The Optimality of Induction
as an Axiomatization of Arithmetic *

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By induction for a formula ϕ we mean the schema

$$\text{IND}_\phi: \quad \phi[0/x] \wedge \forall x(\phi \rightarrow \phi[x+1/x]) \rightarrow \forall x\phi,$$

where $\phi[t/x]$ is the result of simultaneously substituting t for all free occurrences of x in ϕ . If ϕ is Σ_n then IND_ϕ is both Π_{n+2} and Σ_{n+2} , and if ϕ is Π_n , then IND_ϕ is Σ_{n+1} . The universal closure of IND_ϕ is Π_{n+2} in either case, but if ϕ is closed then IND_ϕ is implied by a Π_{n+1} sentence in the first case, and by a Π_n sentence in the latter.

A natural question arises as to whether for some n IND_ϕ can be derived from true or even consistent Π_{n+1} sentences, for all Π_n and Σ_n formulas ϕ . If yes, then an axiomatization T of a consistent extension of First Order Arithmetic exists such that all Π_{n+2} theorems of Arithmetic would be derived from Π_{n+1} axioms of T . For sound extensions of First Order Arithmetic the question was answered negatively by Charles Parsons [70], who proved that for each $n > 0$ there is a Σ_n formula ϕ for which IND_ϕ is not derivable from true Π_{n+1} sentences, and that induction for a Π_n formula is equivalent in Primitive Recursive Arithmetic to induction for a certain Σ_n formula.

We strengthen somewhat Parsons' Theorem in the following

THEOREM. For each $n > 0$ there is a Π_n formula ϕ (whence also a Σ_n formula) that is not derivable from Π_{n+1} sentences consistent with ϵ_0 -induction.

Parsons' proof is essentially model/recursion theoretic. Our proof uses Kreisel's method of partial truth definitions (see e.g. [Kreisel-Wang 55]), and the observation that in treating cut-free proofs of Σ_{n+1} formulas only a truth definition for Σ_{n-1} formulas is needed.

We note in passing that induction can also be expressed by

$$\text{IND}'_{\phi}: \quad \forall x (\forall y < x (\phi[y/x] \rightarrow \phi[y+1/x]) \rightarrow \phi).$$

which, disregarding bounded quantifiers, is less complex than IND_{ϕ} . However, bounded quantifiers must be counted in this context [Parsons 70], and so IND' is just as complex as IND , except for quantifier free ϕ , when IND'_{ϕ} is Π_1 .

Let Z_0 denote Primitive Recursive Arithmetic, with definitional equations for primitive recursive functions and with induction for quantifier free formulas as axioms. $\text{Pr}(x)$ will denote a Σ_1 formula expressing that the formula with code x is provable in Z_0 . Let Tr_n^{Σ} be a Σ_n truth definition for Σ_n formulas, similarly for Tr_n^{Π} [Smorynski 77, 3.5.2]. Let RFN_n^{Σ} be the schema

$$\forall x ["x \text{ codes a } \Sigma_n \text{ formula}" \wedge \text{Pr}(x) \rightarrow \text{Tr}_n^{\Sigma}(x)].$$

Let CE be a Π_2 sentence expressing the Cut Elimination Theorem. Suppose $n > 0$.

LEMMA 1. There is a Σ_n formula ϕ such that

$$Z_0, \text{CE}, \text{IND}_{\phi} \vdash \text{RFN}_{n+1}^{\Sigma}.$$

To see that lemma 1 implies the theorem, suppose that $Z_0, \alpha \vdash \text{IND}_{\phi}$ for some Π_{n+1} formula α . Let $\alpha' \equiv \alpha \wedge \text{CE}$. Then by lemma 1

$$Z_0, \alpha' \vdash \text{Pr}(\ulcorner \neg \alpha' \urcorner) \rightarrow \neg \alpha',$$

where $\lceil \beta \rceil$ denotes the numeral of the canonical code of β ; so

$$Z_0, \alpha' \vdash \neg \text{Pr}(\lceil \neg \alpha' \rceil),$$

and by Godel's Second Incompleteness Theorem $Z_0 \vdash \neg \alpha'$, i.e. $Z_0, \text{CE} \vdash \neg \alpha$. Since CE is equivalent to ϵ_0 -induction, this proves the theorem.

