Real Functions for Representation of Rigid Solids*

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Abstract

A range of values of a real function $f : E^d \rightarrow \Re$ can be used to implicitly define a subset of Euclidean space E^d. Such 'implicit functions' have many uses in geometric and solid modeling. This paper focuses on the properties and construction of real functions for the representation of rigid solids (compact, semi-analytic, and regular subsets of E^d). We review some known facts about real functions defining compact semi-analytic sets, and their applications. The theory of R-functions developed in [Rva82] provides means for constructing real function representations of solids described by the standard (non-regularized) set operations.

But solids are not closed under the standard set operations and such representations are rarely available in modern modeling systems. More generally, assuring that a real function f represents a regular set may be difficult. Until now, the regularity has either been assumed, or treated in an *ad hoc* fashion. We show that topological and extremal properties of real functions can be used to test for regularity, and discuss procedures for constructing real functions with desired properties for arbitrary solids.

1 Introduction

1.1 Complete representations of solids

The origins of current (Western) theory of solid modeling can be traced to 1970s, when various mathematical models for rigid solids were proposed. Notably, a compact, regular, and semi-analytic set of points in E^3 has been accepted as a standard model in solid modeling [Req77], [RT78]; such sets are called r-sets.[1] One important property of r-sets is that they are closed under regularized set union ($\cup^*$), intersection ($\cap^*$), and difference ($-^*$) operations. This facilitates the representation of solids in terms of Boolean operations on simpler solids, which is crucial in user interfaces and many applications.

A representation scheme associates with every r-set a syntactically correct finite symbol structure, or representation, from a particular representation space. Probably the most important property of a representation scheme is its *completeness;* complete representations define solids (r-sets) unambiguously. Given a complete representation of solid S, it should be possible to decide for any point $p \in E^3$ whether p is in S, or out of S. In other words, the characteristic function of S

$$\xi(p, S) = \begin{cases} 1 & \text{if } p \in S \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

can be computed, at least in principle. As a matter of practical necessity, it is sometimes also desired to distinguish between the interior $\mathbf{i}S$ and the boundary ∂S; thus, completeness is often identified with the ability to construct a point membership classification (PMC) function [RV77]

$$\text{PMC}(p, S) = \begin{cases} \textit{in} & \text{if } p \in \mathbf{i}S \\ \textit{on} & \text{if } p \in \partial S, \\ \textit{out} & \text{if } p \in \mathbf{e}S \end{cases} \tag{2}$$

where exterior $\mathbf{e}S$ is defined as the set complement $-S$. Note that the ability to perform PMC does not make a representation scheme any "more complete", because the boundary ∂S is topologically well defined once the characteristic function ξ for S is known.

At least six families of informationally complete representation schemes are currently known [Req80]; two of the most widely used representation schemes have been studied formally: Constructive Solid Geometry (CSG) [RV77] and boundary representation (b-rep) [Sil81]. Other formal properties of various representations, such as validity and uniqueness, seem to be well understood [Req80]. Relationships between distinct representation schemes and conversion algorithms have also been studied [Sha91].

1.2 Implicit real functions

In all representation schemes, PMC sooner or later reduces to a number of simpler PMCs against "primitives" in the representation scheme. The primitives in CSG are typically halfspaces defined by an inequality $f(x, y, z) \geq 0$, in boundary representations the primitives are typically surfaces $f(x, y, z) = 0$ containing a solid's faces, and so on. Primitives may be thought of as the letters in the alphabet of a representation scheme. While the semantics of a representation is usually

[1]There are also alternative mathematical models for solids based on closed two-dimensional manifolds (see [Hof89] for references) and on open regular sets [Arb90], which we shall not discuss in this paper.

determined using set operations, incidence relationships, combinatorial structures, and topological properties, the semantics of a primitive is very simple: it is defined by a range of values of some real function $f(x, y, z)$. Such functions are often called "implicit", because they represent subsets of E^3 that are not specified explicitly by their boundaries or parameterizations.

Many practical uses of real implicit functions representing solids are well documented, and a comprehensive survey is beyond the scope of this paper. Such functions have been used to perform PMC tests in early solid modeling systems [OKK73]. They have been used extensively to model blends and offsets, for example in [Ric73, Roc89] (for a survey on this subject the reader is referred to [Woo87]). Restricted types of real functions have been employed to define surprisingly rich classes of solids whose shape that can be parameterized and manipulated [Bar81, Han88]. Implicit functions defining the geometry of physical environments and obstacles have provided a basis for solving problems of motion planning and control in robotics and manufacturing [Kha86, Kod89, RK90]. The use of such functions has been advocated for interactive modeling and animation by [Bli82], [WMW86], [BW90] and many others. It has been demonstrated in [KK58] that some boundary value problems of mathematical physics can be solved without domain discretization, if appropriate real functions defining the domain are known (see appendix A.1). This has led to the development of a new powerful theory, methods, and systems to approximate the solutions to partial differential equations [Rva67, Rva74, RR79, Rva82]. At the same time, implicit functions facilitate polygonization [Blo88] and computation of simplicial approximations (meshes) of surfaces and domains [AG90, Wid90]. Finally, the ability to encode geometric information in terms of real functions has allowed new formulations and solutions of many geometric placement and optimization problems [Sto75, SY86] (also see appendix A.2).

However, real functions with the desired properties can be difficult to construct for complex objects. Various techniques can be used to obtain sufficiently smooth approximations as in [Ric73] and [BN90]. For animation and visualization purposes [Bli82, WMW86, BW90, Han88], interactive control and "clay-like" deformation properties of represented sets can be more important than the specific function behavior at a point, and many *ad hoc* techniques seem to work well. Such approximations are also important for smoothing and blending applications, but they require much more care and a sophisticated control [Woo87]. Many authors suggest that desired real functions can be defined procedurally, i.e. encoded by an algorithm that returns some (usually heuristically obtained) values [Ric73, WMW86, BW90]. This approach may not be acceptable for many applications, where the formal properties of real functions are important. For example, numerical robustness is addressed in [AG90, Wid90, KB89], issues of convergence arise in [Bli82, Han88], and differential and topological properties are crucial in [Rva82, Kod89, RK90].

The distinction between a primitive and a "non-primitive" object is not entirely clear in this context. What is the class of solids for which functions with desirable properties can be constructed? Are such representations complete in the sense of [Req80]? What are the properties of these real functions and how are they related to other representation schemes? This paper provides answers to some of these questions using both known and new facts about real functions for representing solids.

1.3 Characteristic and PMC defining functions

We say that a real valued function $f : E^d \rightarrow \Re$ (implicitly) defines a set $S \subset E^d$, or f is a *characteristic defining* function for S, if

$$f^{-1}(X) = S, \text{ for some } X \subset \Re. \tag{3}$$

Here X is the range of real values of f that (implicitly) characterizes which points of the Euclidean space E^d belong to S, and f can be trivially transformed into the characteristic function (1), thus completely defining S.[2]

For example, [Ric73] relied on non-negative real functions, called solid defining functions,

$$f(p) \text{ is } \begin{cases} \in (0,1) & \text{if } p \in \mathbf{i}S \\ = 1 & \text{if } p \in \partial S. \\ > 1 & \text{if } p \in \mathbf{e}S \end{cases} \tag{4}$$

According to our definition, f is a characteristic defining function because S is the set of points $f^{-1}((0,1])$. Similar functions are employed in [Bar81, Han88, BN90]. Instead of 1 in Eq. (4), other arbitrary threshold values can be chosen to obtain and control similar defining functions [Bli82, WMW86]. If 1 is chosen as the threshold value, a characteristic defining function for the closure of the complement of the defined set, $\mathbf{k}(-S)$, is conveniently given by the real function $1/f$.

