

Robustly Hedging Variable Annuities with Guarantees Under Jump and Volatility Risks

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Abstract

Accurately quantifying and robustly hedging options embedded in the guarantees of variable annuities is a crucial task for insurance companies in preventing excessive liabilities. Due to sensitivities of the benefits to tails of the account value distribution, a simple Black-Scholes model is inadequate. A model which realistically describes the real world price dynamics over a long time horizon is essential for the risk management of the variable annuities. In this paper, both jump risk and volatility risk are considered for risk management of lookback options embedded in guarantees with a ratchet feature.

We evaluate relative performances of delta hedging and dynamic discrete risk minimization hedging strategies. Using the underlying as the hedging instrument, we show that, under a Black-Scholes model, local risk minimization hedging is significantly better than delta hedging. In addition, we compare risk minimization hedging using the underlying with that of using standard options. We demonstrate that, under a Merton's jump diffusion model, hedging using standard options is superior to hedging using the underlying in terms of the risk reduction. Finally we consider a market model for volatility risks in which the at-the-money implied volatility is a state variable. We compute risk minimization hedging by modeling at-the-money Black-Scholes implied volatility explicitly; the hedging effectiveness is evaluated, however, under a joint underlying and implied volatility model which also includes instantaneous volatility risk. Our computational results suggest that, when implied volatility risk is suitably modeled, risk minimization hedging using standard options, compared to hedging using the underlying, can potentially be more effective in risk reduction under both jump and volatility risks.

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1 Introduction

Until the early 1990's, most variable annuity contracts provided a modest minimum death benefit guarantee of return of premiums. This is the greater of the account balance at death and the sum of the premium deposits, less partial withdrawals, since inception of the contract. The cost of this benefit was considered insignificant and insurance companies did not assess an explicit charge and did not hold an additional reserve for this benefit.

Since the middle of 1990's, variable annuity contracts have offered more substantial minimum death benefit guarantees. These new guarantees of minimum death benefit (GMDB) typically include one or more of the following features:

- **Reset:** the benefit is the periodically automatically adjusted account balance plus the sum of premium deposits less withdrawals since the last reset date.
- **Roll-up:** the death benefit is the larger of the account balance on the date of the death and the accumulation of premium deposits less partial withdrawals accumulated at a specified interest rate. There may be an upper bound on the benefits.
- **Ratchet:** the benefit is the same as a reset benefit except that it is now re-determined at the end of a pre-set number of years (typically annually) to be the larger of the current account balance and the account balance on the prior ratchet date. In other words, the benefit is not allowed to decrease.

In addition to these new benefits, more aggressive equity-indexed annuities (EIA) have been the fastest growing annuity products since 1995. The EIA is an annuity in which the policyholder's rate of return is determined as a defined share in the appreciation of an outside index, e.g., S&P 500 in U.S., with a guaranteed minimum return. While these new insurance contracts provide policyholders with downside risk protection, insurers are faced with the challenging task of hedging against the unfavorable movement of the equity values. Recent research has applied no arbitrage pricing theory from finance to calculate the values of the embedded options in insurance contracts, see for example Brennan and Schwartz (1976,[7]), Boyle Schwartz (1977, [6]), Aase and Persson (1994, [1]), Boyle and Hardy (1997, [5]), Bacinello and Persson (2002, [3]), and Pelsser (2002, [25]).

Although much of the emphasis of the literature has been on pricing, hedging the embedded options in these new insurance policies is of crucial importance for risk management. *Hedging options embedded in the variable annuities is particularly difficult since maturities of insurance contracts are long (typically longer than 10 years), transaction costs limit the rebalancing frequency of hedging portfolios, and liquidity restricts the choice of possible hedging instruments.* In addition, due to the sensitivity of the benefits to the tail distributions, a simple Black-Scholes model for the underlying equity price is not adequate for the observed fat tails of the equity return distributions; a model incorporating jump risk and/or volatility risk is necessary. The long maturity of insurance contracts also makes interest risk modeling necessary; we discuss hedging strategy computation under the interest risk in a separate

paper [10]. Finally there are additional risks such as basis risk, mortality risk, and surrender risk; these risks are not addressed here.

In this paper, we focus on computing and evaluating the effectiveness of hedging strategies using either the underlying (futures) or standard options as hedging instruments, in order to control the market risk embedded in the insurance contracts under different models, including models that are suitable for fat tails of return distributions. We note that this is different from the emphasis of the current literature which focuses on the fair value computation of the insurance contracts. However, the hedging strategy computation does automatically provide an estimate of the hedging cost. To focus on modeling and the hedging strategy computation, we assume in this paper that the underlying account is linked directly to a market index, e.g., S&P 500; the issue of basis risk is not addressed here.

For a derivative contract, the most frequently used hedging strategy in the financial industry is delta hedging. In a delta hedging strategy, the trading position of the underlying is computed from the sensitivity (first order derivative) of a (risk adjusted) option value to the underlying. Hence the delta hedging strategy is determined from the underlying price dynamics under a risk adjusted measure (which is unique under a Black-Scholes model but not under a jump diffusion model).

Unfortunately delta hedging is only instantaneous and continuous rebalancing is clearly impossible in practice. Under the assumption that a hedging portfolio can only be rebalanced at discrete times, the market is incomplete and the intrinsic risk of an option cannot be completely eliminated. For hedging effectiveness and risk management analysis, one is interested in the real world performance: a hedging strategy computed under a risk adjusted measure may not be optimal under the real world price dynamics when the market is incomplete due to discrete hedging, insufficient hedging instruments, jump risk, and volatility risk.

Given that it is impossible to completely eliminate the risk in an option, can we determine a hedging strategy to minimize a chosen measure of risk under a real world price dynamics? A risk minimization hedging method computes an optimal hedging strategy to minimize a specific measure of risk under a real world price model, e.g., [16, 26, 29, 30, 22, 18, 19]. Given a statistical model and assuming trading is done at a discrete set of times, a local risk minimization hedging strategy is computed at each trading time to minimize the variance of the difference between the liquidating portfolio value and the value of the portfolio for the next hedging period.

In order for a discrete hedging strategy using the underlying to be effective, hedging portfolios need to be rebalanced frequently. This can be problematic for hedging the embedded option in an insurance contract due to the transaction costs and long maturity of insurance contracts. In addition, for variable annuities with guarantees, the long maturity and sensitivity of the benefits to the tails of the account value distribution makes the choice of a suitable model difficult but crucial. For example, in order to accurately model the tails of the underlying distribution, jump risk and volatility risk may need to be considered. The presence of such additional risks deteriorates the effectiveness of hedging using the underlying. Since option markets have become increasingly more liquid, standard options are often used as hedging instruments for a complex option. In this paper, we also compare discrete risk minimization hedging using the underlying with that of using liquid standard options.

Using standard options as hedging instruments adds additional complexity in equity return modeling; in this situation it is imperative that stochastic implied volatilities be adequately

modeled. When quantifying and minimizing risk, a statistical model for changes of the underlying and changes of hedging instrument prices between each rebalancing time is needed. In particular, if standard options are used as hedging instruments, it is important to adequately model the stochastic implied volatilities. Unfortunately, estimating a model which is capable of prescribing evolution of implied volatilities for a long time horizon is a challenging task. There are two possible approaches for model estimation. One approach is based on historical prices and the other is based on market calibration. Historical model estimation is faced with a complex decision of the choice of data and statistical methods which lead to stable estimation. Market calibration starts with calibrating a model for the underlying risk from the current liquid option prices. Theoretical values of the option to be hedged (as well as standard options if used as hedging instruments) are computed based on the calibrated model and then either a delta hedging strategy or option hedging strategy is determined based on this risk adjusted valuation. A difficulty of this calibration approach is that the current liquid options have short maturities and the model calibrated from current prices is unlikely to be able to prescribe the option price dynamics for the long time horizon of the insurance contracts; thus, the hedging strategy determined based on market calibration is exposed to significant model risks, particularly when the hedging portfolio needs to be rebalanced frequently. In addition, when the underlying price model is calibrated to the current market option prices, the calibrated model describes price dynamics under a *risk adjusted* measure, not the *objective* probability measure. In an incomplete market, it is necessary to quantify and evaluate hedging effectiveness under the real world price dynamics. A hedging strategy which is determined based on a risk adjusted valuation is typically not optimal when evaluated under the real world price dynamics; the more difficult the option is to hedge, the less optimal will be the hedging strategy determined under a risk adjusted measure.

When standard options are used as hedging instruments, it is crucial that stochastic implied volatilities (which yield the hedging instrument prices) are suitably modeled. This is difficult to accomplish by calibrating a model for the underlying and determining the option prices based on the calibrated model. The increasing autonomy of the option market also leads to a question of whether it is possible to accurately model market standard option evolution in this fashion. We propose to compute the risk minimization hedging using standard options by jointly modeling the underlying price dynamics and the Black-Scholes at-the-money implied volatility explicitly. This modeling approach has many advantages. Firstly, implied volatilities are readily observable and hedging positions can be adjusted according to these observable variables. Secondly, the standard option hedging instruments can be easily and accurately priced. Thirdly, calibration to the current option market is automatically done by setting the implied volatilities to the market implied volatilities. Finally, the statistical information of the past implied volatility evolution can be incorporated in the model.

The main contribution of this paper is hedging effectiveness evaluation for embedded options in variable annuity guarantees, under models for jump and volatility risks. We compare delta hedging, risk minimization hedging using the underlying, and risk minimization hedging using standard options. We illustrate that the risk minimization discrete hedging using the underlying can be significantly more effective than popular delta hedging, particularly when rebalancing is infrequent. We show that, under a jump risk, risk minimization hedging using standard options is more effective in risk reduction than discrete hedging using the

underlying. We propose to model the implied volatilities directly when computing a hedging strategy using standard options as hedging instruments. We evaluate hedging effectiveness under a joint dynamics of the underlying and at-the-money implied volatility and illustrate that risk minimization hedging using standard options can be more effective than using the underlying under both jump and volatility risks.

