Unsolvability Considerations

in

Computational Complexity

by

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Technical Report

70-60

April 1970

Department of Computer Science
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Ithaca New York 14850

UNSOLVABILITY CONSIDERATIONS
IN
COMPUTATIONAL COMPLEXITY

A Thesis

Presented to the Faculty of the Graduate School
of Cornell University for the Degree of
Doctor of Philosophy

by

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June 1970

In this paper several undecidability aspects of complexity classes and sets associated with them will be examined. These sets will be classified by their degree of unsolvability and restrictions will be placed on measures so that these degrees are identical. This gives rise to a new criterion for the "naturalness" of measures and to suggestions for strengthening the measures of complexity.

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Acknowledgements

The author wishes to express his sincere gratitude to the faculty of the Department of Computer Science for their guidance during his graduate work at Cornell.

Special thanks are due to the faculty members in Theory of Computation, especially Professors Hartmanis and Constable for their patient encouragement and assistance during the formulation and completion of this thesis.

Finally, acknowledgements are due to the author's wife, Rosalind, for her patience and assistance during the preparation of this thesis and to Mrs. Camille Taylor for her general efficiency and conscientiousness in typing the final draft of this thesis. This research has been supported in part by National Science Foundation grants GJ 155 and GJ-579.

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Cornell University, 1972

ABSTRACT

The study of Computational Complexity began with the investigation of Turing machine computations with limits on the amounts of tape or time which could be used. Later a set of general axioms for measures of computation was presented and this instigated much study of the properties of these general measures. Many interesting results were shown, but the general axioms allowed measures with undesirable properties and several attempts have been made to tighten up the axioms so that only desirable measures will be defined.

CHAPTER 1

Introduction

1.1 General Background

The aim of computational complexity is to classify and study the functions which are computable (or recursive). This is usually done by placing them into some context using an important characteristic of the function.

Sub-recursive hierarchies were first used to divide these functions into classes and exhibit some of their properties. Examples of this are Grzegorzczuk's hierarchy of the primitive recursive functions [7] and the nested recursive functions of Péter [15]. These methods classify functions by their structure, placing limits on the operations which are used to build functions.

An intuitive characterization of the primitive recursive functions is the loop program hierarchy devised by Meyer and D. Ritchie [12]. A loop program is made up from combinations of the instructions:

$X = X + 1$

$X = 0$

$X = Y$

enclosed in loops which are denoted by the pair of

instructions:

```
LOOP X  
ENDLOOP.
```

The assignment instructions are interpreted as in any standard programming language and the loop instructions cause the segment between them to be executed X times.

The primitive recursive functions are then placed in classes according to how many nested loops are required to compute them. This concept is formalized:

Definition: The class of functions requiring n loops:

$$L_n = \{f \mid \text{There is a loop program which computes } f \text{ with loops nested to depth } n\}$$

A complete history of sub-recursive hierarchies and some results concerning them are contained in [4].

In Automata theory the recursive functions were classified by limiting the basic resources used in computation. This resource-bounded complexity began with the consideration of Turing machine computations using a limited amount of tape [13,20] or time [8].

All of the recursive functions were placed in complexity classes according to how difficult they were to compute, or how much of a "natural" resource they used. For time of computation these classes are defined as follows.

Definition: The class of t-computable functions

$R_t = \{f \mid \text{there is a Turing machine which computes } f(x) \text{ within } t(x) \text{ steps}\}.$

These classes possess many interesting properties and were studied extensively for a number of years. More information on this can be obtained from [2].

Logicians have produced relationships between these two methods of attack on properties of computable functions but the Union Theorem of McCreight and Meyer [10] when combined with previous results [16] finally generalizes this connection. When tape length or time are used as measures of computation, then the sub-recursive hierarchies are reproduced exactly in the complexity hierarchies.

Later a general set of axioms for measures of computation [1] was presented. This involved taking an admissible enumeration [18] of Turing machines (or partial recursive functions) denoted by: $M_0, M_1, M_2, \dots$ and assigning a measure or step counting function $\phi_0, \phi_1, \phi_2, \dots$ to each machine. The set of measures is designated as Φ and these measures must obey the following two rules:

- 1) $M_1(x)$ halts $\Leftrightarrow \phi_1(x)$ halts
- 2) There is a recursive function C such that for all i, m , and n :

$$C(i, m, n) = \begin{cases} 1 & \text{if } \phi_1(m) = n \\ 0 & \text{otherwise} \end{cases}$$

This first axiom indicates that whenever some function computes a value, then a cost of computation can be associated with it. And according to the second rule, the question "Does it cost n to compute $M_1(m)$?" is always recursive.

These axioms allow a very general set of measures for which many interesting results have been derived. The complexity classes formed from the general measures have been studied extensively in regard to their structural [10,3] and naming properties [2]. Also the properties of operators [5,11] have been noted.

Unfortunately, many of the measures allowed under the axioms are so general that not all of the intuitively derivable properties are preserved. Under time and tape as measures, complexity classes are recursively enumerable (r.e.) but measures can be defined so that some of the classes are not. Other properties such as finite invariance and infiniteness are not preserved by measures either.

Therefore some condition must be added to the original two axioms to preserve properties throughout the complexity classes and if possible, to eliminate the undesirable properties. It is reasonable to expect complexity classes to be r.e. and to conform by possessing the same properties. This notion will be formalized below.

1.2 Outline of Thesis

This chapter contained a brief outline of the progress in computational complexity up to this point. A complete review of the literature is not included since this has been done elsewhere [2,4].

In chapter two, several properties of complexity classes and sets associated with them are studied. An open problem in complexity: the enumerability of complexity classes, is finally settled by exhibition of a measure which contains a complexity class that is not enumerable. Several implications and generalizations of this fact are mentioned also.

Next the index sets of the time and tape complexity classes are found to be in the same recursive isomorphism type. Since this insures that they all possess exactly the same properties, this is taken to be a criterion for "natural" measures and called

conformity. Several sets associated with measures are then studied according to conformity and general measures are found to differ remarkably from the natural measures such as tape length or time of computation. The chapter ends with some general results on limits and independence of several kinds of conformity.

In chapter three, restrictions in addition to the two axioms are placed on measures until the measures conform. This is done for both the recursive functions and the primitive recursive functions.

For the convenience of the reader the pages where most of the notation and definitions first appear are given below.

$A \leq_1 B$	16	Ω_{Bound}	19
$A \equiv_1 B$	16	Ω_{Even}	45
$A \equiv B$	15	Ω_{FinSup}	45
$A \leq_T B$	16	Ω_{Zero}	45
$A \equiv_T B$	16	ϕ	3
Arithmetic	17	ϕ_1	3
conform	23	Π_n	18
conform above	31	presentation	10
comesoic	14	productive	14
I_t^ϕ	27	R_t	3
immune	14	R_t^L	9

L_n	2	R_t^ϕ	9
loop programs	1	R_t^P	45
M_1	3	R_t^T	9
matchable	11	Σ_n	17
ΩC	10	speed limit	48
Ω <u>Best</u>	55	ΩC	10
		Ω <u>Finite</u>	19
		ΩN	19

Some of the notation concerning functions must be introduced at this point. The partial recursive functions are usually denoted: $\phi_0, \phi_1, \dots$ where each ϕ_i refers both to the function computed and the particular algorithm in the enumeration of the functions. Naturally this enumeration must be admissible [18]. The symbol ϕ is used so that no particular scheme for computing the partial recursive functions will be implied in the results. In the material which follows a particular enumeration: the Turing machines $M_0, M_1, \dots$ is used. All of the results are valid for any admissible enumeration of the partial recursive functions however.