Many engineering and scientific applications [AG90, Rva82, Wid90, Kha86, Kod89, RK90] take advantage of set representations by functions of the form

$$w(p) \text{ is } \begin{cases} > 0 & \text{if } p \in \mathbf{i}S \\ = 0 & \text{if } p \in \partial S. \\ < 0 & \text{if } p \in \mathbf{e}S \end{cases} \tag{5}$$

Clearly, w is also a characteristic defining function for the set $(w \geq 0) \equiv w^{-1}([0,\infty))$. There is an infinite number of possible characteristic defining functions; all of them are "equally complete" and can be mapped into each other. For example, [Bli82] and [Woo87] take advantage of the fact that the functions in Eqs. (4) and (5) are related by

$$f(p) = e^{-w(p)}, \tag{6}$$

although such a transformation is not unique.

Both functions (4) and (5) also distinguish between the boundary and the interior points of S, which is not required by our definition of a characteristic defining function. Thus both functions actually specify the PMC procedure. Such characteristic defining functions for S will be called *PMC defining* functions for S. When the set S is clear from the context, we may refer to corresponding real functions simply as characteristic (or PMC) defining. It should be apparent that not every characteristic defining function is also PMC defining.

In the sequel, we focus on the properties and construction of characteristic and PMC defining functions for solids, i.e. compact, regular, and semi-analytic sets. We restrict ourselves to characteristic defining functions w for a solid S, such that

$$S = (w \geq 0) \equiv \{p \mid w(p) \geq 0\}. \tag{7}$$

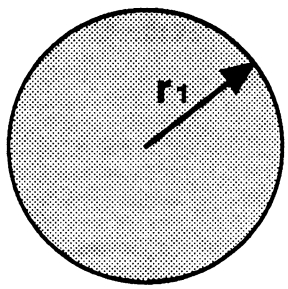
[2] We shall see below that additional properties of real functions could be taken into consideration to construct characteristic functions. We will not call such functions characteristic defining, unless they satisfy the above definition.

Similarly, it will be understood that PMC defining functions for S must satisfy Eq. (5). Analogous results for other characteristic and PMC defining functions can be obtained using simple transformations such as Eq. (6).

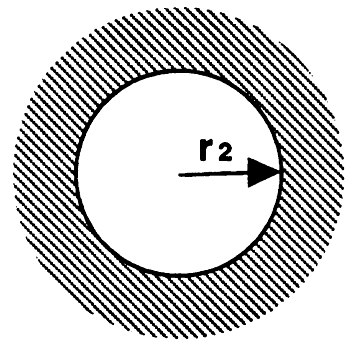
In general, assuring solidity of S may be difficult, though various heuristic arguments have been used (e.g. see [Ric73], [Bar81]). Consider a family of real functions $w_i(x, y) = r_i^2 - x^2 - y^2$, where r_i is a constant. For any finite value r_1, $(w_1 \geq 0)$ defines a two-dimensional solid disk (Figure 1(a)). Figure 1(b) shows that a function $-w_2 = x^2 + y^2 - r_2^2$ defines an unbounded closed subset of E^2 which is not topologically solid. The intersection of the two sets $(w_1 \geq 0)$ and $(-w_2 \geq 0)$ could be defined by a real function $\min(w_1, -w_2)$ [Rva67, Ric73]. The defined set is a solid when $r_1 > r_2$ (Figure 1(c)). If $r_1 < r_2$, this function defines the empty set $\emptyset$, and, when $r_1 = r_2$, the defined set is not regular in E^2 (Figure 1(d)).

In section 2, we review some of the known facts about characteristic defining functions for compact semi-analytic sets, drawing heavily on the theory of R-functions developed in [Rva82]. (This work is published in Russian; an English summary and additional references can be found in [Sha88].) This material is not new, but some established results are rarely acknowledged in the literature.

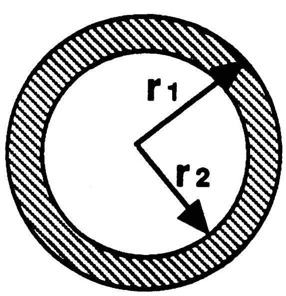
Section 3 focuses on the relationship between topological and extremal properties of characteristic defining functions and the regularity of the represented sets. These results are used to establish necessary and sufficient conditions for a real function to be PMC defining and to derive a systematic procedure for constructing such functions.



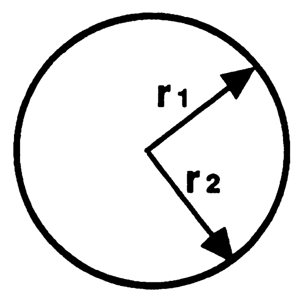
(a) A real function $w_1(x, y) = r_1^2 - x^2 - y^2$ defines a solid ($w_1 \geq 0$) for any value of r_1.



(b) A function $-w_2(x, y) = x^2 + y^2 - r_2^2$ defines an unbounded closed set ($-w_2 \geq 0$).



(c) Whenever $r_1 > r_2$, function $\min(w_1, -w_2)$ defines a solid $(w_1 \geq 0) \cap (-w_2 \geq 0)$.



(d) But if $r_1 = r_2$, then set $(\min(w_1, -w_2) \geq 0)$ is a circle, which is not regular in E^2.

Figure 1: Defining functions for closed sets and solids

2 Real functions for compact semi-analytic sets

2.1 Continuous bounded functions for compact sets

Given two topological spaces A and B, it is a fundamental topological fact that a map $f : A \to B$ is continuous if and only if $w^{-1}(X)$ is closed in A for every closed $X \subset B$. This fact can be used to show that, if $w : E^d \to \Re$ is any continuous real function, then inequality ($w \geq 0$) defines some closed set $S \subseteq E^d$. Conversely, for any closed $S \subset E^d$, let

$$d(p) \equiv \inf_{x \in \partial S} \|p - x\| \tag{8}$$

be the minimum Euclidean distance from a point $p \in E^d$ to the boundary ∂S of the set S. Then it is easy to see that

$$w(p) \equiv \begin{cases} d(p) & \text{if } p \in \mathbf{i}S \\ 0 & \text{if } p \in \partial S \\ -d(p) & \text{if } p \in \mathbf{e}S \end{cases} \tag{9}$$

is a continuous PMC defining function for S. The set of all continuous PMC defining functions for a set S is closed under addition and scalar multiplication, forming a linear vector space. It is apparent that closed sets are naturally defined using continuous real functions. But a stronger result is also known.

Proposition 1 ([RR79]) *Let $S \subset E^d$ be a closed set. For any such S, there exists a PMC defining function $w : E^d \to \Re$ such that $w \in C^\infty$.*

If in addition S is a bounded set, it follows immediately that any such continuous function w is bounded on S. Thus, for every compact $S \subset E^d$ there exists a real, m times continuously differentiable, and bounded PMC defining function w.

2.2 Semi-analytic sets

The class of C^∞ real functions properly contains the class of real *analytic* functions. A real function $f(x)$ is called analytic at a point x_0 if it can be represented as the sum of a convergent power series:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{10}$$

for all x in some neighborhood of x_0. Note that analyticity is defined only at a point and therefore is a local property. Polynomials, the trigonometric functions $\cos x$ and $\sin x$, and the exponential function e^x are all examples of functions that are analytic everywhere. The class of analytic functions is closed under addition, multiplication, inverse, and composition, and possesses other attractive properties.

Unfortunately, analytic functions are not good candidates for the definition of solids because they cannot be used to describe many common objects. For example, there does not exist *any* analytic characteristic defining functions for a simple rectangle [Rva82]: the analyticity breaks down at the corner points. The rectangle (and all solids) belongs to the class of *semi-analytic* sets.