We consider GMDB with a ratchet feature, the most difficult benefit to hedge. We assume that the mortality risk can be diversified and thus consider here the hedging problem with a fixed maturity. We first describe, in §2, computation of dynamic discrete risk minimization hedging strategies, using either the underlying or standard options, for the options embedded in the GMDB with a ratchet feature. We show that the risk minimization hedging using the underlying as the hedging instrument outperforms the delta hedging strategy in §3. We then compare the hedging effectiveness of using the underlying with that of using liquid options in §4 and we illustrate that, in the absence of volatility risks, risk minimization hedging using liquid option instruments can be significantly more effective than hedging using the underlying. Finally, in §5, we compute a risk minimization hedging when at-the-money implied volatility follows an Ornstein-Uhlenbeck process and compare its hedging effectiveness with that of using the underlying under both instantaneous and implied volatility risk. We illustrate that, when implied volatility risk is suitably modeled, risk minimization hedging using options can potentially be more robust and provide better risk reduction than hedging using the underlying.

2 Hedging GMDB in Variable Annuities

For risk management of variable annuities, one is interested in risk quantification and risk reduction under a statistical model for the real price dynamics. In this paper we are interested in computing an optimal hedging strategy, under an assumed statistical market price model, for a variable annuity which has a minimum death guarantee benefit with a ratchet feature. We first describe a delta hedging computation, based on the risk adjusted fair value computation, and risk minimization hedging based on an incomplete market assumption.

Assume that the benefit depends on the value of an equity linked insurance account with anniversaries at

$$0 = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_{\tilde{M}-1} < \tilde{t}_{\tilde{M}} = T.$$

Assume that the account value at time \tilde{t}_i is $S_{\tilde{t}_i}$. Since we assume that the mortality risk can be diversified, we consider here the hedging problem with a fixed maturity T . The option embedded in a variable annuity with a ratchet GMDB is a path dependent lookback option. The payoff of GMDB with a ratchet feature is

$$\begin{aligned} \text{payoff} &= \max(\max\{S_{\tilde{t}_0}, S_{\tilde{t}_1}, S_{\tilde{t}_2}, \cdots, S_{\tilde{t}_{\tilde{M}-1}}\}, S_T) \\ &= S_T + \max(\max\{S_{\tilde{t}_0}, S_{\tilde{t}_1}, S_{\tilde{t}_2}, \cdots, S_{\tilde{t}_{\tilde{M}-1}}\} - S_T, 0) \\ &= S_T + \max(H_{\tilde{t}_{\tilde{M}}} - S_T, 0) \end{aligned} \tag{1}$$

where $H_{\tilde{t}_i}$ denotes the running max account value up to the anniversary time \tilde{t}_{i-1} , i.e.,

$$H_{\tilde{t}_i} = \max\{S_{\tilde{t}_0}, S_{\tilde{t}_1}, S_{\tilde{t}_2}, \cdots, S_{\tilde{t}_{i-1}}\} = \max(H_{\tilde{t}_{i-1}}, S_{\tilde{t}_{i-1}}).$$

Note that $\tilde{t}_{\tilde{M}} = T$. Let Π_T denote the second term in the payoff in (1), i.e., $\Pi_T = \max(H_T - S_T, 0)$ (recall that $H_T = H_{\tilde{t}_{\tilde{M}}}$). The benefit of GMDB with a ratchet feature equals the account value plus a lookback put option with payoff Π_T .

Most hedging strategies used in the financial industry are computed based on a risk adjusted option valuation. In particular, at any time t and underlying price S_t , a delta hedging strategy is determined by first computing the option value $V(S_t, t)$ under a risk adjusted measure and the hedging position in the underlying is given by $\frac{\partial V(S_t, t)}{\partial S}$.

The fair value of a European path dependent option can be computed given a model for the underlying price dynamics under a risk adjusted measure. Consider a European path dependent option whose single payoff at the fixed time $T = T_{\tilde{M}}$ depends on the price path of a single underlying asset S_t :

$$\{S_{\tilde{t}_0}, S_{\tilde{t}_1}, \dots, S_{\tilde{t}_{\tilde{M}}}\}$$

Assume more generally that the option payoff at the maturity T is given by

$$\phi(H_T, S_T)$$

where

$$H_{\tilde{t}_i} = g_i(H_{\tilde{t}_{i-1}}, S_{\tilde{t}_{i-1}}, S_{\tilde{t}_i}), \quad i = 1, \dots, \tilde{M},$$

and $H_{\tilde{t}_0} = S_{\tilde{t}_0} = S_0$. For a lookback option, $H_{\tilde{t}_i} = \max(H_{\tilde{t}_{i-1}}, S_{\tilde{t}_{i-1}})$ and $\phi(H_T, S_T) = \max(H_T - S_T, 0)$.

The time \tilde{t}_i value of such a path dependent option is

$$V(S_{\tilde{t}_i}, H_{\tilde{t}_i}, \tilde{t}_i) = \mathbf{E}(\phi(H_T, S_T) | \tilde{t}_i), \quad i < \tilde{M}$$

where $\mathbf{E}(\cdot)$ denotes the expectation under a risk adjusted measure. Using the recursive property of conditional expectations, for any random variable Z

$$\mathbf{E}(Z | \tilde{t}_i) = \mathbf{E}(\mathbf{E}(Z | \tilde{t}_{i+j}) | \tilde{t}_i)$$

where $0 < j \leq \tilde{M} - i$. The value $V(S_{\tilde{t}_j}, H_{\tilde{t}_j}, \tilde{t}_j)$ can be computed recursively as follows: given $V(S_{\tilde{t}_{\tilde{M}}}, H_{\tilde{t}_{\tilde{M}}}, \tilde{t}_{\tilde{M}}) = \phi(H_T, S_T)$, for $j = \tilde{M} - 1 : -1 : 0$,

$$\begin{aligned} V(S_{\tilde{t}_j}, H_{\tilde{t}_j}, \tilde{t}_j) &= e^{-r(T-\tilde{t}_j)} \mathbf{E} \left(V(S_{\tilde{t}_{\tilde{M}}}, H_{\tilde{t}_{\tilde{M}}}, \tilde{t}_{\tilde{M}}) | \tilde{t}_j \right) \\ &= e^{-r(T-\tilde{t}_j)} \mathbf{E} \left(\mathbf{E}(V(S_{\tilde{t}_{\tilde{M}}}, H_{\tilde{t}_{\tilde{M}}}, \tilde{t}_{\tilde{M}}) | \tilde{t}_{j+1}) | \tilde{t}_j \right) \\ &= e^{-r(\tilde{t}_{j+1}-\tilde{t}_j)} \mathbf{E} \left(V(S_{\tilde{t}_{j+1}}, H_{\tilde{t}_{j+1}}, \tilde{t}_{j+1}) | \tilde{t}_j \right) \end{aligned}$$

Assume that the transitional density function of S_t from \tilde{t}_j to \tilde{t}_{j+1} , under a risk adjusted measure, is $p_{j,j+1}(x)$. Then

$$\mathbf{E}(V(S_{\tilde{t}_{j+1}}, H_{\tilde{t}_{j+1}}, \tilde{t}_{j+1}) | \tilde{t}_j) = \int_0^\infty V(x, H_{\tilde{t}_{j+1}}, \tilde{t}_{j+1}) p_{j,j+1}(x) dx \quad (2)$$

Note that the only random component of $V(S_{\tilde{t}_{j+1}}, H_{\tilde{t}_{j+1}}, \tilde{t}_{j+1})$ comes from $S_{\tilde{t}_{j+1}}$ since $H_{\tilde{t}_{j+1}} = g_j(H_{\tilde{t}_j}, S_{\tilde{t}_j}, S_{\tilde{t}_{j+1}})$.

In practice a hedging portfolio cannot be rebalanced continuously; the rebalancing frequency may be further limited for hedging variable annuities due to the unusually long maturities. Given that a hedging portfolio can be rebalanced only at a limited discrete set of times, what is the optimal hedging strategy? How can this strategy be determined? This is a question of option hedging in an incomplete market and the optimal hedging strategy can be computed to minimize a measure of the hedge risk. In most of the present literature for pricing and hedging in an incomplete market, a hedging strategy is computed to minimize either the quadratic local risk or the quadratic total risk, e.g., [16, 18, 19, 22, 26, 29, 30]. As an alternative to quadratic risk measure, a piecewise linear measure is used in [12, 11, 24] to compute discrete local and total risk minimization hedging strategies. The total risk minimization hedging strategy is computed by solving a stochastic optimization problem in which the risk of failure to match the payoff through the dynamic hedging strategy is minimized, see e.g., [16, 18, 19, 22, 26, 29, 30]. In [16], Föllmer and Schweizer first consider the *local risk* measure. A mean variance *total risk* minimizing strategy is first considered by Schweizer [28] and Duffie [14]. The quadratic criteria for risk minimization in the framework of discrete hedging have been studied in [4, 16, 22, 26, 29].

Assume that $T > 0$ is a hedging time horizon and assume that there are M trading opportunities (it is assumed that, for simplicity, the anniversaries of the variable annuity form a subset of rebalancing times) at

$$0 = t_0 < t_1 < \dots < t_{M-1} < t_M = T.$$

Suppose that we want to hedge a European option whose payoff at the maturity T is denoted by Π_T . Suppose also that the financial market is modeled by a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with filtration $(\mathcal{F}_k)_{k=0,1,\dots,M}$, and the discounted underlying asset price follows a square-integrable process. As discussed before, we compute optimal hedging strategies under a statistical model for the real world price dynamics. This is essential because we want to quantify and reduce the risk of our hedging strategies and this should be done under the real probability measure. Denote by P_k the value of the hedging portfolio at time t_k and by C_k the cumulative cost of the hedging strategy up to time t_k ; this includes the initial cost for setting up the hedging portfolio and the additional costs for rebalancing it at the hedging times t_0, \dots, t_k .