Equality is used quite freely below and is defined as follows. For total recursive f and g :

$$f=g \Leftrightarrow \forall x[f(x) = g(x)]$$

$$f=M_1 \Leftrightarrow \forall x[f(x) = M_1(x)] \text{ and}$$

M_1 is total recursive

$$\begin{aligned} M_i = M_j \iff & \forall x [M_i(x) \text{ halts} \Rightarrow M_j(x) \text{ halts} \\ & \text{and } M_i(x) = M_j(x)] \text{ and} \\ & \forall x [M_i(x) \text{ halts} \iff M_j(x) \text{ halts}]. \end{aligned}$$

All this means is that equality between functions or machines implies that both are exactly the same function. To indicate identical machines equality between their indices will be used.

CHAPTER 2

State of the Art (With two Axioms)

2.1 Complexity Classes

Most of the interesting results in computational complexity have been about the complexity classes or the classes of t -computable recursive functions. These classes are defined as follows:

Definition: The class of t -computable functions is:

$$R_t^\phi = \{ \text{total } f \mid \text{there is some } M_1 = f \text{ such that} \\ \phi_1(x) \leq t(x) \text{ almost everywhere} \}.$$

(when ϕ is time then R_t^T will denote these classes and R_t^L will be used for tape length as a measure).

These classes have been studied in detail for time and tape as measures [8,20] and some of the desirable properties which were found carry over to complexity classes defined from general measures. When measures were restricted slightly, then more properties of the natural measures carried over.

One important property of complexity classes under the "natural" measures was the fact that the classes were r.e. [8]. In [21], Young wondered if this was true for classes defined from general measures. Regretfully it is not, and this result

is presented here and also has been shown independently by Robertson and Landweber [17]. Before this can be shown, however, some preliminaries are in order.

To describe classes and their members more intuitively a set of algorithms for computing them must be given. This set "presents" the class and is defined:

Definition: The set A is a presentation for the class $\underline{C}$ iff A contains an index for each member of $\underline{C}$ and all elements of A are indices for members of $\underline{C}$, or

$$\underline{C} = \{f \mid \exists i [i \in A \wedge M_i = f]\}$$

The presentation which immediately comes to mind is the complete presentation or index set for a class.

Definition: The set A is the index set for the class $\underline{C}$ iff:

$$A = \{i \mid M_i = f \in \underline{C}\}.$$

Usually the index set for the class of functions $\underline{C}$ will be designated $\Omega \underline{C}$. ($\Omega \underline{C}$ is used for classes of functions rather than $\theta \underline{C}$ since θ has been used in the literature for classes of sets.)

Definition: A class is r.c. iff it has an r.c. presentation. (This means that the class is also

recursive via the well known "padding" trick).

An interesting type of class is one where an algorithm for any member of the class can be matched with an element of some standard presentation for the class. These classes are used below and are defined as follows.

Definition: C is a matchable class iff there is a recursive g and recursive presentation A such that:

$$i \in \Omega_C \Rightarrow g(i) \in A. \text{ and } M_i = M_{g(i)}.$$

Examples: a) Const = {the constant functions}. Let

$A = \{a_0, a_1, a_2, \dots\}$ where $\forall x [M_{a_k}(x) = k]$ and for any x define $g(i) = a_{M_i(x)}$.

b) $C = \{f_n\}$ for some recursive set of functions such that $\forall x [f_n(x) < f_{n+1}(x)]$. (e.g. $f_n(x) = n^x$).

c) $C = \{f_n\}$ for some recursive set of functions such that $\exists k \forall n \forall x > k [f_n(x) < f_{n+1}(x)]$. (Functions built from polynomials of degree n would be of this type).

Now a measure will be constructed using a matchable class so that an R_C^Φ from this new measure is not r.e. Any of the examples will work, but (a)

will be used for reasons of clarity.

Theorem 2.1: For any recursive t there is a measure $\hat{\phi}$ such that $R_t^{\hat{\phi}}$ is not r.e.

Proof: Let $A = \{a_0, a_1, \dots\}$ be a recursive presentation of the class Const where for all x , $M_{a_k}(x) = k$. Then for any measure ϕ , consider:

$$\hat{\phi}_1(x) = \begin{cases} 0 & \text{if } i = a_k \text{ and } M_k(k) \text{ does not} \\ & \text{halt in } x \text{ steps} \\ t(x) + \phi_1(x) + 1 & \text{otherwise} \end{cases}$$

Note that $R_t^{\hat{\phi}} = R_0^{\hat{\phi}}$ and $M_{a_k} \in R_0^{\hat{\phi}} \Leftrightarrow M_k(k)$ never halts.

Two things must be shown in order to prove the theorem. $\hat{\phi}$ must be demonstrated to be a measure and the non-enumerability of $R_t^{\hat{\phi}}$ must be shown. This proceeds as follows:

a) $\hat{\phi}$ is a measure since:

- 1) $\hat{\phi}_1(x)$ halts $\Leftrightarrow M_1(x)$ halts since all M_{a_k} are total and ϕ is a measure
- 2) $\hat{C}(i, m, n) \equiv \hat{\phi}_1(m) = n$ is recursive since it can be described:

$$C(i, m, n) = \begin{cases} 0 & \text{if } i \geq t(m)+1 \text{ and} \\ & \text{a) } i \neq a_k \text{ or} \\ & \text{b) } i = a_k \text{ and } M_{b_1}(k) \text{ halts in } \tau \text{ steps} \\ 1 & \text{if } n=0, i=a_k \text{ and } M_{b_1}(k) \text{ doesn't halt} \\ & \text{in } m \text{ steps} \\ 0 & \text{otherwise} \end{cases}$$

b) $R_t^{\hat{\phi}} = R_n^{\hat{\phi}}$ is not r.e.

Assume $R_t^{\hat{\phi}}$ is r.e. and let $\{a_0, a_1, \dots\}$ be an enumeration of it. Due to the construction of $\hat{\phi}$ all b_k must be indices for constant functions and in fact:

$$M_{b_1}(x) = k \Leftrightarrow M_{b_1} = a_k$$

Therefore the set $\{a_{M_{b_0}(x)}, a_{M_{b_1}(x)}, \dots\}$ is an r.e.

set and is exactly the set

$$\{k | M_k(k) \text{ never halts}\}.$$

This set is obviously recursively isomorphic to the well known set $\bar{K} = \{k | M_k(k) \text{ never halts}\}$ which is not r.e.

So from this contradiction, it can be concluded that $R_t^{\hat{\phi}}$ is not r.e.

In the construction of the previous proof the productive set $\bar{K}$ was used in order to produce a non-r.e.

class of total functions. This particular class of functions happens to be a productive class, of more formally as in Dekker and Myhill [6]:

Definition: $\underline{C}$ is a productive class iff there is a recursive f such that for every r.e. presentation W_1 of a subclass $\underline{D}$, the set $W_{f(1)}$ is a presentation of a class in $\underline{C-D}$.

