SLOPE AND GEOMETRY IN VARIATIONAL MATHEMATICS

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Structure permeates both theory and practice in modern optimization. To make progress, optimizers often presuppose a particular algebraic description of the problem at hand, namely whether the functional components are affine, polynomial, smooth, sparse, etc., and a qualification (transversality) condition guaranteeing the components do not interact wildly. This thesis deals with structure as well, but in an intrinsic and geometric sense, independent of functional representation.

On one hand, we emphasize the *slope* — the fastest instantaneous rate of decrease of a function — as an elegant and powerful tool to study nonsmooth phenomenon. The slope yields a verifiable condition for existence of exact error bounds — a Lipschitz-like dependence of a function's sublevel sets on its values. This relationship, in particular, will be key for the convergence analysis of the method of alternating projections and for the existence theory of steepest descent curves (appropriately defined in absence of differentiability).

On the other hand, the slope and the derived concept of subdifferential may be of limited use in general due to various pathologies that may occur. For example, the subdifferential graph may be large (full-dimensional in the ambient space) or the critical value set may be dense in the image space. Such pathologies, however, rarely appear in practice. Semi-algebraic functions — those functions whose graphs are composed of finitely many sets, each defined by finitely many polynomial inequalities — nicely represent concrete functions

arising in optimization and are void of such pathologies. To illustrate, we will see that semi-algebraic subdifferential graphs are, in a precise mathematical sense, small. Moreover, using the slope in tandem with semi-algebraic techniques, we significantly strengthen the convergence theory of the method of alternating projections and prove new regularity properties of steepest descent curves in the semi-algebraic setting. To illustrate, under reasonable conditions, bounded steepest descent curves of semi-algebraic functions have finite length and converge to local minimizers — properties that decisively fail in absence of semi-algebraicity.

We conclude the thesis with a fresh new look at active sets in optimization from the perspective of representation independence. The underlying idea is extremely simple: around a solution of an optimization problem, an "identifiable" subset of the feasible region is one containing all nearby solutions after small perturbations to the problem. A quest for only the most essential ingredients of sensitivity analysis leads us to consider identifiable sets that are "minimal". In the context of standard nonlinear programming, this concept reduces to the active-set philosophy. On the other hand, identifiability is much broader, being independent of functional representation of the problem. This new notion lays a broad and intuitive variational-analytic foundation for optimality conditions, sensitivity, and active-set methods. In the last chapter of the thesis, we illustrate the robustness of the concept in the context of eigenvalue optimization.

BIOGRAPHICAL SKETCH

Dmitriy Drusvyatskiy was born in Minsk, Belarus in 1987. He and his family emigrated to the United States in 1998, settling in Brooklyn, New York. Dmitriy received his undergraduate degree from Polytechnic Institute of NYU in Computer Science and began his Ph.D. studies in the School of Operations Research and Information Engineering at Cornell University in 2008. Upon completing his doctoral studies, he will spend a year as a postdoctoral student at University of Waterloo, followed by a tenure-track Assistant Professor appointment in the Mathematics Department at University of Washington in Seattle.

This thesis is dedicated to my family and friends.

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I would like to thank my unofficial advisor Alex D. Ioffe. I have learned by far more mathematics from him than from any other source. Alex's mathematical brilliance and intuition have made our collaboration a truly humbling experience for me. I remember fondly when Alex and his wife Rosa warmly welcomed me in their apartment in Haifa. There, working over soup with Alex, I witnessed the excitement and sheer joy that is discovery of new mathematics. Collaborating with someone of Alex's stature so early in my career has been priceless, and I am looking forward to our future work together.

Next, I would like to thank my committee members, James Renegar and Michael J. Todd, for always providing deep insight and sound advice. I owe my committee a great deal of gratitude for persevering through this not-so-short thesis. I thank David B. Shmoys, who is an important reason why I came to ORIE in the first place. His seemingly limitless enthusiasm and his wise advice never ceases to amaze me. I also owe a debt of gratitude to my undergraduate advisers Joel M. Wein and Boris Aranov, not only for encouraging me to pursue graduate studies, but also for making it possible through undergraduate research projects.

My gratitude goes out to my family for supporting me through this long journey, and especially to my brother Slava Libman. He unfortunately was unable to complete his doctoral studies because he had to support our family when we emigrated to the United States. Slava instilled in me an interest in mathematics very early on, and my success as a doctoral student is in great part his own.

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CHAPTER 1
INTRODUCTION

Nonsmoothness is prevalent throughout variational mathematics, but not pathologically so. On the contrary, structure permeates both theory and practice of modern optimization. To make progress, optimizers often presuppose a particular algebraic description of the problem at hand, namely whether the functional components are affine, polynomial, smooth, sparse, etc., and a qualification (transversality) condition guaranteeing the components do not interact wildly. This thesis is centered around structure as well, but in an intrinsic and geometric sense, independent of functional representation. We will elucidate this distinction through the discussion.

To fix notation, consider a function f on $\mathbf{R}^n$ taking values in the extended-real-line $\overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$. In particular, constraints (independent of representation) are easy to incorporate into f by declaring f to be $+\infty$ outside the feasible region. With such a function, we associate a simple and yet powerful object, the *slope*:

$$|\nabla f|(\bar{x}) := \limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{(f(\bar{x}) - f(x))^+}{\|\bar{x} - x\|},$$

where we use the convention $r^+ := \max\{0, r\}$. Thus the slope $|\nabla f|(\bar{x})$ is simply the fastest instantaneous rate of decrease f at $\bar{x}$. The slope of a smooth function simply coincides with the norm of the gradient, and hence the notation. For more details on this construction see for example [43]. Even though the definition is deceptively simple, we will see that slope is a precise and convenient tool with many far-reaching applications.

1.1 An illustration: method of alternating projections

A first illustration, which is the focus of Chapter 3, concerns an intuitive and widely-used method for finding a point in the intersection of two closed subsets A and B of $\mathbf{R}^n$ — an ubiquitous problem in computational mathematics. The *method of alternating projections* presupposes that the nearest point mappings to A and to B can be easily computed — a reasonable assumption in a variety of applications. The algorithm then proceeds by projecting a starting point onto the first set A, then projecting the resulting point onto B, and then projecting back onto A, and so on and so forth; see Figure 1.1.

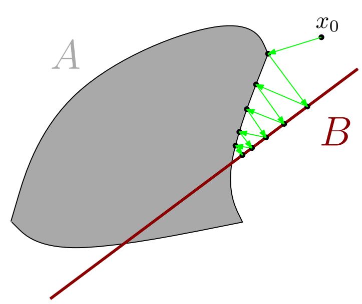


Figure 1.1: Illustration of alternating projections

To see the applicability of the method, consider for example the setting of *compressed sensing*, where we may be interested in finding a sparse vector $x \in \mathbf{R}^n$ satisfying the affine system $Ax = b$, or in other words, we seek a point in the intersection

$$\{x : \operatorname{rank} x \leq r\} \cap \{x : Ax = b\},$$

where r is a small positive integer. (Here, $\operatorname{rank} x$ denotes the number of nonzero coordinates of x.) Projecting a point y onto the affine space $\{x : Ax = b\}$ is a simple operation, while projecting y onto the nonconvex set $\{x : \operatorname{rank} x \leq r\}$ simply amounts to setting the $n - r$ smallest coordinates of y in absolute value to zero, thereby making the method applicable. Another important example comes from *low order control*, where one seeks a positive semi-definite matrix X of low rank satisfying an affine relation $\mathcal{A}(X) = b$, or more precisely a symmetric matrix X in the intersection

$$\{X \succeq 0 : \operatorname{rank} X \leq r\} \cap \{X : \mathcal{A}(X) = b\}.$$

Again projecting a matrix onto either of these sets can be done with ease.

The method of alternating projections has been used extensively; e.g. inverse eigenvalue problems [28, 29], pole placement [120], information theory [113], control design problems [62, 63, 95], and phase retrieval [6, 118]. Observe that the method can be interpreted as minimizing the function

$$\psi(x, y) := \begin{cases} |x - y| & \text{if } x \in A, y \in B \\ +\infty & \text{otherwise} \end{cases}$$

alternately in the variable x and y. This rather trivial observation, on the other hand, pays great dividends since it immediate hints that it is the relationship between the partial slopes $|\nabla\psi_x|(y)$ and $|\nabla\psi_y|(x)$ that drives the convergence. This realization, in turn, leads to a short and elegant proof of the following theorem (see Chapter 3), superseding the earlier results [84, 85] and nicely complementing the newer manuscripts [7, 8, 69].

Theorem 1.1.1 (Local convergence of alternating projections)**.** *Consider two closed subsets A and B of $\mathbf{R}^n$ satisfying the transversality condition*

$$N_A(\bar{x}) \cap (-N_B(\bar{x})) = \{0\}, \tag{1.1}$$

where $\bar{x}$ is a point in the intersection $A \cap B$. Then the method of alternating projections, when started from a point sufficiently close to $\bar{x}$, converges R-linearly to a point in the intersection $A \cap B$.

Here $N_A(\bar{x})$ and $N_B(\bar{x})$ refer to the *limiting normal cones* to A and B, respectively, at a point $\bar{x}$ in the intersection of A and B. To illustrate, in the case that say A is presented classically as the solution set of finitely many smooth inequalities

$$A = \{x : g_i(x) \leq 0 \text{ for all } i = 1, \ldots, k\},$$

the normal cone $N_A(\bar{x})$ has an appealing form: it is simply the convex cone generated by the active gradients

$$N_A(\bar{x}) = \text{cone}\,\{\nabla g_i(\bar{x}) : g_i(\bar{x}) = 0\},$$

whenever the active gradients are positive-linearly independent. At this point, we should stress that it is the geometric condition (1.1) that guarantees the local R-linear convergence of the method; interpretation of this condition in a representation-dependant setting is secondary, depending only on a robust calculus of normals cones. Indeed, appealing to a specific representation of the sets A and B too early would only make the convergence analysis more opaque.

The definition of slope above has a minor drawback: the slope $|\nabla f|$ is generally not a lower-semicontinuous function of its argument (e.g. $f(x) = \min\{x, 0\}$). Hence it is prudent to introduce the *limiting slope*

$$\overline{|\nabla f|}(\bar{x}) := \liminf_{x \xrightarrow[f]{} \bar{x}} |\nabla f|(x),$$

where the convergence $x \xrightarrow[f]{} \bar{x}$ means $(x, f(x)) \to (\bar{x}, f(\bar{x}))$.

