \geq 0$ for any d satisfying $a_i^T d = 0$ for all i with $a_i^T x_* - b_i = 0$, we only need to show that the projected Hessian $Z_*^T \hat{H}_* Z_*$ is positive semidefinite.

- (a) If zero is a limit point of $\{\nu_k\}$, the result immediately follows. We prove that zero is a limit point of $\{\nu_k\}$ by contradiction. Assume that $\nu_k \geq \epsilon > 0$ for sufficiently large k . First we show that this implies that $\{x_k\}$ converges and $\{\Delta_k\}$ converges to zero. Using Lemma 3.9 for sufficiently large k ,

$$(3.20) \quad -(\psi_k(s_k) + \frac{1}{2} s_k^T A^T D_k^{-1} C_k A s_k) \geq \frac{\beta_q \theta_0^2}{2} \min\{1, \beta_k^2\} \epsilon \Delta_k^2.$$

Hence, for sufficiently large k and a successful step,

$$(3.21) \quad f(x_k) - f(x_{k+1}) \geq \frac{\beta_q \theta_0^2 \mu}{2} \min\{1, \beta_k^2\} \epsilon \Delta_k^2.$$

If there are finite number of successful steps, $\{x_k\}$ converges and $\{\Delta_k\}$ converges to zero. Otherwise, let $\{k_i\}$ be the infinite sequence of successful iterations. Using Lemma 3.10,

$$(3.22) \quad \beta_k \geq \frac{\nu_k}{\chi_0(\chi_1 + (\chi_2 + \nu_k)\Delta_k)}.$$

Inequality (3.21), (3.22) and $\{f(x_k) - f(x_{k+1})\}$ converges to zero imply that $\{\Delta_k\}$ is bounded, i.e., there exists $\chi_\Delta > 0$ such that $\Delta_k \leq \chi_\Delta$. Hence

$$(3.23) \quad \beta_k \geq \frac{\nu_k}{\chi_0(\chi_1 + (\chi_2 + \nu_k)\Delta_k)} \geq \frac{1}{\chi_0(\frac{\chi_1}{\epsilon} + (\frac{\chi_2}{\epsilon} + 1)\chi_\Delta)}.$$

This inequality, (3.21), and $\{f(x_k) - f(x_{k+1})\}$ converging to zero imply that

$$\sum_{i=1}^{\infty} \Delta_{k_i}^2 < \infty.$$

The trust region updating rules of TRAM then imply that

$$\sum_{k=1}^{\infty} \Delta_k^2 \leq (1 + \frac{\gamma_2^2}{1 - \gamma_1}) \sum_{i=1}^{\infty} \Delta_{k_i}^2.$$

Hence $\{x_k\}$ converges and $\{\Delta_k\}$ converges to zero. Since $\|s_k\|_2 \leq \beta_s \Delta_k$ and $\|p_k\|_2 \leq \Delta_k$, both $\{s_k\}$ and $\{p_k\}$ converge to zero. From (3.23), (3.20) and $-\psi_k(s_k) \geq -(\psi_k(s_k) + \frac{1}{2}s_k^T A^T D_k^{-1} C_k A s_k)$, there exists $\chi_3 > 0$ such that

$$-\psi_k(s_k) \geq \chi_3 \Delta_k^2 \geq \frac{\chi_3}{\beta_s^2} \|s_k\|_2^2.$$

A standard estimate is that

$$|f(x_k + s_k) - f(x_k)| \leq \|s_k\|_2^2 \max_{0 \leq \xi_k \leq 1} \|\nabla^2 f(x_k + \xi_k s_k) - \nabla^2 f(x_k)\|_2.$$

The last two inequalities, convergence of $\{x_k\}$ and $\{s_k\}$ converging to zero imply that $\{|\rho_k - 1|\}$ converges to zero. Therefore the entire sequence $\{\rho_k\}$ converges to unity. The trust region updating rules of TRAM imply that $\{\Delta_k\}$ cannot converge to zero, which is a contradiction. Hence there is a limit point with $Z_*^T \hat{H}_* Z_*$ positive semidefinite.