In other words the function f provides a witness to the fact that the class $\underline{C}$ is not an r.e. class. Other kinds of non-r.e. classes are the immune (do not contain any infinite r.e. subclasses), and comeasurable (neither productive nor immune) classes.

By selecting the proper set (in place of $\bar{K}$) an immune or comeasurable class $R_t^{\hat{\phi}}$ could be formed. Another interesting aspect of the construction is that if $\hat{\phi}$ had been formed from $\hat{\phi}$ -time then $\hat{\phi}$ would have many natural properties but not all of the complexity classes would be r.e.

This non-enumerability of complexity classes points out one of the problems of general measures. No important properties are possessed by all complexity classes of all measures. In order to remove pathologies from complexity classes and isolate the

"natural" measures it seems that properties must be uniform throughout the complexity classes. The remaining results hopefully help to provide this uniformity.

2.2 Unsolvability Preliminaries

Most of the important properties in Automata theory are preserved under isomorphisms. These are the recursively invariant properties, and when sets are classified under recursive isomorphisms, all the sets in any isomorphism type possess the same properties. These concepts are defined:

Definition: P is a recursively invariant property iff for any 1-1, onto, recursive function f , if the set A has P then so does $f(A)$.

Properties such as recursive enumerability and most cardinality properties are recursively invariant. In order to study sets effectively they should be grouped by the properties they possess. The obvious classification is by equivalence under recursive isomorphisms.

Definition: A is recursively isomorphic to B ($A \equiv B$) iff there is a 1-1, onto, recursive f such that $B = f(A)$.

(The function f is also known as a recursive permutation)

By a theorem of Myhill [14], the isomorphism types (sets equivalent under isomorphisms) are the same as the equivalence classes (or 1-degrees) under 1-1 reducibility. The reducibilities used here are defined as follows.

Definition: A is 1-1 reducible to B ($A \leq_1 B$) iff there is a 1-1, recursive function g such that for all x :

$$x \in A \iff g(x) \in B.$$

Definition: A is 1-1 equivalent to B ($A \equiv_1 B$) iff $A \leq_1 B$ and $B \leq_1 A$.

Definition: A is Turing reducible to B ($A \leq_T B$) iff there is a Turing machine with B written on one tape which can decide membership in A .

Definition: A is Turing equivalent to B ($A \equiv_T B$) iff $A \leq_T B$ and $B \leq_T A$.

Hierarchies which result from the reducibilities outlined above can be used to describe sets very precisely, but almost no intuitive information about

a set is given by its reducibility information. Therefore another hierarchy, the Arithmetical Hierarchy, will be used in conjunction with the 1-degrees. This hierarchy reflects the structure of a set according to the number of alternating quantifiers in the expression of its membership problem. The membership problem for a Σ_n set will begin with a " $\exists$ " and contain n alternating quantifiers, while a Π_n set begins with a " $\forall$ ".

This is stated more formally as follows:

Definition: A predicate is arithmetical if it is constructed from a recursive predicate by quantification of number variables.

Definition: A set is arithmetical if it is of the form $\{x | P(x)\}$ for some arithmetical predicate P .

The arithmetical sets are then placed in the hierarchy by the structure of their membership predicates which can always be expressed as a recursive predicates with a prefixes of quantifiers.

Definition: a) A predicate is in Σ_n iff its prefix begins with $\exists$ and contains n alternating quantifiers.

b) A predicate is in Π_n iff its prefix begins with $\forall$ and contains n alternating quantifiers.

A pictorial representation of the Arithmetical Hierarchy appears below as figure 1. The lines slanting down towards the right denote the upper boundaries of the Σ_n areas, while the lines slanting down to the left form the top of the Π_n areas.

Figure 1

The intersection of Σ_1 and Π_1 is where the recursive sets are found and is called Σ_0 or Π_0 . The r.e. sets all belong to Σ_1 and the locations of the following well known sets are indicated in figure 1.

- a) $\emptyset = \{i | M_1 \text{ never accepts}\}$
- b) $\emptyset_{\text{Finite}} = \{i | M_1 \text{ accepts a finite set}\}$
- c) $\emptyset_N = \{i | M_1 \text{ accepts everything}\} = \Omega_{\text{Total}}$
 $= \{i | M_1 \text{ is a total function}\}$
- d) $\Omega_{\text{Bound}} = \{i | \text{the range of } M_1 \text{ is bounded}\}$
 $= \{i | \exists n \text{ and } \forall x [M_1(x) \leq n]\}$
- e) $\emptyset_{\text{Cofinite}} = \{i | M_1 \text{ accepts a cofinite set}\}$

These sets are all complete or maximal in their respective locations in the hierarchy when ordered by 1-degrees. Only the proof for Ω_{Bound} will be given here since the others may be found in [9].

Fact: Ω_{Bound} is in $\Sigma_3 \cap \Pi_3$

Proof: a) First the prenex normal form of the membership problem will be written out.

$$\begin{aligned} i \in \Omega_{\text{Bound}} &\Leftrightarrow M_1 \text{ is total and } \exists n \forall x, t \\ &\quad [M_1(x) \leq n \text{ and halts in } t \text{ steps} \\ &\quad \text{or } M_1(x) \text{ does not halt in } t \text{ steps}] \\ &\Leftrightarrow \forall x \wedge \exists t \end{aligned}$$

which can be expressed as $\exists W$ or $\forall W$. Therefore $\Omega_{\text{Bound}} \in \Sigma_3 \cap \Pi_3$.

b) Now ΩBound must be shown to be above L_2 (or $\Omega\text{Finite} \leq_1 \Omega\text{Bound}$). This is done by taking any M_1 and constructing the machine $M_{g(1)}$:

$M_{g(1)}(x)$ = the number of inputs less than x which M_1 accepts in x or less steps.

Obviously if M_1 accepts a finite set then $M_{g(1)}$ will be bounded by the cardinality of that set and so:

$$i \in \Omega\text{Finite} \iff g(i) \in \Omega\text{Bound}.$$

c) Finally ΩBound must be shown to be above Π_2 . This will be done by reducing ON to it. Again, taking any machine M_1 consider $M_{g(1)}$:

$$M_{g(1)}(x) = \begin{cases} 0 & \text{if } M_1(x) \text{ halts} \\ & \text{do not halt otherwise} \end{cases}$$

If M_1 is a total function then $M_{g(1)}$ will compute the zero function (which is bounded), otherwise $M_{g(1)}$ will be a partial function.

2.3 A Criterion for Measures

First, the index sets of complexity classes formed from the "natural" measures will be classified. This will be done for time, (the number of steps in a

computation). The proof may be done more elegantly via two theorems in the next chapter but it is presented here to emphasize that the r.e.-ness of R_t^T and the fact that each R_t^T contains all finite variants for some function are important properties of time as a measure. This classification will be used to suggest a criterion for measures and possibly to help find a method for strengthening the axiom.

