IMPLICATIONAL COMPLEXITY
IN INTUITIONISTIC ARITHMETIC

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0. Introduction. In classical arithmetic, a natural measure for the complexity of relations is provided by the number of quantifier alternations in an equivalent prenex normal form. However, the proof of the Prenex Normal Form Theorem uses the following intuitionistically invalid rules for permuting quantifiers with propositional constants.

$$(1) \quad A \rightarrow \exists x Bx \quad . \leftrightarrow \quad \exists x [A \rightarrow Bx]$$

$$(2) \quad \forall x Bx \rightarrow A \quad . \leftrightarrow \quad \exists x [Bx \rightarrow A]$$

$$(3) \quad \forall x [A \vee Bx] \leftrightarrow . \quad A \vee \forall x Bx$$

Each one of these schemas, when added to Intuitionistic (Heyting's) Arithmetic IA, generates full Classical (Peano's) Arithmetic. Schema (3) is of little interest here, since one can obtain a formula intuitionistically equivalent to $A \vee \forall x Bx$, which is prenex if A and B are: $A \vee \forall x Bx \quad . \leftrightarrow .$

$\exists y [(y \neq 0 \rightarrow A) \wedge (y \neq 0 \rightarrow \forall x Bx)] \quad . \leftrightarrow . \quad \exists y \forall x [(y = 0 \rightarrow A) \wedge (y \neq 0 \rightarrow Bx)] \quad .$ For the two conjuncts on the r.h.s. (1) may be successively applied, since $y = 0$ is decidable.

We shall readily verify that there is no way of similarly going around (1) or (2). This fact calls for counting implication (though not conjunction or disjunction) in measuring, in IA, the complexity of arithmetic relations. The natural implicational measure, for our purpose, is the depth of negative nestings of implication, defined as follows. $I(F) := 0$ if F is atomic; $I(F \wedge G) = I(F \vee G) := \max [I(F), I(G)]$; $I(\forall x F) = I(\exists x F) := I(F)$; $I(F \rightarrow G) := \max [I(F) + 1, I(G)]$.

This measure is also the minimal one satisfying the following properties. (1) One can define I , as above, for functional types. Then, for each type τ , if $N = I(\tau)$, then τ is canonically embedded in the n 'th pure type. An explicit connection between implicational complexity and functional complexity is provided by Gödel's (so-called "Dialectica") interpretation. To isolate the role of implication, consider a propositional schema F in Σ_1^0 sentences. The interpretation of F is of the form $\exists x^{\tau} \forall y^{\sigma} F_0$, where F_0 is quantifier free, $I(\tau) = I(F)$ and $I(\sigma) < I(F)$. (2) Let IA_n stand for IA with induction restricted to formulas F with $I(F) \leq n$; the consistency of IA_n can be proved in IA_{n+1} (Le[75]).

Note that positive nesting of implications may be essential in other contexts. E.g., in $IA + \text{Church's Thesis}$ there exists a truth definition for formulas with a given bound on the depth of all implication nestings. (I am indebted to the referee for this remark.)

In treating intuitionistic implicational complexity, one should avoid literal analogies with classical treatment of arithmetical complexity. For instance, an arithmetic relation R cannot be distinguished from its prenex form R_p simply by refuting $\forall \vec{x}. R\vec{x} \leftrightarrow R_p \vec{x}$, as the refutation is underivable in any classically sound theory. As on similar occasions, the problem is to state intuitionistic invalidity without using refutation. We confine ourselves here to the proof theoretic analogues of classical theorems. Given a theory A we say that two relations R, Q are A -equivalent (notation: $R \equiv_A Q$), if $\vdash_A \forall \vec{x}. R\vec{x} \leftrightarrow Q\vec{x}$. The stronger theory A is, the larger the equivalence classes.

In order to discuss syntactic forms more general than the prenex ones we refer to first-order predicate formulas (without equality), which we call frames. An arithmetic formula F is a case of the frame ϕ (notation: $F \in \phi$) if F comes from ϕ by substituting atomic formulas (i.e., equations between primitive recursive terms) for the predicate letters. E.g., the Π_2^0 unary relations are exactly the cases of $\forall x \exists y P(x, y, z)$. For frames ϕ, ψ we write $\phi \leq_A \psi$, if for any $F \in \phi$ there is a $G \in \psi$ with $F \equiv_A G$. Also, $\phi \equiv_A \psi$ if both $\phi \leq_A \psi$ and $\psi \leq_A \phi$. E.g., $\neg \exists x Px \equiv_{IA} \forall x \neg Px$. Using these notations, we may describe our subject as the study of the partial ordering $\leq_A$ on frames, for suitable (intuitionistic arithmetic theories A).