- (b) If $\{x_k\}$ converges to x_* , the result follows from (a). If $\{x_k\}$ does not converge then Lemma 3.8 applies. Thus, if $\{x_{l_j}\}$ is the subsequence guaranteed by Lemma 3.8, then $\Delta_{l_j} \geq \frac{1}{\beta_s}\epsilon$. From Lemma 3.9,

$$-(\psi_k(s_{l_j}) + \frac{1}{2}s_{l_j}^T A^T D_{l_j}^{-1} C_{l_j} A s_{l_j}) \geq \frac{\beta_q \theta_0^2 \min\{1, \beta_k^2\}}{2} \nu_{l_j} \Delta_{l_j}^2.$$

Using Lemma 3.10, there exist $\chi_0, \chi_1, \chi_2 > 0$ such that, for sufficiently large j ,

$$\beta_{l_j} \geq \frac{\nu_{l_j}}{\chi_0(\chi_1 + (\chi_2 + \nu_k)\Delta_{l_j})}.$$

Hence

$$f(x_{l_j}) - f(x_{l_j+1}) \geq \frac{\mu\beta_q\theta_0^2}{2} \min\{1, (\frac{\nu_{l_j}}{\chi_0(\chi_1 + (\chi_2 + \nu_{l_j})\Delta_{l_j})})^2\} \nu_{l_j} \Delta_{l_j}^2.$$

From $\lim_{j \rightarrow \infty} f(x_{l_j}) - f(x_{l_j+1}) = 0$ and $\Delta_{l_j} \geq \frac{1}{\beta_s}\epsilon$, the above inequality implies that $\{\nu_{l_j}\}$ converges to zero. Thus $Z_*^T \hat{H}_* Z_*$ is positive semidefinite.

We have established (a)-(b). $\square$

4. Local Convergence. Let $F(x, \lambda) = 0$ denote the complementarity conditions, i.e.,

$$F(x, \lambda) \stackrel{\text{def}}{=} \begin{bmatrix} \nabla f(x) - A^T \lambda \\ D(x) \lambda \end{bmatrix},$$

where $D(x) = \text{diag}(Ax - b)$ as in (2.1). Then the Jacobian matrix $J(x, \lambda)$ of $F(x, \lambda)$ is

$$J(x, \lambda)^T \stackrel{\text{def}}{=} \begin{bmatrix} \nabla^2 f(x) & -A^T \\ \text{diag}(\lambda)A & D(x) \end{bmatrix}.$$

From (2.4), a Newton step p_k^N of the trust region subproblem (2.5) is an approximate Newton step for the complementarity conditions $F(x, \lambda) = 0$.

To establish the quadratic convergence of the proposed TRAM, we prove that, asymptotically, the Newton step p_k^N of the trust region subproblem is a sufficiently accurate approximation to the exact Newton step of $F(x, \lambda) = 0$. Moreover, asymptotically, the Newton step p_k^N is in the interior of the trust region (hence $p_k = p_k^N$) and (AS.1), (AS.2) and (AS.3) can be satisfied by a damped Newton step p_k^N .

First, we prove that the Jacobian matrix $J(x, \lambda)$ is nonsingular near a local minimizer satisfying strict complementarity and linear independence assumptions.

THEOREM 4.1. *Assume that $f(x) : \mathcal{F} \rightarrow \mathfrak{R}$ is twice continuously differentiable and the second order sufficiency conditions of (1.1) are satisfied at $(x_*; \lambda_*)$. Assume further that strict complementarity holds at x_* and $\{a_i : a_i^T x_* - b_i = 0\}$ are linearly independent. Then*

- (a) *The Jacobian matrix $J(x_*, \lambda_*)$ is nonsingular;*

(b) The symmetric matrix $A^T(\text{diag}(|\lambda|)D(x)^{-1})A + \nabla^2 f(x)$ is positive definite when $x \in \text{int}(\mathcal{F})$ and $(x; \lambda)$ is sufficiently close to $(x_*; \lambda_*)$.

Proof. Assume that there exists $(v; w)$, $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, such that

$$\begin{bmatrix} \nabla^2 f_* & -A^T \\ \text{diag}(\lambda_*)A & D_* \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = 0.$$

Without loss of generality, partition $A = [A_0; A_1]$ where the rows of A_0 correspond to the indices $\{i : a_i^T x_* - b_i = 0\}$ and the rows of A_1 correspond to the indices $\{i : a_i^T x_* - b_i \neq 0\}$. Since strict complementarity holds at $(x_*; \lambda_*)$, $A_0 v = 0$, $w_1 = 0$, and $\nabla^2 f_* v - A_0^T w_0 = 0$. Hence $v^T \nabla^2 f_* v = 0$. Since the second order sufficiency conditions are satisfied at $(x_*; \lambda_*)$ and $A_0 v = 0$, $v = 0$ holds. This implies that $A^T w = 0$. Therefore $A_0^T w_0 = 0$. Applying linear independence assumption of the columns of A_0^T , $w = 0$. Hence the Jacobian matrix $J(x_*, \lambda_*)$ is nonsingular.