Semi-analytic sets were first suggested and studied in [Loj64] as a natural generalization of semi-algebraic sets. They can be defined locally (at every point) as a finite Boolean combination (i.e. a finite sequence of unions, intersections, and complements) of sets $\{x \in E^d \mid f_i \geq 0\}$, where f_i are real analytic functions. If S is a semi-analytic set, then the closure $\mathbf{k}S$, interior $\mathbf{i}S$, boundary ∂S, and connected components of S are also semi-analytic. Furthermore, all semi-analytic sets have "well-behaved" boundaries and can be triangulated. In that sense, *bounded* semi-analytic sets are finitely describable. These attractive properties were observed by Requicha [Req77], who suggested that semi-analytic sets constitute the appropriate class of mathematical objects for solid modeling.

In summary, while not all semi-analytic sets can be defined by real analytic functions, a PMC defining C^∞ function exists for any closed semi-analytic set. Of course, such an existential statement is not very useful unless we can actually supply a method for constructing appropriate functions for any semi-analytic S. By definition, any semi-analytic set can be defined, at least in principle, using set operations on some "primitives"; each primitive would be specified implicitly by the sign of some real analytic function f_i. If a method for constructing characteristic defining functions from such set-theoretic expressions can be found, we would indeed achieve the goal of representing semi-analytic sets using real functions. It turns out that such a method is well known from the theory of R-functions developed since the 1960's [Rva67, Rva82]. A detailed survey of this theory and its applications is beyond the scope of this paper (see [Sha88]); the following discussion has the rather limited goal of establishing the relationship between Boolean (logic or set) operations and certain real functions.

2.3 R-functions

Some real-valued functions of real variables have the property that their signs are completely determined by the signs of their arguments and are independent of the magnitude of the arguments. For example, the function $W_1 = xyz$ can be negative only when the number of its negative arguments is odd. A similar property is possessed by functions $x + y + \sqrt{xy + x^2 + y^2}$ and $xy + z + |z - yx|$, and so on. In contrast, the signs of many functions (like $xyz + 1$ and $\sin xy$) depend not only on the sign of the arguments but also on their magnitude.

Besides the partition of real numbers according to their sign, there are many other choices for partitions of real numbers (e.g. into all real numbers in interval $[0, 1]$, and the rest of the real numbers). In general, any such partition of the real line is based on some criterion, which also determines a set of those real functions that in some sense "inherit" the partition criterion. Such functions are called ***R*-functions**. Here we will only consider R-functions defined by the partition of the real axis into negative and non-negative numbers $\{(-\infty, 0),\ [0, +\infty)\}$.

To formalize the notion of such R-functions, consider function $B : \Re \to \{\text{true}, \text{false}\}$ defined on the real axis as follows:

$$B(x) = \begin{cases} \text{false} & \text{if } x < 0, \\ \text{true} & \text{if } x \geq 0. \end{cases} \tag{11}$$

A real function $f(x_1, \ldots, x_n)$ is an R-function if and only if there exists a Boolean logic function $\Phi(X_1, \ldots, X_n)$ such that

$$B[f(x_1, x_2, \ldots, x_n)] = \Phi[B(x_1), B(x_2), \ldots, B(x_n)]. \tag{12}$$

The Boolean function Φ is called the *companion* function of a given R-function. Many interesting properties of R-functions have been studied in [Rva67], [Rva82]. Every Boolean function is a companion to an infinite number of R-functions, which form a *branch* of the set of R-functions. For example, $\min(x_1, x_2)$ is an R-function whose companion Boolean function is logical "and" ($\wedge$), and $\max(x_1, x_2)$ is an R-function whose companion Boolean function is logical "or" ($\vee$). Thus, functions

$$\begin{aligned} x_1 \wedge_1 x_2 \equiv \min(x_1, x_2) &= \tfrac{1}{2}[x_1 + x_2 - \sqrt{(x_1 - x_2)^2}]; \\ x_1 \vee_1 x_2 \equiv \max(x_1, x_2) &= \tfrac{1}{2}[x_1 + x_2 + \sqrt{(x_1 - x_2)^2}] \end{aligned} \tag{13}$$

are called R-conjunction and R-disjunction respectively. The operation of $\sqrt{x^2}$ can be replaced with $|x|$ which is convenient for computational purposes. The R-functions in (13) are not differentiable along the lines $x_1 = x_2$, but the same branches of R-functions contain many other functions, e.g.

$$\begin{aligned} x_1 \wedge_\alpha x_2 &\equiv \tfrac{1}{1+\alpha}(x_1 + x_2 - \sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2}); \\ x_1 \vee_\alpha x_2 &\equiv \tfrac{1}{1+\alpha}(x_1 + x_2 + \sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2}), \end{aligned} \tag{14}$$

where $\alpha(x_1, x_2)$ is an arbitrary symmetric function such that $-1 < \alpha(x_1, x_2) \le 1$. The precise value of α often may not matter, and it can be set to constant. For example, setting $\alpha = 1$ yields the system (13). Similarly, setting $\alpha = 0$ results in:

$$\begin{aligned} x_1 \wedge_0 x_2 &\equiv x_1 + x_2 - \sqrt{x_1^2 + x_2^2}; \\ x_1 \vee_0 x_2 &\equiv x_1 + x_2 + \sqrt{x_1^2 + x_2^2}. \end{aligned} \tag{15}$$

This system is sometimes preferable to the system (13), because the defined R-functions are differentiable unless $x_1 = x_2 = 0$. Finally, R-functions

$$\begin{aligned} x_1 \wedge_\alpha^m x_2 &\equiv (x_1 \wedge_\alpha x_2)(x_1^2 + x_2^2)^{\frac{m}{2}}; \\ x_1 \vee_\alpha^m x_2 &\equiv (x_1 \vee_\alpha x_2)(x_1^2 + x_2^2)^{\frac{m}{2}} \end{aligned} \tag{16}$$

are analytic everywhere except the origin ($x_1 = x_2 = 0$), where they are m times differentiable (i.e. they are in C^m). Other systems of R-functions are studied in [Rva82]. The choice of an appropriate system of R-functions is dictated by many considerations, including simplicity, continuity, differential properties, and computational convenience.

Just as Boolean functions, R-functions are closed under composition. In particular, suppose $\Phi(X_1, \ldots X_n)$ is a Boolean expression constructed from n logical variables X_i and logical connectives $\vee$, $\wedge$. A corresponding R-function is immediately obtained by formally replacing logical variables X_i with real variables x_i and logical connectives with some R-conjunctions and R-disjunctions respectively.

Let us construct an R_0-function whose companion Boolean function is given by logical expression $\Phi(X_1, X_2) = (X_1 \wedge X_2) \wedge (X_1 \vee X_2)$. The desired R_0-function is

$$\begin{aligned} f(x_1, x_2) &= (x_1 \wedge_0 x_2) \wedge_0 (x_1 \vee_0 x_2) = \\ &= (x_1 + x_2 - \sqrt{x_1^2 + x_2^2}) + (x_1 + x_2 + \sqrt{x_1^2 + x_2^2}) - \\ &- \left[(x_1 + x_2 - \sqrt{x_1^2 + x_2^2})^2 + (x_1 + x_2 + \sqrt{x_1^2 + x_2^2})^2\right]^{\frac{1}{2}}. \end{aligned}$$

After simplifying and dividing by a positive multiplier, we get

$$f(x_1, x_2) = x_1 + x_2 - \sqrt{x_1^2 + x_2^2 + x_1 x_2}.$$

A simpler R_0-function could be obtained by noticing that the original logical expression Φ is equivalent to $X_1 \wedge X_2$. Some R-function simplification techniques have been studied in [Rva82], but in general, optimization of R-functions remains a challenging open problem.