When A is Classical (Peano's) Arithmetic CA , each frame is $\leq_A \Pi_n^0$ or $\leq_A \Sigma_n^0$ for some n . However, $\leq_A$ cannot be replaced here by $\equiv_A$. E.g., each propositional combination of Σ_n^0 relations is in $\Delta_{n+1}^0 := \Sigma_{n+1}^0 \cap \Pi_{n+1}^0$, but not vice versa. In other words, there are interpolant frames under $\leq_{CA}$ between the frames (for) Σ_n^0 and Σ_{n+1}^0 (or Π_{n+1}^0). Such frames have not drawn much interest, since their position in the neat prenex hierarchy is easily identified. Also, these frames arise in connection with a "separation of variables"; e.g., $\exists x f(x)=0 \wedge \forall y g(y)=0$ has a prenex form $\exists x \forall y [f(x)=0 \wedge g(y)=0]$, in the matrix of which the variables x and y appear in distinct propositional constituents.

1. Hierarchy theorems for implicational complexity

Our first aim is to show that, in Intuitionistic Arithmetic, the effect of implication goes beyond generating interpolating frames in the prenex hierarchy.

We shall make use of the propositional formulas in one letter, $N_1[p]$, characterized by Nishimura [60]: $N_0 := \perp$ (absurdity);

$N_1 := p$; $N_2 := \neg p$; $N_{2k+1} := N_{2k} \vee N_{2k-1}$ ($k \geq 1$); $N_{2k+2} := N_{2k+1} \rightarrow N_{2k-1}$ ($k \geq 1$).

Note that $I(N_{2k}) = I(N_{2k+1}) = k$ ($k \geq 0$).

From the definitions, the following properties follow.

$$(1) \quad N_{2k+2} \equiv N_{2k} \rightarrow N_{2k-1} \quad (k \geq 1) \\ \equiv (N_{2k-2} \rightarrow N_{2k-3}) \rightarrow (N_{2k-2} \vee N_{2k-3}) \\ (k \geq 2, \text{ by (1)})$$

$$(2) \quad \equiv N_{2k} \rightarrow N_{2k-3} \quad (k \geq 2).$$

$$(3) \quad N_{2k+2} \rightarrow N_{2k} \equiv N_{2k} \quad (k \geq 1)$$

This is immediate for $k = 1$; for $k \geq 1$ observe that

$$(N_{2k+4} \rightarrow N_{2k+2}) \rightarrow \dots (N_{2k+2} \rightarrow N_{2k-1}) \rightarrow N_{2k+2} \\ \rightarrow \dots (N_{2k+2} \rightarrow N_{2k-1}) \rightarrow N_{2k-1} \\ \rightarrow \dots ((N_{2k} \rightarrow N_{2k-1}) \rightarrow N_{2k-1}) \rightarrow N_{2k-1} \quad \text{by (1)} \\ \rightarrow \dots (N_{2k} \rightarrow N_{2k-1}) \equiv : N_{2k+2}.$$

$$(4) \quad N_{2k+4} \rightarrow N_{2k-1} \rightarrow \dots (N_{2k+2} \rightarrow N_{2k-1}) \rightarrow N_{2k-1} \quad \text{by (2)} \\ \rightarrow N_{2k+2} \quad \text{as above.}$$

$$(5) \quad N_{2k+4} \rightarrow N_{2k} \rightarrow \dots N_{2k-1} \rightarrow N_{2k} \quad \text{by (2)} \\ \rightarrow \dots N_{2k-1} \rightarrow (N_{2k-1} \rightarrow N_{2k-3}) \\ \rightarrow \dots N_{2k-1} \rightarrow N_{2k-3} \\ \equiv : N_{2k}.$$

Let A be IA augmented with transfinite induction over certain recursive well-orderings $\{R_i\}_i$. Clearly, when $\{R_i\}_i$ is an r.e. set (under some canonical indexing) then so is the theory A . When $\{R_i\}_i$ consists of all recursive well-orderings, then A , denoted by IA^* (or HA^*), is a rather strong theory; e.g., $A \subset IA^*$ as above may consist of the arithmetic theorems of the intuitionistic impredicative theory of types (Le[75a]). Also, IA^* is complete for Π_2^0

sentences, and IA^* augmented by classical logic is complete for classical truth (in the standard model) (Kr, Sh, Wa[60]).