A point $\bar{x}$ is *critical* for f whenever the equality $\overline{|\nabla f|}(\bar{x}) = 0$ holds. Generalized critical points of smooth functions f are, of course, simply critical points in the classical sense. However, the more general notion is particularly interesting to optimization specialists because critical points of convex functions are just minimizers [108, Proposition 8.12], and more generally, for a broader class of functions (for instance, those that are Clarke regular [31]), a point is critical exactly when the directional derivative is non-negative in every direction.

Criticality, in turn, naturally yields a generalized derivative construction — an invaluable tool in nonsmooth optimization: the *limiting subdifferential* of f at a point $\bar{x}$, denoted $\partial f(\bar{x})$, simply consists of all vectors $v \in \mathbf{R}^n$ such that $\bar{x}$ is a critical point of the perturbed function $x \mapsto f(x) - \langle v, x\rangle$. Thus $\bar{x}$ is a critical point of f if and only if the inclusion $0 \in \partial f(\bar{x})$ holds.

1.2 Tame optimization

A principal goal of nonsmooth optimization is the search for critical points of nonsmooth functions $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. More generally, given a smooth mapping $G\colon \mathbf{R}^m \to \mathbf{R}^n$, we might be interested in solutions $(x, y) \in \mathbf{R}^m \times \mathbf{R}^n$ to the system

$$(G(x), y) \in \operatorname{gph} \partial f \ \text{ and } \ \nabla G(x)^* y = v \tag{1.2}$$

(where $*$ denotes the adjoint) and

$$\operatorname{gph} \partial f := \{(x, v) : v \in \partial f(x)\}$$

is the *subdifferential graph*. Such systems arise naturally when we seek critical points of the composite function $x \mapsto f(G(x)) - \langle v, x\rangle$.

The system (1.2) could, in principle, be uninformative if the subdifferential graph $\operatorname{gph} \partial f$ is large. In particular, if the dimension (appropriately defined) of the graph is larger than n, then we could not typically expect the system to be a very definitive tool, since it involves $m + n$ variables constrained by only m linear equations and the inclusion. Alarmingly, there exist Lipschitz continuous functions f on $\mathbf{R}^n$ whose subdifferential graphs are $2n$-dimensional [15]. Moreover, in a precise mathematical sense, this property is actually typical for such functions [19].

Notwithstanding this pathology, concrete functions f on $\mathbf{R}^n$ encountered in practice have subdifferentials ∂f whose graphs are, in some sense, small and this property can be useful practically (see e.g. [102]).

For what functions then is the subdifferential a definitive tool?

The class of *semi-algebraic* functions — those functions whose graphs are semi-algebraic, meaning composed of finitely-many sets, each defined by finitely-many polynomial inequalities — is an excellent candidate, and it will play a central role in the thesis. This class of functions subsumes neither the simple case of a smooth function, nor the case of a convex function, neither of which is necessarily semi-algebraic. Nonetheless, it has a certain appeal: semi-algebraic functions are common, they serve as an excellent model for "concrete" functions in nonsmooth optimization [74], and in marked contrast with many other classes of favorable functions, such as amenable functions, they may not even be Clarke regular. Furthermore, semi-algebraic functions are easy to recognize (as a consequence of the Tarski-Seidenberg theorem on preservation of semi-algebraicity under projection). For instance, observe that the spectral radius function on $n \times n$ matrices is neither Lipschitz nor convex, but it is easy to see that it is semi-algebraic.

It is important to note that any particular choice of defining polynomials of a semi-algebraic function will be inconsequential for us; it is merely the existence of such a polynomial representation that endows the class of semi-algebraic functions with far-reaching geometric and analytic properties, and renders them free of most pathologies obscuring variational analysis. To illustrate, coming back to the size of subdifferential graphs, we will show in Chapters 5 and 6 that closed semi-algebraic functions $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ always have n-dimensional subdifferential graphs, in both a global [52] and local sense [45, 48]. Dimensional considerations, in turn, have direct consequences on *generic* properties of semi-algebraic optimization problems: given a semi-algebraic function f on $\mathbf{R}^n$, the perturbed function $f_v(x) := f(x) - \langle v, x\rangle$, for a *generic* vector v, has only finitely many critical points, each of which is "nondegenerate", and lies on a unique "active smooth manifold". Moreover, under such generic perturbations, all local minimizers satisfy a uniform quadratic growth condition [46].

The importance of slope as a technical tool is due to its connection to exact error bounds — an ubiquitous concept in nonsmooth optimization. Roughly speaking, whenever the slope of a function f is lower-bounded away from zero, one is guaranteed that the sublevel sets of f behave in a Lipschitz-like manner. The following theorem appears as [12, Theorem 2]. To state it, for any function f on $\mathbf{R}^n$, we define $[a < f < b] := \{x \in \mathbf{R}^n : a < f(x) < b\}$.

Theorem 1.2.1 (Slope and error bounds)**.** *Consider a lower-semicontinuous function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, and suppose that the slice $[0 < f < r_0]$ is nonempty. Then the following statements are equivalent.*

Non-criticality: *For all $x \in [0 < f < r_0]$, the inequality*

$$|\nabla f|(x) \geq \frac{1}{k} \qquad \textit{holds.}$$

Error-bound: *For all* $r \in (0, r_0)$ *and* $x \in [0 < f < r_0]$, *the inequality*

$$d(x, [f \leq r]) \leq k(f(x) - r)^+ \qquad \textit{holds}.$$

The importance of the theorem above, much like that of the classical inverse function theorem, stems from the fact that the slope — a computable quantity — characterizes a Lipschitz-like behavior of sublevel sets — objects that are in general wild and difficult to handle. This deep relationship, in particular, plays a crucial role in the convergence analysis of the method of alternating projections (Theorem 1.1.1).

On a sobering note, observe that the connection between the slope and error bounds, established in Theorem 1.2.1, is only fruitful when there exist intervals of non-critical values, a property that can easily fail even for $\mathbf{C}^\infty$-smooth functions. Indeed, even though the celebrated Sard's theorem shows that the critical-value set of a $\mathbf{C}^\infty$-smooth function $f \colon \mathbf{R}^n \to \mathbf{R}$ is Lebesgue null, the critical-value set can be dense (e.g. $f(x) = e^{-\frac{1}{|x|}} \sin(\frac{1}{x})$). Reassuringly, such pathologies do not occur in the semi-algebraic setting [11, Corollary 5].

Theorem 1.2.2 (Semi-algebraic Morse-Sard theorem)**.** *Any semi-algebraic function* $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ *has at most finitely many critical values.*

In fact, a much stronger statement holds. The following theorem, based on the work of [78], extends the classical Łojasiewicz inequality for analytic functions to a nonsmooth semi-algebraic setting [11, Theorem 14].

Theorem 1.2.3 (Kurdyka-Łojasiewicz inequality)**.** *Consider a lower-semicontinuous semi-algebraic function* $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ *and a point* $\bar{x}$ *with* $f(\bar{x}) = 0$. *Then for any bounded neighborhood* U *of* $\bar{x}$, *there exists a real number* $\rho > 0$ *and a non-negative function* $\psi \colon [0, \rho) \to \mathbf{R}$ *satisfying*

- *ψ is continuous on $[0, \rho)$ with $\psi(0) = 0$,*
- *The restriction of ψ to $(0, \rho)$ is $\mathbf{C}^1$-smooth with $\psi' > 0$,*
- *The inequality*

$$|\nabla(\psi \circ f)|(x) \geq 1,$$

  holds for all $x \in U \cap [0 < f < \rho]$.

In a nutshell, the theorem above states that semi-algebraic functions, up to a reparametrization of the image, always admit error bounds! This supports our claim that variational analysis works with full efficiency in the semi-algebraic setting. For example, in Chapter 3, we will use the *Kurdyka-Łojasiewicz inequality* to prove that for the problem of finding a point in the intersection of two closed semi-algebraic sets A and B, any limit point of the iterates generated by the method of alternating projections lies in the intersection $A \cap B$, provided that the initial point is sufficiently close to the intersection. That is, the transversality condition (1.1) only serves to establish the R-linear convergence. This is particularly important in practice, since verification of transversality requires one to already know a point in the intersection — an impossible task.

We should note in passing that all results in the thesis pertaining to semi-algebraic functions have direct analogues for functions definable in an "o-minimal structure" and, more generally, for "tame" functions. In particular, our results hold for globally subanalytic functions, discussed in [110]. For a quick introduction to these concepts in an optimization context, see [74]. To ease the exposition, however, we stay within the semi-algebraic setting.

1.3 Curves of descent

One of the foundational notions in optimization is the intuitive idea of a steepest descent curve. In the smooth case, such curves are rigorously defined as solutions of the gradient dynamical system $\dot{x} = -\nabla f(x)$. In absence of differentiability, however, an analogous notion revolves around the slope. To motivate the discussion, consider a function f on $\mathbf{R}^n$ and a 1-Lipschitz curve $\gamma\colon (a,b) \to \mathbf{R}^n$. One can then readily verify that the inequality

$$|\nabla (f \circ \gamma)|(t) \le |\nabla f|(\gamma(t)) \qquad \text{holds for a.e. } t \in (a,b). \tag{1.3}$$

Supposing that f is continuous (for technical reasons), it is then natural to call γ a *steepest descent curve* if $f \circ \gamma$ is nonincreasing and the reverse inequality holds in (1.3). Such curves, up to a reparametrization and an integrability condition, are the curves of maximal slope studied in [2, 42, 44, 88]. Requiring the weaker inequality

$$|\nabla (f \circ \gamma)|(t) \ge \overline{|\nabla f|}(\gamma(t)) \qquad \text{to hold for a.e. } t \in (a,b).$$

defines *near-steepest descent curves*. See Figure 1.2 for an illustration.

In Chapter 4, which is based on the forthcoming manuscript [49], we show that such curves always exist for lower-semicontinuous functions $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, satisfying a mild continuity condition. The key idea of our construction is to discretize the range of f and then build a piecewise-linear curve by projecting iterates onto successive sublevel sets. Passing to the limit as the mesh of the partition tends to zero, under reasonable conditions and a reparametrization, yields a near-steepest descent curve. Not surprisingly, it is the relationship between the slope and error bounds that drives the analysis (and motivates the construction).