Without loss of generality, assume that $\|u + v\|_2 = 1$, $u, v \in \mathbb{R}^n$, with $u^T v = 0$ and $A_0 v = 0$. Since the second order sufficiency conditions are satisfied at $(x_*; \lambda_*)$ with strict complementarity, there exist $\chi_0, \sigma_{\min}, \sigma_{\max} > 0$, such that in a neighborhood of $(x_*; \lambda_*)$,

$$(4.1) \quad \min(\lambda_0) \geq \chi_0, \quad v^T \nabla^2 f(x) v \geq \sigma_{\min} \|v\|_2^2, \quad \text{and} \quad \|\nabla^2 f(x)\|_2 \leq \sigma_{\max}.$$

Using $\|v\|_2^2 = 1 - \|u\|_2^2$, $\|v\|_2 \leq 1$ and $\|u\|_2 \leq 1$,

$$\begin{aligned} (u + v)^T \nabla^2 f(x) (u + v) &= v^T \nabla^2 f(x) v + 2u^T \nabla^2 f(x) v + u^T \nabla^2 f(x) u \\ &\geq \sigma_{\min} \|v\|_2^2 - 2\sigma_{\max} \|u\|_2 \|v\|_2 - \sigma_{\max} \|u\|_2^2 \\ &= \sigma_{\min} \|v\|_2^2 - \sigma_{\max} \|u\|_2 (2\|v\|_2 + \|u\|_2) \\ &\geq \sigma_{\min} - (\sigma_{\min} + 3\sigma_{\max}) \|u\|_2. \end{aligned}$$

Let $M = A^T(\text{diag}(|\lambda|)D(x)^{-1})A + \nabla^2 f(x)$. From the above inequality and (4.1),

$$\begin{aligned} (u + v)^T M (u + v) &\geq v^T \nabla^2 f(x) v + 2u^T \nabla^2 f(x) v + u^T \nabla^2 f(x) u + u^T A_0^T D_0^{-1} \text{diag}(\lambda_0) A_0 u \\ &\geq \sigma_{\min} - (\sigma_{\min} + 3\sigma_{\max}) \|u\|_2 + \frac{\chi_0}{\|A_0 x - b_0\|_\infty} \|A_0 u\|_2^2. \end{aligned}$$

If $\|u\|_2 < \chi_1 \stackrel{\text{def}}{=} \frac{\sigma_{\min}}{2(\sigma_{\min} + 3\sigma_{\max})}$, then $(u + v)^T M (u + v) \geq \frac{1}{2} \sigma_{\min}$. If $\|u\|_2 \geq \chi_1$, there exists $\chi_2 > 0$ such that $\|A_0 u\|_2 \geq \chi_2 \|u\|_2^2 \geq \chi_1^2 \chi_2$ since A_0 is of full row rank. Hence

$$(u + v)^T M (u + v) \geq \frac{\chi_0 \chi_1^2 \chi_2}{\|A_0 x - b_0\|_\infty} - (\sigma_{\min} + 3\sigma_{\max}).$$

When $\|A_0 x - b_0\|_\infty < \frac{2\chi_0 \chi_1^2 \chi_2}{\sigma_{\min} + 3\sigma_{\max}}$, $(u + v)^T M (u + v) > \sigma_{\min} + 3\sigma_{\max}$. Hence there exists a neighborhood of $(x_*; \lambda_*)$ in which the symmetric matrix M is positive definite. The proof is completed. $\square$

We have established in Theorem 3.7 and 3.12 that, assuming sufficient decrease and strict complementarity, there exists a limit point satisfying the first order and second order optimality conditions. We prove next that, assuming the second order

sufficiency assumption at this limit, convergence occurs and trust region size $\{\Delta_k\}$ is asymptotically bounded away from zero.

THEOREM 4.2. *Assume that (AS.0) and (AS.1) hold and $f(x) : \mathcal{F} \rightarrow \mathfrak{R}$ is twice continuously differentiable. Let $\{x_k\}$ be the sequence generated by TRAM. Assume further that (AS.3) holds for sufficiently large k and the second order sufficiency conditions and strict complementarity are satisfied at a limit x_* of $\{x_k\}$. Then $\{x_k\}$ converges to x_* , $Z_k^T \hat{H}_k Z_k$ is positive definite for sufficiently large k , all iterations are eventually successful, and $\{\Delta_k\}$ is bounded away from zero.*