Theorem 2.2: $\Omega R_t^T \equiv \Omega \text{Bound}$

Proof: a) $\Omega R_t^T \leq \Omega \text{Bound}$

Since R_t^T is an r.e. class [8] for time as a measure, let $A = \{a_0, a_1, \dots\}$ be an r.e. presentation for R_t^T . Then for any M_i define the machine $M_{g(i)}$ as follows:

$$M_{g(i)}(x) = \begin{cases} \text{the least } k \in \{0, \dots, x\} \text{ such that} \\ M_i(0) = M_{a_k}(0), \dots, M_i(x) = M_{a_k}(x) \\ x \text{ if the above condition is not} \\ \text{satisfied} \\ \text{diverge if any of } M_i(0), \dots, M_i(x) \\ \text{diverge} \end{cases}$$

Therefore if $i \in \Omega R_t^T$ then for some $a_k \in A$, $M_i = M_{a_k}$ and so $\forall x [M_{g(i)}(x) \leq k]$. If $i \notin \Omega R_t^T$ then no a_k

will be found and so $M_{g(1)}$ will be unbounded. Thus:

$$1 \in \Omega R_t^T \Leftrightarrow g(1) \in \Omega \text{Bound}.$$

$$b) \Omega \text{Bound} \leq_1 \Omega R_t^T$$

For any r.e. R_t^T there is a function b which majorizes the class [2]. Also, every R_t^T contains at least one function and all its finite variants. Therefore, select some f in R_t^T so that all of its finite variants are also in R_t^T , and for any M_1 define:

$$M_{g(1)}(x) = \begin{cases} f(x) & \text{if } M_1(x) \leq \max [M_1(0), \dots, M_1(x-1)] \\ b(x)+1 & \text{if } M_1(x) > \max [M_1(0), \dots, M_1(x-1)] \\ \text{diverge} & \text{if any of } M_1(0), \dots, M_1(x) \\ & \text{diverge} \end{cases}$$

Therefore $M_{g(1)}$ is total if M_1 is total, and if M_1 is bounded then $M_{g(1)}$ will be a finite variant of f and therefore a function in R_t^T .

From this theorem and a result of McCreight and Meyer [10] which states that for Φ -time there is a t such that the loop program class L_n is exactly the same as R_t^T , the following result may be easily proven.

Corollary: $\Omega L_n \equiv \Omega \text{Bound}$.

An interesting property of time as a measure is indicated by this theorem. All of the index sets of the complexity classes fall into the same 1-degree; in fact, the 1-degree of ΩBound . This means that all of the ΩR_n^L and ΩR_n^T possess exactly the same properties. The following condition on measures is suggested by this fact.

Definition: ϕ conforms (on ΩR_n^ϕ) iff for all recursive t_1 and t_2 , $\Omega R_{t_1}^\phi \equiv \Omega R_{t_2}^\phi$.

This definition can be easily extended to Turing-reducibility in the obvious manner. (Since all index sets are cylinders, m-conformity is equivalent to l-conformity.)

2.4 Index Sets of Complexity Classes

Now the index sets of the complexity classes (ΩR_n^ϕ) for general measures will be classified by their degrees of unsolvability. The only positive result in this section is that all ΩR_n^ϕ fall into the same Turing degree of unsolvability and this is shown below.

Theorem 2.3: Every Φ T-conforms (on ΩR_t^Φ).

Proof: The "oracle machine" $M^{\Omega Equal}$ which has:

$$\Omega Equal = \{ \langle i, j \rangle \mid \forall x [M_i(x) = M_j(x)] \}$$

(or all pairs of identical machines)

written on its reference tape will be used in this proof. $\Omega Equal$ is a well known Π_2 -complete set, and therefore can decide membership problems in the Turing-degree containing (b), (c), and (d) in Figure 1.

Also, the following two presentations are used:

$$A = \{a_0, a_1, \dots\} = \text{presentation of } R_t^\Phi$$

$$B = \{b_0, b_1, \dots\} = \text{presentation of } PR - R_t^\Phi$$

where PR is the class of all partial recursive functions.

The set A used here will be defined later as I_t^Φ , which is Σ_2 set. Robertson and Landweber have shown the set B to be r.e. [17]. Therefore both A and B can be enumerated or listed by $M^{\Omega Equal}$.

This machine operates (when given input i) by listing the sequence $a_0, b_0, a_1, b_1, \dots$ and checking the pairs $\langle a_0, i \rangle$, $\langle b_0, i \rangle$, etc. against the pairs on the reference list.

Sooner or later some $\langle a_m, i \rangle$ or $\langle b_n, i \rangle$ will match a pair on the reference tape and then the machine halts and outputs:

$$\chi_{\Omega\text{Equal}}(i) = \begin{cases} 1 & \text{if } \langle a_m, i \rangle \in \Omega\text{Equal} \\ 0 & \text{if } \langle b_n, i \rangle \in \Omega\text{Equal}. \end{cases}$$

Therefore $\Omega R_{t-T}^\phi \leq \Omega\text{Equal}$ and since $\Omega\text{Equal} \equiv \Omega N_{-1}^\phi \Omega R_t^\phi$

(shown below), the result follows.

The previous Theorem is quite interesting because it shows that all ΩR_t^ϕ are confined to the Turing degree of (b), (c), and (d) on the chart given earlier. Also it indicates that with the original axioms the index sets for the complexity classes conform in a rough way. Unfortunately Turing-reducibility is rather crude and allows sets with differing properties to be included within its degrees. On the other hand, 1-reducibility is much stricter and as was pointed out above, sets in the same 1-degree possess the same recursively invariant properties. If all measures possessed 1-conformity then pathological properties might be excluded. However, general measures do not conform, and this fact is shown in the following sequence of results.

Fact: For any recursive function t and r.e. class of

total function $\underline{C}$, there is a measure $\hat{\phi}$ such that:

$$R_t^{\hat{\phi}} = \underline{C}.$$

Proof: Let $A = \{a_0, a_1, \dots\}$ be an r.e. presentation of the class $\underline{C}$ and ϕ be any measure. The measure $\hat{\phi}$ is defined:

$$\hat{\phi}_1(x) = \begin{cases} 0 & \text{if } i \in \{a_0, \dots, a_x\} \\ t(x) + \phi_1(x) + 1 & \text{otherwise} \end{cases}$$

Obviously $\hat{\phi}$ is a measure (since ϕ is a measure and $t(x)$ is recursive) and $R_0^{\hat{\phi}} = R_t^{\hat{\phi}} = \underline{C}$.

Corollary: There are measures ϕ which do not conform (on ΩR_t^{ϕ}).

Proof: Immediate from the above fact and the existence of r.e. classes of total functions whose index sets are not in the same 1-degree. (For example ΩBound and $\Omega\text{Const} \equiv \Omega\text{N}$).

These results point out some differences between index sets for complexity classes. The T-conformity theorem indicates that if a Turing machine is given a listing of an index set for one complexity class then it can generate the index set for any other class. If the natural measures are used then this can be done by a recursive function.

2.3 Other Sets in Complexity Theory

Another interesting set defined from measures is the set of algorithms which are t -computable; the efficient presentation for R_t^ϕ .