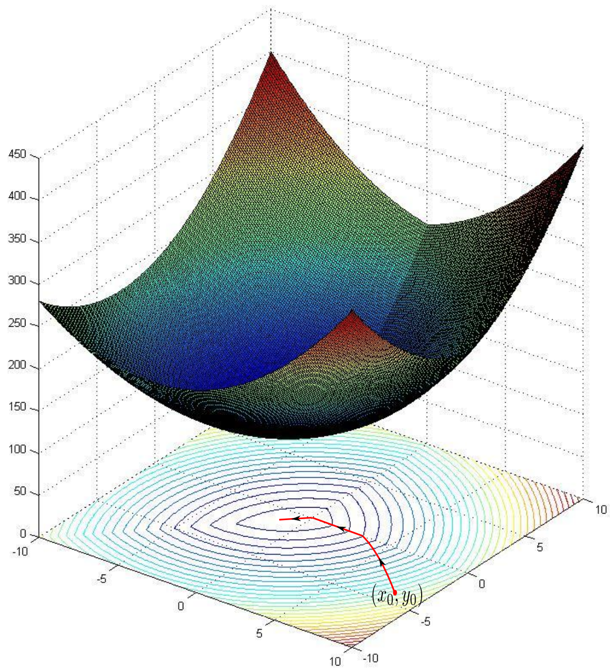


Figure 1.2: $f(x, y) = \max\{x + y,\, |x - y|\} + x(x + 1) + y(y + 1) + 100$

Again, much stronger results hold in the semi-algebraic setting. For semi-algebraic functions $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that are locally Lipschitz continuous on their domains, near-steepest descent curves, up to a reparametrization, are precisely the solutions of the subgradient dynamical system

$$\dot{x}(t) \in -\partial f(x(t)) \quad \text{a.e.}$$

Moreover, combining our construction with the Kurdyka-Łojasiewicz inequality we prove that any near-steepest descent curve $\gamma\colon (a, b) \to \mathbf{R}^n$ for a semi-algebraic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ satisfies the following regularity properties.

- γ is either unbounded or necessarily has finite length, and
- the domain of γ can always be extended so that γ remains a near-steepest descent curve, and so that γ is either unbounded or it necessarily converges to a local minimizer of f (and not just to a critical point).

All of these regularity properties easily fail even for $\mathbf{C}^\infty$-smooth functions.

1.4 Foundations of active sets in nonsmooth optimization

In Chapter 7 we discuss the foundations of active sets in nonsmooth optimization. To ground the discussion, consider minimizing the classically studied max-type function

$$f(x) = \max_{i \in I} f_i(x), \qquad \text{where } I \text{ is a finite index set and } f_i \text{ are } \mathbf{C^2}\text{-smooth.}$$

Then in terms of the *active index set* $I(x) := \{i : f(x) = f_i(x)\}$, the subdifferential $\partial f(x)$ has an appealing form: it is simply the convex hull of the active gradients,

$$\partial f(x) = \text{conv}\,\{\nabla f_i(x) : i \in I(x)\}.$$

Consequently, the criticality condition $0 \in \partial f(\bar{x})$ amounts to classical *first-order necessary conditions* for optimality. Now fix a point $\bar{x} \in \mathbf{R}^n$ and assume that the active gradients $\{\nabla f_i(\bar{x}) : i \in I(\bar{x})\}$ are affinely independent. Then the active manifold,

$$\mathcal{M} = \{x : I(x) = I(\bar{x})\}, \tag{1.4}$$

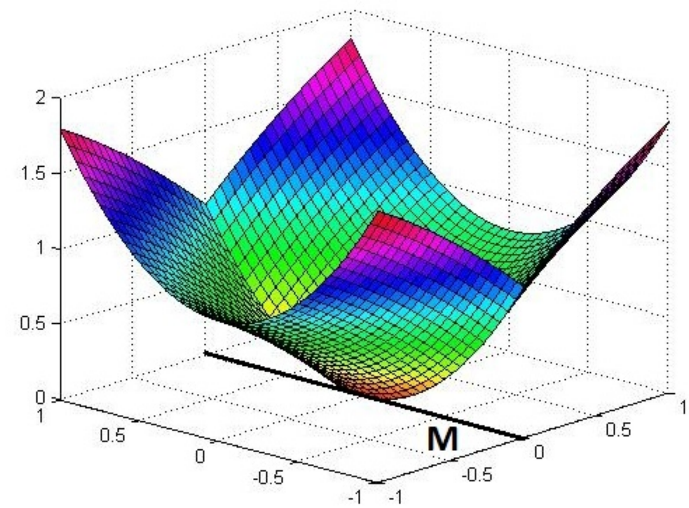
plays a crucial role both in theory and in practice. To illustrate, assuming strict complementarity — meaning that zero lies in the relative interior of $\partial f(\bar{x})$ — classical arguments yield the following far-reaching consequences.

Activity: Locally $\mathcal{M}$ consists of all nearby "approximate" critical points, meaning that for any small neighborhood U of zero, $\mathcal{M}$ coincides with the set $\{x : \partial f(x) \cap U \neq \emptyset\}$ locally around $\bar{x}$ — motivation for active-set methods.

Second-order growth: Quadratic growth of f around $\bar{x}$ is equivalent to quadratic growth of the smooth function $f\big|_{\mathcal{M}}$, and hence is recognizable using classical calculus.

The active manifold $\mathcal{M}$ is interesting geometrically because the epigraph of f has a distinctive "ridge" along the graph of $f\big|_{\mathcal{M}}$; see Figure 1.3.

Figure 1.3: $f(x,y) := \max\{x(1-x)+y^2,\, -x(1+x)+y^2\}$.



The two results above are well-known to those experienced in standard nonlinear programming. However, many important functions in optimization cannot be reformulated as max-type functions (with a finite index set): the Euclidean norm $\|\cdot\|$ is an obvious example. To illustrate briefly, consider the function

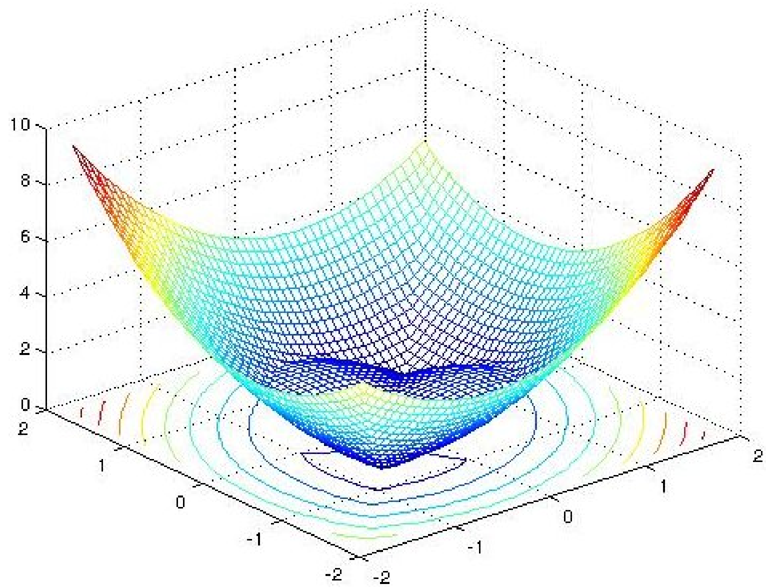
$$f(x) = \sum_{i=1}^{k} \|F_i(x)\|,$$

where the mappings $F_i \colon \mathbf{R}^n \to \mathbf{R}^{m_i}$ are $\mathbf{C}^2$-smooth. With any point x, we may associate an *active index set* $I(x) := \{i : F_i(x) = 0\}$. Fixing a putative local minimizer $\bar{x}$ of f, under reasonable conditions, the set

$$\mathcal{M} = \{x : I(x) = I(\bar{x})\},$$

is a smooth manifold satisfying the same two key properties as in the case of the max-type function. See Figure 1.4 for an illustration.

Figure 1.4: $f(x, y) := |x^2 + y^2 - 1| + |x - y|$.



Contemporary interest in semi-definite programming and eigenvalue optimization leads to many more examples of the same flavor.

So, is there a natural way to model "active sets" (not necessarily manifolds) in a way independent of algebraic description?

Remarkably, the answer is yes! The idea is extremely simple and algorithmic: many numerical methods minimizing a function f on $\mathbf{R}^n$ (e.g. proximal point, subgradient projection, and Newton-like methods) not only produce iterates x_i converging to a critical point $\bar{x}$, but also subgradients $v_i \in \partial f(x_i)$ converging to the zero vector, and thereby serving as certificates of approximate criticality. See for example [68]. This observation naturally leads to the notion of an *identifiable set* introduced in [53]: A set $\mathcal{M}$ is *identifiable* at a critical point $\bar{x}$ if whenever there are sequences $x_i \to \bar{x}$ and $v_i \in \partial f(x_i)$ with $v_i \to 0$, the points x_i lie in $\mathcal{M}$ for all sufficiently large i. We should mention that the intuitive idea of identifiability is rather old, having roots in the notion of an "identifiable surface" [119] and its precursors [1,23,26,27,54,57,59].

The computational promise of identifiable sets is immediate: the problem of minimizing f near $\bar{x}$ is equivalent to minimizing $f\big|_{\mathcal{M}}$ — a potentially easier problem — because the identifiability property allows convergent algorithms to find $\mathcal{M}$. Furthermore, such sets have a natural appeal for optimality conditions and sensitivity analysis: quadratic growth of f around $\bar{x}$ is equivalent to quadratic growth of $f\big|_{\mathcal{M}}$ near $\bar{x}$ — a potentially easier condition to check.

The case when an identifiable set $\mathcal{M}$ is a smooth manifold and the function $f\big|_{\mathcal{M}}$ is smooth is particularly interesting, since it directly leads to a smooth reduction in the problem instance. In particular, the manuscript [87] performs (a much simplified) sensitivity analysis in this setting. Remarkably, this situation is equivalent to a powerful but seemingly stringent list of properties known as partial smoothness [83], nondegeneracy and prox-regularity, thereby giving these sophisticated notions an intuitive interpretation and unifying a number of earlier results in the aforementioned articles.

1.5 Active sets in eigenvalue optimization

The identifiability concept adapts perfectly to eigenvalue optimization problems. To illustrate, consider the space of symmetric matrices $\mathbf{S}^n$ endowed with the trace inner-product and the eigenvalue mapping $\lambda \colon \mathbf{S}^n \to \mathbf{R}^n$ assigning to each matrix A a vector of its eigenvalues $(\lambda_1(A), \ldots, \lambda_n(A))$ in increasing order.

Many functions of interest, such as the spectral radius for instance, only depend on a matrix through its eigenvalues. Such functions necessarily have the composite form $f \circ \lambda$, where $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is *permutation-invariant*. The idea is then to study identifiable sets of the complicated spectral function $f \circ \lambda$, using identifiable sets of the potentially simpler function f. This strategy for studying variational properties of spectral functions goes back to [82].