Proof. For any $w \in \mathfrak{R}^m$, let $(d; \hat{d}) = Z_k^T w$ where $d \in \mathfrak{R}^n$ and $\hat{d} \in \mathfrak{R}^m$. Then $Ad - D_k^{-\frac{1}{2}} \hat{d} = 0$, i.e., $\hat{d} = D_k^{-\frac{1}{2}} Ad$. Let $M_k = A^T (\text{diag}(|\lambda_k|) D_k)^{-1} A + \nabla^2 f_k$. Using Theorem 4.1 and $C_k = \text{diag}(\lambda_k)$,

$$w^T Z_k^T \hat{H}_k Z_k w = (d; \hat{d})^T \hat{H}_k (d; \hat{d}) = d^T M_k d > 0, \text{ if } w \neq 0.$$

Hence $Z_k^T \hat{H}_k Z_k$ is positive definite when $(x_k; \lambda_k)$ is sufficiently close to $(x_*; \lambda_*)$.

Since the second order sufficiency conditions are satisfied at x_* , the point x_* is an isolated limit point. Let $\hat{s}_k = D_k^{-\frac{1}{2}} A s_k$. Then $Z_k Z_k^T (s_k; \hat{s}_k) = (s_k; \hat{s}_k)$. Since (AS.3) is satisfied for sufficiently large k , $\psi_k(s_k) + \frac{1}{2} s_k^T A^T D_k^{-1} C_k A s_k = \nabla f_k^T s_k + \frac{1}{2} (s_k; \hat{s}_k)^T \hat{H}_k (s_k; \hat{s}_k) < 0$ asymptotically. Hence

$$\begin{aligned} 0 &\leq \frac{1}{2} (s_k; \hat{s}_k)^T Z_k (Z_k^T \hat{H}_k Z_k) Z_k^T (s_k; \hat{s}_k) < -[\nabla f_k, 0]^T Z_k Z_k^T (s_k; \hat{s}_k) \\ &\leq \|Z_k^T (s_k; \hat{s}_k)\|_2 \|Z_k^T (\nabla f_k; 0)\|_2, \end{aligned}$$

whenever $Z_k^T \hat{H}_k Z_k$ is positive definite and k is sufficiently large. But

$$(s_k; \hat{s}_k)^T Z_k (Z_k^T \hat{H}_k Z_k) Z_k^T (s_k; \hat{s}_k) \geq \frac{1}{\|(Z_k^T \hat{H}_k Z_k)^{-1}\|_2} \|Z_k^T (s_k; \hat{s}_k)\|_2^2.$$

This implies that, for sufficiently large k ,

$$(4.2) \quad \frac{1}{2} \|Z_k^T (s_k; \hat{s}_k)\|_2 \leq \|(Z_k^T \hat{H}_k Z_k)^{-1}\|_2 \|Z_k^T (\nabla f_k; 0)\|_2,$$

whenever $Z_k^T \hat{H}_k Z_k$ is positive definite. From Theorem 3.7 and

$$\|Z_k Z_k^T (\nabla f_k; 0)\|_2^2 = -\nabla f_k^T g_k,$$

$\{Z_k^T (\nabla f_k; 0)\}$ converges to zero. Hence $\{(p_k; \hat{p}_k)\}$ and $\{s_k\}$ converge to zero. Following Lemma 3.8, $\{x_k\}$ converges to x_* .

Let κ be an upper bound on the condition number of $Z_k^T \hat{H}_k Z_k$. From $Z_k^T (\nabla f_k; 0) = -(Z_k^T \hat{H}_k Z_k)^T (p_k^N; \hat{p}_k^N)$ and (4.2),

$$(4.3) \quad \frac{1}{2} \|(s_k; \hat{s}_k)\|_2 \leq \kappa \|(p_k^N; \hat{p}_k^N)\|_2,$$

whenever $\{Z_k^T \hat{H}_k Z_k\}$ is positive definite and k is sufficiently large.

Since $\{x_k\}$ converges to x_* satisfying the second order sufficiency conditions with strict complementarity and $\{(p_k; \hat{p}_k)\}$ converges to zero, using Lemma 3.9 and 3.10, there exists $\chi_0 > 0$ such that, for sufficiently large k ,

$$-(\psi_k(p_k^*) + \frac{1}{2} p_k^{*T} A^T D_k^{-1} C_k A p_k^*) \geq \frac{\beta_q \theta_0^2 \chi_0}{2} \|R_k Z_k^T (p_k; \hat{p}_k)\|_2^2.$$