When trading is limited to a set of discrete times using a finite number of instruments, it is impossible to eliminate the intrinsic option risk. Based on the risk minimization principle, there are two main quadratic approaches for choosing an optimal hedging strategy. One possibility is to control the total risk by minimizing $\mathbf{E}((\Pi_T - P_M)^2)$, where $\mathbf{E}(\cdot)$ denotes the expected value with respect to the objective probability measure \mathcal{P} and P_M is portfolio value at t_M associated with a self financed trading strategy; thus P_M is the initial portfolio value P_0 plus the cumulative gain. This is the total risk minimization criterion. A total risk minimizing strategy exists under the additional assumption that the discounted underlying asset price has a bounded mean-variance tradeoff. In this case, the strategy is given by an analytic formula. The existence and the uniqueness of a total risk minimizing strategy have been extensively studied in [29].

Computing an optimal total risk minimization requires solving a dynamic stochastic programming problem which is, in general, computationally very difficult. In addition, a total risk minimization strategy is weakly dynamically consistent [23]. Local risk minimization

hedging, on the other hand, assumes that the hedging portfolio value P_M equals the liability Π_T and computes the hedging strategy to minimize each incremental cost $\mathbf{E}((C_{k+1} - C_k)^2 | \mathcal{F}_k)$ for $k = M - 1, M - 2, \dots, 0$; here C_k denotes the cumulative cost of the hedging strategy. Compared to the total risk minimization hedging strategy, computation of a local risk minimization strategy is simpler. In addition, a local risk minimization hedging strategy is strongly dynamically consistent [23]. The same assumption that the discounted underlying asset price has a bounded mean-variance tradeoff is sufficient for the existence of an explicit local risk minimizing strategy (see [26]). This strategy is no longer self-financing, but it is mean-self-financing, i.e., the cumulative cost process is a martingale. In general, the initial costs for the local risk minimizing and total risk minimizing strategies are different. However, as Schäl noticed in [26], the initial costs agree in the case when the discounted underlying asset price has a deterministic mean-variance tradeoff.

In this paper, we focus on the local risk minimization hedging. In the current literature on pricing and hedging in an incomplete market, the hedging instruments are typically the underlying assets. In this paper, we also consider standard options as hedging instruments; thus, the traded instruments may exist for a sub-period of the entire hedging horizon.

Assume that at time t_k , n risky hedging instruments with values $U_k \in \mathfrak{R}^n$ can be traded; we assume that U_k are normalized by riskless bonds. At time t_{k+1} , these instruments have values $U_k(t_{k+1})$; we have omitted dependence on the underlying value for notational simplicity.

A hedging strategy is a sequence of the trading positions $\{(\xi_k, \eta_k), k = 0, 1, \dots, M\}$ in the risky hedging instruments U_k and riskless bond respectively. The hedging position (ξ_k, η_k) at time t_k is liquidated at time t_{k+1} ; at which time a new hedging position (ξ_{k+1}, η_{k+1}) is formed.

The *initial cost* of the trading portfolio is

$$C_0 = P_0 = U_0 \cdot \xi_0 + \eta_0$$

where U_0 is a n -row vector of the initial hedging instrument values. Similarly, the value of the trading portfolio P at any time t_k is

$$P_k = U_k \cdot \xi_k + \eta_k$$

The *cumulative gain* of the trading strategy at time t_k is

$$G_k = \sum_{j=0}^{k-1} (U_j(t_{j+1}) - U_j) \cdot \xi_j$$

The *cumulative cost* of the trading strategy at t_k is

$$C_k = P_k - G_k$$

A trading strategy is *self-financed* if

$$C_{k+1} - C_k = U_{k+1} \cdot \xi_{k+1} - U_k(t_{k+1}) \cdot \xi_k + \eta_{k+1} - \eta_k = 0, \quad k = 0, 1, \dots, M - 1$$

When a market is incomplete, a risk minimization hedging strategy is fundamentally different from hedging strategies based on risk adjusted option values, e.g., delta hedging

and semi-static hedging which is based on the principal of replicating a complex option with standard options [8, 9]. Consider hedging a lookback option using the underlying as an example. Delta hedging, which is based on continuously rebalancing, is computed from the sensitivity of risk adjusted option values to the underlying. Thus delta hedging requires a price dynamics under a risk adjusted measure which is typically obtained by calibration to the standard option market. For hedging lookback options embedded in insurance benefits, there are potentially serious difficulties due to the requirement of market calibration to the option market. Firstly, for risk management and hedging purposes, real world price dynamics models are needed. Such models are essential to accurately quantify risks, for example, in emerging cost analysis and calculating reserves. They are equally essential for hedging strategy computation and analysis. A model for the market price evolution (in contrast to a risk adjusted price dynamics) is needed, particularly when hedging is difficult due to market incompleteness. Secondly, any parsimonious model inevitably leads to calibration errors. Thirdly, if standard options are used as hedging instruments, it is crucial for a model to calibrate to the forward implied volatilities (due to the long maturity). Unfortunately, this is difficult to accomplish. For risk minimization hedging, one typically assumes statistical models for the *market* value of the underlying and the *market* values of the hedging instruments at trading times; risk adjusted liability values are not needed. It may be possible to estimate such a model from the historical observable prices of the underlying and hedging instruments. We further discuss this possibility of computing risk minimization hedging strategies using standard options by explicitly model implied volatilities in §5. Given a statistical model for both the underlying asset and hedging instruments, a risk minimization hedging strategy is computed to minimize hedge risk. Finally a risk minimization hedging achieves *optimality* under a chosen risk measure. For example, a local risk minimization hedging strategy is optimal with respect to the specified trading requirement and the quadratic increment cost. Delta hedging and semi-static hedging are typically not optimal with respect to the discrete trading specification.

3 Hedging Using the Underlying Asset

In practice, hedging options embedded in the variable annuities is a risk management problem in an incomplete market. In our discussion so far, we have emphasized the theoretical difference between a risk minimization hedging strategy based on an incomplete market assumption and a hedging strategy based on risk adjusted valuation. In this section, we evaluate, computationally, the effectiveness of delta hedging and risk minimization hedging using the underlying asset for a GMDB with a ratchet feature. As discussed in §2, the payoff of the option embedded in the variable annuity with a ratchet benefit can be expressed as $\Pi_T = \max(H_T - S_T, 0)$ where the path dependent (ratchet) value $H_T = H_{\tilde{t}_M}$ and

$$H_{\tilde{t}_i} = \max\{S_{\tilde{t}_0}, S_{\tilde{t}_1}, S_{\tilde{t}_2}, \dots, S_{\tilde{t}_{i-1}}\} = \max(H_{\tilde{t}_{i-1}}, S_{\tilde{t}_{i-1}}), \quad 1 \leq i \leq \tilde{M}.$$

We evaluate hedging performance in terms of the *total risk* and *total cost* at the maturity T for the entire hedging horizon from $t = 0$ to T . Let P_M^{sf} denote the time T self-financed hedging portfolio value corresponding to a hedging strategy. For example, if $\{(\xi_k, \eta_k), k = 0, \dots, M-1\}$ represents the optimal holdings computed from the risk minimization hedging

	annually			monthly			biweekly		
	Π_T	RM	delta	Π_T	RM	delta	Π_T	RM	delta
C_0	0	18.1	20	0	19.8	20	0	19.9	20
$\text{mean}(\Pi_T - P_M^{\text{sf}})$	24.4	0.387	-8.45	24	0.0638	-0.762	24.9	0.2	-0.186
$\text{std}(\Pi_T - P_M^{\text{sf}})$	36.9	24	36.2	36.3	7.96	11.4	37.7	5.62	7.98
$\sqrt{\mathbf{E}((\Pi_T - P_M^{\text{sf}})^2)}$	44.3	24	37.2	43.6	7.96	11.4	45.2	5.62	7.99
$\text{mean}(\text{total cost})$	24.4	30.2	24.5	24	32.7	32.2	24.9	33	32.8
$\text{VaR}(95\%)$	95.2	40.7	45	94.8	12.5	16.6	97.6	9	12.1
$\text{CVaR}(95\%)$	136	63.9	73.9	134	19.5	26.4	140	13.8	18.9

$$\#\text{scenarios} = 20000 \quad \sigma = 0.2 \quad r = 0.05 \quad \mu = 0.1$$

Table 1: Delta Hedging versus RM Hedging Using the Underlying (under a BS Model)

computation, then

$$P_M^{\text{sf}} = P_0 + G_M = U_0 \cdot \xi_0 + \eta_0 + \sum_{j=0}^{M-1} (U_{j+1}(t_{j+1}) - U_j) \cdot \xi_j$$

The total risk $\Pi_T - P_M^{\text{sf}}$ measures the amount of money that the hedging strategy is short of meeting the liability at maturity T . The total cost $\Pi_T - G_M$ is the time T lookback option payoff plus the cumulative loss, $-G_M$, of the hedging strategy.

We compare the expected total cost and the first two moments of the total risk. Information about liability (payoff) at maturity T is listed under the column Π_T for comparison; it corresponds to no hedging. Results obtained using the underlying for biweekly hedging, monthly hedging, and annually hedging are given under the column *biweekly*, *monthly*, and *annually* respectively.

In addition, for a given confidence level, we report the Value-at-Risk (VaR) and the conditional Value-at-Risk (CVaR) of the total risk at the maturity T . For example, $\text{VaR}(95\%)$ is the minimum amount of money, that the self-financed hedging portfolio P_M^{sf} is short of meeting the liability Π_T , with 5% probability; it is the cost at T , for the hedger, that is exceeded with 5% probability. The value $\text{CVaR}(95\%)$ is, with a 95% confidence level, the conditional expected cost that a hedging strategy requires at T , conditional on the cost is greater than $\text{VaR}(95\%)$.