To see how the identifiability notion adapts to this setting, consider for example the function of two variables

$$f(x, y) := \max\{|x|, |y|\}$$

and the corresponding spectral function

$$(f \circ \lambda)(X) := \max\{|\lambda_1(X)|, |\lambda_2(X)|\}.$$

It is easy to see that the set

$$\mathcal{M} = \{(x, x) : x > 0\}$$

is an identifiable manifold relative to f at the point $(1, 1)$. It would be ideal then if the preimage

$$\lambda^{-1}(\mathcal{M}) = \{X \succ 0 : \lambda_1(X) = \lambda_2(X)\}$$

were to be an identifiable manifold relative to $f \circ \lambda$ at the identity matrix $I_{2\times 2}$. This is indeed the case. In Chapter 8, we show an elegant fact: given a matrix $\overline{X} \in \mathbf{S}^n$, any identifiable manifold $\mathcal{M} \subset \mathbf{R}^n$ (relative to a permutation-invariant function f) at $\lambda(\overline{X})$ lifts to a spectral set $\lambda^{-1}(\mathcal{M})$ that is an identifiable manifold at $\overline{X}$ relative to the spectral function $f \circ \lambda$. This direction of research, in particular, shows great promise for the identifiability concept in eigenvalue optimization.

CHAPTER 2

PRELIMINARIES

In this chapter, we establish the basic notation and record some preliminary results that we will use throughout the thesis. We should emphasize that none of the material in this section is new; rather the results presented here are the basic tools of variational analysis and semi-algebraic geometry.

2.1 Introduction

We will let $(\mathcal{X}, d)$ be a complete metric space. An open ball of radius ϵ around a point $\bar{x}$ will be denoted by $B_\epsilon(\bar{x})$, while the open unit ball will be denoted by $\mathbf{B}$. For any set $Q \subset \mathcal{X}$, the symbols $\mathrm{cl}\, Q$, $\mathrm{int}\, Q$, and $\mathrm{bd}\, Q$ will denote the closure, interior, and boundary of Q, respectively. Consider the extended real line $\overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$. We say that an extended-real-valued function is proper if it is never $\{-\infty\}$ and is not always $\{+\infty\}$. For a function $f\colon \mathcal{X} \to \overline{\mathbf{R}}$, the *domain* of f is

$$\mathrm{dom}\, f := \{x \in \mathcal{X} : f(x) < +\infty\},$$

and the *epigraph* of f is

$$\mathrm{epi}\, f := \{(x, r) \in \mathcal{X} \times \mathbf{R} : r \geq f(x)\}.$$

A function $f\colon \mathcal{X} \to \overline{\mathbf{R}}$ is *lower-semicontinuous* (or *lsc* for short) at $\bar{x}$ if the inequality $\mathrm{liminf}_{x \to \bar{x}} f(x) \geq f(\bar{x})$ holds. For a set $Q \subset \mathcal{X}$ and a point $x \in \mathcal{X}$, the *distance* of x from Q is

$$d(x, Q) := \inf_{y \in Q} d(x, y),$$

and the *metric projection* of x onto Q is

$$P_Q(x) := \{y \in Q : d(x,y) = d(x,Q)\}.$$

2.2 Slope and error bounds

A fundamental notion in local variational analysis is that of *slope* — the "fastest instantaneous rate of decrease" of a function. For more details about slope and its relevance to the theory of metric regularity, see [71].

Definition 2.2.1 (Slope)**.** Consider a function $f\colon \mathcal{X} \to \overline{\mathbf{R}}$, and a point $\bar{x} \in \mathcal{X}$ with $f(\bar{x})$ finite. The *slope* of f at $\bar{x}$ is

$$|\nabla f|(\bar{x}) := \limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{(f(\bar{x}) - f(x))^+}{d(\bar{x}, x)}.$$

The *limiting slope* is

$$\overline{|\nabla f|}(\bar{x}) := \liminf_{x \xrightarrow[f]{} \bar{x}} |\nabla f|(x),$$

where the convergence $x \xrightarrow[f]{} \bar{x}$ means $(x, f(x)) \to (\bar{x}, f(\bar{x}))$.

Slope allows us to define generalized critical points.

Definition 2.2.2 (Critical points)**.** Consider a function $f\colon \mathcal{X} \to \overline{\mathbf{R}}$. We will call any point $\bar{x}$ satisfying $|\nabla f|(\bar{x}) = 0$ a *Fréchet critical point* of f. On the other hand, if $\bar{x}$ satisfies $\overline{|\nabla f|}(\bar{x}) = 0$ then we will say that $\bar{x}$ is a *limiting critical point* of f.

To shorten notation, we will refer to limiting critical points simply as *critical points*. For $\mathbf{C}^1$-smooth functions f on a Hilbert space, the equation $\overline{|\nabla f|}(\bar{x}) = |\nabla f|(\bar{x}) = \|\nabla f(\bar{x})\|$ holds, and hence the notation. In particular, critical points of such functions are critical points in the classical sense. The following local version of Theorem 1.2.1 will play a crucial role in our work [71, Basic Lemma, Chapter 1].

Lemma 2.2.3 (Slope and local error bound)**.** *Consider a lsc function $f\colon \mathcal{X} \to \overline{\mathbf{R}}$. Assume that for some point $x \in \operatorname{dom} f$, there are constants $\alpha < f(x)$ and $r, K > 0$ so that the implication*

$$\alpha < f(u) \leq f(x) \quad \textit{and} \quad d(u,x) \leq K \quad \Longrightarrow \quad |\nabla f|(u) \geq r, \quad \textit{holds}.$$

If in addition the inequality $f(x) - \alpha < Kr$ is valid, then the sublevel set $[f \leq \alpha]$ is nonempty and we have the estimate $d(x, [f \leq \alpha]) \leq r^{-1}(f(x) - \alpha)$.

2.3 Subdifferentials and set-valued mappings

In the setting of linear spaces, a primary method of studying nonsmooth functions is by means of generalized derivatives, or *subdifferentials*. For simplicity, we stay within a finite dimensional setting. We refer the reader to the monographs of Rockafellar-Wets [108], Borwein-Zhu [20], Clarke-Ledyaev-Stern-Wolenski [32], and Mordukhovich [93, 94], for more details. Unless otherwise stated, we follow the terminology and notation of [108].

Throughout, the symbol $\mathbf{R}^n$ will denote an n-dimensional Euclidean space. A *set-valued mapping* F from $\mathbf{R}^n$ to $\mathbf{R}^m$, denoted by $F\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$, is a mapping from $\mathbf{R}^n$ to the power set of $\mathbf{R}^m$. Thus for each point $x \in \mathbf{R}^n$, $F(x)$ is a subset of

$\mathbf{R}^m$. The *domain*, *graph*, and *range* of F are defined to be

$$\begin{aligned} \operatorname{dom} F &:= \{x \in \mathbf{R}^n : F(x) \neq \emptyset\}, \\ \operatorname{gph} F &:= \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^m : y \in F(x)\}, \\ \operatorname{rge} F &:= \bigcup_{x \in \mathbf{R}^n} F(x), \end{aligned}$$

respectively. Observe that $\operatorname{dom} F$ and $\operatorname{rge} F$ are images of $\operatorname{gph} F$ under the projections $(x,y) \mapsto x$ and $(x,y) \mapsto y$, respectively.

The following definition extends in two ways the classical notion of continuity to set-valued mappings.

Definition 2.3.1 (Continuity)**.** Consider a set-valued mapping $F\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$.

1. F is *outer semicontinuous* at a point $\bar{x} \in \mathbf{R}^n$ if for any sequence of points $x_i \in \mathbf{R}^n$ converging to $\bar{x}$ and any sequence of vectors $v_i \in F(x_i)$ converging to $\bar{v}$, we must have $\bar{v} \in F(\bar{x})$.
2. F is *inner semicontinuous* at $\bar{x}$ if for any sequence of points x_i converging to $\bar{x}$ and any vector $\bar{v} \in F(\bar{x})$, there exist vectors $v_i \in F(x_i)$ converging to $\bar{v}$.

If both properties hold, then we say that F is *continuous* at $\bar{x}$. We will say that F is inner-semicontinuous at $\bar{x}$, *relative* to a certain set $Q \subset \mathbf{R}^n$, if the condition above for inner-semicontinuity is satisfied for sequences $x_i \to \bar{x}$ in Q.

Henceforth $o(\|x - \bar{x}\|)$ will denote a term with the property

$$\frac{o(\|x-\bar{x}\|)}{\|x-\bar{x}\|} \to 0, \qquad \text{when } x \to \bar{x} \text{ with } x \neq \bar{x}.$$

The symbols $\operatorname{conv} Q$, $\operatorname{cone} Q$, and $\operatorname{aff} Q$ will denote the convex hull, the (nonconvex) conical hull, and the affine span of Q respectively. The symbol $\operatorname{par} Q$ will denote the parallel subspace of Q, namely the set $\operatorname{par} Q := \operatorname{aff} Q - \operatorname{aff} Q$.

Definition 2.3.2 (Fréchet and limiting subdifferentials)**.** Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a point $\bar{x}$ with $f(\bar{x})$ finite.

1. The *Fréchet subdifferential* of f at $\bar{x}$, denoted $\hat{\partial} f(\bar{x})$, is the set of vectors defined by

$$v \in \hat{\partial} f(\bar{x}) \quad \Longleftrightarrow \quad \bar{x} \text{ is a Fréchet critical point of } f - \langle v, \cdot \rangle.$$

2. The *limiting subdifferential* of f at $\bar{x}$, denoted $\partial f(\bar{x})$, is the set of vectors defined by

$$v \in \partial f(\bar{x}) \quad \Longleftrightarrow \quad \bar{x} \text{ is a limiting critical point of } f - \langle v, \cdot \rangle.$$

We say that f is *subdifferentiable* at $\bar{x}$ whenever $\partial f(\bar{x})$ is nonempty.

A few comments are in order. First, one can easily verify that for a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, the inclusion $v \in \hat{\partial} f(\bar{x})$ holds if and only if the inequality

$$f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(\|x - \bar{x}\|) \quad \text{holds.}$$

In turn, the subdifferential $\partial f(\bar{x})$ consists of all vectors $v \in \mathbf{R}^n$ for which there exist sequences $x_i \in \mathbf{R}^n$ and $v_i \in \hat{\partial} f(x_i)$ with $(x_i, f(x_i), v_i)$ converging to $(\bar{x}, f(\bar{x}), v)$. Moreover, as the following proposition shows, the limiting slope measures the norm of the "shortest" element of the subdifferential [71, Propositions 1 and 2, Chapter 3].

Proposition 2.3.3 (Slope and subdifferentials)**.** *Consider an lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, and a point $\bar{x} \in \mathbf{R}^n$ with $f(\bar{x})$ finite. Then we have $|\nabla f|(\bar{x}) \leq d(0, \hat{\partial} f(\bar{x}))$, and furthermore the equality*

$$\overline{|\nabla f|}(\bar{x}) = d(0, \partial f(\bar{x})), \quad \textit{holds.}$$

In particular, the two conditions $\overline{|\nabla f|}(\bar{x}) = 0$ and $0 \in \partial f(\bar{x})$ are equivalent.