Since the eigenvalues of $\{Z_k^T \hat{H}_k Z_k\}$ are positive and bounded away from zero asymptotically, there exists $\chi_1 > 0$ such that, for sufficiently large k ,

$$-(\psi_k(p_k^*) + \frac{1}{2} p_k^{*T} A^T D_k^{-1} C_k A p_k^*) \geq \chi_1 \min\{\Delta_k^2, \|(p_k^N; \hat{p}_k^N)\|_2^2\}.$$

Using (4.3) and (AS.3), there exists $\chi_2 > 0$ such that

$$(4.4) \quad -(\psi_k(s_k) + \frac{1}{2} s_k^T A^T D_k^{-1} C_k A s_k) \geq \chi_2 \|(s_k; \hat{s}_k)\|_2^2 \geq \chi_2 \|s_k\|_2^2.$$

Therefore $-\psi_k(s_k) \geq \chi_2 \|s_k\|_2^2$ for sufficiently large k . But $f(x)$ is twice continuously differentiable on $\mathcal{F}$,

$$\begin{aligned} & |f(x_k + s_k) - f(x_k) - \psi_k(s_k)| \\ & \leq \|s_k\|_2^2 \max_{0 \leq \xi_k \leq 1} \|\nabla^2 f(x_k + \xi_k s_k) - \nabla^2 f(x_k)\|_2. \end{aligned}$$

Since $\{x_k\}$ converges and $\{s_k\}$ converges to zero, using (4.4), $\lim_{k \rightarrow \infty} \rho_k = 1$. This implies that $\rho_k > \eta$ for k sufficiently large. Therefore $\{\Delta_k\}$ is bounded away from zero. $\square$

Recall that p_k^N denotes the Newton step of the trust region subproblem and satisfies (2.4). Lemma 4.3 shows that the stepsize α_k^* converges to one.

LEMMA 4.3. *Assume that $f(x) : \mathcal{F} \rightarrow \Re$ is twice continuously differentiable and $\{x_k\}$ converges to x_* . Moreover, $\{a_i : a_i^T x_* - b_i = 0\}$ are linearly independent and the second order sufficiency conditions and strict complementarity are satisfied at x_* . Then, both $\{\lambda_k\}$ and $\{\lambda_k^p\}$ converge to λ_* and*

$$\begin{aligned} \|\lambda_k - \lambda_*\|_2 &= O(\|x_k - x_*\|_2), \\ \|\lambda_k - \lambda_k^p\|_2 &= O(\|x_k - x_*\|_2 + \|\lambda_k^p - \lambda_*\|_2), \\ \|\lambda_{k+1}^p - \lambda_k\|_2 &= O(\|(x_k; \lambda_k^p) - (x_*; \lambda_*)\|_2), \\ \|\alpha_k^* - 1\|_2 &= O(\|(x_k; \lambda_k^p) - (x_*; \lambda_*)\|_2), \end{aligned}$$

where $\alpha_k^* = \min\{1, \beta_k\}$ and β_k is the stepsize to the boundary of the feasible region $\mathcal{F}$ along p_k^N .

Proof. By definition (2.12) of λ_k , $(AA^T + D_k^2)\lambda_k = A\nabla f_k$. Since $\{a_i : a_i^T x_* - b_i = 0\}$ are linearly independent, $[A, -D_*]$ has full row rank and $(AA^T + D_*^2)$ is nonsingular. But $\nabla f(x)$ is continuous in $\mathcal{F}$. Thus $\|\lambda_k - \lambda_*\|_2 = O(\|x_k - x_*\|_2)$. Since the coefficient matrix (2.4) is equivalent to $J(x, \lambda)^T$ at $(x_*; \lambda_*)$, applying Theorem 4.1, the linear system (2.4) is nonsingular at $(x_*; \lambda_*)$. Since $\{x_k\}$ converges to x_* and complementarity is satisfied at x_* , $\{D_k \lambda_k\}$ and $\{\nabla f_k - A^T \lambda_k\}$ converge to zero. From (2.4) and $\{\lambda_k\}$ converges to λ_* , $\{\lambda_k^p\}$ converges to λ_* and $\lim_{k \rightarrow \infty} p_k^N = 0$. Moreover, $\|\lambda_k - \lambda_k^p\|_2 \leq \|\lambda_k - \lambda_*\|_2 + \|\lambda_k^p - \lambda_*\|_2 = O(\|x_k - x_*\|_2 + \|\lambda_k^p - \lambda_*\|_2)$. From (2.4),

$$\begin{aligned} \|\lambda_{k+1}^p - \lambda_k\|_2 &= O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2) \\ &= O(\|(x_k; \lambda_k^p) - (x_*; \lambda_*)\|_2) + O(\|\lambda_k^p - \lambda_k\|_2) \\ &= O(\|(x_k; \lambda_k^p) - (x_*; \lambda_*)\|_2). \end{aligned}$$