THEOREM I (implicational complexity for $\leq_A$). For each $k \geq 1$ there is a frame Φ_k with $I(\Phi_k) = k$, s.t. $\Phi_k \not\leq_A \Psi$ for any frame Ψ with $I(\Psi) < k$ (regardless of the quantificational complexity of Ψ). Explicitly, we may take $\Phi_k := N_{2k}[\exists x \forall y P(x, y)]$ for an r.e. theory A , and $\Phi_k := N_{2k}[\forall x \exists y \forall z P(x, y, z)]$ for $A = IA^*$.

A weaker form of theorem I, for a measure counting all nestings of implication, can be proved by the method of D.H.J. de Jongh [73] (for an exposition, see Tr[73], 5.3.20-5.3.22).

Let Σ_n^* denote the propositional combinations of Σ_n^0 sentences, and let $\Sigma_n^{(j)} := \{ F \in \Sigma_n^* : I(F) \leq j \}$. $\Sigma_n^{(j)}$ frames are defined analogously.

THEOREM II (2-step hierarchy for Σ_1^*). Let A be r.e.. For each $k \geq 1$ there is a $\Sigma_1^{(k+1)}$ frame Φ_k s.t. $\Phi_k \not\leq_A \Psi$ for any frame Ψ with $I(\Psi) < k$.

The following theorem will be used in the proofs of theorems I and II.

Theorem III Let A be r.e. ($A=IA^*$), and let $\phi: \exists \neg \forall x P x$ ($\neg \forall x \exists y \forall z P(x,y,z)$), respectively). Then $\phi \notin_A \Psi$ for any prenex frame Ψ (and hence, for any frame Ψ with $I(\Psi) = 0$). Explicitly: if B is any Π_1^0 (resp., Π_3^0) sentence undecidable in A , then $\neg B$ is not equivalent in A to any prenex sentence.

PROOF: Case (I) A is r.e.. Let B be a Π_1^0 A -undecidable sentence, and let

$$C \equiv \forall y_0 \exists z_0 \dots \forall y_n \exists z_n C_0(\vec{y}, \vec{z})$$

be a prenex sentence. Assume

$$(1) \quad \vdash_A \neg B \rightarrow C, \text{ and}$$

$$(2) \quad \vdash_A C \rightarrow \neg B.$$

By (2) and the soundness of A , the sentence C is false. From (1) we conclude, by the closure of A under the extended Church's Rule ECR_0

(cf. Tr[73] 3.7.2), that for certain $e_0, \dots, e_n$

$$(3) \quad \vdash_A \bigwedge_{i \leq n} \forall a \exists b T(\bar{e}_i, a, b) \wedge \forall y_0 \dots \forall y_n \forall w_0 \dots \forall w_n [T(\bar{e}_0, y_0, w_0) \wedge T(\bar{e}_1, \langle y_0, y_1 \rangle, w_1) \wedge \dots \wedge T(\bar{e}_n, \langle y_0, \dots, y_n \rangle, w_n) \wedge \neg B \rightarrow C_0(\vec{y}, U(w_0), \dots, U(w_n))]$$

But C is false, so for suitable $y_0, \dots, y_n, w_0, \dots, w_n$

$$(4) \quad T(\bar{e}_0, \bar{y}_0, \bar{w}_0) \wedge \dots \wedge T(\bar{e}_n, \langle \bar{y}_0, \dots, \bar{y}_n \rangle, \bar{w}_n) \wedge \neg C_0(\bar{y}, U(\bar{w}_0), \dots, U(\bar{w}_n)).$$

The expression (4) is closed and quantifier-free, so it is provable in

$IA \subset A$. We therefore conclude from (3) that $\vdash_A \neg B$, and hence $\vdash_A B$,

contradicting the choice of B .

Case (II) $A=IA^*$. We first note that there is a Π_3^0 sentence

$B \equiv \forall x \exists y \forall z B_0(x, y, z)$ undecided in IA^* . Let B be any Π_3^0 sentence s.t.