When working with the two subdifferentials we have defined, it is often convenient to introduce a subdifferential which can in principle be smaller.

Definition 2.3.4 (Proximal subdifferential)**.** Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a point $\bar{x}$ with $f(\bar{x})$ finite. The *proximal subdifferential* of f at $\bar{x}$, denoted $\partial_P f(\bar{x})$, consists of all vectors $v \in \mathbf{R}^n$ for which there exist real numbers $r, \epsilon > 0$ satisfying

$$f(x) \geq f(\bar{x}) + \langle v, x - \bar{x}\rangle - \frac{r}{2}\|x - \bar{x}\|^2 \quad \text{for all } x \in B_\epsilon(\bar{x}).$$

It is worth noting that for any lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and any limiting subgradient $v \in \partial f(\bar{x})$ there exist sequences $x_i \in \mathbf{R}^n$ and $v_i \in \partial_P f(x_i)$ with $(x_i, f(x_i), v_i)$ converging to $(\bar{x}, f(\bar{x}), v)$, just as in the Fréchet case. Often one needs to require a kind of uniformity in the subgradients, leading to the notion of *prox-regularity*. This concept has been discovered and rediscovered by various authors, notably by Federer [56] and Rockafellar-Poliquin [98]. We follow the notation of [98] and [99].

Definition 2.3.5 (Prox-regularity)**.** An lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is *prox-regular at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$* if f is finite at $\bar{x}$ and there are constants $r > 0$ and $\epsilon > 0$ such that for all $x, u \in B_\epsilon(\bar{x})$ with $|f(u) - f(\bar{x})| \leq \epsilon$ we have

$$f(x) \geq f(u) + \langle v, x - u\rangle - \frac{r}{2}\|x - u\|^2, \qquad \text{whenever } v \in \partial f(u) \cap B_\epsilon(\bar{v}).$$

We will say that f is *prox-regular at $\bar{x}$* if f is prox-regular at $\bar{x}$ for every $v \in \partial f(\bar{x})$.

The final two derivative-like objects that we will need arise from directional limits of subgradients and from a convexification procedure.

Definition 2.3.6 (Horizon & Clarke subdifferentials)**.** Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a point $\bar{x}$ with $f(\bar{x})$ finite.

1. The *horizon subdifferential* of f at $\bar{x}$, denoted $\partial^{\infty} f(\bar{x})$, consists of all vectors $v \in \mathbf{R}^n$ for which there exists a sequence of real numbers $\tau_i \downarrow 0$ and a sequence of points $x_i \in \mathbf{R}^n$, along with subgradients $v_i \in \hat{\partial} f(x_i)$, so that $(x_i, f(x_i), \tau_i v_i)$ converge to $(\bar{x}, f(\bar{x}), v)$.

2. The *Clarke subdifferential* of f at $\bar{x}$, denoted $\partial_c f(\bar{x})$, is obtained by the convexification

$$\partial_c f(\bar{x}) := \operatorname{cl} \operatorname{co} [\partial f(\bar{x}) + \partial^{\infty} f(\bar{x})].$$

For x such that $f(x)$ is not finite, we follow the convention that all the aforementioned subdifferentials there are empty. Clearly, the inclusions

$$\partial_P f(\bar{x}) \subset \hat{\partial} f(\bar{x}) \subset \partial f(\bar{x}) \subset \partial_c f(\bar{x}) \quad \text{hold.}$$

The subdifferentials $\partial_P f(\bar{x}), \hat{\partial} f(\bar{x})$, $\partial f(\bar{x})$, and $\partial_c f(\bar{x})$ generalize the classical notion of gradient. In particular, for $\mathbf{C}^2$-smooth functions f on $\mathbf{R}^n$, these four subdifferentials consist only of the gradient $\nabla f(x)$ for each $x \in \mathbf{R}^n$. For convex f, these subdifferentials coincide with the convex subdifferential. The horizon subdifferential $\partial^{\infty} f(\bar{x})$ plays an entirely different role; namely, it detects horizontal "normals" to the epigraph. In particular, a lsc function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is locally Lipschitz continuous around $\bar{x}$ if and only if we have $\partial^{\infty} f(\bar{x}) = \{0\}$.

For a set $Q \subset \mathbf{R}^n$, we define the *indicator function* of Q, denoted δ_Q, to be zero on Q and plus infinity elsewhere. The geometric counterparts of subdifferentials are normal cones.

Definition 2.3.7 (Normal cones)**.** Consider a set $Q \subset \mathbf{R}^n$. Then the *proximal*, *Fréchet*, *limiting*, and *Clarke normal cones* to Q at any point $\bar{x} \in \mathbf{R}^n$ are defined by $N^P_Q(\bar{x}) := \partial_P \delta(\bar{x}), \hat{N}_Q(\bar{x}) := \hat{\partial}\delta(\bar{x})$, $N_Q(\bar{x}) := \partial \delta(\bar{x})$, and $N^c_Q(\bar{x}) := \partial_c \delta(\bar{x})$ respectively.

In particular, the *proximal normal cone* to Q at $\bar{x}$ consists of all vectors $v \in \mathbf{R}^n$ such that $\bar{x} \in P_Q(\bar{x} + \frac{1}{r}v)$ for some $r > 0$. Furthermore, this condition amounts to requiring

$$\langle v, x - \bar{x} \rangle \leq O(|x - \bar{x}|^2) \quad \text{as } x \to \bar{x} \text{ in } Q.$$

On the other hand, any Frechét normal $v \in \hat{N}_Q(\bar{x})$ is characterized by the inequality

$$\langle v, x - \bar{x} \rangle \leq o(|x - \bar{x}|) \quad \text{as } x \to \bar{x} \text{ in } Q.$$

In turn, the a vector v lies in $N_Q(\bar{x})$ whenever there exist sequences $x_i \to \bar{x}$ and $v_i \to \bar{v}$ with $v_i \in \hat{N}_Q(x)$. The Clarke normal cone $N_Q^c(\bar{x})$ is then simply the closed convex hull of $N_Q(\bar{x})$.

We will then say that a closed set Q is *prox-regular at a point* $\bar{x}$ *for* $\bar{v} \in N_Q(\bar{x})$ whenever the indicator function δ_Q is prox-regular at $\bar{x}$ for $\bar{v}$. Geometrically, this amounts to requiring that there exist real numbers $\epsilon > 0$ and $r > 0$ such that the implication

$$\left. \begin{array}{cc} x \in Q, & v \in N_Q(x) \\ |x - \bar{x}| < \epsilon, & |v - \bar{v}| < \epsilon \end{array} \right\} \Rightarrow P_{Q \cap B_\epsilon(\bar{x})}(x + r^{-1}v) = x,$$

holds.

Similarly, Q is *prox-regular at* $\bar{x}$ (with no regard to direction) whenever δ_Q is prox-regular at $\bar{x}$. This, in turn, holds if and only if there exists a neighborhood U of $\bar{x}$ so that the the projection mapping P_Q is single-valued on U [108, Exercise 13.38].

The following standard theorem yields a concise relationship between subdifferentials and normal cones to epigraphs.

Theorem 2.3.8 (Subdifferentials & normals to epigraphs)**.** *Consider a function*

$f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a point $\bar{x} \in \mathbf{R}^n$, with $f(\bar{x})$ finite. Then we have

$$\partial_P f(\bar{x}) = \{v \in \mathbf{R}^n : (v, -1) \in N^P_{\text{epi}\, f}(\bar{x}, f(\bar{x}))\}.$$

$$\hat{\partial} f(\bar{x}) = \{v \in \mathbf{R}^n : (v, -1) \in \hat{N}_{\text{epi}\, f}(\bar{x}, f(\bar{x}))\}.$$

$$\partial f(\bar{x}) = \{v \in \mathbf{R}^n : (v, -1) \in N_{\text{epi}\, f}(\bar{x}, f(\bar{x}))\}.$$

$$\partial^\infty f(\bar{x}) = \{v \in \mathbf{R}^n : (v, 0) \in N_{\text{epi}\, f}(\bar{x}, f(\bar{x}))\}.$$

$$\partial_c f(\bar{x}) = \{v \in \mathbf{R}^n : (v, -1) \in N^c_{\text{epi}\, f}(\bar{x}, f(\bar{x}))\}.$$

A particularly nice situation occurs when all the normal cones coincide.

Definition 2.3.9 (Clarke regularity of sets)**.** A set $Q \subset \mathbf{R}^n$ is said to be *Clarke regular* at a point $\bar{x} \in Q$ if it is locally closed at $\bar{x}$ and every limiting normal vector to Q at $\bar{x}$ is a Fréchet normal vector, that is the equation $N_Q(\bar{x}) = \hat{N}_Q(\bar{x})$ holds.

The functional version of Clarke regularity is as follows.

Definition 2.3.10 (Subdifferential regularity)**.** A function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is called *subdifferentially regular* at $\bar{x}$ if $f(\bar{x})$ is finite and epi f is Clarke regular at $(\bar{x}, f(\bar{x}))$ as a subset of $\mathbf{R}^n \times \mathbf{R}$.

In particular, if $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is subdifferentially regular at a point $\bar{x} \in \text{dom}\, f$, then equality $\hat{\partial} f(\bar{x}) = \partial f(\bar{x})$ holds ([108, Corollary 8.11]).

Given any set $Q \subset \mathbf{R}^n$ and a mapping $F\colon Q \to \widetilde{Q}$, where $\widetilde{Q} \subset \mathbf{R}^m$, we say that F is $\mathbf{C}^p$*-smooth* $(p \geq 1)$ if for each point $\bar{x} \in Q$, there is a neighborhood U of $\bar{x}$ and a $\mathbf{C}^p$-smooth mapping $\hat{F}\colon \mathbf{R}^n \to \mathbf{R}^m$ that agrees with F on $Q \cap U$.

Definition 2.3.11 (Smooth Manifolds)**.** We say that a set M in $\mathbf{R}^n$ is a $\mathbf{C}^p$ *manifold* (for $p = 1, \ldots, \infty$) of dimension r if for each point $\bar{x} \in M$, there is an open neighborhood U around $\bar{x}$ and a mapping $F\colon \mathbf{R}^n \to \mathbf{R}^{n-r}$ that is $\mathbf{C}^p$-smooth with $\nabla F(\bar{x})$ of full rank, satisfying $M \cap U = \{x \in U : F(x) = 0\}$.

A good reference on smooth manifold theory is [79].