For computational results in this paper, analytic formula for the standard option prices and transitional density functions under a Black-Scholes model or a Merton's jump diffusion model are used to compute delta hedging strategies and risk minimization hedging strategies. For delta hedging, the analytic formula for the transitional density function is used to compute values of lookback options based on (2). For risk minimization hedging strategies, analytic formula for the transitional density function and standard option values are used to compute optimal hedging positions on a finite grid; the hedging strategies corresponding to

independent simulations are computed using spline interpolations from the hedging positions on the grid.

Although a Black-Scholes model is not adequate for modeling a return distribution with a fat tail, we first assume a Black-Scholes model and consider using the underlying as the hedging instrument; this is interesting since the market incompleteness in this case comes entirely from our inability to hedge continuously. Under a Black-Scholes model, the real world underlying price S_t is modeled as a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu \cdot dt + \sigma \cdot dW_t \quad (3)$$

where μ is the expected rate of asset return, σ is the volatility, and W_t is a standard Brownian motion. Assume that the instantaneous risk free rate is $r > 0$. Note that positions in a delta hedging strategy are independent of the actual expected rate of return μ .

In our computational results, it is assumed that the initial account value $S_0 = 100$. Table 1 compares hedging performance of delta hedging with that of the risk minimization hedging using the underlying. Table 1 illustrates that, under a Black-Scholes model,

- The effectiveness of both risk minimization hedging using the underlying and delta hedging improves significantly as the portfolio is rebalanced more frequently. In particular, the extreme risk of biweekly rebalancing, measured in VaR and CVaR, is noticeably smaller than that of the monthly rebalancing.
- For risk minimization hedging, the initial cost C_0 decreases as the rebalancing frequency decreases. The initial cost of delta hedging, on the other hand, does not change with the frequency of rebalancing; it equals the unique initial lookback option price. The initial hedging cost of the risk minimization hedging is smaller compared to that of the delta hedging.
- Compared to delta hedging, risk minimization hedging using the underlying is significantly more effective in reducing risk, measured either in standard deviation, VaR, or CVaR.
- For both delta hedging and risk minimization hedging, the expected total cost increases as the hedging portfolio is rebalanced more frequently. For annual rebalancing, the average total cost of the risk minimization hedging is larger than that of delta hedging. For monthly and biweekly rebalancing, the average total costs of risk minimization are close to that of delta hedging; however the risk from the risk minimization hedging is significantly smaller than that from delta hedging.

Table 1 clearly indicates that, under a Black-Scholes model, risk minimization hedging using the underlying is better than delta hedging since the former minimizes risk under the assumption that hedging portfolio is rebalanced at the specified times.

Unfortunately, a Black-Scholes model (3) is not appropriate for the risk management of variable annuities since the embedded options are sensitive to tails of the underlying distribution; we consider the Black-Scholes model here to illustrate the difference between risk minimization hedging using the underlying and delta hedging when the incompleteness comes

from discrete rebalancing, not additional risks such as jump or volatility. Determining a suitable model is usually a challenging task; it is even more difficult for variable annuities due to the long maturities and sensitivities of the payoff to the extreme price movements. The Canadian Insurance Association on Segregated Funds (SFTF) recommends that model calibration be adjusted to give a sufficiently accurate fit in the left tail of the distribution [17]. A Black-Scholes model certainly seems a poor choice in this regard. Next we consider Merton's jump diffusion model which is more suitable to model fat tails of return distributions.

Let us assume that the real world price dynamics is modeled by a jump diffusion model with a constant volatility, i.e.,

$$\frac{dS_t}{S_t} = (\mu - q - \kappa\lambda) \cdot dt + \sigma \cdot dW_t + (J - 1) \cdot d\pi_t \quad (4)$$

where r is the risk free rate, q is the continuous dividend yield, σ is a constant volatility. In addition, π_t is a Poisson counting process, $\lambda > 0$ is the jump intensity, and J is a random variable of jump amplitude with $\kappa = \mathbf{E}(J - 1)$. For simplicity, $\log J$ is assumed here to be normally distributed with a constant mean μ_J and variance σ_J^2 ; thus $\mathbf{E}(J) = e^{\mu_J + \frac{1}{2}\sigma_J^2}$. Under this assumption, we compute the transitional density function for an underlying price process (4) using an analytic formula [21].

While a risk minimization hedging strategy is computed directly from the original process (4), delta hedging is computed from a risk adjusted value of the option. When jump risk exists, no arbitrage option value is no longer unique. Thus a delta hedging strategy and its associated hedging cost depends on how the risk is adjusted from the real world price process (4).

From the utility-based equilibrium theory, assuming $\gamma \leq 1$ is the risk aversion parameter, a risk-adjusted price process corresponding to (4) is

$$\frac{dS_t}{S_t} = (r - q - \kappa^Q\lambda^Q) \cdot dt + \sigma \cdot dW_t^Q + (J^Q - 1) \cdot d\pi_t^Q \quad (5)$$

where W_t^Q is a standard Brownian motion and π_t^Q is a Poisson counting process and $\log J^Q$ is normally distributed with

$$\begin{aligned} \sigma_J^Q &= \sigma_J \\ \mu_J^Q &= \mu_J - (1 - \gamma)\sigma_J^2 \\ \lambda^Q &= \lambda e^{-(1-\gamma)(\mu_J + \frac{1}{2}\sigma_J^2)} \end{aligned}$$

where $\kappa^Q = \mathbf{E}(J^Q - 1)$. As an investor becomes more risk averse, the jump frequency and the expected magnitude of jumpsize are adjusted to larger quantities. In other words, the options are priced according to a larger (combined diffusion and jump) risk.

Our objective is to compute a hedging strategy under the assumption that a real world price dynamics (4) is given. To determine a delta hedge one needs to first choose a risk aversion parameter and then evaluate the risk adjusted lookback option values $V(S, t)$. Choosing a risk aversion parameter is ad-hoc in nature and much effort has been spent by economists to find appropriate risk aversion parameters. Alternatively, one can calibrate with the current market standard option prices to determine a risk adjusted process from these prices.

	annually			monthly			biweekly		
	Π_T	RM	delta	Π_T	RM	delta	Π_T	RM	delta
C_0	0	22.7	20.8	0	24.3	20.8	0	24.4	20.8
$\text{mean}(\Pi_T - P_M^{\text{st}})$	27.9	0.427	-7.77	27.4	0.042	-0.746	28.2	0.0777	-0.274
$\text{std}(\Pi_T - P_M^{\text{st}})$	43.6	28.5	44.9	43.5	20.8	27.4	45.3	19.7	25.7
$\sqrt{\mathbf{E}((\Pi_T - P_M^{\text{st}})^2)}$	51.7	28.5	45.5	51.4	20.8	27.4	53.4	19.7	25.7
$\text{mean}(\text{total cost})$	27.9	37.8	26.6	27.4	40.1	33.6	28.2	40.2	34.1
$\text{VaR}(95\%)$	116	49.9	62.4	115	35.3	44.9	120	33.7	42.6
$\text{CVaR}(95\%)$	162	77.8	97.8	162	54.9	68.9	169	51.4	64.5

MJD Model: #scenarios = 20000 $\sigma = 0.15$ $r = 0.05$ $\mu = 0.1$
risk aversion $\gamma = 1$ $\lambda = 0.2$ $\mu_J = -0.3$ $\sigma_J = 0.15$

Table 2: Delta Hedging versus RM Hedging Using the Underlying (under a MJD Model)

When the risk aversion parameter $\gamma = 1$, options are priced risk neutrally; the investors price the asset under the risk free interest rate. In our example, we assume that the jump frequency $\lambda = 12\%$, i.e., approximately a jump once every 8.3 years on average, a mean of -30% and a volatility of 15% for the logarithm of the jumpsize. Table 2 compares, under the risk neutral assumption, the effectiveness of delta hedging with the risk minimization hedging using the underlying. From Table 2, it can be observed that

- Both hedging strategies are more effective as one rebalances more frequently. However, due to jump risk, the improvement of more frequent rebalancing is less dramatic compared to that under the Black-Scholes model. Specifically, compared to annual rebalancing, monthly or biweekly rebalancing, under a Merton's jump diffusion model, improves the hedging performance only slightly further.
- Compared to delta hedging, risk minimization hedging using the underlying is significantly more effective in reducing risk under a jump diffusion model. Not surprisingly, risk minimization hedging incurs a larger average cost.
- For annual rebalancing, delta hedging can perform worse than not hedging at all as measured by the standard deviation of hedging error.
- The monthly risk minimization hedging using the underlying significantly reduces the risk, relative to an un-hedged position.

If delta hedging is determined from a risk adjusted model which is calibrated from the option market and has a risk aversion parameter $\gamma < 1$, the option is not priced risk neutrally. Thus the delta hedging strategy depends on how the option market or the hedger adjusts the jump risk; risk minimization hedging, on the other hand, is typically determined under the objective probability measure.

4 Hedging Using Standard Options

With the rapid growth of derivative markets, it is now a common practice in the financial industry to use liquid vanilla options to hedge an exotic option. The S&P 500 index options traded on the exchange, for example, are natural instruments for hedging variable annuities with guarantees linked directly to or correlated closely to the S&P 500 index.

In this section, we assume that there is no transaction cost and compare risk minimization hedging using the underlying with risk minimization hedging using standard options under the Black-Scholes model as well as Merton's jump model; we discuss volatility risks in the next section.

To illustrate, we assume that hedging instruments are standard options with one year maturity; more liquid options with a shorter maturity can similarly be used. We consider hedging using 6 options, 3 calls with strikes $[100\%, 110\%, 120\%] \cdot S_t$ and 3 puts with strikes $[80\%, 90\%, 100\%] \cdot S_t$, and hedging using 2 options (one call and one put with strike equal the current underlying value).