(i) B is classically true, i.e.: $B' \equiv \forall x \neg \exists y \forall z B_0$ is true; (ii) no

recursive function f satisfies $\forall x \forall z B_0(x, f(x), z)$. Since B' is true, $\nVdash_{IA^*} \neg B$. On the other hand, if $\vdash_{IA^*} B$, then B is recursively realizable, contradicting (ii).

The proof in (I) goes through now, with the aid of the following:

LEMMA 1.1: If $\vdash_{IA^*} \neg\neg B$ for a prenex sentence B , then weakly (i.e., $\rightsquigarrow$) $\vdash_{IA^*} B$. We postpone the proof of this lemma to §4.

2. Unbounded width of the implicational hierarchy.

From theorem I we easily obtain k frames incomparable under $\leq_A$, simply by taking frames with increasing I-complexity and decreasing quantificational complexity. However, the frames obtained in theorem I are loaded with the kind of separated variables mentioned above, and one might be led to see this as the reason of the incomparability (as in the classical case). In order to focus on the more interesting cases, let us call a frame unified if it contains a single predicate letter.

THEOREM IV: Let A be r.e. ($A = IA^*$, resp.). For any $k \geq 3$ there are k unified frames ϕ_i ($i=1, \dots, k$), incomparable under $\leq_A$, and with $I(\phi_i) = 1$.

PROOF: (I) A is r.e.. Given $k \geq 3$ (or $k \geq 1$, for that matter), let

$$\phi_i := \exists z_1 \dots \forall z_{2i} \neg \forall u \exists w_1 \dots Q w_{k-i} P(\vec{z}, u, \vec{w}),$$

where the dots stand for alternating quantifiers. ϕ_i is, classically, the same as Σ_{k+i+1}^0 .

Let $j < i \leq k$, and let $A_r := A$ augmented with all classically true Π_r^0 sentences. Let F be a case of ϕ_i which is classically equivalent to a Σ_{k+j+1}^0 (relativized) Gödel sentence G for A_{k+j} . Then G is not equivalent in A_{k+j} to any case of ϕ_j , and so F is not equivalent in A to any case of ϕ_j . Hence $\phi_i \not\equiv_A \phi_j$.

For the converse, we observe that theorem III may be relativized to A_{k-j-1} , yielding a Σ_{k-j}^0 formula $B(x)$ s.t. the sentence $\neg \forall x B$ is not equivalent in A_{k-j-1} to any sentence

prenex in a classically Σ_{k-j}^0 matrix. Since $\neg \forall x B$ is a case of ϕ_j , and each case of ϕ_i is prenex in a classically Σ_{k-j}^0 matrix, we conclude that

$$\phi_j \not\equiv_A \phi_i.$$

(II) $A = IA^*$. Let

$$\phi_i := Qz_1 \dots Vz_{\gamma_i} \neg \forall u \exists w_1 \dots Q'w_{3k-3i} P(\vec{z}, u, \vec{w}) \quad (i = 1, \dots, k)$$

and let $j < i \leq k$. ϕ_i is, classically, the same as $T_i := \Sigma_{3k+4i+1}^0$ or $\Pi_{3k+4i+1}^0$ (according to the evenness of i). As at the end of the proof of theorem III, we now take a classically true T_i sentence which is not implied by any true T_j sentence. Thus $\phi_i \not\equiv_{IA^*} \phi_j$.

For the converse, we proceed as in part (I), using the case $A = IA^*$ of theorem III. □

3. Application to truth definitions.

The classical arithmetical hierarchy is intimately related to enumeration theorems. In particular, it follows from Kleene's Hierarchy Theorem that there is

no Π_k^0 or Σ_k^0 relation enumerating all (unary, say) Π_{k+1}^0 relations. The proof-theoretic analogue of an enumerating relation is a truth-definition.

Given an arithmetic theory A , and a set δ of arithmetic sentences, a unary predicate $V(x)$ is a truth-definition (t.d.) in A for δ , if

$$\vdash_A V(\overline{F}) \leftrightarrow F \quad \text{for each } F \in \delta.$$

In analogy to the classical Enumeration Theorem, Troelstra ([74] 1.5.4) shows that for each frame ϕ there is a case V of ϕ which is a truth-definition in A for all closed cases of ϕ . On the other hand, no truth definition exists in any consistent extension of IA for all sentences (Tarski's Theorem).