Theorem 2.3.12. *[108, Example 6.8] Consider a $\mathbf{C}^2$-manifold $M \subset \mathbf{R}^n$. Then at every point $x \in M$, each type of normal cone is equal to the normal space to M at x, in the sense of differential geometry.*

2.4 Semi-algebraic geometry

A *semi-algebraic* set $S \subset \mathbf{R}^n$ is a finite union of sets of the form

$$\{x \in \mathbf{R}^n : P_1(x) = 0, \ldots, P_k = 0, Q_1(x) < 0, \ldots, Q_l(x) < 0\},$$

where $P_1, \ldots, P_k, Q_1, \ldots, Q_l$ are polynomials in n variables. In other words, S is a union of finitely many sets, each defined by finitely many polynomial equalities and inequalities. A map $F\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ is *semi-algebraic* if $\operatorname{gph} F$ is a semi-algebraic set, while a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is semi-algebraic if $\operatorname{epi} f$ is a semi-algebraic set. Semi-algebraic sets enjoy many nice structural properties. We discuss some of these properties in this section. See the monographs of Basu-Pollack-Roy [4], Lou van den Dries [114], and Shiota [110]. For a quick survey, see the article of van den Dries-Miller [115] and the surveys of Coste [33, 34]. Unless otherwise stated, we follow the notation of [115] and [34].

A fundamental fact about semi-algebraic sets is provided by the Tarski-Seidenberg Theorem [34, Theorem 2.3]. It states that the image of any semi-algebraic set $S \subset \mathbf{R}^n$, under a projection to any linear subspace of $\mathbf{R}^n$, is a semi-algebraic set. From this result, it follows that a great many constructions preserve semi-algebraicity. In particular, for a semi-algebraic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, it is easy to see that the slopes $|\nabla f|$, $\overline{|\nabla f|}$ and all the subdifferential set-valued mappings are semi-algebraic. See for example [74, Proposition 3.1].

The most striking and useful fact about semi-algebraic sets is that they can be partitioned into finitely many semi-algebraic manifolds that fit together in a regular pattern. The particular stratification that we are interested in is defined below.

Definition 2.4.1 (Whitney (a)-regular stratification)**.** Consider a semi-algebraic set Q in $\mathbf{R}^n$. A *Whitney (a)-regular stratification* of Q is a finite partition of Q into semi-algebraic $\mathbf{C}^1$ manifolds M_i (called strata) with the following properties:

1. For distinct i and j, if $M_i \cap \operatorname{cl} M_j \neq \emptyset$, then $M_i \subset \operatorname{cl} M_j \setminus M_j$.
2. For any sequence of points x_k in a stratum M_j converging to a point x in a stratum M_i, if the corresponding normal vectors $y_k \in N_{M_j}(x_k)$ converge to a vector y, then the inclusion $y \in N_{M_i}(x)$ holds.

Observe that property 1 of Definition 2.4.1 gives us topological information on how the strata fit together, while property 2 gives us control over how sharply the strata fit together. Property 1 is called the *frontier condition* and property 2 is called *Whitney condition (a)*. We should note that a Whitney (a)-regular stratification, as defined above, is normally referred to as a $\mathbf{C^1}$ *Whitney (a)-regular stratification*. One simple example of this type of a stratification to keep in mind throughout the discussion is the partition of a polytope into its open faces.

Definition 2.4.2 (Compatibility)**.** Given finite collections $\{B_i\}$ and $\{C_j\}$ of subsets of $\mathbf{R}^n$, we say that $\{B_i\}$ is *compatible* with $\{C_j\}$ if for all B_i and C_j, either $B_i \cap C_j = \emptyset$ or $B_i \subset C_j$.

The notion of a stratification being compatible with some predefined sets might not look natural; in fact, it is crucial since this property enables us to construct refinements of stratifications. We will have occasion to use the following result [115, Theorem 4.8].

Theorem 2.4.3 (Existence of Whitney (a)-regular stratifications)**.** *Consider a semi-algebraic set S in $\mathbf{R}^n$ and a semi-algebraic map $f\colon S \to \mathbf{R}^m$. Let $\mathcal{A}$ be a finite collection of semi-algebraic subsets of S and $\mathcal{B}$ a finite collection of semi-algebraic subsets of $\mathbf{R}^m$. Then there exists a Whitney (a)-regular stratification $\mathcal{A}'$ of S that is compatible with $\mathcal{A}$ and a Whitney (a)-regular stratification $\mathcal{B}'$ of $\mathbf{R}^m$ compatible with $\mathcal{B}$ such that for every stratum $Q \in \mathcal{A}'$, we have that the restriction $f|_Q$ is smooth and $f(Q) \in \mathcal{B}'$.*

In particular, it follows that semi-algebraic maps are "generically" (in a sense about to be made clear) smooth.

Definition 2.4.4 (Dimension)**.** Let $A \subset \mathbf{R}^n$ be a nonempty semi-algebraic set. Then we define the *dimension* of A, $\dim A$, to be the maximal dimension of a stratum in any Whitney (a)-regular stratification of A. We adopt the convention that $\dim \emptyset = -\infty$.

It can be easily shown that the dimension does not depend on the particular stratification. See [114, Chapter 4] for more details. In various fields of mathematics, a set $U \subset \mathbf{R}^n$ is said to be "generic", if it is large in some precise mathematical sense, depending on context. Two popular choices are that of U

being a *full-measure* set, meaning its complement has Lebesgue measure zero, and that of U being *topologically generic*, meaning it contains a countable intersection of dense open sets. In general, these notions are very different. However for semi-algebraic sets, the situation simplifies drastically. Indeed, if $U \subset \mathbf{R}^n$ is a semi-algebraic set, then the following are equivalent.

- U is full-measure.
- U is topologically generic.
- The dimension of U^c is strictly smaller than n.

We will say that a certain property holds for a generic vector $v \in \mathbf{R}^n$ if the set of vectors for which this property holds is generic in the sense just described. Generic properties of semi-algebraic optimization problems will be discussed in Section 5.3.

Observe that the dimension of a semi-algebraic set only depends on the maximal dimensional manifold in a stratification. Hence, dimension is a crude measure of the size of the semi-algebraic set. This motivates a localized notion of dimension.

Definition 2.4.5 (Local dimension)**.** Consider a semi-algebraic set $Q \subset \mathbf{R}^n$ and a point $\bar{x} \in Q$. We let the *local dimension* of Q at $\bar{x}$ be

$$\dim_Q(\bar{x}) := \inf_{\epsilon>0} \dim(Q \cap B_\epsilon(\bar{x})).$$

It is not difficult to see that there exists a real number $\bar{\epsilon} > 0$ such that for every real number $0 < \epsilon < \bar{\epsilon}$, we have $\dim_Q(\bar{x}) = \dim(Q \cap B_\epsilon(\bar{x}))$.

Semi-algebraic methods have recently found great uses in set-valued analysis. See for example [40, 52, 72–74]. A fact that will be particularly useful for us is that semi-algebraic set-valued mappings are "generically" inner-semicontinuous.

Proposition 2.4.6. *[52, Proposition 2.28, 2.30] Consider a semi-algebraic, set-valued mapping $G\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$. Then there exists a stratification of* $\operatorname{dom} G$ *into finitely many semi-algebraic manifolds $\{M_i\}$ such that on each stratum M_i, the mapping G is inner-semicontinuous and the dimension of the images $F(x)$ is constant. If in addition F is closed-valued, then we can ensure that the restriction $G\big|_{M_i}$ is also outer-semicontinuous for each index i.*

Finally, we end the chapter with the celebrated Kurdyka-Łojasiewicz inequality [12], which has already been alluded to in the introductory chapter.

Definition 2.4.7 (Kurdyka-Łojasiewicz inequality)**.**

- A function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is said to satisfy the *upper Kurdyka-Łojasiewicz inequality* if for any bounded open set $U \subset \mathbf{R}^n$ and any real τ, there exists $\rho > 0$ and a non-negative continuous function $\psi\colon [\tau, \tau+\rho) \to \mathbf{R}$ satisfying
  - ψ is continuous on $[\tau, \tau+\rho)$ with $\psi(\tau) = \tau$,
  - The restriction of ψ to $(\tau, \tau+\rho)$ is $\mathbf{C}^1$-smooth with $\psi' > 0$,
  - The inequality
    $$|\nabla(\psi \circ f)|(x) \geq 1,$$
    holds for all $x \in U$ with $\tau < f(x) < \tau + \rho$.
- A function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is said to satisfy the *lower Kurdyka-Łojasiewicz inequality* if for any bounded open set $U \subset \mathbf{R}^n$ and any real τ, there exists $\rho > 0$ and a non-negative continuous function $\psi\colon (\tau-\rho, \tau] \to \mathbf{R}$ satisfying
  - ψ is continuous on $(\tau-\rho, \tau]$ with $\psi(\tau) = \tau$,

- The restriction of ψ to $(\tau - \rho, \tau)$ is $\mathbf{C}^1$-smooth with $\psi' > 0$,
- The inequality

  $$|\nabla(\psi \circ f)|(x) \geq 1,$$

  holds for all $x \in U$ with $\tau - \rho < f(x) < \tau$.

In particular, all analytic and all semi-algebraic functions satisfy both the upper and lower KŁ-inequalities [11, Theorem 14].

CHAPTER 3

ILLUSTRATION: METHOD OF ALTERNATING PROJECTIONS

3.1 Introduction

Finding a point in the intersection of two closed, convex subsets Q and S of $\mathbf{R}^n$ is a common problem in optimization. A conceptually simple and widely used method for solving feasibility problems of this form is the *method of alternating projections* — discovered and rediscovered by a number of authors, notably by John von Neumann and Norbert Wiener. It consists of projecting a starting point onto the first set Q, then projecting the resulting point onto S, and then projecting back onto Q, and so on and so forth. The convergence of the method for two intersecting closed convex sets was shown in [21], and linear convergence under a *regular intersection* assumption, $\operatorname{ri} Q \cap \operatorname{ri} S \neq \emptyset$, was proved in [5].

The method of alternating projections makes sense even for non-convex feasibility problems (e.g. intersection of a sphere and hyperplane), and has been used extensively. It is enough to mention inverse eigenvalue problems [28, 29], pole placement [120], information theory [113], control design problems [62, 63, 95], and phase retrieval [6, 118].

This chapter — excerpted from an upcoming paper [47] — will put many of the aforementioned tools of variation analysis and semi-algebraic geometry to use in a concrete setting — local convergence analysis of the nonconvex method of alternating projections. In particular, the analysis presented here supersedes that of [84, 85] and nicely complements that of the newer manuscripts [7, 8, 69]. More refined results will appear in [47].