2.4 Characteristic defining functions for closed semi-analytic sets

Recall that here we are only interested in *closed* semi-analytic sets. The family of all closed semi-analytic subsets of E^d forms a Boolean lattice with the standard set union ($\cup$) and intersection ($\cap$) operations. This means that every closed semi-analytic set can be represented using the two operations of $\cup$ and $\cap$ on some primitives, where a primitive is defined by a real analytic function inequality ($f_i \geq 0$). Accordingly, the theory of R-functions suggests a practical method for constructing characteristic defining functions for any closed semi-analytic sets.

Proposition 2 ([Rva74]) *Let $f(x_1, \ldots, x_n)$ be an R-function whose Boolean companion function $\Phi(X_1, \ldots, X_n)$ maps closed sets into closed sets. If a closed set S is defined using n primitives $(\phi_i \geq 0)$ as*

$$S = \Phi[(\phi_1 \geq 0), \ldots, (\phi_n \geq 0)], \tag{17}$$

then S is also defined by

$$f(\phi_1, \ldots, \phi_n) \geq 0. \tag{18}$$

In other words, we assume that a closed semi-analytic set $S \subset E^3$ is defined by a Boolean expression Φ in Eq. (17) using the standard set operation on n primitives $(\phi_i(x, y, z) \geq 0)$. To obtain a characteristic defining real function $f(x, y, z)$ in inequality (18), it suffices to construct an R-function f whose companion Boolean function is Φ and substitute for arguments of f the primitive characteristic defining functions $\phi_i(x, y, z)$.[3]

Let us construct a C^m characteristic defining function for solid S shown in Figure 2. S can be represented as the union of two solid blocks B_1 and B_2 intersected with the exterior cylindrical halfspace $C = (g(x, y, z) \geq 0)$. Each block B_i is an intersection of six linear halfspaces: $B_i = (f_{i1}(x, y, z) \geq 0) \cap \ldots \cap (f_{i6}(x, y, z) \geq 0)$. Thus S can be defined by the following Boolean expression:

$$S = (B_1 \cup B_2) \cap C = \{[(f_{11} \geq 0) \cap \ldots \cap (f_{16} \geq 0)] \cup [(f_{21} \geq 0) \cap \ldots \cap (f_{26} \geq 0)]\} \cap (g \geq 0).$$

Using the procedure outlined in proposition 2, we get the following characteristic defining function for S:

$$S = \{[(f_{11} \wedge_\alpha^m \ldots \wedge_\alpha^m f_{16}) \vee_\alpha^m (f_{21} \wedge_\alpha^m \ldots \wedge_\alpha^m f_{26})] \wedge_\alpha^m g \geq 0\}.$$

Functions $f_{11}, \ldots, f_{26}, g$ and operations $\wedge_\alpha^m, \vee_\alpha^m$ could be further replaced by their respective definitions, yielding a (rather cumbersome) characteristic defining function for S in terms of the cartesian coordinates x, y, z and the usual arithmetic operations.

The properties of the constructed function f in (18) are determined by the properties of the chosen R-functions and of the primitive real functions ϕ_i. In particular, if all primitives are defined

[3] A similar result holds for constructing real-function inequalities $f > 0$ defining open semi-analytic sets.

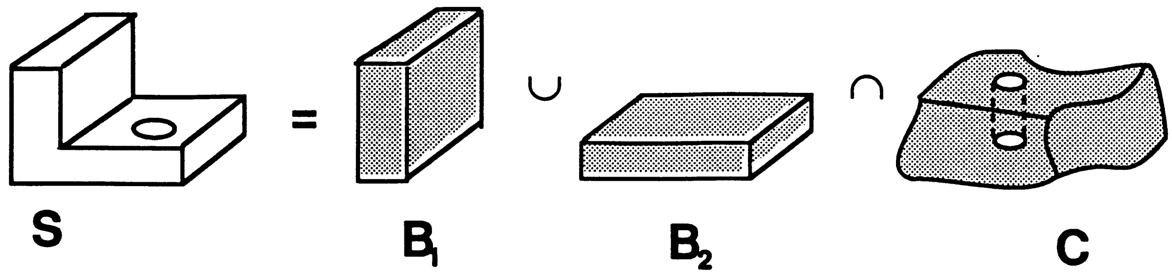


Figure 2: Solid S can be defined using standard set operations on primitive halfspaces.

by analytic functions and the constructed R-function f is a composition of $\vee_{\alpha}^{m}$ and $\wedge_{\alpha}^{m}$ given in Eqs. (16), then f will belong to the class C^m. (It also turns out that, once a continuous characteristic defining function for S is obtained, a C^{∞} characteristic defining function can be also constructed [RR79, pp. 50-51].)

Since all solids are closed semi-analytic sets, it follows from the theory of R-functions that C^m characteristic defining functions can be constructed for any solid, at least in principle. The theory also gives means for constructing such functions. The relationship between set operations and functions {min, max} was observed independently by Ricci [Ric73], who used this fact to find approximate smooth defining functions. Often other and more convenient methods for obtaining characteristic defining functions are available for many solids with special properties, such as symmetry or periodicity [Rva82], desired parametric definitions [Han88, Bar81], and so on.

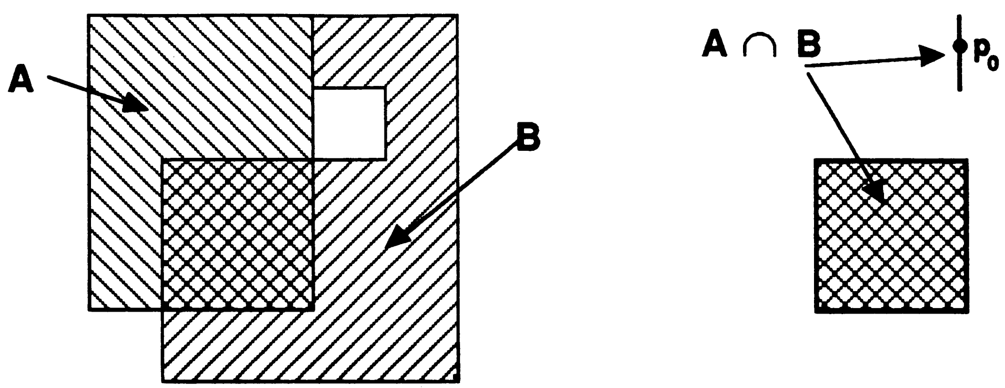
However, we shall see below that the characteristic defining functions constructed using R-functions are *not* PMC defining, because they do not explicitly distinguish between interior and boundary points. Furthermore, modern solid modeling systems seldom represent solids using standard set operations on analytic primitives. Such representations are clearly absent in systems that rely primarily on boundary or "sweep" representations. And, because solids are not closed under standard set operations [RT80], even CSG systems cannot support the construction of characteristic defining functions.

More generally, little attention has been paid to the regularity of the sets defined by real functions. This important property is usually treated in an *ad hoc* fashion, and neither [Ric73] nor [Rva82] can guarantee the regularity of the represented semi-analytic sets. The following section considers relationships between the properties of characteristic defining functions and the regularity of the represented sets. In particular, we determine under what conditions such functions are PMC defining for solids, and suggest a method for their construction.

3 Functions defining closed regular sets

3.1 Closed regular sets

By definition, a set S is closed regular if $\mathbf{ki}S = S$.[4] Properties of closed regular sets have been studied in [MT46], [KM76], [RT78], and are well understood. It is well known that closed regular sets are closed under set union ($\cup$), but *not* under intersection ($\cap$) [RT80]. For example, Figure 3 shows that intersection of two-dimensional solids can define a set that is not regular in E^2.



(a) Sets A and B are regular **(b)** Set $A \cap B$ is not regular

Figure 3: Intersection of solids A and B is not a regular set.