The rôle of implication in intuitionistic theories may now be enunciated in yet another manner by the following immediate consequence of theorems I and II and corollary I.

THEOREM V. Let A be an r.e. theory as above, and let $k \geq 0$.

- (a) There is no t.d. V in A to $\Pi_1^{(k)}$ or to $\Sigma_2^{(k)}$ with $I(V) \leq k$.
- (b) There is no t.d. V in A to $\Sigma_1^{(k)}$ with $I(V) \leq k+1$. Hence:
- (c) (D.H.J. de Jongh [73]) There is no t.d. in A to Σ_1^* .
- (d) There is no t.d. V in IA^* to $\Pi_3^{(k)}$ with $I(V) \leq k$.
- (e) There is no t.d. in IA^* to Π_3^* .

4. Proof of lemma 1.1.

In the remaining of this paper we will use the sequential calculus IA^∞ of recursively presented cut-free infinitary derivations of $Le[A]$ and the terminology and notational conventions thereof. The use of infinitary derivations is motivated mainly by the subformula property: each (positive, respectively - negative, occurrence of a) formula in an infinitary (cut-free) derivation Δ for the sequent $\Gamma:F$ is a (positive, respectively - negative)

subformula of $(\Gamma) \rightarrow F$. Here (Γ) stands for the conjunction of all elements of Γ . Another important property of the calculus is this. Let A be an extension of IA as above. A sentence F is a theorem of A if there is a derivation Δ of IA^∞ deriving F , and s.t. "bar induction over Δ " is provable in A . In particular, IA^* consists exactly of the theorems of IA^∞ .

To prove lemma 1.1, let B be a prenex sentence, and let Δ be a derivation for $\neg B : \perp$. Since IA^* is consistent, the premiss $\neg B$ must be active at some inference in Δ , and therefore

- (1) $\neg \neg \exists u [\text{"} \perp \notin \Gamma^u \text{" and } \neg B \text{ is active at the node } u \text{ of } \Delta, \text{ but at no node } v \succ u]$.

If Δ^u is a sub-derivation of Δ above u , as in (1), then Δ^u has the form

$$\frac{\begin{array}{cc} \Delta^{u*\hat{0}} & \Delta^{u*\hat{1}} \\ \Gamma:B & \Gamma,\perp: F^u \end{array}}{\Gamma, \neg B : F^u}$$

Each $G \in \Gamma$ occurs negatively in $\neg B : \perp$, so $G \equiv \perp$. Hence, by the condition on u , $\Gamma = \emptyset$, and so $\Delta^{u*\hat{0}}$ derives B . Thus (1) implies

$\sim \vdash_{IA^*} B$, as required. □

5. A reduction lemma.

5.1 LEMMA: Let $\sigma = (\Gamma, G:H)$ be a sequent in a derivation for $\Theta:F$. If G is an active premiss of $\rightarrow R$, then $I(G) \leq \max[I(F)-1, I(E)-2 \mid E \in \Theta]$.

PROOF: $G \rightarrow H$ is a positive subformula of $(\Theta) \rightarrow F$, so G is a negative subformula of F , or a positive, but not strictly positive, subformula of some $E \in \Theta$. The inequality follows. □

Let B be a prenex sentence. We write B_1 for $N_1[B]$.

5.2. REDUCTION LEMMA. Let $n \geq 1$, and let P be a sentence with $I(P) \leq n+1$. The following is provable in IA : given a derivation Δ for $P \leftrightarrow B_{2n+4}$, we have $Q \leftrightarrow B_{2n+2}$ for some Q with $I(Q) \leq n$.

5.3. SUBLEMMA: If Δ is a derivation for $P, B_{2n+2} : B_{2n-1}$, then there is a sentence G with $I(G) \leq n-1$, satisfying

- (1) $P \wedge G \rightarrow B_{2n}$
- (2) $(P \wedge G \rightarrow B_{2n-1}) \rightarrow B_{2n-1}$

5.4. The sublemma implies the reduction lemma. If Δ derives $P : B_{2n+4}$, i.e., $P : B_{2n+2} \rightarrow B_{2n-1}$, then a variant Δ' of Δ derives $P, B_{2n+2} : B_{2n-1}$. Apply now 5.3, and let $Q := G \rightarrow B_{2n-1}$.