3.2 Convergence analysis

Throughout we will consider two closed subsets Q and S of $\mathbf{R}^n$, along with a point $\bar{x} \in Q \cap S$. We define a function $\varphi\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ by

$$\varphi(x,y) := \delta_Q(x) + \|x-y\| + \delta_S(y).$$

In terms of φ, the *method of alternating projections* can be written as follows: if the iterate x_i lies in Q, then define the next iterates by the inclusions

$$x_{i+1} \in \operatorname{argmin}\, \varphi(x_i,\cdot) \qquad \text{and} \qquad x_{i+2} \in \operatorname{argmin}\, \varphi(\cdot, x_{i+1}).$$

Therefore it is not surprising that convergence analysis for the method will revolve around variational properties of the function φ. In particular, the partial slopes of φ separately in the variables x and y will play a key role. For ease of notation, the symbols φ_y and φ_x will denote the functions $x \mapsto \varphi(x,y)$ and $y \mapsto \varphi(x,y)$, respectively. We now make some key observations about the function φ. One can easily check using Theorem 2.3.3 that for any points $x \in Q$ and $y \in S$ with $x \neq y$, we have

$$\overline{|\nabla \varphi_y|}(x) = d(0, \partial \varphi_y(x)) = d\Big(\frac{y-x}{\|y-x\|}, N_Q(x)\Big), \tag{3.1}$$

$$\overline{|\nabla \varphi_x|}(y) = d(0, \partial \varphi_x(y)) = d\Big(\frac{y-x}{\|y-x\|}, -N_S(y)\Big). \tag{3.2}$$

Moreover, for such points x and y, the equation

$$\partial \varphi(x,y) = \partial \varphi_y(x) \times \partial \varphi_x(y) \qquad \text{holds,}$$

and hence we deduce

$$\begin{aligned} \overline{|\nabla \varphi|}(x,y) = d(0, \partial\varphi(x,y)) &= \sqrt{\Big(d(0,\partial\varphi_y(x)\Big)^2 + \Big(d(0,\partial\varphi_x(y)\Big)^2} \\ &= \sqrt{\Big(\overline{|\nabla\varphi_y|}(x)\Big)^2 + \Big(\overline{|\nabla\varphi_x|}(y)\Big)^2}. \end{aligned}$$

Given two closed cones $C_i,\ i = 1, 2$ in $\mathbf{R}^n$, we define the *opening* between C_1 and C_2 to be

$$\rho(C_1, C_2) = \sup\{\langle u_1, u_2\rangle :\ u_i \in C_i,\ \|u_i\| = 1\}.$$

Clearly $\rho(C_1, C_2)$ is the cosine of the minimal angle between nonzero elements of the sets C_1 and C_2. In particular, if we have $\rho(C_1, C_2) = 1$, then the cones C_1 and C_2 have a common nonzero element. The following are the two key notions that will drive the convergence analysis.

Definition 3.2.1 (Transversality)**.** Consider two closed subsets Q and S of $\mathbf{R}^n$ and a point $\bar{x} \in Q \cap S$. We define the following notions.

- Q and S are *transversal* at $\bar{x}$ if the equation

$$N_Q(\bar{x}) \cap (-N_S(\bar{x})) = \{0\} \qquad \text{holds.}$$

- Q and S are *intrinsically transversal at $\bar{x}$ with modulus κ* if there exists a real number $\epsilon > 0$ so that for any points $x \in Q \cap B_\epsilon(\bar{x})$ and $y \in S \cap B_\epsilon(\bar{x})$ we have

$$\max\left\{\ d\Big(\frac{y-x}{\|y-x\|}, N_Q(x)\Big),\ d\Big(\frac{y-x}{\|y-x\|}, -N_S(y)\Big)\ \right\} \geq \kappa.$$

Observe that in light of equations (3.1) and (3.2), intrinsic transversality with modulus κ amounts to requiring that there exist $\epsilon > 0$ so that the inequality

$$\max\{|\nabla\varphi(\cdot, y)|(x), |\nabla\varphi(x, \cdot)|(y)\} \geq \kappa,$$

holds for any points $x \in Q \cap B_\epsilon(\bar{x})$ and $y \in S \cap B_\epsilon(\bar{x})$.

Transversality is a central notion in variational analysis and the theory of metric regularity. In particular, it guarantees that the two sets Q and S cannot easily be pulled apart. For more details, see for example [108, Proposition 9.69].

The following proposition shows that transversality entails intrinsic transversality.

Proposition 3.2.2 (Transversality & intrinsic transversality). *If two closed subsets Q and S of $\mathbf{R}^n$ are transversal at a point $\bar{x} \in Q \cap S$, then they are intrinsically transversal at $\bar{x}$ with modulus κ satisfying*

$$\kappa \geq \frac{1-\sqrt{\theta}}{1+\sqrt{\theta}} - \epsilon,$$

where $\theta = \rho(N_Q(\bar{x}), -N_S(\bar{x}))$ and $\epsilon > 0$ can be chosen to be arbitrarily small.

Proof. Fix a certain $\kappa > 0$ and assume that there are sequences $x_i \in Q$ and $y_i \in S$, with $x_i \neq y_i$ for each i, satisfying $x_i \to \bar{x}$ and $y_i \to \bar{x}$, and such that

$$|\nabla \varphi_{y_i}|(x_i) < \kappa, \quad |\nabla \varphi_{x_i}|(y_i) < \kappa \qquad \text{for each } i.$$

Hence the functions

$$x \mapsto \varphi_{y_i}(x) + \kappa\|x - x_i\| \quad \text{and} \quad y \mapsto \varphi_{x_i}(y) + \kappa\|y - y_i\| \tag{3.3}$$

attain local minima respectively at x_i and y_i. We deduce

$$0 \in w_i + \frac{x_i - y_i}{\|x_i - y_i\|} + \kappa\mathbf{B}, \qquad 0 \in z_i + \frac{y_i - x_i}{\|x_i - y_i\|} + \kappa\mathbf{B} \tag{3.4}$$

for some vectors $w_i \in N_Q(x_i)$ and $z_i \in N_S(y_i)$. Thus, for any limit point (w, z) of (w_i, z_i), we have

$$w = e + a, \qquad z = -e + b,$$

for some vectors e and a satisfying $\|e\| = 1,\ \|a\| \leq \kappa,\ \|b\| \leq \kappa$. Consequently we obtain

$$\theta \geq \frac{\langle e + a, e - b\rangle}{\|e + a\|\|e - b\|} \geq \frac{(1-\kappa)^2}{(1+\kappa)^2}.$$

The result follows immediately. □

The converse of the theorem above is decisively false. For example, two distinct lines in $\mathbf{R}^3$ are never transversal, though they are intrinsically transversal.

Theorem 3.2.3 (Alternating projections and intrinsic transversality)**.** *Consider two closed subsets Q and S of $\mathbf{R}^n$ that are intrinsically transversal at a point $\bar{x} \in Q \cap S$ with modulus κ. Then for any real number $c \in (0, \kappa)$, there exists $\delta > 0$ such that the method of alternating projections, when started from a point lying in $B_\delta(\bar{x})$, converges R-linearly with rate $\sqrt{1-c^2}$ to a point $\hat{x}$ in the intersection $Q \cap S$.*

Proof. Since Q and S are intrinsically transversal at $\bar{x}$, there exists $\epsilon > 0$ so that the inequality

$$\max\{|\nabla\varphi(\cdot, y)|(x), |\nabla\varphi(x, \cdot)|(y)\} \geq \kappa,$$

holds for any points $x \in Q \cap B_\epsilon(\bar{x})$ and $y \in S \cap B_\epsilon(\bar{x})$.

Consider two points $u \in Q$ and $v \in S$, with $u \neq v$, and satisfying $v \in P_S(u)$. Consider the set

$$\mathcal{K} = \left\{x \in \mathbf{R}^n \setminus \{v\} :\ \left\langle \frac{u-v}{\|u-v\|}, \frac{x-v}{\|x-v\|} \right\rangle < \sqrt{1-\kappa^2} \right\}.$$

Observe that $\mathcal{K}$ is an ice cream cone with vertex v and $u - v$ being the direction of its axis. Define the real number $\rho := \|u - v\|$ and observe that the distance from u to the boundary of $\mathcal{K}$ is precisely $\kappa\rho$.

Suppose now that both u and v lie in the ball $B_{\epsilon-\kappa\rho}(\bar{x})$. Observe that for any point $x \in Q \cap B_{\kappa\rho}(u)$ we have $\overline{|\nabla\varphi(x, \cdot)|}(v) = d\left(\frac{v-x}{\|v-x\|}, -N_S(v)\right) < \kappa$. Consequently for such points x we have $|\nabla\varphi(\cdot, v)|(x) \geq \kappa$. Now define $\alpha = (1-c^2)\rho$. We can apply Lemma 2.2.3 with $f = \varphi(\cdot, v)$, r replaced by κ, and and K replaced by $c\rho$ (note that u and x have switched roles). According to the lemma there is an $\tilde{x} \in Q$ such that $\varphi(\tilde{x}, v) \leq \alpha = (1-c^2)\rho$. It follows that for any $x \in P_Q(v)$, we have $\|x - v\| \leq \|\tilde{x} - v\| = \varphi(\tilde{x}, v) \leq (1-c^2)\|u - v\|$.

Hence if the iterates $u = x_n$ and $v = x_{n+1}$ lie in the ball $B_{\epsilon-\kappa\|x_{n+1}-x_n\|}(\bar{x})$, we have the inequality

$$\|x_{n+2} - x_{n+1}\| \leq (1 - c^2)\|x_{n+1} - x_n\|. \tag{3.5}$$

Observe that if for each index n we had $x_{n+1} \in B_{\epsilon-\kappa\|x_{n+1}-x_n\|}(\bar{x})$, then we would deduce

$$\sum_{i=0}^{\infty} \|x_{i+1} - x_i\| \leq \frac{\|x_1 - x_0\|}{c^2}.$$

Suppose now that the initial point x_0 lies within $\frac{\epsilon}{(1+\kappa)+\frac{1}{c^2}}$ of $\bar{x}$. We deduce

$$\begin{aligned}\|x_{n+1} - \bar{x}\| &\leq \sum_{i=0}^{n} \|x_{i+1} - x_i\| + \|x_0 - \bar{x}\| \leq (1 + c^{-2})\|x_0 - \bar{x}\| \\ &\leq \epsilon - \kappa\|x_0 - \bar{x}\| \leq \epsilon - \kappa\|x_{n+1} - x_n\|.\end{aligned}$$

Consequently, the inequality (3.5) holds for all indices n. A standard induction argument (see for example the proof of [84, Theorem 5.2]) then implies that the sequence x_n converges R-linearly with rate $\sqrt{1-c^2}$ to a point $\hat{x}$ in $Q \cap S$. $\square$

The following now follows immediately by appealing to Proposition 3.2.2 and Theorem 3.2.3.