Table 3 compares, under a Black-Scholes model, monthly and annually rebalancing using the underlying with annual rebalancing using 6 options and 2 options respectively. Table 4 presents a similar comparison but under a Merton's jump diffusion model (4).

Table 3 and Table 4 illustrate that, compared to hedging using the underlying, option hedging leads to better risk reduction. More specifically,

- Under a Black-Scholes model, monthly rebalancing using the underlying and annual rebalancing using two options have similar initial costs and average total costs. However, better risk reduction (measured either in standard deviation, VaR, or CVaR of the total risk) is achieved using two options compared to monthly hedging using the underlying. In addition, compared to hedging using 2 options, hedging using 6 options incurs slightly larger initial and average total costs but achieves significantly better risk reduction.
- Under a Merton's jump model, hedging using 6 options incurs slightly more initial and average total costs than hedging using 2 options. However, hedging using 6 options achieves better risk reduction than hedging using 2 options: contrast a CVaR(95%) value of 4.55 using 6 options with a CVaR(95%) value of 15.2 using 2 options. In addition, hedging using 2 options is much better than monthly rebalancing using the underlying in risk reduction; the initial cost and average total cost of option hedging is larger than that of hedging using the underlying. This result suggests that, under jump risk, option hedging leads to better risk reduction even though it may incur slightly larger costs.

5 Hedging Under Volatility Risks

Computational results in §4 clearly suggest that standard options have potential use in hedging risks embedded in variable annuities. Specifically, Table 3 and 4 suggest that, hedging using standard options produces significantly greater risk reduction than using the underlying, particularly under a jump diffusion model.

	Π_T	monthly underlying	annually underlying	annually option(6)	annually option(2)
C_0	0	17.3	13.9	17.7	17.4
$\text{mean}(\Pi_T - P_M^{\text{sf}})$	11.4	0.0766	0.326	0.00956	-0.0927
$\text{std}(\Pi_T - P_M^{\text{sf}})$	20.9	5.46	15.9	1.6	4.61
$\sqrt{\mathbf{E}((\Pi_T - P_M^{\text{sf}})^2)}$	23.8	5.47	15.9	1.6	4.61
$\text{mean}(\text{total cost})$	11.4	23.5	19.1	23.9	23.4
VaR (95%)	55.4	8.91	28.5	2.42	7.02
CVaR (95%)	77.3	13	43.2	3.81	11.9

BS Model: #scenarios = 20000 $\sigma = 0.15$ $r = 0.03$ $q = 0$ $\mu = 0.1$

Table 3: RM Hedging Using the Underlying versus RM hedging Using Options (a BS Model)

	Π_T	monthly underlying	annually underlying	annually option(6)	annually option(2)
C_0	0	22.8	19.5	24.5	24
$\text{mean}(\Pi_T - P_M^{\text{sf}})$	16.3	0.115	0.49	0.0183	-0.0976
$\text{std}(\Pi_T - P_M^{\text{sf}})$	29.1	12.9	21.5	1.83	6.03
$\sqrt{\mathbf{E}((\Pi_T - P_M^{\text{sf}})^2)}$	33.4	12.9	21.5	1.83	6.03
$\text{mean}(\text{total cost})$	16.3	30.9	26.8	33.1	32.4
VaR (95%)	75.9	23.3	39.8	2.79	8.56
CVaR (95%)	109	38	59.8	4.55	15.2

MJD Model: #scenarios = 20000 $\sigma = 0.15$ $r = 0.03$ $q = 0$ $\mu = 0.1$
 $\gamma = -1.5$ $\lambda = 0.1$ $\mu_J = -0.2$ $\sigma_J = 0.15$

Table 4: RM Hedging Using the Underlying versus RM hedging Using Options (a MJD Model)

Does this imply that one should hedge using options? To make this decision, we need to evaluate hedging strategies under more realistic market price evolution assumptions. In particular, when standard options are used as hedging instruments, quantifying risk and hedging performance depends on accurately modeling the *market option price dynamics*, in addition to the underlying dynamics. Moreover, one needs to consider other factors such as transaction costs, liquidity risk, and default risk. In this section, we focus on evaluating hedging effectiveness under more realistic market price models which account for volatility risks.

Given the current market convention of quoting implied volatilities instead of option prices, accurately modeling evolution of market implied volatilities is necessary. Following market practice, implied volatility here is defined by inverting the Black-Scholes formula from an option price.

Implied volatilities have been observed to display a curvature across moneyness and a term structure across time to maturity. In addition to modeling this static implied volatility structure, the evolution of the implied volatilities over time needs to be accurately modeled when standard options are used as hedging instruments. There has been active research in recent years on the evolution of the market implied volatilities over time based on historic implied volatility data, e.g., [13, 15, 31]. Zu and Avellaneda [31] construct a statistical model for the term structure of the implied volatilities for currency options. Cont et al [13] observe that, for major indices including S&P 500, the implied volatility surface changes dynamically over time in a way that is not taken into account by current modeling approaches, giving rise to "Vega" risk in option portfolios. They believe that option markets may have become increasingly autonomous and option prices are driven, in addition to movements in the underlying, also by internal supply and demand in the options market. They study the implied volatility time series for major indices, including S&P500 from March 2, 2002 and February 2, 2003, and observe that:

- Implied volatilities are not static; they fluctuate around their means. The daily standard deviation of the implied volatility can be as large as a third of its typical value for out-of-the money options.
- The variance of daily log-variations in the implied volatility surface can be satisfactorily explained in terms of two or three principal components.
- The first principal component reflects an overall shift in the level of all implied volatilities which accounts for around 80% of the daily variance [13]. The second principal component explains the opposite movement of the out-of-the-money call and put implied volatilities while the third principal component reflects change in the convexity of the volatility surface. In addition, there is a strong negative correlation between the index return and the change of the implied volatility level; the correlations between the underlying return and the other components of implied volatility changes are either significantly smaller or negligible.

Hence, in order to accurately quantify the hedging effectiveness when the hedging instruments consist of standard options, implied volatility risk needs to be suitably modeled. Unfortunately, under both the Black-Scholes model and Merton's jump diffusion model, im-

plied volatilities are constant over time. This clearly is a substantial departure from the observed dynamics of the market implied volatilities.

How should the dynamics of the implied volatilities be modeled? To hedge a derivative contract, the typical approach is to assume a model for the underlying price evolution and derive implied volatilities from the assumed underlying price model; an underlying model can be estimated through calibration to the option market. Since both the Black-Scholes model and Merton's jump model lead to a static volatility surface, they are inadequate in modeling the observed stochastic implied volatilities. Amongst the typical models for underlying prices, only a stochastic instantaneous volatility underlying model can generate stochastic implied volatilities.

Unfortunately, the fact that instantaneous volatility is not directly observable presents many challenges for hedging and model estimation under an instantaneous volatility model.

- Firstly, it is difficult to adjust hedging positions according to an unobservable value.
- Secondly, it is not clear how to estimate an instantaneous volatility price model, from historical data, given that the instantaneous volatility is not observable; a stochastic instantaneous volatility underlying model can only be calibrated from liquid option prices. Moreover, the model obtained by calibrating to the market option prices gives the price model under a risk adjusted measure. As previously discussed, for hedging and risk management assessment we need a model for the real world market price dynamics.
- Thirdly it is difficult to calibrate the current market option prices sufficiently accurately using a parsimonious stochastic volatility model with a small number of model parameters, e.g., the Hestons model [20]. In addition, there is no evidence that such a stochastic volatility model for the underlying is capable of modeling the *evolution* of the implied volatilities over time.
- Fourthly, even if a model calibrates the market prices by allowing a sufficient number of model parameters, e.g., a jump diffusion model with a local volatility function [2], it is difficult to accurately model the implied volatility evolution for a long time horizon based on market calibration. A model that sufficiently calibrates the market implied volatilities today may give a poor fit for the implied volatilities tomorrow. This can be problematic for hedging and risk management purposes.

In addition to the cautionary remarks provided above, there are also arguments about whether an option pricing model derived from any underlying price model can adequately model option price evolution because of the autonomy of option market, e.g., [13]. This suggests that it is reasonable and possibly better to consider a joint model for the underlying price and implied volatility evolutions.

In [27], Schönbucher jointly models, under a risk adjusted measure, the underlying and implied volatility evolution in which the instantaneous volatility and implied volatility both become state variables. No arbitrage restriction is established for the joint processes under a risk adjusted measure.

Since our focus is risk management and hedging for options embedded in variable annuities, we are interested in modeling *the real world price evolution* for both the underlying and

hedging options. To evaluate hedging effectiveness of discrete dynamic hedging strategies for variable annuities, a joint discrete time model for real-world underlying and hedging instrument price changes between each rebalancing time is needed. In particular, this model needs to accurately describe the tails of the underlying price distribution and (mean reverting) stochastic implied volatility dynamics.

More specifically, we are interested in modeling the most liquid implied volatilities. We assume that the time to maturity of the option hedging instruments is fixed at one year (a shorter maturity can similarly be used) and far out-of-the money options are not used as hedging instruments due to liquidity considerations. Since change in the implied volatility level explains 80% of the daily variation in implied volatilities, as a first improvement, we model the at-the-money implied volatility evolution and assume that the ratios of implied volatilities to the at-the-money implied volatility are constant over time.

Let $\check{\sigma}_t$ denote the at-the-money implied volatility with a time to maturity of one year. Using an Ornstein-Uhlenbeck process, this at-the-money implied volatility $\check{\sigma}_t$ is modeled as follows

$$d \log(\check{\sigma}_t) = \check{\alpha} \cdot (\overline{\log \check{\sigma}} - \log(\check{\sigma}_t)) \cdot dt + \check{\sigma}_{\text{vol}} \cdot dZ_t \quad (6)$$

where Z_t is a standard Brownian, $\overline{\log \check{\sigma}}$ denotes the long term average value (of the logarithm of the at-the-money implied volatility level). Figure 1 displays a sample path and distribution of the implied volatility $\check{\sigma}_t$ at time $t = 1$ year under the assumed model parameters; this model is used for at-the-money implied volatility in our subsequent computational investigation.