From (2) and $B_{2n+4} \rightarrow P$ we get $(G \rightarrow B_{2n-1}) \rightarrow (B_{2n+4} \rightarrow B_{2n-1})$. But, by 1(4), $(N_{2n+4} \rightarrow N_{2n-1}) \rightarrow N_{2n+2}$, so $Q \rightarrow B_{2n+2}$.

To prove the converse observe that, from (1) and $B_{2n+4} \rightarrow P$, we have $G \rightarrow (B_{2n+4} \rightarrow B_{2n})$. But $(N_{2n+4} \rightarrow N_{2n}) \rightarrow N_{2n}$ by 1(5), so $G \rightarrow B_{2n}$, and thus $(B_{2n} \rightarrow B_{2n-1}) \rightarrow (G \rightarrow B_{2n-1})$. Since $N_{2n+2} \equiv N_{2n} \rightarrow N_{2n-1}$, by 1(1), this yields $B_{2n+2} \rightarrow Q$.

□

5.5. Proof of 5.3. Let Δ derive $P, B_{2n+2} : B_{2n-1}$. For a node u in Δ , let $\Gamma_0^u := \Gamma^u \setminus \{B_{2n+2}\}$, and $\Gamma_1^u := \{E \in \Gamma^u \mid I(E) \geq n\}$. Let $V(x)$ be a truth-definition for finite conjunctions of subformulas H of the derived sequent, with $I(H) < n$. It is easily seen that V may be taken so that $I(V) = n-1$.

We show that there is a recursive function f (depending on Δ), with the following property. Let T, V be Kleene's computation relation and result-extracting function, respectively. Define

$$G(u) := \exists x \exists y [T(f(u), x, y) \wedge V(U(y))] \\ \equiv \exists z \in R_u V(z) \quad (\text{where } R_u \text{ is the range of } \varphi_{f(u)}).$$

Then, for each node u in Δ ,

$$(3) \quad (\Gamma_1^u, G(u)) \rightarrow B_{2n} \\ (4) \quad (\Gamma_0^u, G(u) \rightarrow B_{2n-1}) \rightarrow F^u.$$

Also, $I(G) = I(V) = n-1$. Taking, in particular, $u = \langle \rangle$, the root of Δ , we have $\Gamma^u = (P, B_{2n+2})$ and $F^u \equiv B_{2n-1}$. Letting $G := G(\overline{\langle \rangle})$, we get then from (3) $PAG \rightarrow (B_{2n+2} \rightarrow B_{2n})$; but $(N_{2n+2} \rightarrow N_{2n}) \equiv N_{2n}$ by 1(3), so

$$PAG \rightarrow B_{2n}. \quad \text{On the other hand, we get from (4) } (P, G \rightarrow B_{2n-1}) \rightarrow B_{2n-1},$$

as required.

The function f is defined by giving recursive instructions for obtaining the r.e. set R_u from $\{R_{u*\hat{m}}\}_m$. I.e., $G(u)$ is defined recursively from $G(u*\hat{m})$, $m \geq 0$, thus yielding a Herbrand-Gödel algorithm for f . Simultaneously with the definition we also verify that (3) and (4) are satisfied, by "bar induction on Δ ".

Case 1 (Basis): u is a top-node in Δ . B_{2n+2} cannot occur positively in the derived sequent $P, B_{2n+2} : B_{2n-1}$; hence $F^u \neq B_{2n+2}$, and so both (3) and (4) hold for $G(u) := 1$ (i.e., $R_u = \{\ulcorner 1 \urcorner\}$).

Case 2 (Induction Step): u is not a top-node. We consider cases for the inference rule of Δ at u . The only non-trivial cases are the following:

$$\underline{\wedge R}, \text{ say } \frac{\Gamma^u : F_0 \quad \Gamma^u : F_1}{\Gamma^u : F_0 \wedge F_1}$$

By ind. ass., $F(u*\hat{0})$ and $f(u*\hat{1})$ are defined so that $(\Gamma_1^{u*\hat{j}}, G(u*\hat{j})) \rightarrow B_{2n}$ and $(\Gamma_0^{u*\hat{j}}, G(u*\hat{j}) \rightarrow B_{2n-1}) \rightarrow F_j$, $j = 0, 1$. Define $f(u)$ so that

$R_u = R_{u*\hat{0}} \cup R_{u*\hat{1}}$; i.e., $G(u) \leftrightarrow G(u*\hat{0}) \vee G(u*\hat{1})$. We have $\Gamma_1^{u*\hat{0}} = \Gamma_1^{u*\hat{1}} = \Gamma_1^u$ ($i=0,1$), so $(\Gamma_1^u, G(u)) \rightarrow B_{2n}$ and $(\Gamma_0^u, G(u) \rightarrow B_{2n-1}) \rightarrow F_0 \wedge F_1$.