In solid modeling, the problem is addressed by introducing the regularized set operations, defined by

$$A \cap^* B = \mathbf{ki}(A \cap B); \qquad A \cup^* B = \mathbf{ki}(A \cup B); \qquad A -^* B = \mathbf{ki}(A - B).$$

Note that the regularized union $\cup^*$ is identical to the non-regularized union $\cup$. These operations provide the basis for the popular CSG representation of solids [RV77].

Suppose we are given a CSG representation Φ of S; Φ is an expression involving regularized operations on primitives defined as $(\phi_i \geq 0)$. Can we use Φ to construct a characteristic defining function for S? It is tempting to misinterpret proposition 2 and to replace all regularized set operations in Φ by the "corresponding" R-functions [AB90]. But this may lead to an erroneous result, because the R-conjunction $\wedge_\alpha^m$ corresponds to the standard set intersection $\cap$, and not to the regularized $\cap^*$. Consider the two solids in Figure 3(a) that are defined as $A = (f_A \geq 0)$ and $B = (f_B \geq 0)$ respectively. The function $f_A \wedge_\alpha^m f_B$ is characteristic defining for the set shown in Figure 3(b), which is not regular. The results of regularized set operations depend on the local behavior of their arguments; this behavior can be detected by the regularized set membership classification

[4] According to [KM76], closed regular sets were first defined by Lebesque.

algorithms [Til80], but is not captured by R-functions. A proper application of proposition 2 requires *rerepresenting* solid S using standard set operations $\cap$, $\cup$ alone. This latter problem is not trivial; a conversion algorithm for general semi-algebraic solids is described in [Sha91].[5]

The theory of R-functions suggests that real function representations should be viewed as secondary representations derived from (primary) set representations of solids. This view makes direct manipulation of characteristic defining functions difficult, because it is not clear that regularity can be guaranteed or even tested for. Even if a solid S is represented by a characteristic defining function as $(w \geq 0)$, we cannot guarantee that w is strictly positive in $\mathbf{i}S$, as for example is required in approximating solutions to boundary value problems (see Appendix A.1). More generally, such a function w is not PMC defining, because the interior $\mathbf{i}S$ and the boundary ∂S are not explicitly distinguished.

3.2 Regularity, boundaries, and zero sets

Suppose w is a continuous characteristic defining function for a closed set $S \subset E^d$, i.e. $S = (w \geq 0)$. Irrespectively of how w is constructed, if $w(p_0) > 0$ for some point $p_0 \in E^d$, then p_0 must belong to the interior $\mathbf{i}S$. (By continuity of w, there is a neighborhood of p_0 where $w > 0$.) It follows that, if $p_0 \in \partial S$, then $w(p_0)$ must be equal to zero, and so $\partial S \subseteq (w = 0)$. Note that w could be identically zero anywhere (even everywhere) in S; thus, knowing that $w(p_0) = 0$ does not imply anything more than $p_0 \in S$. Henceforth we will call the set $(w = 0)$ a *zero set* of w and points $p_0 \in (w = 0)$ *zero points* of w. These simple observations allow to express the condition for regularity of the represented set S in terms of zero points of a characteristic defining function w.

Proposition 3 *Let $w : E^d \rightarrow \Re$ be a continuous characteristic defining function for a closed set $S = (w \geq 0)$. Set S is regular if and only if every neighborhood of every zero point $p_0 \in (w = 0)$ contains interior points of S.* □

The proposition states an intuitively obvious fact that, given an arbitrary real function w, the regularity of set $S = (w \geq 0)$ is determined by the boundary points of S, which are also zero points of w. The properties of w at interior points of S (including those where $w > 0$) are not important for regularity of S. Indeed, all interior points of S are automatically included in $\mathbf{ki}S$, and need not be considered. Thus, it is easy to see that the proposition is true for any regular set S, since neighborhood of every boundary point $p_0 \in S$ contains interior points of S. Similarly, requiring that neighborhoods of all zero points of w contains interior points of S translates into requirement that all boundary points of S are in $\mathbf{ki}S$; hence S must be regular.

Now consider two characteristic defining functions w_1 and w_2 for the same set S; it is clear that $w_1 + w_2$ is also characteristic defining for S. In view of the above discussion, a stronger statement is possible for regular sets.

Proposition 4 *Suppose S is a closed regular set and w_1, w_2 are two real functions such that $S = (w_1 \geq 0) = \mathbf{ki}(w_2 \geq 0)$. Then $S = (w_1 + w_2 \geq 0)$.* □

Clearly, for any point $p \in S$, $w_1(p) + w_2(p) \geq 0$. If $p \notin S$, then w_1 must be negative and w_2 cannot be positive at p. And so in this case $w_1(p) + w_2(p) < 0$. In other words, the regularity of

[5] Roughly, the conversion may require construction of additional primitives $(\phi_i \geq 0)$, followed by a decomposition of the space E^d into appropriately defined "cells," and classification of these cells against the given CSG representation of solid S.

set ($w_1 \geq 0$) "absorbs" any non-regularity of set ($w_2 \geq 0$). This fact will be used in section 3.5 for construction of PMC defining functions for solids.

3.3 Regularity and extrema of characteristic defining functions

Proposition 3 does not help in the construction of characteristic defining functions, but it does suggest how the "non-regular" points of a set $S = (w \geq 0)$ can be identified from the properties of w. Well known regularization algorithms [RT78] examine the neighborhoods of certain boundary points of S. Suppose a closed set S is not regular. By proposition 3, there exists a zero point p_0 (for example, see Figure 3) whose neighborhood contains only boundary points (w is zero), and exterior points of S (w is negative). Therefore p_0 must be a point where $w(p)$ has a local maximum. Thus all neighborhood information needed to decide the regularity of S is encoded in terms of the local extremum properties of the characteristic defining function w.

The above analysis has two implications. First, any characteristic defining function w for a closed (but not necessarily regular) set $S = (w \geq 0)$ can be also used to represent the regularized set $\mathbf{ki}S$. The corresponding characteristic function can be constructed as

$$\xi(p, \mathbf{ki}S) = \begin{cases} 1 & \text{if } w(p) > 0 \text{, or } w(p) = 0 \text{ and } w(p) \text{ is not a maximum} \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

However, such a function w does not qualify as characteristic defining according to our definition in section 1.3, because Eq. (19) relies on extremal properties of w. Second, the relationship between the extremal properties of w and regularity of the defined set S suggests the possibility of alternative regularization algorithms.

As an example, consider a real function $f(x, y) = y^3 - y^2 - x^2$, and the set $S \subset E^2$ defined by ($f \geq 0$) (see Figure 4). The regularity of S fails at the origin $p_0 = (0, 0)$; clearly, $p_0 \in S$ but $p_0 \notin \mathbf{ki}S$, because there are no interior points in the neighborhood of p_0. The same conclusion can be reached by considering derivatives of f:

$$f_x = -2x; \qquad f_y = 3y^2 - 2y; \qquad f_{xx} = -2; \qquad f_{yy} = 6y - 2; \qquad f_{xy} = 0.$$

Both f_x and f_y vanish at p_0, indicating a local extremum. Since both f_{xx} and f_{yy} are negative at the origin, the Hessian determinant is positive; thus f must have a local maximum at p_0, and $p_0 \notin \mathbf{ki}S$. It is easy to check that f does not take on extremum values at any other zero points f.

In general, however, deciding whether a real function w attains a maximum value at a point may be problematic. For example, w may be continuous but not differentiable at some points. Even if $w \in C^m$, its derivatives may not provide the needed information. Such difficulties are easily observed if w is constructed using R-functions. At $x_1 = x_2 = 0$, partial derivatives of R_0-functions (15) do not exist, while partial derivatives of all orders (through m) of R_α^m-functions (16) are identically zero.