It is important to note that, by modeling implied volatilities directly, the standard option prices are directly given by the implied volatilities via the Black-Scholes formula; these liquid option values are no longer determined by the underlying price model. There are many advantages gained by computing a risk minimization hedging using standard options based on a model with implied volatilities directly as state variables. Firstly, the implied volatilities are directly observable from the market. This makes it possible to estimate a model from historic data and adjust hedging positions based on the implied volatilities. Secondly, by modeling the market implied volatility directly, calibration to market is automatically accomplished by setting the initial implied volatilities to the market implied volatilities. Thirdly, the hedging instruments are valued exactly using the Black-Scholes formula according to the market practice. For simplicity, we have modeled here only change in the level of the implied volatilities; but the approach can be extended (with additional computational complexity) to model the opposite movement of the implied volatilities for out-of-the money calls and puts and change in the convexity of the implied volatility curve.

Since the instantaneous volatility is not directly observable in practice, it is difficult to estimate a model with an instantaneous volatility as a state variable and adjust hedging positions based on this unobservable variable. Thus, when computing hedging strategies, we only assume a crude (constant) approximation σ_0 of the instantaneous volatility (for example σ_0 can be an estimation of the average of the instantaneous volatility over time). In other words, we compute risk minimization option hedging strategies under the joint model (7) below for the underlying price and the implied volatility $\check{\sigma}_t$:

$$\begin{cases} \frac{dS_t}{S_t} = (\mu - q - \kappa\lambda) \cdot dt + \sigma_0 \cdot dW_t + (J - 1) \cdot d\pi_t \\ d \log(\check{\sigma}_t) = \check{\alpha} \cdot (\overline{\log \check{\sigma}} - \log(\check{\sigma}_t)) \cdot dt + \check{\sigma}_{\text{vol}} \cdot dZ_t, \end{cases} \quad (7)$$

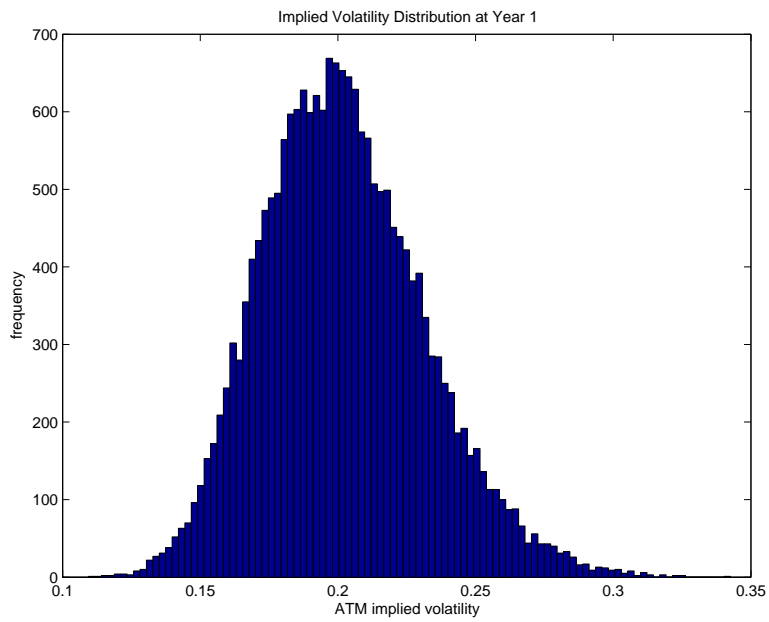
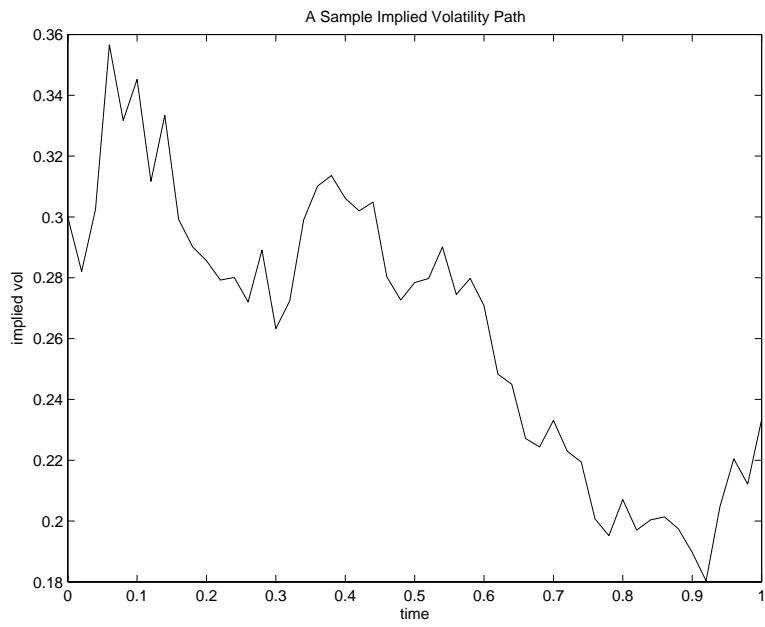


Figure 1: Implied Volatility $\check{\sigma}$: $\check{a} = 0.6$, $\check{\sigma}_{\text{VOL}} = 0.45$, $\check{\sigma}_0 = 30\%$, and $\overline{\log \check{\sigma}} = \log(20\%)$

where W_t and Z_t can be correlated in general. Note that the approach of determining option prices by modeling the underlying price evolution as described in §3 and §4 can be considered as a special case of a model (7) with $\check{\alpha} = \check{\sigma}_{\text{vol}} = 0$. In addition, jumps can be introduced in the mean reverting implied volatility dynamics as well.

Presently, for computational simplicity, we compute hedging strategies based on a joint model (7) assuming that W_t and Z_t are independent. *However, we evaluate hedging effectiveness under a joint model with both instantaneous volatility risk and implied volatility risk, and possibly correlation between risks.* Specifically, we evaluate hedging performance under the joint price model (8) below:

$$\begin{cases} \frac{dS_t}{S_t} = (\mu - q - \kappa\lambda) \cdot dt + \sigma_t \cdot dW_t + (J - 1) \cdot d\pi_t \\ d \log(\sigma_t) = a \cdot (\overline{\log \sigma} - \log(\sigma_t)) \cdot dt + \sigma_{\text{vol}} \cdot dX_t \\ d \log(\check{\sigma}_t) = \check{\alpha} \cdot (\overline{\log \check{\sigma}} - \log(\check{\sigma}_t)) \cdot dt + \check{\sigma}_{\text{vol}} \cdot dZ_t, \end{cases} \quad (8)$$

where $\overline{\log \sigma}$ denotes the long term average value (of the logarithm of the instantaneous volatility), and W_t , X_t , and Z_t can be correlated. Evaluating hedging effectiveness under this joint dynamics provides an assessment of the impact of ignoring correlation and instantaneous volatility risk in the hedging strategy computation.

We now compare effectiveness of the risk minimization hedging using the underlying with risk minimization hedging using six standard options. In addition to monthly hedging using the underlying, we compare the option hedging strategy computed by explicitly modeling the implied volatility risk in the risk minimization hedging computation (under column *option(6)- $\check{\sigma}$*) with the risk minimization option hedging strategy for which the hedging positions at time t_k are computed assuming that the implied volatility remains at the time t_k level from t_k to the maturity T (under column *option(6)-reset*); this is similar to computing a hedging strategy by recalibrating implied volatilities. We note that the risk minimization hedging *option(6)- $\check{\sigma}$* computes holdings on a 3-dimensional grid, along the directions of the underlying, running max, and at-the-money implied volatility.

Tables 5 & 6 compare the hedging performance of these three strategies. In these computations, the initial implied volatilities are set to the Black-Scholes implied volatilities corresponding to the option prices computed under the underlying dynamics in the model (7); parameters for the model (7) are $a = \check{\alpha} = 0.6$, $\overline{\log \sigma} = \overline{\log \check{\sigma}} = \log(20\%)$, with other parameters given explicitly in the table. The only difference in computational setup between Table 5 and Table 6 is that the constant volatility σ_0 in the model (7) is set to different values; thus the initial implied volatilities $\check{\sigma}_0$ are different as well. From Table 5 & 6, we observe the following:

- Hedging using the underlying is sensitive to instantaneous volatility risk which is difficult to model; this is indicated by the different cost and risk under the column underlying- σ . In Table 5, hedging using the underlying leads to an *under* hedge of the payoff since a constant volatility of 15%, which is lower than the average of 20%, is used in the hedging computation. In Table 6, hedging using the underlying leads to an *over* hedge since a constant volatility of 22%, higher than the average of 20%, is used in the hedging computation.
- For hedging strategy *option(6)-reset* computed by resetting the implied volatility level at time t_k to $\check{\sigma}_{t_k}$ from t_k to the maturity T , the hedging strategy and its performance is

	Π_T^σ	monthly underlying- σ	annually option(6)-reset	annually option(6)- $\check{\sigma}$
C_0	0	23.7	21.5	28.8
mean($\Pi_T - P_M^{\text{sf}}$)	31.4	20.5	10.8	0.612
std($\Pi_T - P_M^{\text{sf}}$)	46.4	18.8	8.1	6.78
$\sqrt{\mathbf{E}(\Pi_T - P_M^{\text{sf}})^2}$	56.1	27.8	13.5	6.81
mean(total cost)	31.4	52.5	39.9	39.4
VaR (95%)	118	54.4	24.3	11.4
CVaR (95%)	174	76.8	31.5	18.6

MJD Model: #scenarios = 20000 $\sigma_0 = 0.15$ $r = 0.03$ $q = 0$ $\alpha = 0.1$
 $\lambda = 0.12$ $\mu_J = -0.2$ $\sigma_J = 0.15$ $\gamma = 1$

Table 5: Hedging Comparison Under a Volatility Model (8) : $\sigma_{vol} = 0.45$, $\check{\sigma}_{vol} = 0.45$

similarly sensitive to the initial implied volatility level. For example, hedging strategy *option(6)-reset* under-estimates the initial hedging cost in Table 5, while the initial hedging cost is over-estimated in Table 6.