Definition: The set of t -computable algorithms is defined:

$$I_t^\phi = \{i | M_i \text{ is total and } \phi_i(x) \leq t(x), \text{ almost everywhere}\}.$$

This set is contained in Σ_2 since its membership problem is easily written in Σ_2 form as follows.

$$i \in I_t^\phi \Leftrightarrow \phi_i(x) \leq t(x) \text{ almost everywhere}$$

$$\Leftrightarrow \exists n [\forall x (x \geq n \Rightarrow \phi_i(x) \leq t(x))$$

$$\wedge \forall x \leq n - z (M_i(x) \text{ halts in } z \text{ steps})]$$

This is obviously a Σ_2 form since bounded quantifiers can be eliminated.

When tape and time are used as measures, the I_t^ϕ for $t(x) = \text{constant}$ are recursive (this allows minimal growth rate [2]), while for increasing $t(x)$, I_t^ϕ becomes Σ_2 -complete (or $I_t^\phi \equiv \Theta\text{Finite}$). By slight modification both of these measures can be forced to conform on I_t^ϕ . The changes are quite reasonable and involve making each machine read or copy its input.

The following theorem is proven for tape length but time can be used also.

Theorem 2.4: For M_1 which copy their inputs, L conforms (on I_t^L).

Proof: The construction of the membership problem for I_t^Φ shown above indicates that I_t^L is in Σ_2 and therefore $I_t^L \leq_1 \underline{OFinite}$.

In order to prove conformity on I_t^L then the reduction: $\underline{OFinite} \leq_1 I_t^L$ must be demonstrated. This reduction of $\underline{OFinite}$ to I_t^L is achieved by constructing a machine $M_{g(1)}$ which copies down its input, then simulates M_1 upon this amount of tape. If M_1 accepts anything new, then $M_{g(1)}$ exceeds $t(x)$ tape. Therefore, if M_1 accepts a finite set, then $M_{g(1)}$ stays within the amount of tape the input takes up almost everywhere.

More formally, this is done by taking any M_1 and constructing $M_{g(1)}$ such that: when given an input x , $M_{g(1)}$ lays off x squares on the work tape, and simulates M_1 on each of the inputs $0, \dots, x$ until for each one:

a) x steps of the computation have been completed or

b) $M_1(x)$ halts or

c) $M_1(x)$ requires more than x squares of tape.

Then if M_1 halts on some input for the first time, $M_{g(i)}$ uses more than $t(x)$ amount of tape. Otherwise $M_{g(i)}$ halts.

Therefore:

$$\begin{aligned} i \in \text{Finite} &\iff M_1 \text{ halts on finite number of inputs} \\ &\iff M_{g(i)} \text{ exceeds } t(x) \text{ tape a finite} \\ &\quad \text{number of times} \\ &\iff g(i) < i_t^L. \end{aligned}$$

Since there are measures which have recursive and Σ_2 -complete I_t^ϕ even T-conformity is out of the question for general measures. In fact, as the next sequence of results shows, the I_t^ϕ are spread out through Σ_2 .

Lemma: For any set A and infinite, recursive set $B = \{b_0, b_1, \dots\}$ the set

$$C = \{b_i \mid i \in A\} \in \mathcal{I}_m.$$

Proof: a) $A \subseteq C$ via $g(x) = b_x$

b) Assume that $x \notin A$ and then $C \leq_m A$ via:

$$g(x) = \begin{cases} x & \text{if } x \in B \\ b_i & \text{if } x = b_i \end{cases}$$

and therefore $C \equiv_m A$.

Theorem 2.5: For any recursive t and r.e. set A there is a measure ϕ such that $I_t^\phi \equiv_m A$.

Proof: Select an infinite set $B = \{b_0, b_1, \dots\}$ such that all M_{b_i} are total functions. Then define the r.e. set $C \equiv_m A$ from the Lemma.

Let $C = \{c_0, c_x, \dots\}$ and for any ϕ define:

$$\hat{\phi}_1(x) = \begin{cases} 0 & \text{if } i \in \{c_0, \dots, c_x\} \\ \phi_1(x) + t(x) + 1 & \text{otherwise} \end{cases}$$

$\hat{\phi}$ is a measure since t is a recursive function, C is an r.e. set, and ϕ is a measure. Also, from the lemma:

$$I_t^{\hat{\phi}} = C \equiv_m A.$$

An extension of the techniques used in Theorem 2.1 and 2.5 can be used to produce a measure where for every m -degree in $\Sigma_1 \cup \Pi_1$ there is some t such that I_t^ϕ is in that m -degree. This gives a rather ugly, layered structure to the complexity classes.

2.6 Conformity Relationships

Even though measures exist with irregularities or undesirable properties, these phenomena exist only in the complexity classes at the bottom of the hierarchy. An example of this is that as soon as the functions of

finite support become t -computable (as they must sooner or later), then the complexity classes become r.e. [2]. As one would expect, all measures exhibit conformity above some point in the complexity hierarchy.

Definition: ϕ conforms above t_0 (on ΩR_t^ϕ) iff for all recursive $s > t_0$: $\Omega R_s^\phi = \Omega R_{t_0}^\phi$.

Theorem 2.6: For every ϕ there is a recursive function t_0 such that ϕ conforms above t_0 (on ΩR_t^ϕ).

Proof: The desired t_0 is that such that $R_{t_0}^\phi$ contains the functions of finite support. Then the proof proceeds like Theorem 2.2.

In order to show conformity on I_t^ϕ the recursive relativity between measures (due to Blum [1]) is required.

Theorem 2.7 (Blum): For any measures ϕ and $\hat{\phi}$ there is a recursive function f such that for almost all x and all i :

$$\phi_i(x) \leq f(x, \hat{\phi}_i(x)) \text{ and } \hat{\phi}_i(x) \leq f(x, \phi_i(x)).$$

Theorem 2.8: For every ϕ there is a recursive function t_ϕ such that ϕ conforms above t_ϕ (on I_t^ϕ).

Proof: This proof proceeds in the same manner as Theorem 2.4 with the required $t_\phi(x) = f(x, x)$ where f is the function in the Theorem 2.7 with $\hat{\phi}$ -tape.

This forces the relationship:

$$i \in \Omega_{\text{finite}} \Rightarrow g(i) \in I_{f(x, x)}^\phi = I_{t_\phi}^\phi \subseteq I_t^\phi$$

In order to make the machine $M_{g(i)}$ take more than $t_\phi(x)$ steps in ϕ an r.e. presentation for a class of total functions containing R_t^ϕ is selected and whenever M_i accepts a new input, then $M_{g(i)}$ computes a value different from another member of the presentation. Thus:

$$g(i) \in I_t^\phi \Rightarrow g(i) \notin I_{t_\phi}^\phi \Rightarrow i \notin \Omega_{\text{finite}}.$$

At first, one might expect to find some relationship between the two kinds of conformity which have been examined so far. A close look at time and tape length as measures reveals that conformity on ΩR_t^ϕ does not imply conformity on I_t^ϕ . This seems reasonable since I_t^ϕ ranges over a larger area than ΩR_t^ϕ . However the reverse of this is not true either and is demonstrated in the next theorem.

Theorem 2.9: There is a measure which conforms on I_t^ϕ but does not conform on ΩR_t^ϕ .