3.4 Interior zero points

If a set S is represented using a characteristic defining function w as $S = (w \geq 0)$, the corresponding characteristic function $\xi(p, S)$ is trivially obtained by checking the sign of $w(p)$. However, constructing a PMC function (2) for S is problematic. Points of $\mathbf{i}S$ and ∂S cannot be distinguished without an examination of their neighborhoods. Given the existence of the signed distance function

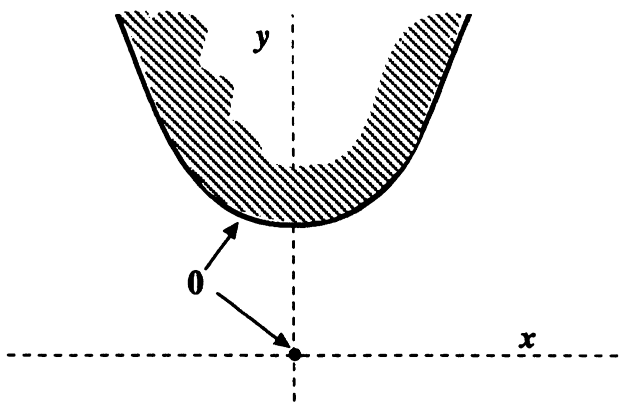


Figure 4: The set $S = (y^3 - y^2 - x^2 \geq 0)$ is not regular.

(9) for any solid S, it is natural to seek a PMC defining function of the form (5) that explicitly distinguishes between the interior and the boundary points of S. In addition, some applications (for example, see appendix A.1) specifically require a real function that is strictly positive at every point $p \in \mathbf{i}S$.

It is well known that all closed sets have nowhere dense[6] boundaries [RT78]. Since the zero set of a PMC defining function w represents the boundary of a closed set, it is necessary that $(w = 0)$ is a nowhere dense set. But it is clear from Figure 5(a) that this condition is not sufficient. (This problem is observed in [Rva82] and is treated in an *ad hoc* fashion, but no systematic solution is offered.) Proposition 3 implies that a characteristic defining function w with a nowhere dense zero set is strictly positive somewhere in the neighborhood of every interior point. Thus, if p is an interior point and $w(p) = 0$, p must be a local minimum for w. The situation is *dual* to the regularization problem studied above. Formally, we can make the following statement.

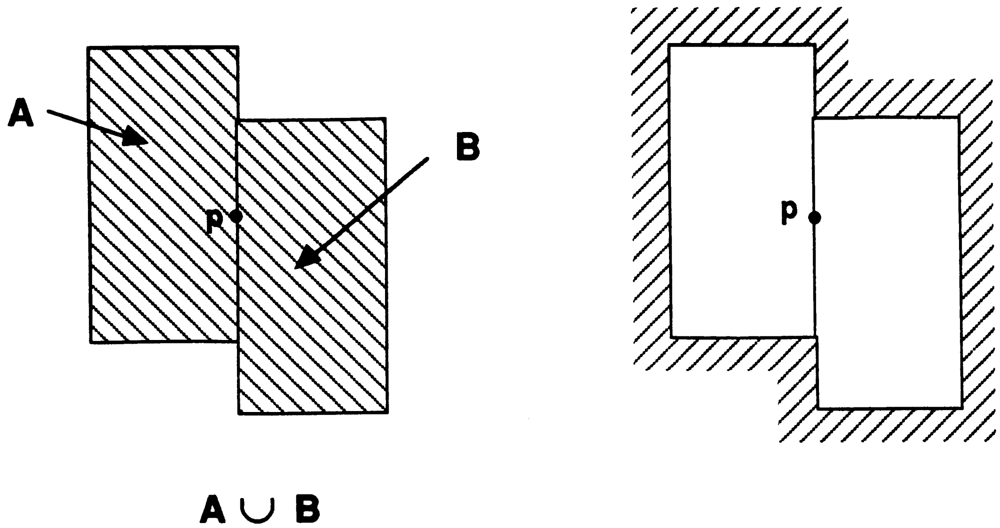
Proposition 5 *Let w be a real function with a nowhere dense zero set* $(w = 0)$, *and* $S = \mathbf{ki}(w \geq 0)$. *Then the regularized complement of S is given by* $-^*S = \mathbf{ki}(w \leq 0)$.

Proof By definition and properties of complement, interior, and closure [KM76],

$$-^*S = \mathbf{k}(-S) = \mathbf{k}(-(\mathbf{ki}(w \geq 0))) == \mathbf{k}(-(\mathbf{k}(-\mathbf{k}(-(w \geq 0))))) = \mathbf{k}(-(w \geq 0) = \mathbf{k}(w < 0).$$

Thus we only need to show that $\mathbf{ki}(w \leq 0) = \mathbf{k}(w < 0)$. For points p such that $w(p) < 0$, the statement is trivial. So consider a zero point p of w, where $w(p) = 0$. If $p \in \mathbf{k}(w < 0)$, then $p \in \mathbf{ki}(w \leq 0)$, because $(w < 0) \subseteq \mathbf{i}(w \leq 0)$. Conversely, if $p \in \mathbf{ki}(w \leq 0)$, then by proposition 3, every neighborhood of p must contain interior points of the regular set $\mathbf{ki}(w \leq 0)$. Since $(w = 0)$ is nowhere dense, w must strictly negative at these interior points. Thus $p \in \mathbf{k}(w < 0)$. □

[6]A set $X \subset E^d$ has a nowhere dense boundary if ∂X is nowhere dense in E^d, i.e. $\mathbf{i}(\partial X) = \emptyset$.



(a) $A = (f_A \geq 0)$, $B = (f_B \geq 0)$.
$f_A \vee_\alpha^m f_B$ has zeros in interior of $A \cup B$.

(b) Set $(f_A \vee_\alpha^m f_B \leq 0)$ is not regular.

Figure 5: Relationship between interior zeros and regularity.

Propositions 5 and 3 together imply that the presence of interior zero points is equivalent to the non-regularity of set $(w \leq 0)$ (see Figure 5(b)). If $S = (w \geq 0)$ and $w(p) = 0$ for some point $p \in \mathbf{i}S$, then set $(w \leq 0)$ cannot be regular: the regularity fails at p whose neighborhood contains no points of $\mathbf{i}(w \leq 0)$.

Suppose we are given a solid S, and we seek a real function f that is strictly positive inside S and is zero on the boundary ∂S. Following the arguments in section 2.4 (and for example using R-functions), we can construct characteristic defining function g for $-^*S = (g \geq 0)$. The desired function f is then obtained by a simple change of sign, i.e. $f = -g$. Note that f may now take on zero values in the exterior $\mathbf{e}S$. While this may be acceptable for some applications, strictly speaking, f is not a characteristic defining function for S.

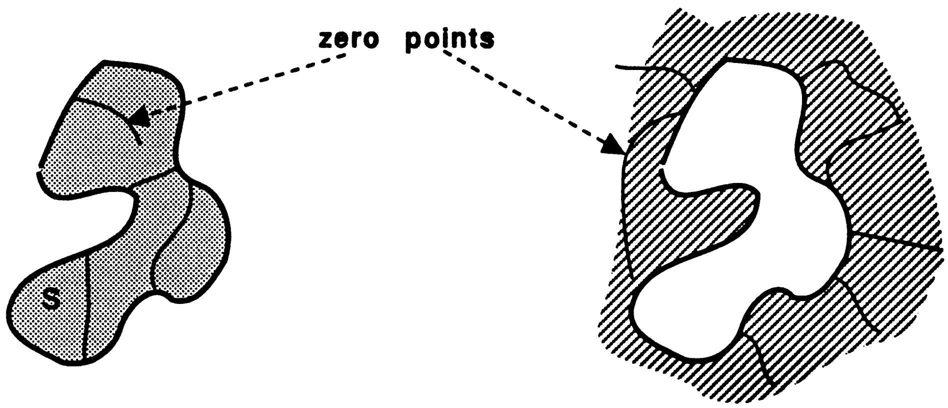
3.5 Constructing PMC defining functions for solids

It may seem that the above discussion did little to advance our goal of constructing PMC defining functions of the form of Eq. (5). However, propositions 3 and 5 lead directly to conditions that must be satisfied by such functions.