- When the implied volatility $\check{\sigma}_t$ is explicitly modeled in the risk minimization hedging framework, column *option(6)- $\check{\sigma}$* , the hedging strategy computation takes the future implied volatility dynamics into consideration and the initial hedging cost is significantly less sensitive to the initial implied volatility level; note the striking similarities of the performance assessment under column *option(6)- $\check{\sigma}$* for Table 5 & 6. Moreover, we note that, compared to hedging using the underlying, hedging using options remains significantly more effective in risk reduction under both jump and volatility risks.

Table 7 displays the hedging performance when there is no instantaneous volatility risk (instantaneous volatility is a constant). We recall that the implied volatility evolution is modeled in the option hedging strategy *option(6)- $\check{\sigma}$* . Comparing Table 7 with Table 5, the hedging results indicate sensitivities to the instantaneous volatility risk. It can be observed that hedging effectiveness of the risk minimization hedging using options under the implied volatility model (corresponding to the hedging strategy *option(6)- $\check{\sigma}$*) is relatively insensitive to instantaneous volatility risk. This suggests that hedging using standard options may potentially be an effective way of reducing instantaneous volatility risk. Hedging using the underlying, on the other hand, is sensitive to the instantaneous volatility risk.

Table 5 and Table 6 indicate that hedging using options by simply resetting the implied volatility level at the rebalancing time, i.e., *option(6)-reset*, either over- or under- estimates the initial hedging costs. Figure 2 display the relative frequency (frequency divided by the number of simulations) of the increment cost $(U_{k+1}\xi_{k+1} + \eta_{k+1}) - (U_k(t_{k+1})\xi_k + \eta_k)$. The top plot displays the relative frequency of the incremental cost for the hedging strategy *option(6)- $\check{\sigma}$* computed by explicitly modeling the implied volatility evolution; the bottom plot is for the hedging strategy *option(6)-reset* which simply assumes that the future implied volatility stays at the current level at each rebalancing time. The hedging strategy, *option(6)-reset*, leads to a significantly larger average and variance of the incremental cost at each rebalancing

	Π_T^σ	monthly underlying- σ	annually option(6)-reset	annually option(6)- $\check{\sigma}$
C_0	0	35.6	35	28.9
mean($\Pi_T - P_M^{\text{sf}}$)	31.5	-6.26	-8.65	0.374
std($\Pi_T - P_M^{\text{sf}}$)	46.7	15.6	7.72	6.48
$\sqrt{\mathbf{E}(\Pi_T - P_M^{\text{sf}})^2}$	56.3	16.8	11.6	6.49
mean(total cost)	31.5	41.8	38.5	39.4
VaR (95%)	119	18.4	3.08	10.8
CVaR (95%)	175	33.9	8.64	16.8

MJD Model: #scenarios = 20000 $\sigma_0 = 0.22$ $r = 0.03$ $q = 0$ $\alpha = 0.1$
 $\lambda = 0.12$ $\mu_J = -0.2$ $\sigma_J = 0.15$ $\gamma = 1$

Table 6: Hedging Comparison Under a Volatility Model (8): $\sigma_{vol} = 0.45$, $\check{\sigma}_{vol} = 0.45$

	Π_T^σ	monthly underlying- σ	annually option(6)-reset	annually option(6)- $\check{\sigma}$
C_0	0	23.7	21.5	28.8
mean($\Pi_T - P_M^{\text{sf}}$)	17.7	0.583	10.8	0.627
std($\Pi_T - P_M^{\text{sf}}$)	30.8	13.9	6.25	5.71
$\sqrt{\mathbf{E}(\Pi_T - P_M^{\text{sf}})^2}$	35.5	13.9	12.5	5.74
mean(total cost)	17.7	32.5	39.9	39.4
VaR (95%)	80.4	25.1	21.5	10.2
CVaR (95%)	116	40.7	26	14.8

MJD Model: #scenarios = 20000 $\sigma_0 = 0.15$ $r = 0.03$ $q = 0$ $\alpha = 0.1$
 $\lambda = 0.12$ $\mu_J = -0.2$ $\sigma_J = 0.15$ $\gamma = 1$

Table 7: Hedging Comparison Under a Volatility Model (8): $\sigma_{vol} = 0$, $\check{\sigma}_{vol} = 0.45$

	Π_T^σ	monthly underlying- σ	annually option(6)-reset	annually option(6)- $\check{\sigma}$
C_0	0	23.7	21.5	28.8
mean($\Pi_T - P_M^{\text{sf}}$)	31.4	20.5	10.1	-0.401
std($\Pi_T - P_M^{\text{sf}}$)	46.4	18.8	7.89	6.5
$\sqrt{\mathbf{E}(\Pi_T - P_M^{\text{sf}})^2}$	56.1	27.8	12.8	6.51
mean(total cost)	31.4	52.5	39.1	38.4
VaR (95%)	118	54.4	22.8	8.89
CVaR (95%)	174	76.8	29.8	14.6

MJD Model: #scenarios = 20000 $\sigma_0 = 0.15$ $r = 0.03$ $q = 0$ $\alpha = 0.1$
 $\lambda = 0.12$ $\mu_J = -0.2$ $\sigma_J = 0.15$ $\gamma = 1$

Table 8: Hedging Comparison Under a Volatility Model (8): $\text{corr}(Z_t, W_t) = -.6$, $\sigma_{vol} = 0.45$, $\check{\sigma}_{vol} = 0.45$

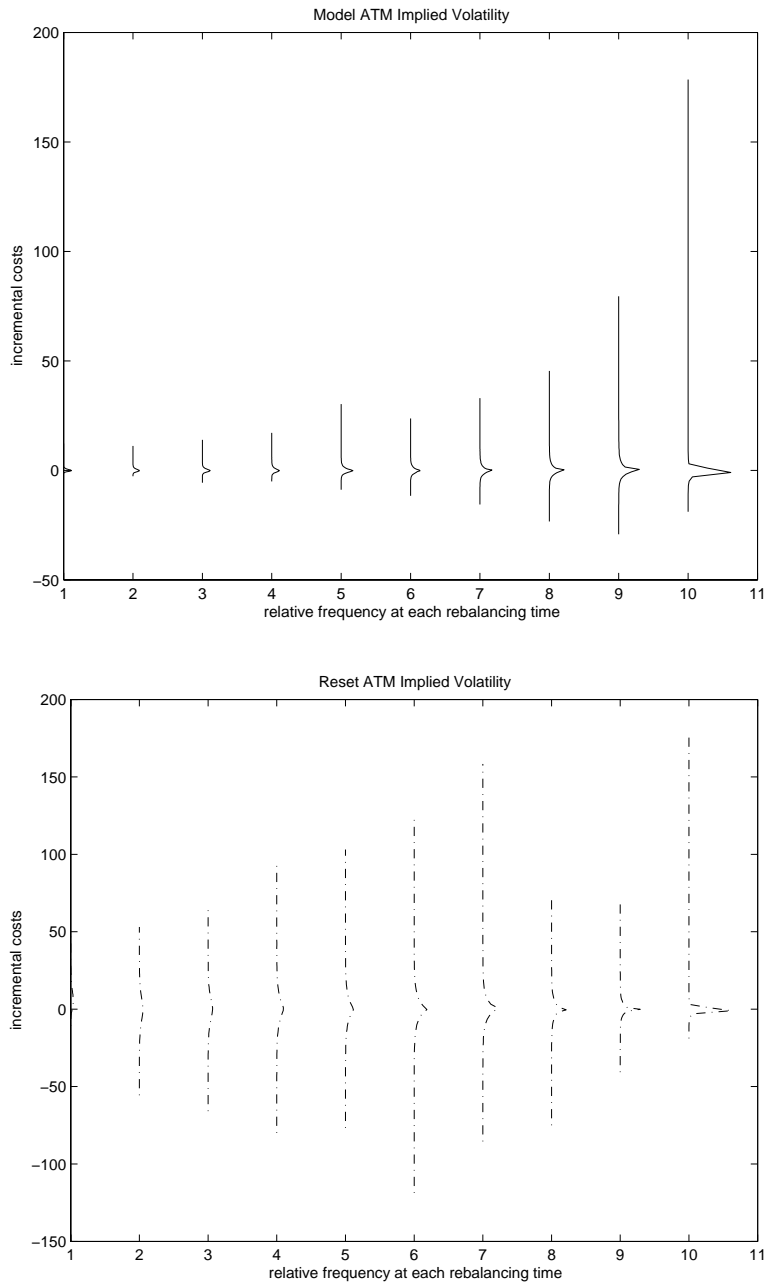
time compared to the risk minimization hedging *option(6)- $\check{\sigma}$* for which the implied volatility risk is properly modeled.

Finally, Table 8 illustrates the hedging effectiveness of the three strategies evaluated under a correlation assumption between the change in $\log \check{\sigma}_t$ and the Brownian innovation of the asset return, specifically $\text{corr}(W_t, Z_t) = -0.6$ in the model (8). Comparing Table 8 with Table 5, it can be observed that hedging effectiveness using standard options is not significantly affected by the correlation risk in terms of the standard deviation of the hedging error. However, VaR and CVaR become slightly smaller for this example.