Proof: Let $\{z_0, z_1, \dots\}$ be an infinite, recursive set of indicies for the zero function. Consider the measure which is defined:

$$\phi_i(x) = \begin{cases} 0 & \text{if } i = z_k \text{ and the same} \\ & \text{number of } M_k(0), \dots, M_k(x) \\ & \text{halt in } x \text{ steps as } M_k(0), \dots, M_k(x-1) \\ & \text{halted in } x-1 \text{ steps} \\ \text{time (read input) otherwise} \end{cases}$$

Obviously $R_0^\phi = \{\text{zero function}\}$ since no machine which reads its input can operate in zero steps.

Also, since the zero function can be computed in x steps, all of the remaining R_t^ϕ are the same as time. Therefore

$$\Omega R_0^\phi = \{i \mid \forall x [M_i(x) = 0]\} \equiv_1 \Omega N$$

while every other $\Omega R_t^\phi \equiv_1 \Omega$ empty. Therefore ϕ does not conform on ΩR_t^ϕ .

But ϕ does conform on I_t^ϕ and this is accomplished for I_0^ϕ by the following reduction:

$$\begin{aligned} x \in \text{Finite} & \Leftrightarrow \exists n \forall y > n [M_k(y) \text{ does not halt}] \\ & \Leftrightarrow \exists n \forall y > n [\phi_{z_k}(y) = 0] \\ & \Leftrightarrow z_k \in I_0^\phi \end{aligned}$$

A recursive function mapping k into z_k is rather easily found and so all $I_t^\phi \equiv_1 \underline{\text{OFinite}}$.

For natural measures such as tape and time the set I_t^ϕ is 1-1 reducible to ΩR_t^ϕ . This means that there is a recursive function which maps I_t^ϕ into ΩR_t^ϕ . This is not true however for the measures constructed in theorems 2.1 and 2.9. Another interesting aspect of tape and time (when inputs are read and copied) is that both I_t^ϕ and ΩR_t^ϕ are in the same Turing degree of unsolvability. The following trilogy points out some of the reasons for this. Unfortunately the converses are false and this is easily seen by examining some of the previous theorems.

Theorem 2.10: For every measure ϕ , $I_t^\phi \leq_T \Omega R_t^\phi$.

Proof: Since $I_t^\phi \in \Sigma_2$ then $I_t^\phi \leq_T \underline{\text{OFinite}}$ and as was pointed out in the T-conformity theorem, $\underline{\text{OFinite}} \equiv_T \Omega R_t^\phi$.

Theorem 2.11: If ϕ conforms (on I_t^ϕ) then $I_t^\phi \equiv_T \Omega R_t^\phi$.

Proof: Conformity on I_t^ϕ implies that $I_t^\phi \equiv_T \underline{\text{OFinite}}$.

Theorem 2.12: If ϕ conforms (on ΩR_t^ϕ) then $I_t^\phi \leq_1 \Omega R_t^\phi$.

Proof: Conformity on ΩR_t^ϕ implies that $\Omega R_t^\phi \equiv_1 \underline{\Omega\text{Bound}}$ and $\underline{\text{OFinite}} \leq_1 \underline{\Omega\text{Bound}}$.

Since the work of Blum [1] the definition of complexity classes have been defined with an "almost everywhere" clause. This has been justified by stating that for most natural measures, the classes are equivalent to those defined with an "everywhere" clause if output operations are ignored. However there are some differences and several interesting facts are noted here.

Definition: The class of t-computable functions:

$$S_t^\phi = \{f \mid \exists M_1 - f \text{ and } \forall x [\phi_1(x) \leq t(x)]\}$$

Definition: The set of t-computable algorithms:

$$H_t^\phi = \{i \mid \forall x [\phi_1(x) \leq t(x)]\}$$

One thing which should be noted immediately is that H_t^ϕ is a Π_1 set and NS_t^ϕ (like NR_t^ϕ) is a $\Sigma_3 \cap \Pi_3$ set. Several other obvious facts are stated in the following corollary.

Corollary:

- a) $H_t^\phi \leq_1 NS_t^\phi$
- b) $S_t^\phi \subseteq R_t^\phi$
- c) $H_t^\phi \subseteq I_t^\phi$

Most other correspondences between the two definitions are false. There are measures and recursive t for which $H_{t-1}^\phi <_1 I_t^\phi$, $I_{t-1}^\phi <_1 H_t^\phi$, $NS_{t-1}^\phi <_1 NR_t^\phi$, and

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Definition: The class of t-computable functions:

$$S_t^\phi = \{f | \exists M_1 = f \text{ and } \forall x [\phi_1(x) \leq t(x)]\}$$

Definition: The set of t-computable algorithms:

$$H_t^\phi = \{i | \forall x [\phi_1(x) \leq t(x)]\}$$

One thing which should be noted immediately is that H_t^ϕ is a Π_1 set and NS_t^ϕ (like NR_t^ϕ) is a $\Sigma_2 \cap \Pi_3$ set. Several other obvious facts are stated in the following corollary.

Corollary: a) $H_t^\phi \leq_1 NS_t^\phi$

b) $S_t^\phi \subseteq R_t^\phi$

c) $H_t^\phi \leq_1 I_t^\phi$

Most other correspondences between the two definitions are false. There are measures and recursive t for which $H_{t-1}^\phi \leq_1 I_t^\phi$, $I_{t-1}^\phi \leq_1 H_t^\phi$, $NS_{t-1}^\phi \leq_1 NR_t^\phi$, and

$$\Omega R_{t-1}^\phi \leq \Omega S_t^\phi.$$

As before, there are several limits on the irregularities present in measures. The following theorem corresponds to several previous results.

Theorem 2.13: For any ϕ there are recursive t_1 and t_2 such that:

- a) $H_{t-1}^\phi \leq I_t^\phi$ for all $t > t_1$
- b) $\Omega S_t^\phi \equiv \Omega R_t^\phi$ for all $t > t_2$.

One of the important questions in Automata theory has always been whether things are recursively enumerable. However answering this question does not provide much information, and more specific attributes of enumeration must be studied. In [21], Young looked at the speed of enumeration and provided some additional properties of computation.

Of interest here are the relationships between presentations of complexity classes and the sets of t -computable algorithms. The first result in this area was exhibited by McCreight and Meyer [10].

Theorem 2.14: For any ϕ there is a recursive g such that for sufficiently large t there is a subset of I_h^ϕ [where $h(x) = g(t(x), x)$] which is an r.e. presen-

tation of R_t^ϕ .

For some measures this function g could conceivably grow quite quickly, so these presentations may contain many algorithms which are nowhere near t -computable. But viewed with a version of the Gap Theorem [2] an immediate corollary insures that some complexity classes are presentable in t -computable form.

Theorem 2.15: (Weak Gap): For any ϕ and every recursive g there are arbitrarily large t such that:

$$I_t^\phi = I_h^\phi$$

[where $h(x) = g(t(x), x)$].

Corollary: For any ϕ there are arbitrarily large recursive t such that there is a subset of I_t^ϕ which is an r.e. presentation for R_t^ϕ .

The next step is to ask whether all measures produce complexity classes which have no r.e. presentations that contain only t -computable algorithms. This is not true, since Robertson and Landweber [17] produce an algorithm based upon [10] which generates r.e. subsets of I_t^L which present R_t^L . This proof (with minor modifications) is also valid for time as a measure.