If $\beta_k = \infty$, $\alpha_k^* - 1 = 0$. Consider the case when $\beta_k < \infty$. Using (2.4) again, for $1 \leq i \leq m$,

$$a_i^T p_k^N = -\frac{(a_i^T x_k - b_i)(\lambda_{k+1}^p)_i}{|(\lambda_k)_i|}.$$

Since $\{p_k^N\}$ converges to zero and strict complementarity is satisfied at x_* , for sufficiently large k ,

$$\beta_k = \frac{|(\lambda_k)_j|}{(\lambda_{k+1}^p)_j}, \text{ for some } (\lambda_{k+1}^p)_j > 0 \text{ and } a_j^T x_* - b_j = 0.$$

From strict complementarity, $\lim_{k \rightarrow \infty} (\lambda_k^p)_j = (\lambda_*)_j > 0$ for any $a_j^T x_* - b_j = 0$. Hence

$$|1 - \alpha_k^*| = O(|1 - \beta_k|) = O(\|\lambda_{k+1}^p - \lambda_k\|_2) = O(\|(x_k; \lambda_k^p) - (x_*; \lambda_*)\|_2).$$

The proof is completed. $\square$

THEOREM 4.4. *Assume $0 < \mu < 1$ and $f(x) : \mathcal{F} \rightarrow \Re$ is twice continuously differentiable on $\mathcal{F}$. Assume that $\{x_k\}$ converges to x_* , the second order sufficiency conditions and strict complementarity are satisfied at $(x_*; \lambda_*)$ and $\{a_i : a_i^T x_* - b_i = 0\}$ are linearly independent. Moreover, $|\theta_k - 1| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2)$. Then $|1 - \alpha_k| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|)$, and*

$$f(x_k + \alpha_k p_k^N) - f(x_k) < \mu \psi_k(\alpha_k p_k^N), \quad \text{where } \alpha_k = \theta_k \alpha_k^*, \quad 0 < \theta_0 \leq \theta_k < 1,$$

$\alpha_k^* = \min\{1, \beta_k\}$ and β_k is the stepsize to the boundary of the feasible region $\mathcal{F}$ along p_k^N . In addition, if $x_{k+1} = x_k + \alpha_k p_k^N$ for sufficiently large k then $\{(x_k; \lambda_k^p)\}$ converges quadratically to $(x_*; \lambda_*)$.

Proof. Let $d_k = \alpha_k p_k^N$. Since $f(x)$ is twice continuously differentiable in $\mathcal{F}$, there exists $0 \leq \xi_k \leq 1$ such that

$$f(x_k + d_k) - f(x_k) = \psi_k(d_k) + \frac{1}{2} d_k^T (\nabla^2 f(x_k + \xi_k d_k) - \nabla^2 f_k) d_k.$$

Since $\{x_k\}$ converges to x_* at which the second order sufficiency conditions are satisfied, using (2.4), $\{p_k^N\}$ and $\{d_k\}$ converge to zero. Therefore, there exists $\{\epsilon_k\}$ converging to zero such that, for sufficiently large k ,

$$f(x_k + d_k) - f(x_k) = \psi_k(d_k) + \epsilon_k \|d_k\|_2^2.$$

From $0 < \mu < 1$,

$$\begin{aligned} & f(x_k + d_k) - f(x_k) - \mu \psi_k(d_k) \\ &= (1 - \mu)(\nabla f_k^T d_k + \frac{1}{2} d_k^T \nabla^2 f_k d_k) + \epsilon_k \|d_k\|_2^2 \\ &\leq (1 - \mu)(\nabla f_k^T d_k + \frac{1}{2} (d_k; \hat{d}_k)^T \hat{H}_k(d_k; \hat{d}_k)) + \epsilon_k \|d_k\|_2^2. \end{aligned}$$

Since the columns of Z_k form an orthonormal basis for the null space of $[A, -D_k^{\frac{1}{2}}]$, and $A p_k^N - D_k^{\frac{1}{2}} \hat{p}_k^N = 0$, $(p_k^N; \hat{p}_k^N) = Z_k Z_k^T (p_k^N; \hat{p}_k^N)$ and

$$\begin{aligned} \nabla f_k^T p_k^N &= -(p_k^N; \hat{p}_k^N)^T \hat{H}_k(p_k^N; \hat{p}_k^N) \\ &= -(p_k^N; \hat{p}_k^N)^T Z_k (Z_k^T \hat{H}_k Z_k) Z_k^T (p_k^N; \hat{p}_k^N). \end{aligned}$$