Proposition 6 *Let $S \subset E^d$ be a closed regular set, and $w : E^d \to \Re$ be a real function such that $\partial S \subseteq (w = 0)$. Then w is PMC defining for S (i.e. it satisfies Eq. (5)) if and only if*

(a) *set $(w = 0)$ is nowhere dense, and*

(b) *sets $(w \geq 0)$ and $(w \leq 0)$ are both regular.*



(a) $S = (w_1 \geq 0)$; w_1 has zero points in $\mathbf{i}S$.

(b) $-^*S = (w_2 \geq 0)$; w_2 has zero points in $\mathbf{i}(-^*S)$.

Figure 6: Function $w = w_1 - w_2$ defines S with $w > 0$ in $\mathbf{i}S$ and $w < 0$ in $\mathbf{e}S$.

Proof If w satisfies Eq. (5), by assumption $S = (w \geq 0)$ and $\mathbf{k}(-S) = (w \leq 0)$ are regular sets, and $\partial S = (w = 0)$ is nowhere dense. Suppose now that conditions (a) and (b) hold. It is clear that $w(p) > 0$ implies $p \in \mathbf{i}S$, and and $w(p) < 0$ implies $p \in \mathbf{e}S$. Finally, if $w(p) = 0$, proposition 3 and proposition 5 imply that $p \in \partial S$, because every neighborhood of p contains points where $w > 0$ and points where $w < 0$. □

One method of constructing such a function w is suggested by proposition 4 and is demonstrated in Figure 3.5. Given a solid S, let us construct two functions w_1 and w_2 such that

$$S = (w_1 \geq 0), \qquad -^*S = (w_2 \geq 0).$$

For example, this can be done using R-functions as suggested by proposition 2, provided that primitives real functions ϕ_i have nowhere dense zero sets. In this case, sets $(w_1 = 0)$ and $(w_2 = 0)$ are nowhere dense as well [Rva82]. While $(w_1 \leq 0)$ and $(w_2 \leq 0)$ are not regular sets, it is true that

$$S = \mathbf{ki}(w_2 \leq 0), \qquad -^*S = \mathbf{ki}(w_1 \leq 0).$$

Applying proposition 4, we see that

$$S = (w_1 - w_2 \geq 0), \qquad -^*S = (w_1 - w_2 \leq 0).$$

It follows that $w = w_1 - w_2$ is a PMC defining function for S satisfying Eq. (5). Note that this construction essentially doubles the size of the representation for S. Thus better methods for constructing PMC defining functions remain of interest.

4 Conclusion

It is clear that real-function representations are useful in solid modeling. It is also clear that use of these representations is accompanied by a number of complications: they may possess unpleasant numerical properties, do not explicitly represent solid boundaries, may not accommodate needed parameterizations, and may be difficult to construct and manipulate. If we hope to use these representations in practical systems, the formal properties, advantages, and limitations of characteristic and PMC defining functions must be thoroughly understood. This paper is intended to be a step towards that goal.

The theory of R-functions suggests a close connection between the set-theoretic (non-regularized) and the real-function representations of geometric objects. Such set representations are usually not available in either b-rep or CSG geometric modeling systems, and so additional representation conversions may be required. Algorithms to perform such representation conversions for semi-algebraic solids are studied in [Sha91]. While in principle the conversion process can be completely automated, many difficult questions remain open.

Regularity of solids is sometimes dismissed as a "technicality" that can be dealt with using various "hacks". The presented results underscore the importance of regularity once again, by showing the close coupling between the properties of defining functions and regularity. We have seen that extremal properties of a characteristic defining function can be used to identify non-regular points of the represented set S, as well as the zero points in the interior of S. At the same time, regularity is the key to eliminating the interior zero points, which is important in many applications. Dual results can be formulated for open regular sets used in [Arb90].

Using the properties of continuous functions, the discussion in this paper can be generalized to other defining functions. If w is a continuous real function such that $S = w^{-1}(X)$, $X \subset \Re$, and $g : \Re \rightarrow \Re$ is a homeomorphism, then it is easy to show that $S = [g \circ w]^{-1}(g(X))$. For example, since e^x is a homeomorphic mapping, Eq. (6) extends proposition 3 to real functions of the type (4) used by [Ric73], [Bar81], and others.[7]

[7]The "zero set" of such defining functions needs to be redefined appropriately as the image of the original zero set under the homeomorphism. Thus for functions of the type (4) a "zero point" is a point p where $w(p) = 0$ and $f(p) = e^{w(p)} = e^0 = 1$.

A Some applications

A.1 Approximate solutions of boundary value problems

Consider a classical example from [Tim70] of finding an approximate solution to a Saint-Venant torsion problem. Specifically, suppose we would like to determine the torsional rigidity of a straight bar with rectangular cross section (Figure 7). The problem can be reduced to finding a stress

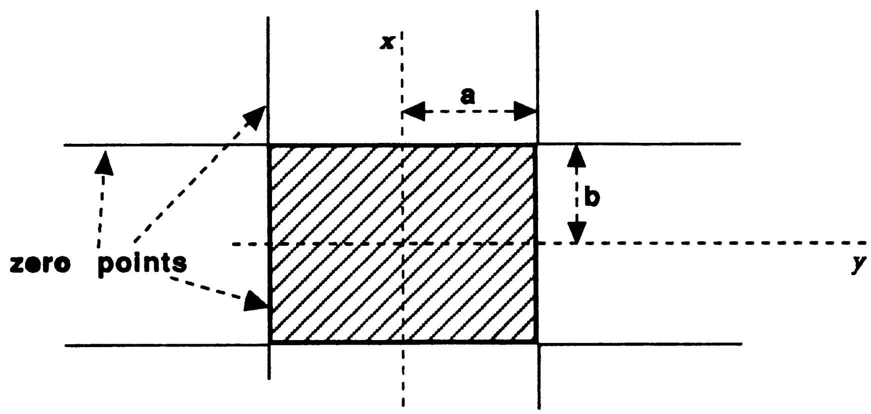


Figure 7: Function $(x^2 - a^2)(y^2 - b^2)$ is strictly positive in the interior and zero on the boundary of the rectangular cross section.

function ϕ, such that $\phi = 0$ on the boundary of the cross section, and minimizes the energy integral

$$U = \iint \left\{ \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] - 2G\theta\phi \right\} dx\, dy, \tag{20}$$

where θ is twist, and G is modulus of rigidity. We *assume* that ϕ is of the form

$$\phi = w \sum_{m=0}^{p} \sum_{n=0}^{q} c_{mn} x^m y^n, \tag{21}$$

where $w = (x^2 - a^2)(y^2 - b^2)$. Substituting expression (21) in Eq. (20), we need to solve for the unknown coefficients c_{mn} which minimize the integral. This translates into the requirement that the partial derivatives of this integral with respect to all c_{mn} must vanish, yielding a system of linear equations for c_{mn}.

For example, suppose we have a square cross section with $a = b$. Using only the first term in expression (21) we get

$$\phi = c_0(x^2 - a^2)(y^2 - a^2). \tag{22}$$

Substituting this expression into integral (20), integrating over the square region, and setting the derivative of the integral with respect to c_0 equal to zero, we solve to obtain

$$c_0 = \frac{5}{8}\frac{G\theta}{a^2}.$$

Substituting the value of c_0 back in the expression (22) allows for the computation of the approximate torque and torsional rigidity. (See [Tim70] for additional details.)