$\rightarrow R$, say $\frac{\Gamma^u, F_0 : F_1}{\Gamma^u : F_0 \rightarrow F_1}$.

By lemma 5.1 $F_0 \notin \Gamma_1^{u*\hat{0}}$, so $\Gamma_1^u = \Gamma_1^{u*\hat{0}}$, while $\Gamma_0^u = \Gamma_0^{u*\hat{0}}$ since

$F_0 \notin B_{2n+2}$ (by the subformula property). Set $f(u) = f(u*\hat{0})$, i.e.

$G(u) \leftrightarrow G(u*\hat{0})$.

$\rightarrow L$, say $\frac{\Gamma^u : H_0 \quad \Gamma^u, H_1 : F^u}{\Gamma^u, H_0 \rightarrow H_1 : F^u}$.

Subcase (a): $(H_0 \rightarrow H_1) \notin B_{2n+2}$. Define $f(u)$ so that

$G(u) \leftrightarrow G(u*\hat{0}) \vee G(u*\hat{1}) \wedge H_0$. Note that $I(H_0) < I(H_0 \rightarrow H_1) \leq n+1$.

Subcase (b): $(H_0 \rightarrow H_1) \equiv B_{2n+2} \equiv (B_{2n} \rightarrow B_{2n-1})$. This is the core of the

proof. We have

$\frac{\Gamma^u : B_{2n} \quad \Gamma^u, B_{2n-1} : F^u}{\Gamma^u, B_{2n+2} : F^u}$

Let $G(u)$ be the conjunction of all formulas in $\Gamma^u \setminus \Gamma_1^u$, i.e., of all $E \in \Gamma^u$ with $I(E) < n$. From the l.h.s. premiss we trivially have $(\Gamma_1^u, G(u)) \rightarrow B_{2n}$.

From the r.h.s. premiss and ind.ass. $(\Gamma_0^u, G(u*\hat{1}) \rightarrow B_{2n-1}, B_{2n-1}) \rightarrow F^u$. But, by the definition of $G(u)$, $(\Gamma_0^u) \rightarrow G(u)$, so $(\Gamma_0^u, G(u) \rightarrow B_{2n-1}) \rightarrow F^u$.

All other cases for propositional rules and for $\forall L$ and $\exists R$ are treated like the case for $\wedge R$.

$\forall R$, say $\frac{\{\Gamma^u : F(\bar{m})\}_m}{\Gamma^u : \forall x F(x)}$.

We let $f(u)$ be s.t. $R_u = \bigcup_m R_{u*\hat{m}}$, i.e., $G(u) \leftrightarrow \exists m G(u*\hat{m})$.

$\exists L$ - similar to $\forall R$. The presence of these infinitary rules is the reason for defining $G(u)$ via $\vee \dots$

6. Proof of theorem II. For $k = 1$, let $\phi_k := \neg \exists x P x$, and apply theorem III.

For $k \geq 2$, let $\phi_k := N_{2k+4} [\exists x P(x)]$, so $\phi_k \in \Sigma_1^{(k)}$, and B_{2k+4} is a case of ϕ_k . Using induction on k , we prove that for a suitable Σ_1^0

sentence B $\nVdash_A B_{2k+4} \leftrightarrow C$ whenever C is a sentence with $I(C) < k$.

Basis: By theorem III there is a Σ_1^0 sentence B s.t. $\nVdash_A \neg B \leftrightarrow C$ whenever

$I(C) = 0$. Induction step: this is the reduction lemma 5.2. $\square$

7. Proof of theorem I.

7.1 LEMMA: Assume that B and C are sentences with $I(B) = 0$, $I(C) = 1$.

The following is provable in IA : if Δ is a derivation for $\neg B : C$,

then there is a sentence G , with $I(G) = 0$, s.t. $G \rightarrow \neg B$ and $\neg G \rightarrow C$.

PROOF: For a node u in Δ , let $\Gamma_0^u := \Gamma^u \setminus \{\neg B\}$, $\Gamma_1^u := \{E \in \Gamma^u \mid I(E) > 0\}$.