An interesting fact is that this result does not apply to S_t^L and R_t^L . This is shown in the following sequence.

Lemma: If an r.e. presentation for R_t^L (or S_t^L) is a subset of H_t^L then there is a very small recursive g such that:

$$R_{g+t}^L - R_t^L \neq \emptyset.$$

Proof: This will be basically a diagonalization procedure. The function g mentioned above will be $g(x)=x$ for all inputs x .

Let $A = \{a_0, a_1, \dots\}$ be the r.e. presentation of R_t^L and let all a_k be members of I_t^L . Then define the machine M which will diagonalize over the complexity class R_t^L as follows.

When presented with the input x , the machine M :

- a) Lays off x squares of tape.
- b) Produces the integer k : the x^{th} number in the sequence:
0,0,1,0,1,2,0,1,2,3,0,...
- c) Writes down a description of machine M_{a_k} if this is possible within x tape squares. If not, then halts with an output of zero.

- d) If step (c) was completed successfully then simulate M_k on the tape adjoining the description and output a different answer.

This procedure obviously produces a function which is not in R_t^L since sooner or later step (c) will be completed for every integer. This machine computes within $t(x)+x$ space for all inputs, so therefore it is in R_{t+g}^L where $g(x)=x$.

This lemma also holds for time as a measure with a few minor modifications. When coupled with the Gap theorem the following result emerges.

Theorem 2.16: There are arbitrarily large recursive t such that no subset of R_t^L is an r.e. presentation of R_t^L .

Proof: Immediate from the lemma and the Gap theorem.

It would seem reasonable to suspect that the last two results might hold for general measures rather than only tape and time. This can be accomplished by application of the Blum relativity result which was given as Theorem 2.7.

Chapter 3

Evolution of a New Axiom

From the evidence in the last section and in the literature it could be assumed that the original two axioms are too weak to characterize the natural measures of computation. Some new requirement must be added to the axioms in order to eliminate measures with undesirable properties.

In this section constraints will be placed upon measures to force conformity. This is done for the complexity hierarchies of recursive functions and primitive recursive functions.

3.1 Measures for Recursive Functions

These results are suggested by Theorem 2.2 and progress gradually towards conformity.

Theorem 3.1: If $\underline{C}$ is a non-empty class of total functions, then $\text{ON}_{\leq 1} \underline{NC}$.

Proof: For some function $f \in \underline{C}$ and any machine M_1 consider:

$$M_{g(1)}(x) = \begin{cases} f(x) & \text{if } M_1(x) \text{ halts} \\ \text{diverge} & \text{otherwise} \end{cases}$$

Therefore $M_{g(1)}$ is total and equal to f only if M_1 is total.

Or, more formally:

$$\begin{aligned} \text{if ON} &\Leftrightarrow \forall x [M_1(x) \text{ halts}] \\ &\Leftrightarrow \forall x [M_{g(1)}(x) = f(x)] \\ &\Leftrightarrow \Omega(1) \in \Omega. \end{aligned}$$

When paired with the T-conformity theorem (Theorem 2.3), this result provides maximal and minimal bounds for the location of Ω_t^ϕ in the arithmetical hierarchy. This is pictured in Figure 2 and represents where the Ω_t^ϕ are located for general measures.

Figure 2

The next step is to require that all complexity classes be r.e., which seems to be a reasonable restriction.

Theorem 3.4: If $\underline{C}$ is an r.e. class of total functions, then $R_{\underline{C}}^{\phi}$ is recursive.

Proof: Let $\{M_{a_0}, M_{a_1}, \dots\}$ be an r.e. presentation for the class $\underline{C}$ and for any M_i define:

$$M_{g(i)}(x) = \mu k [M_{a_k}(0) = M_i(0) \wedge \dots \wedge M_{a_k}(x) = M_i(x)]$$

or equivalently, if M_{a_k} is the first machine on the list A which acts the same as M_i on inputs less than or equal to x then set $M_{g(i)}(x) = k$.

Obviously if M_i is a function in the class $\underline{C}$ then some M_{a_k} will be found to be the same as M_i and

$$\forall x [M_{g(i)}(x) \leq k].$$

Otherwise machines further and further along on the list will have to be examined and $M_{g(i)}$ will be unbounded.

Combined with Theorem 3.1, this result restricts the location of $R_{\underline{C}}^{\phi}$ to the shaded area in Figure 3. Conformity has not been achieved yet, so just making the complexity classes r.e. does not seem to be an adequate restriction.

Figure 3

Theorem 3.3: If $\underline{C}$ is an r.e. class of total functions containing all finite variations of at least one function, then $\text{Bound}_{\leq 1} \Omega \underline{C}$.

Proof: Since $\underline{C}$ is r.e. there is a recursive b such that $f \in \underline{C} \rightarrow f(x) < b(x)$ almost everywhere. So if f is the function whose finite variants are in $\underline{C}$, for any M_1 define:

$$M_{g(1)}(x) = \begin{cases} f(x) & \text{if } M_1(x) \leq \max\{M_1(0), \dots, M_1(x-1)\} \\ b(x) & \text{if } M_1(x) > \max\{M_1(0), \dots, M_1(x-1)\} \\ \text{diverge} & \text{if for some } y \leq x M_1(y) \text{ diverges} \end{cases}$$

Thus if M_1 is total then $M_{g(1)}$ is total and $M_{g(1)}$ is a finite variant of f iff M_1 is bounded. More formally:

$$\begin{aligned} 1 \in \Omega \text{Bound} &\Leftrightarrow \exists k \forall x [M_1(x) \leq k] \\ &\Leftrightarrow \exists k \text{ values of } x \text{ where } M_{g(1)}(x) = b(x) \\ &\Leftrightarrow g(1) \in \Omega C. \end{aligned}$$

When these classes are complexity classes, then containing finite variations implies recursive enumerability. ΩR_c^ϕ has been confined to a single 1-degree as is shown in Figure 4. Of course this degree contains all of the ΩR_c^ϕ for tape and time as measures.

Figure 4

3.2 Measures for Primitive Recursive Functions

Much of the work in Automata Theory has been concerned with the functions which are frequently computed; not with all of the computable functions. In this section the primitive recursive functions (or loop programs) are denoted $p_0, p_1, \dots$ and the step counting functions $P_0, P_1, \dots$ are assigned to them. The complexity classes R_c^p are defined similarly to the R_c^0 .

Several important index sets for classes of primitive recursive functions are defined:

- a) $\Omega_{Zero} = \{\text{functions with zero in range}\}$
 $= \{i | \exists x [p_i(x) = 0]\}$
- b) $\Omega_{Even} = \{\text{even functions}\}$
 $= \{i | \forall x [p_i(x) \text{ is even}]\}$
- c) $\Omega_{Zero} \cap \Omega_{Even}$
- d) $\Omega_{FinSup} = \{\text{functions of finite support}\}$
 $= \{i | \exists k \forall x [x \geq k \Rightarrow p_i(x) = 0]\}$

and the location of each of these sets is indicated in Figure 5.