The above procedure is essentially a well known classical Ritz's method that does not require domain decomposition (meshing), and [KK58] observed that it can be generalized to arbitrary boundary value problem with homogeneous boundary conditions, if a real function w can be found for a domain Ω such that

$$w > 0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

For example, the same approximation procedure can be carried out for the torsion problem with arbitrary cross section, if such a defining real function w is available. Yet further generalization of this idea is a notion of the *structure* of the solution of a boundary value problem, proposed in [Rva82]: it is an expression

$$u = B(\Phi, w, \varphi),$$

which satisfies boundary conditions φ prescribed on the boundary defined by w, and where Φ is the undetermined component of the solution. The concept of the solution structure seems to unify various direct methods for solving boundary value problems (Ritz/Galerkin, Finite Element, etc.) and to explicate their differences. Structures for many common boundary value problems have been constructed [Rva82] using the theory of R-functions. This allowed development of software systems that generate solutions of field problems in engineering and physics from high-level mathematical descriptions (see [Sha88] for additional information and references).

A.2 Null-object and interference analyses

Consider a representation of a solid S by a real-function inequality ($f \geq 0$). If f is continuous, then f is also bounded in S, and achieves its global maximum value somewhere in the interior of S. To test if $S = \emptyset$, observe [Rva67] that

$$S = \emptyset \qquad \Longleftrightarrow \qquad \max_{p \in S} f(p) \geq 0. \tag{23}$$

Computing the global maximum of an arbitrary function f may be difficult, but relation (23) has been used successfully in a number of special situations. Often function f possesses many additional properties, and knowing how f is constructed may aid in determining the number and location of its extrema.

Probably the main utility of the null-object detection test is in interference analysis: two solids A and B intersect if and only if $A \cap^* B = \emptyset$ [Til84]. Suppose that $A = (f_A \geq 0)$ and $B = (f_B \geq 0)$, and solid defining functions f_A, f_B are strictly positive in the interiors of solids A and B respectively. Then it easy to see that $A \cap^* B = \emptyset$ if and only if $f_A \wedge_\alpha^m f_B > 0$, where $\wedge_\alpha^m$ is any R-conjunction.

More generally, R-functions (see section 2.3) can be used to formulate relative position criteria for multiple geometric objects (such as minimum distance, non-overlap, etc.) as conditions on some defining real functions. As a result, various geometric placement, optimization, and motion planning problems are reduced to corresponding problems of mathematical programming. This approach to problems of geometric design and accompanying optimization techniques (including relevant numerical, combinatorial, and stochastic algorithms) are suggested and developed in [Sto75, SY86].

References

[AB90] D. V. Altiparmakov and P. Belicev. R-function solid modeling for Monte Carlo calculations. In *PHYSOR'90 - Int. Conf. Physics of Reactors: Operation, Design, and Computation*, pages XII–69 - XII–78, Marseille, April 23–27 1990.

[AG90] E. L. Allgower and K. Georg. *Numerical Continuation Methods: An introduction.* Springer-Verlag, 1990.

[Arb90] F. Arbab. Set models and Boolean operations for solids and assemblies. *IEEE Computer Graphics and Applications*, pages 76–86, November 1990.

[Bar81] A. H. Barr. Superquadrics and angle-preserving transformations. *IEEE Computer Graphics and Applications*, 1:11–23, 1981.

[Bli82] J. Blinn. A generalization of algebraic surface drawing. *ACM Transactions on Graphics*, 1(3):235–256, July 1982.

[Blo88] J. Bloomenthal. Polygonization of implicit surfaces. *Computer Aided Geometric Design*, 5(4):341–355, November 1988.

[BN90] J. L. Blechschmidt and D. Nagasuru. The use of algebraic functions as a solid modeling alternative: and investigation. In *16th ASME Design Automation Conference*, pages 33–41, Chicago, IL, September 16–19 1990.

[BW90] J. Bloomenthal and B. Wyvill. Interactive techniques for implicit modeling. *Computer Graphics*, 24(2):109–116, March 1990.

[Han88] A. J. Hansen. Hyperquadrics: smoothly deformable shapes with convex polyhedral bounds. *Computer Vision, Graphics, and Image Processing*, 44:191–210, 1988.

[Hof89] C. H. Hoffmann. *Geometric and Solid Modeling.* Morgan Kaufmann Publishers, 1989.

[KB89] D. Kalra and A. H. Barr. Guaranteed ray intersections with implicit surfaces. *Computer Graphics*, 23(3):297–306, July 1989.

[Kha86] O. Khatib. Real-time obstacle avoidance for manipulators and mobile robots. *The International Journal of Robotics Research*, 5(1):90–98, 1986.

[KK58] L. V. Kantorovich and V. I. Krylov. *Approximate Methods of Higher Analysis.* Interscience Publishers, 1958.

[KM76] K. Kuratowski and A. Mostowski. *Set theory.* North-Holland Publishing Co., 1976.

[Kod89] D. E. Koditschek. Natural control in manufacturing. In *Proceedings of the 15th NSF Conference on Production Research and Technology*, January 9-13 1989.

[Loj64] S. Lojasiewicz. Triangulations of semi-algebraic sets. *Annali della Scuola Normale Superiore di Pisa*, 18:449–474, 1964.

[MT46] J. C. C. Mckinsey and A. Tarski. On closed elements in closure algebras. *Annals of Mathematics*, 47(1):122–162, January 1946.

[OKK73] N. Okino, Y. Kakazu, and H. Kubo. TIPS-1: technical information processing system for computer-aided design, drawing and manufacturing. In J. Hatvany, editor, *Computer Languages for Numerical Control, Proceedings of the Second International IFIP/IFAC PROLAMAT Conference*, pages 141–150, Budapest, 1973.

[Req77] A. A. G. Requicha. Mathematical models of rigid solid objects. Tech. Memo 28, Production Automation Project, University of Rochester, Rochester, NY, November 1977.

[Req80] A. A. G. Requicha. Representations for rigid solids: Theory, methods, and systems. *ACM Computing Surveys*, 12(4):437–464, December 1980.

[Ric73] A Ricci. A constructive geometry for computer graphics. *Computer Journal*, 16(3):157–160, May 1973.

[RK90] E. Rimon and D. E. Koditschek. Exact robot navigation in geometrically complicated but topologically simple environment. In *IEEE International Conference on Robotics and Automation*, May 13-18 1990.

[Roc89] A. P. Rockwood. The displacement method for implicit blending surfaces in solid modeling. *ACM Transactions on Graphics*, 8(4):279–297, 1989.

[RR79] V. L. Rvachev and V. A. Rvachev. *Nonclassical Methods of Approximation Theory in Boundary Value Problems*. Naukova Dumka, 1979. In Russian.

[RT78] A. A. G. Requicha and R. B. Tilove. Mathematical foundations of constructive solid geometry: General topology of closed regular sets. Tech. Memo 27a, Production Automation Project, University of Rochester, June 1978.

[RT80] A. A. G. Requicha and R. B. Tilove. Closure of Boolean operations on geometric entities. *Computer-Aided Design*, 12(5):219–220, September 1980.

[RV77] A. A. G. Requicha and H. B. Voelcker. Constructive solid geometry. Tech. Memo 25, Production Automation Project, University of Rochester, November 1977.

[Rva67] V. L. Rvachev. *Geometric Applications of Logic Algebra*. Naukova Dumka, 1967. In Russian.

[Rva74] V. L. Rvachev. *Methods of Logic Algebra in Mathematical Physics*. Naukova Dumka, 1974. In Russian.

[Rva82] V. L. Rvachev. *Theory of R-functions and Some Applications*. Naukova Dumka, 1982. In Russian.

[Sha88] V. Shapiro. Theory of R-functions and applications: A primer. Tech. Report CPA88-3, Cornell Programmable Automation, Cornell University, Ithaca, NY, November 1988.