Substitute $d_k = \alpha_k p_k^N$,

$$\alpha_k \nabla f_k^T d_k = -(d_k; \hat{d}_k)^T Z_k (Z_k^T \hat{H}_k Z_k) Z_k^T (d_k; \hat{d}_k).$$

Hence

$$\begin{aligned} & f(x_k + d_k) - f(x_k) - \mu \psi_k(d_k) \\ & \leq -\frac{2 - \alpha_k}{2\alpha_k} (1 - \mu) (d_k; \hat{d}_k)^T Z_k (Z_k^T \hat{H}_k Z_k) Z_k^T (d_k; \hat{d}_k) + \epsilon_k \|d_k\|_2^2. \end{aligned}$$

From Theorem 4.2, the projected Hessian $Z_k^T \hat{H}_k Z_k$ is positive definite for sufficiently large k . Hence there exists $\epsilon > 0$ such that, for k sufficiently large,

$$(d_k; \hat{d}_k)^T Z_k (Z_k^T \hat{H}_k Z_k) Z_k^T (d_k; \hat{d}_k) \geq \epsilon \|Z_k^T (d_k; \hat{d}_k)\|_2.$$

Hence $\|Z_k^T (d_k; \hat{d}_k)\|_2^2 = \|(d_k; \hat{d}_k)\|_2^2$. Hence

$$f(x_k + d_k) - f(x_k) - \mu \psi_k(d_k) \leq -\frac{2 - \alpha_k}{2\alpha_k} (1 - \mu) \epsilon \|(d_k; \hat{d}_k)\|_2^2 + \epsilon_k \|d_k\|_2^2.$$

From $\{\theta_k\}$ converges to 1, $\{\alpha_k\}$ converges to 1. But $\{\epsilon_k\}$ converges to zero. Hence, for sufficiently large k ,

$$f(x_k + d_k) - f(x_k) < \mu \psi_k(d_k).$$

Let $\lambda_{k+1}^p = \lambda_k^p + w_k$. Similar to (2.4), $(d_k; w_k)$ satisfies

$$\tilde{J}_k^T (d_k; w_k) = -(\nabla f_k - A \lambda_k^p; D_k \lambda_k^p),$$

where

$$\tilde{J}_k^T = \begin{bmatrix} \frac{1}{\alpha_k} \nabla^2 f_k & -A^T \\ \frac{1}{\alpha_k} C_k A & D_k \end{bmatrix},$$

and

$$\|J(x_k, \lambda_k^p)^T - \tilde{J}_k^T\|_2 \leq \|(\text{diag}(\lambda_k^p) - C_k)A\|_2 + |1 - \frac{1}{\alpha_k}| \|\nabla^2 f_k\|_2 + |1 - \frac{1}{\alpha_k}| \|C_k A\|_2.$$

Using Lemma 4.3, $|\alpha_k^* - 1| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2)$. In addition, $|\theta_k - 1| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2)$ and $0 < (1 - \theta_k \alpha_k^*) < (1 - \theta_k) + \theta_k(1 - \alpha_k^*)$. Hence $|\alpha_k - 1| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2)$. Since the first order necessary conditions are satisfied at $(x_*; \lambda_*)$ with strict complementarity, using Lemma 4.3,

$$\|\text{diag}(\lambda_k^p) - C_k\|_2 = O(\|(x_k; \lambda_k^p) - (x_*; \lambda_*)\|_2).$$

Thus

$$\|J(x_k, \lambda_k^p)^T - \tilde{J}_k^T\|_2 = O(\|(x_k; \lambda_k^p) - (x_*; \lambda_*)\|_2).$$

Applying Theorem 3.4 in [11], $\{(x_k; \lambda_k^p)\}$ converges quadratically to $(x_*; \lambda_*)$. $\square$

THEOREM 4.5. *Assume that (AS.0) holds and $f(x) : \mathcal{F} \rightarrow \Re$ is twice continuously differentiable on $\mathcal{F}$. Assume (AS.1) is satisfied at every iteration, and (AS.2)*

and (AS.3) hold for sufficiently large k . Let $\{x_k\}$ be generated by TRAM in FIG. 1 and x_* be a limit point. Assume further that the second order sufficiency conditions are satisfied at $(x_*; \lambda_*)$ with strict complementarity, and $|\theta_k - 1| = O(\|(x_k; \lambda_k) - (x_*; \lambda_*)\|_2)$. Let $\alpha_k = \theta_k \min\{1, \beta_k\}$ where β_k is the stepsize along p_k to the boundary of $\mathcal{F}$. Then, for sufficiently large k , $\alpha_k p_k$ satisfies (AS.1), (AS.2) and (AS.3). If $x_{k+1} = x_k + \alpha_k p_k$ for sufficiently large k , then $\{(x_k; \lambda_k^p)\}$ converges to $(x_*; \lambda_*)$ quadratically.