In proving lemma 1 we use Gentzen's sequential calculus for Classical First Order Logic (see e.g. [Kleene 52]), for sequents of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite sets of formulas and cut is the only structural rule. We also use a rule of repetition (with a single premise, identical to the conclusion), so as to be able to transform a given proof into a proof π satisfying:

(*) if ρ is a subproof of π of height h with two immediate subproofs, then the immediate subproofs of ρ are both of height $h-1$.

Let $n > 0$. Let $\alpha \equiv \exists x \forall y \alpha_0[x, y]$ be a prenex Σ_{n+1} formula, where α_0 is Σ_{n-1} . Define the set S of direct subformulas of α inductively: $\alpha \in S$; if $\forall u \beta \in S$ then $\beta \in S$; if $\exists u \beta \in S$ and t is a term free for u in β , then $\beta[t/u] \in S$. An α -sequent is a sequent of the form $Z_1, \Gamma \Rightarrow \Delta$, where Z_1 is a set of Z_0 axioms, all formulas in Γ are quantifier free, and all formulas in Δ are quantifier free or direct subformulas of α . Clearly, if σ is an α -sequent derived by an inference rule other than cut from sequents σ_1, σ_2 then σ_1, σ_2 are also α -sequents.

LEMMA 2. Assume $\neg \alpha$. Let π be a proof deriving an α -sequent

$$\sigma \equiv Z_1, \Gamma \Rightarrow (\alpha), \beta_1, \dots, \beta_k, \gamma_1, \dots, \gamma_r$$

where α may occur or not, where each β_i is $\forall y \alpha_0[t, y]$ for some term t , and where each γ_i is Σ_{n-1} . Then the formula

$\eta[\sigma] \equiv$ the universal closure of $\wedge \Gamma \rightarrow (Y_1 \vee \dots \vee Y_r)$

is true (where $\wedge \Gamma$ is the conjunction of all formulas in Γ). In particular, if π derives $Z_1 \Rightarrow \alpha$ then $\neg \wedge Z_1$.

PROOF. By induction on the height h of π , where we assume without loss of generality that π satisfies (*). If $h=1$ then $Y_i \in \Gamma \cup Z_1$ for some i , so $\eta[\sigma]$ is trivially true. If $h>1$ then the immediate subproofs π_1, π_2 of π are of height $h-1$, and since π is cut-free they both derive α -sequents σ_1, σ_2 , respectively. By induction assumption $\eta[\sigma_1]$ and $\eta[\sigma_2]$ are true.

Consider cases for the active conclusion-formula θ of the inference deriving σ from σ_1 and σ_2 . If θ is in Γ, Z_1 or is one of $Y_1, \dots, Y_r$ then it is easy to verify that (in Z_0) $\eta[\sigma_1] \wedge \eta[\sigma_2] \rightarrow \eta[\sigma]$, so $\eta[\sigma]$ is true. If θ is α then $\eta[\sigma] = \eta[\sigma_1]$ (and π_2 is empty). Finally, if θ is some $\beta_i \equiv \forall y \alpha_0[t, y]$, where σ is derived by $\forall$ -right, then $\eta[\sigma_1] \Leftrightarrow \eta[\sigma] \vee \alpha_0[t, y]$. By the usual restriction on $\forall$ -right y is not free in $\eta[\sigma]$, so $\eta[\sigma_1] \Leftrightarrow \eta[\sigma] \vee \beta_i$. But $\beta_i \rightarrow \alpha$ and $\neg \alpha$ is assumed, so $\neg \beta_i$, and hence $\eta[\sigma_1] \rightarrow \eta[\sigma]$. $\square$

PROOF OF LEMMA 1. Let f be a primitive recursive function that assigns to the code of an α -sequent σ the code of the formula $\eta[\sigma]$ defined above. Observe that $\eta[\sigma]$ is Π_n . Clearly, lemma 2 can be formally proved in Z_0 using induction for the Π_n formula

$$\begin{aligned} \phi(h) \equiv \forall p, s ["p \text{ codes a cut-free proof of height } h \text{ deriving} \\ \text{an } \alpha\text{-sequent with code } s" \\ \rightarrow \text{Tr}_n^{\Pi}(f(s))]. \end{aligned}$$

(The code of α is a free variable of ϕ .) So

$Z_0, CE, IND, \vdash \neg\alpha \rightarrow \neg Pr(\ulcorner\alpha\urcorner)$.

Since the proof is uniform in α , this establishes lemma 1. $\square$

References.

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