6 Concluding Remarks

Accurate quantification and robust hedging of the market risk embedded in a guaranteed minimum death benefit of a variable annuity contract is a challenging task. This is mainly due to the long maturity of the insurance contract and the sensitivity of the benefit to the tails of the account value distribution. For this type of contracts, risk quantification and risk reduction require accurately modeling of the evolution of both the underlying account value and hedging instrument prices. In addition to a good model, effectiveness of a hedging strategy depends on the method from which hedging positions are determined. A delta hedging strategy is computed based on option values under a risk adjusted measure. A risk minimization hedging strategy is computed to minimize risk based on a model for the real world price evolution and rebalancing specification. We analyze and compare these different hedging methods using either the underlying or standard options to hedge a lookback option embedded in variable annuity contracts for which the minimum death guarantee has a ratchet feature.

Due to sensitivity of a variable annuity benefit to the tails of the account value distribution, jump risk and volatility risks need to be appropriately modeled. These additional risks, and the fact that dynamic hedging can only be implemented at discrete times with a limited choice of hedging instruments, suggest that hedging options embedded in variable



MJD Model: #scenarios = 20000 $\sigma_0 = 0.15$ $r = 0.03$ $q = 0$ $\alpha = 0.1$
 $\lambda = 0.12$ $\mu_J = -0.2$ $\sigma_J = 0.15$ $\gamma = 1$

Figure 2: Incremental Costs Under a Volatility Model (8): $\sigma_{vol} = 0.45$, $\check{\sigma}_{vol} = 0.45$, $\sigma_0 = 0.15$

annuities are hedging problems in an incomplete market. Thus it may be more appropriate to compute a hedging strategy to directly minimize a measure of the hedge risk given a set of trading times and a set of hedging instruments; this is the risk minimization hedging.

We evaluate and compare effectiveness of delta hedging with risk minimization hedging using the underlying under a Black-Scholes model as well as a Merton's jump diffusion model. We first assume that there is no volatility risk and illustrate that risk minimization hedging using the underlying is superior to delta hedging. In addition, monthly rebalancing risk minimization hedging using the underlying is relatively effective under a Black-Scholes model; the hedging effectiveness further improves as hedging portfolio is rebalanced more frequently. Moreover, under a Merton's jump model, risk reduction in both delta hedging and risk minimization hedging using the underlying is less effective. In particular, for both methods, improvement of biweekly hedging over monthly hedging is less significant.

Due to the increasing liquidity of standard options, we compare risk minimization hedging using standard options to that of using the underlying. We observe that the risk minimization hedging using standard options (even with only two at-the-money options) is significantly more effective than that of using the underlying, particularly when jump risk is considered.

Maturities of variable annuities are usually long; this makes modeling of relevant risks especially challenging. For hedging a long term derivative, volatility risk clearly cannot be ignored. Hedging using the underlying asset is susceptible to instantaneous volatility risk. When standard options are used as hedging instruments, hedging effectiveness needs to be evaluated under the implied volatility risk. Since the instantaneous volatility is not directly observable, it is difficult to model instantaneous volatility risk and implement hedging strategies which adjust hedging positions according to this unobservable variable. The typical approach of calibrating a model for the underlying evolution from the option prices is faced with the need for a complex model to match the current option prices and the need for a parsimonious model to stably model future option prices evolution. Moreover, the underlying price dynamics calibrated to the option market is under a risk adjusted measure rather than the real price dynamics necessary for risk quantification and hedging.

We compute a risk minimization hedging using standard options by explicitly modeling evolutions in the underlying as well as the at-the-money implied volatility. The ratios of the implied volatilities of hedging instruments to the at-the-money implied volatility is assumed to be constant over time for simplicity. Since instantaneous volatility is not observable, we compute hedging strategy assuming a constant approximation to the instantaneous volatility in a model (7). By evaluating hedging performance under a joint model (8) of the underlying account value evolution, which includes instantaneous volatility risk, and implied volatilities, we illustrate that, when implied volatility risk is suitably modeled, risk minimization hedging using standard options is effective in reducing lookback option risk embedded in variable annuity with a ratchet guaranteed minimum death benefit. In particular, unlike hedging using the underlying, which yields larger hedging error due to failure to model instantaneous volatility risk, effectiveness of the risk minimization hedging using standard options is relatively insensitive to the presence of instantaneous volatility risk. We also illustrate hedging performance when the change of the implied volatility is correlated to the change of the underlying.

Hedging variable annuities is a complex and challenging task. Investigation and analysis in this paper are based on models which have been used to describe evolution of asset

price with fat tails in the return distribution. For future investigation, it is important to evaluate how well a joint model (8) for the underlying and implied volatility can describe the underlying price and implied volatility evolution for a long time horizon. In addition to jump risk, volatility risks, other risks such as mortality risk, basis risk, and surrender risk also need to be properly analyzed. In a separate paper [10], we analyze sensitivity of risk minimization hedging to interest risk and compute the optimal risk minimization hedging strategy under both equity and interest risks.

References

- [1] K. Aase and S. Persson. Pricing of unit-linked life insurance policies. *Scandinavian Actuarial Journal*, 1:26–52, 1994.
- [2] L. Andersen and J. Andreasen. Jump-diffusion processes: Volatility smile fitting and numerical methods for option pricing. *Review of Derivative Research*, 4:231–262, 2000.
- [3] A. Bacinello and S. Persson. Design and pricing of equity-linked life insurance under stochastic interest rates. *Journal of Risk Finance*, 3:6–21, 2002.
- [4] D. Bertsimas, L. Kogan, and A. Lo. Hedging derivative securities and incomplete markets: An ϵ -arbitrage approach. *Operations Research*, 49:372–397, 2001.
- [5] P. Boyle and M. Hardy. Reserving for maturity guarantees: Two approaches. *Insurance: Mathematics and Economics*, 21:113–127, 1997.
- [6] P. Boyle and E. Schwartz. Equilibrium prices of guarantees under equity-linked contracts. *Journal of Risks and Insurance*, 44:639–680, 1977.
- [7] M. Brennan and E. Schwartz. The pricing of equity-linked life insurance policies with an asset value guarantee. *Journal of Financial Economics*, 3:195–213, 1976.
- [8] Peter Carr. Semi-static hedging. Technical Report February, Courant Institute, NYU, 2002.
- [9] Peter Carr and Liuren Wu. Static hedging of standard options. Technical report, Courant Institute, New York University, November 26, 2002.
- [10] T. F. Coleman, Y. Li, and M. Patron. Hedging guarantees in variable annuities using risk minimization (under both equity and interest risks). Technical Report in preparation, Cornell University, 2004.
- [11] T. F. Coleman, Y. Li, and M. Patron. Total risk minimization using Monte-carlo simulations. Technical Report in preparation, Cornell University, 2004.
- [12] Thomas F. Coleman, Yuying Li, and M. Patron. Discrete hedging under piecewise linear risk minimization. *The Journal of Risk*, 5:39–65, 2003.
- [13] R. Cont and J. D. Fonseca. Dynamics of implied volatility surfaces. *Quantitative Finance*, 2:45–60, 2002.

- [14] D. Duffie and H. R. Richardson. Mean-variance hedging in continuous time. *Ann. Applied Probab.*, pages 1–15, 1991.
- [15] M. Fengler, W. Härdle, and C. Villa. The dynamics of implied volatilities: a common principal component approach. *review of Derivative Research*, 6, 2003.
- [16] H. Föllmer and M. Schweizer. Hedging by sequential regression: An introduction to the mathematics of option trading. *The ASTIN Bulletin*, 1:147–160, 1989.
- [17] M. Hardy. *Investment Guarantees: Modeling and risk management for equity-linked life insurance*. John Wiley & Sons, Inc., 2003.
- [18] D. Heath, E. Platen, and M. Schweizer. A comparison of two quadratic approaches to hedging in incomplete markets. *Mathematical Finance*, 11:385–413, 2001a.
- [19] D. Heath, E. Platen, and M. Schweizer. Numerical comparison of local risk-minimisation and mean-variance hedging. In *Option pricing, interest rates and risk management*, pages 509–537. (ed. E. Jouini, J. Cvitanic and, M. Musiela), Cambridge Univ. Press, 2001b.
- [20] S.L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.
- [21] G. Labahn. Closed form PDF for Merton’s jump diffusion model. Technical report, School of Computer Science, University of Waterloo, Waterloo, Ont., Canada N2L 3G1, 2003.
- [22] F. Mercurio and T. C. F. Vorst. Option pricing with hedging at fixed trading dates. *Applied Mathematical Science*, 3:135–158, 1996.
- [23] F. Mercurio and T. C. F. Vorst. Option pricing with hedging at fixed trading dates. *Applied Mathematical Science*, 3, 1996.
- [24] M. Patron. *Risk Measures and Optimal Strategies for Discrete Hedging*. Cornell University Ph.D thesis, 2003.
- [25] Antoon Pelsser. Pricing and hedging guaranteed annuity options via static option replication. Technical report, Head of ALM Dept, Nationale-Nederlanden Actuarial Dept, PO Box 796, 300 AT Rotterdam, The Netherlands, 15-May-2002.
- [26] M. Schäl. On quadratic cost criteria for option hedging. *Mathematics of Operation Research*, 19(1):121–131, 1994.
- [27] P. J. Schönbucher. A market model for stochastic implied volatility. *Phil. Trans. R. Soc.*, 357, 1999.
- [28] M. Schweizer. Mean variance hedging for general claims. *Ann. Appl. Probab.*, 2:171–179, 1992.
- [29] M. Schweizer. Variance-optimal hedging in discrete time. *Mathematics of Operation Research*, 20:1–32, 1995.

- [30] M. Schweizer. A guided tour through quadratic hedging approaches. In *Option pricing, interest rates and risk management*, pages 538–574. (ed. E. Jouini, J. Cvitanic and, M. Musiela), Cambridge Univ. Press, 2001.
- [31] Y. Zhu and M. Avellaneda. An E-ARCH model for the term structure of implied volatility of FX options. *Appl. Math. Finance*, 4, 1997.