Proof. From Theorem 4.2, $\{x_k\}$ converges to x_* and $\{\Delta_k\}$ is bounded away from zero. Since p_k^N satisfies

$$(Z_k^T \hat{H}_k Z_k) Z_k^T (p_k^N; \hat{p}_k^N) = -Z_k^T (\nabla f_k; 0),$$

and $\{\nabla f_k^T g_k\}$ converges to zero, the sequence $\{(p_k^N; \hat{p}_k^N)\}$ converges to zero. Hence, for sufficiently large k , $\|(p_k^N; \hat{p}_k^N)\|_2 < \Delta_k$ and $p_k = p_k^N$.

Using Theorem 4.4, $\{\alpha_k\}$ converges to 1. From $0 \leq \alpha_k \leq 1$ and $(p_k; \hat{p}_k) \hat{H}_k (p_k, \hat{p}_k) \geq 0$ for sufficiently large k ,

$$\psi_k(\alpha_k p_k) + \frac{\alpha_k^2}{2} p_k^T A^T D_k^{-1} C_k A p_k \leq \alpha_k (\psi_k(p_k) + \frac{1}{2} p_k^T A^T D_k^{-1} C_k A p_k).$$

Clearly (AS.3) is satisfied with $\alpha_k p_k$ for sufficiently large k . From $0 < \beta_{cs} < 1$ and $\|g_k^*\| \leq \Delta_k$, for k sufficiently large,

$$\psi_k(\alpha_k p_k) + \frac{\alpha_k^2}{2} p_k^T A^T D_k^{-1} C_k A p_k < \beta_{cs} (\psi_k(g_k^*) + \frac{1}{2} g_k^{*T} A^T D_k^{-1} C_k A g_k^*),$$

i.e., $\alpha_k p_k$ satisfies (AS.1). Since the first order necessary conditions are satisfied at $(x_*; \lambda_*)$ with strict complementarity, $g_k = \tilde{g}_k$ for sufficiently large k . Thus $\alpha_k p_k$ satisfies (AS.2) asymptotically as well.

Following Theorem 4.4, $\alpha_k p_k$ yields a successful step for sufficiently large k . Therefore $x_{k+1} = x_k + \alpha_k p_k$ asymptotically. Using Theorem 4.4 again, $\{(x_k; \lambda_k^p)\}$ converges quadratically to $(x_*; \lambda_*)$. $\square$

It is straightforward to verify that the convergence analysis and results hold if, in the computation of the projected gradient g_k , $D_k^{\frac{1}{2}}$ is replaced by D_k^l where $l \geq \frac{1}{2}$. Similar remarks can be made for $\tilde{g}_k$.

5. Conclusion. In this paper, global and local convergence properties of a trust region and affine scaling interior point method (TRAM) are established for minimizing a general nonlinear (nonconvex) function with linear inequality constraints.

Under compactness and full row rank assumption (AS.0), it is established that, if a computed step leads to a sufficient decrease with respect to the projected gradient g_k using affine scaling D_k , every limit point satisfies the complementarity conditions. If (AS.2) is satisfied asymptotically, then there exists a limit x_* with $\nabla f_*^T \tilde{g}_* = 0$. If, in addition, strict complementarity holds at x_* . Then $\nabla f_*^T \tilde{g}_* = 0$ at every limit point of $\{x_k\}$. Thus the Kuhn-Tucker conditions are satisfied at any limit x_* satisfying strict complementarity. Moreover, if (AS.3) is satisfied asymptotically, then there exists a limit point at which both the first and second order necessary conditions are satisfied. Finally, if the second order sufficiency conditions and strict complementarity are satisfied at a limit point, then $\{(x_k; \lambda_k^p)\}$ converges quadratically.

In the proposed method, a 2-norm trust region subproblem can be approximately solved. This trust region subproblem uses an affine scaling so that a damped step

of the subproblem solution generates sufficient decrease of the objective function. Moreover, asymptotically, the solution of the trust region subproblem corresponds to an approximate Newton step for complementarity conditions. Preliminary computational results in [8] suggest that the proposed method is capable of solving many large linear-inequality-constrained nonlinear minimization problems in small numbers of iterations.

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