Figure 5

The original Blum axioms for measures do not translate freely to the primitive recursive functions since they are all total functions and so a series of restrictions will be placed on the measures. Several facts are in order at this point.

Facts: a) There are no non-trivial index sets which are finite or cofinite since any index set contains an infinite number of elements.

b) Ω_{τ}^3 is a Σ_2 set, since its membership problem can be expressed: $\exists k \forall x [x > k \Rightarrow P_1(x) \leq t(x)]$.

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Figure 6

Initially the measures considered are required to be primitive recursive. This is reasonable since a function's cost of computation should be the same kind of function as the one being computed. This permits ΩR_t^P to fall in the shaded area of Figure 6. Unfortunately, this restriction does not necessarily even produce a hierarchy, since all functions can be assigned to the same complexity class. Also the classes R_t^P do not have to be r.e. and almost any undesirable property can be found in the classes.

The next restriction to be placed on measures is that each R_t^P must contain at least one function such that for some $g \notin R_t^P$ if

$$n(x) = \begin{cases} g(x) & \text{for } x \leq n \\ f(x) & \text{for } x > n \end{cases}$$

for some n then $n \in R_t^P$. This provides the reduction $\Omega_{Zero < 1} \Omega R_t^P$ via the function g which is defined:

$$M_{g(1)}(x) = \begin{cases} g(x) & \text{if } 0 \in \{M_1(0), \dots, M_1(x)\} \\ f(x) & \text{if } 0 \notin \{M_1(0), \dots, M_1(x)\} \\ \text{diverge} & \text{otherwise} \end{cases}$$

This forces the index sets for complexity classes to be in the area designated in Figure 7. No hierarchy is

assured, and R_C^T does not have to be r.e. as of yet.

Figure 7

Making a relationship between a function and its cost mandatory as the next restriction to be placed on measures.

Definition: f is a speed limit for P iff for all i , $f(x) \geq p_i(x)/P_i(x)$ almost everywhere.

This speed limit can be measured in digits printed per step if the measure is thought of as steps of computation. For time and tape as measures the speed

limit is somewhere between $\log(x)$ and 1 depending on how they are defined.

Theorem 3.4: If P has a primitive recursive speed limit, then $\Omega_{\text{Even}} \leq_1 \Omega_{\text{t}}^P$.

Proof: Assume that $q \in \mathbb{R}_{\text{t}}^P$ and that s is the speed limit for P . Then for any p_1 define:

$$p_{g(1)}(x) = \begin{cases} q(x) & \text{if } \forall y \leq x [p_1(y) \text{ is even}] \\ t(x) \cdot s(x) + 1 & \text{otherwise} \end{cases}$$

If p_1 is an even function the $p_{g(1)} = q$ and $g(1) \in \Omega_{\text{t}}^P$.

But if for some z $p_1(z) \neq z$ then:

$$p_{g(1)}(x) = \begin{cases} q(x) & \text{for } x < z \\ t(x) \cdot s(x) + 1 & \text{for } x \geq z \end{cases}$$

which is obviously a primitive recursive function (can be computed by a loop program) and due to the definition of speed limits:

$$\forall x \geq z [P_1(x) = p_1(x)/s(x) > t(x)].$$

This new restriction allows Ω_{t}^P to range within the shaded area indicated on Figure 8. At last a hierarchy is assured, because the speed limit places a bound on the values of the functions in each complexity

class. But, even though this is true the complexity classes still do not have to be r.e.

Figure 8

Figure 9 indicates the result of combining the three restrictions mentioned above. This follows from the reduction $\Omega_{\text{Even}} \cap \Omega_{\text{Zero}} \leq_1 \Omega_{\text{c}}^P$. The complexity classes

Figure 9

are not all r.e. yet, but naturally there is a hierarchy.

Theorem 3.5: If P has a primitive recursive speed limit and R_t^P contains all finite variants of some function, then $\Omega\text{FinSup} \leq_1 \Omega R_t^P$.

Proof: Let q be a function in R_t^P and s be the speed limit for P . Then for any primitive recursive p_1 define:

$$p_{g(1)}(x) = \begin{cases} q(x) & \text{if } p_1(x) = 0 \\ s(x) \cdot t(x) + 1 & \text{otherwise} \end{cases}$$

If $i \in \Omega\text{FinSup}$ then $p_{g(1)}$ is a finite variant of q and therefore $g(1) \in \Omega R_t^P$. Otherwise $p_{g(1)} = s \cdot t + 1$ infinitely often and due to the definition of speed limits:

$$p_{g(1)}(x) = p_{g(1)}(x) / s(x) = [s(x) \cdot t(x) + 1] / s(x) > t(x).$$

Therefore $g(1) \notin \Omega R_t^P$ if $i \notin \Omega\text{FinSup}$.

The speed limit restriction is necessary to have a hierarchy. At last conformity has been provided for the primitive recursive functions, as is indicated in Figure 10.

Figure 10

In fact, the complexity classes are r.e. also.

CHAPTER 4

Conclusion

4.1 General Remarks

The major problem with general measures is that the desirable properties which some measures possess are not found in all measures. Therefore, it is desirable to isolate those measures which are natural and do not have any pathological properties.

One way to do this is to require that the complexity classes defined from measures all have identical properties. This was the rationale behind the definition of conformity, and therefore conformity seems to be a reasonable criterion for measures or any other axiom system.

From the phenomena exhibited previously, it would seem that a new axiom is needed. If this axiom were that every complexity class contains all the finite variants of at least one function, then conformity will be achieved. However, this is a restriction on complexity classes and is not an axiom in the spirit of the Blum axioms for measures. Possibly some further restriction on measures themselves could provide com-

plexity classes with the finite variants that are required.

Whenever pathological problems exist in complexity hierarchies, it has been shown that they exist only in the lower levels of the hierarchy. This means that almost all of the complexity classes for any measure belong to the same 1-degree, and that the measure conforms "almost everywhere". Conditions that occur in all but a finite number of places are accepted in Automata theory as being desirable in most cases. In fact, the definition of the complexity classes R_t^ϕ contains an "almost everywhere" clause.

But, in complexity hierarchies, the functions which are easiest to compute, and that are computed most often, occur at the bottom. These very functions are the ones computed in "real life" and therefore are quite important. Facts about their complexity should be meaningful, and so measures used should not have any pathological properties, even for a few classes at the bottom of the hierarchy.

4.2 The Direction of future research

Some open problems and areas for further study are as follows.

a) Conformity on I_c^ϕ and I_c^p should be studied by placing restraints on the axioms for measures.

b) Define the class:

$$\begin{aligned} \underline{Best} &= \{f \mid \text{there is a "fastest" program for } f\} \\ &= \{f \mid \exists i [M_i = f \wedge \forall j (M_i = M_j \Rightarrow \phi_i \leq \phi_j)]\} \end{aligned}$$

and study Ω_{Best} with respect to conformity.

c) Speed limits, defined in the last section, should be investigated with regard to their effect on conformity and other properties of measures.

d) Possibly some properties of classes of recursive functions (along the lines of those in Dekker and Myhill) could be formulated which would have some significance when applied to R_c^ϕ .

e) The criterion of conformity might be profitably applied to axiom systems in other areas of Automata theory.

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