Exactly as in 5.5, we prove that for a suitable formula $G(u)$, with $I(G) = 0$,

the following holds for any node u in Δ : $(\Gamma_0^u, G(u)) \rightarrow \neg B$ and

$(\Gamma_1^u, G(u) \rightarrow \perp) \rightarrow F^u$. For the case $\rightarrow L$, subcase (b), note that $(\neg B \rightarrow \neg B) \leftrightarrow \neg B$,

in analogy to the equivalence $(B_{2n+2} \rightarrow B_{2n}) \leftrightarrow B_{2n}$ used in 5.5. The lemma

follows by taking $u = \langle \rangle$. $\square$

7.2. LEMMA: Let B be a prenex sentence, s.t. $\neg \neg B \rightarrow B$ is not provable

in $A_1 := A$ augmented with all true Π_1^0 sentences. Then $\nVdash_A \neg \neg B \leftrightarrow C$

for any sentence C with $I(C) \leq 1$. (Note that if $A = IA^*$ then $A_1 = A$).

PROOF: If $\vdash_A \neg\neg B \rightarrow C$, then there is a derivation, probably correct in A , for $\neg\neg B:C$. By 7.1 this implies, in IA and hence in A , that for some implication-free sentence G , both $G \rightarrow \neg B$ and $\neg G \rightarrow C$. Since we also assume $\vdash_A C \rightarrow \neg\neg B$, this yields $\vdash_A \neg G \leftrightarrow \neg\neg B$.

Let Δ be a derivation for $\neg G, \neg B:\perp$ (provably correct in A).

Consider the Π_1^0 statement

$S := "$ $\neg B$ is not an active conclusion at any node ^{u} in Δ where $\perp \notin \Gamma^u "$.

If S is true, then $\neg G \rightarrow \perp$, so $\neg B$, and therefore $\neg\neg B \rightarrow B$. If not, then there (weakly) exists a node u for which the following Π_1^0 sentence is true: B is active at u but at no node $v \succ u$, and $\perp \notin \Gamma^u$. Say, we have

$$\frac{\Theta, \neg G:B \quad \Theta, \neg G, \perp:F^u}{\Theta, \neg B, \neg G:F^u}$$

where we may assume w.l.g. that $\neg B \notin \Theta$, since $\neg B$ is not active at any node $v \succ u$. Each sentence $H \in \Theta = \Gamma^u \setminus \{\neg B, \neg G\}$ occurs negatively in the derived sequent $\neg B, \neg G:\perp$. Since $\perp \notin \Gamma^u$, we must have $\Theta = \emptyset$, and so the subderivation of Δ above $u \hat{=} \hat{0}$ is known in A_1 to derive $\neg G \rightarrow B$.

We have shown that $\sim\sim \vdash_{A_1} \neg\neg B \rightarrow B$, contradicting the choice of B . Thus $\vdash_A \neg\neg B \leftrightarrow C$ is impossible. $\square$

(Note that 7.2 implies lemma 1.1.)

7.3 Proof of theorem I - concluded.

Case (I): A is r.e.. If $k = 1$, let ϕ_k be $\neg\forall xPx$. Then, by theorem III, $\phi_1 \not\vdash_A \psi$ for any frame ψ with $I(\psi) = 0$. For $k \geq 2$ we proceed by induction.

Basis: Fix a false Σ_2^0 sentence B consistent with A_1 . Then, in a manner similar to the proof of theorem III, one concludes that $\forall_{A_1} \neg\neg B \rightarrow B$.

Let $\phi_2 := \neg\exists x\forall yP(x,y)$, the frame corresponding to $\neg\neg B$. Then, by 7.2,

$\phi_2 \not\vdash_A \psi$ whenever $I(\psi) = 0$. Induction step: this is the reduction lemma 5.2.

Case (II): $A = IA^*$. For $k = 1$ proceed as in (I). For $k \geq 2$ proceed,

again, by induction. Basis: Let B be a Π_3^0 sentence which is true, but underivable in IA^* (cf. the proof of theorem III). Then, as in the proof of theorem III, $\forall_{IA^*} \neg\neg B \rightarrow B$. The statement of the theorem follows now, as in

(I), from 7.2. Induction step: again, this is the reduction lemma 5.2. $\square$

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