Corollary 3.2.4 (Alternating projections and transversality). *Consider two closed subsets Q and S of $\mathbf{R}^n$ that are transversal at a point $\bar{x} \in Q \cap S$. Then for any real number c satisfying*

$$0 < c < \frac{1-\sqrt{\theta}}{1+\sqrt{\theta}},$$

where $\theta = \rho(N_Q(\bar{x}), -N_S(\bar{x}))$, there exists $\delta > 0$ such that the method of alternating projections, when started from a point lying in $B_\delta(\bar{x})$, converges R-linearly with rate $\sqrt{1-c^2}$ to a point $\hat{x}$ in the intersection $Q \cap S$.

It is interesting to ask whether convergence of alternating projections is guaranteed (albeit sublinear) without intrinsic transversality. This is particularly important, since in practice to check whether transversality holds requires one to already know a point in the intersection. It is easy to come up with pathological examples showing that without intrinsic transversality even limit points of the iterates generated by the method are not guaranteed to lie in the intersection. Nevertheless, such pathologies do not occur in the semi-algebraic setting.

Theorem 3.2.5 (Semi-algebraic intersections)**.** *Consider two closed semi-algebraic subsets Q and S of $\mathbf{R}^n$ with nonempty intersection, and suppose that Q is compact. If the initial point $x_0 \in S$ is sufficiently close to Q, then every limit point of the iterates produced by the method of alternating projections lies in the intersection $Q \cap S$.*

Proof. Let U be any bounded open set containing Q. Observe that the function

$$\varphi(x, y) = \delta_Q(x) + \|x - y\| + \delta_S(y),$$

is semi-algebraic. Clearly the set $\{(x, x) : x \in Q \cap S\}$ is comprised of critical points of φ. Then the KŁ-inequality implies that there is a constant $\rho > 0$ and a nonnegative continuous function $\psi \colon [0, \rho) \to \mathbf{R}$ that is $\mathbf{C}^1$-smooth on $(0, \rho)$ with $\phi' > 0$, and satisfies

$$\overline{|\nabla \varphi|}(x, y) \geq \frac{1}{\psi'(\|x - y\|)}$$

whenever $(x, y) \in (U \times U) \cap (Q \times S)$ and $0 < \|x - y\| < \rho$. Consequently

$$\max\left\{\overline{|\nabla \varphi_x|}(y), \overline{|\nabla \varphi_y|}(x)\right\} \geq \frac{1}{\sqrt{2}} \cdot \frac{1}{\psi'(\|x - y\|)}, \tag{3.6}$$

whenever we have $(x, y) \in (U \times U) \cap (Q \times S)$ and $0 < \|x - y\| < \rho$.

We may now decrease ρ so as to ensure that ψ' is bounded on compact subsets of $(0, \rho)$ and so that the set

$$W = \{y \in \mathbf{R}^n : d(y, Q) < \rho\},$$

is contained in U. Let the initial point x_0 lie in $S \cap W$. Observe then that any successive iterates x_n, x_{n+1} produced by the method of alternating projections would still lie in W and hence would satisfy $\|x_n - x_{n+1}\| < \rho$. Thus the KŁ-inequality applies.

Consider now a pair $x_n \in Q$ and $x_{n+1} \in S$ with $x_{n+1} \in P_S(x_n)$. Define $\alpha(x, x_n)$ to be the angle between the vectors $x - x_{n+1}$ and $x_n - x_{n+1}$. Observe

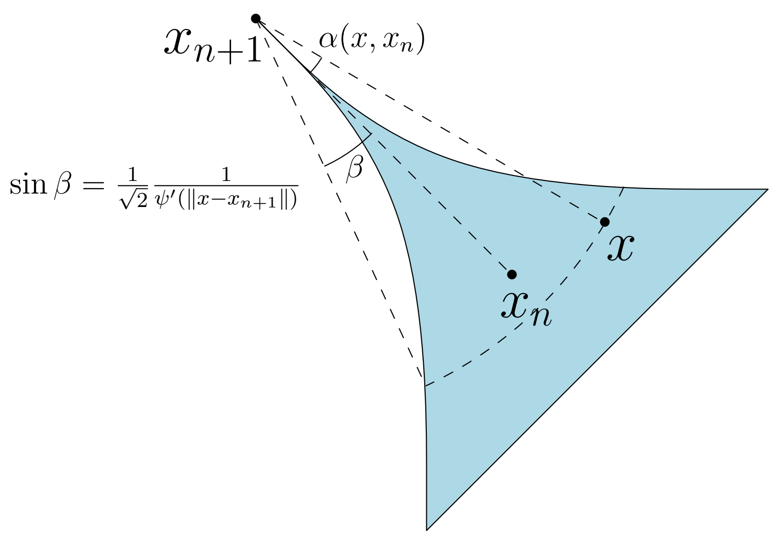
$$\overline{|\nabla \varphi(\cdot, x_{n+1})|}(x) = d\Big(0, \frac{x_{n+1} - x}{\|x_{n+1} - x\|} + N_S(x_{n+1})\Big) \leq \sin \alpha(x, x_{n+1}).$$

Consider the set

$$\mathcal{K} := \Big\{x \in B_\rho(x_{n+1}) \setminus \{x_{n+1}\} : \sin \alpha(x, x_{n+1}) < \frac{1}{\sqrt{2}} \cdot \frac{1}{\psi'(\|x - x_{n+1}\|)}\Big\}.$$

See Figure 3.1 below for an illustration.

Figure 3.1: The set $\mathcal{K}$



For any point $x \in \mathcal{K} \cap Q$, using inequality (3.6), we deduce

$$\overline{|\nabla \varphi(x, \cdot)|}(x_{n+1}) \geq \frac{1}{\sqrt{2}} \cdot \frac{1}{\psi'(\|x - x_{n+1}\|)}.$$

Observe that x_n is contained in $\mathcal{K}$ since ψ' is bounded on compact subsets of $(0, \rho)$. Now define

$$r_n := \sup\{r > 0 : B_r(x_n) \subset \mathcal{K}\} \qquad \text{and} \qquad K_n = \inf_{x \in B_{r_n}(x_n)} \frac{1}{\sqrt{2}} \cdot \frac{1}{\psi'(\|x - x_{n+1}\|)}.$$

Observe that both quantities r_n and K_n are strictly positive. Using Lemma 2.2.3 as in the proof of Theorem 3.2.3, we deduce

$$\|x_{n+2} - x_{n+1}\| \leq \|x_{n+1} - x_n\| - K_n r_n. \tag{3.7}$$

Suppose now for the sake of contradiction that the quantity $\|x_{n+1} - x_n\|$ does not tend to zero. We claim that the constants $K_n r_n$ are uniformly bounded away from zero

To see this, fix a real number $\epsilon > 0$ satisfying $\epsilon < \min\{\|x_n - x_{n+1}\|, \rho - \|x_n - x_{n+1}\|\}$ and consider the set $\mathcal{Z} := \mathcal{K} \setminus B_\epsilon(x_{n+1})$. Continuity of ψ' on $(0, \rho)$ then implies that there exists some real $\delta > 0$ such that a δ-tube around the segment

$$\{\lambda(x_n - x_{n+1}) : \epsilon < \lambda \|x_n - x_{n+1}\| < \rho - \epsilon\}$$

is contained in $\mathcal{Z}$. We conclude then that r_n is bounded away from zero. Moreover, since the quantity $\psi'(\|x - x_{n+1}\|)$ is bounded over all points $x \in \mathcal{Z}$, the numbers K_n are bounded away from zero too. Thus we have arrived at a contradiction. It easily from equation (3.7) that every limit point of the iterates x_n lies in the intersection $Q \cap S$. $\square$

CHAPTER 4

CURVES OF DESCENT

4.1 Introduction

The intuitive notion of *steepest descent* plays a central role in theory and practice. So what are steepest descent curves in an entirely nonsmooth setting? Observe that for any 1-Lipschitz curve $\gamma\colon (a,b) \to \mathbf{R}^n$, the inequality

$$|\nabla(f \circ \gamma)|(t) \leq |\nabla f|(\gamma(t)) \qquad \text{holds for a.e. } t \in (a,b). \tag{4.1}$$

Supposing that f is continuous (for technical reasons), it is then natural to call γ a *steepest descent curve* if $f \circ \gamma$ is nonincreasing and the reverse inequality holds in (4.1). Such curves, up to a reparametrization and an integrability condition, are the curves of maximal slope studied in [2, 42, 44, 88]. Replacing the slope $|\nabla f|$ with its lower-semicontinuous envelope in equation (4.1) defines *near-steepest descent curves*. See Definition 4.2.5 for a more precise statement.

The question concerning existence of near-steepest descent curves is at the core of the subject. Roughly speaking, there are two strategies in the literature for constructing such curves for a function f on $\mathbf{R}^n$. The first one revolves around minimizing f on an increasing sequence of balls around a point until the radius hits a certain threshold, at which point one moves the center to the next iterate and repeats the procedure. Passing to the limit as the thresholds tend to zero, under suitable conditions and a reparametrization, yields a near-steepest descent curve [88, Section 4]. The second approach is based on De Georgi's generalized movements [42]. Namely, one builds a piecewise-constant curve by declaring the next iterate to be a minimizer of the function f plus a scaling of the squared distance from the previous iterate [2, Chapter 2]. The analysis, in both cases, is highly nontrivial and moreover does not give an intuitive meaning to the parametrization of the curve used in the construction.

In the current work, we propose an alternate transparent strategy for constructing near-steepest descent curves. The key idea of our construction is to discretize the range of f and then build a piecewise-linear curve by projecting iterates onto successive sublevel sets. Passing to the limit as the mesh of the partition tends to zero, under reasonable conditions and a reparametrization, yields a near-steepest descent curve. Moreover, the parametrization of the curve used in the construction is entirely intuitive: the values of the function parametrize the curve. From a technical viewpoint, this type of a parametrization allows for the deep theory of metric regularity to enter the picture [71, 107], thereby yielding a simple and elegant existence proof.

The question concerning when solutions of subgradient dynamical systems and near-steepest descent curves are one and the same has been studied as well. However a major standing assumption that has so far been needed to establish positive answers in this direction is that the slope of the function f is itself a lower-semicontinuous function [2, 88] and hence it coincides with the limiting slope — an assumption that many common functions of nonsmooth optimization (e.g. $f(x) = \min\{x, 0\}$) do not satisfy. In the current work, we study this question in absence of such a continuity condition. As a result, *semi-algebraic functions* — those functions whose epigraph can be written as a finite union of sets, each defined by finitely many polynomial inequalities [34, 115] — come to the fore.

For semi-algebraic functions that are locally Lipschitz continuous on their domains, solutions of subgradient dynamical systems are one and the same as curves of near-maximal slope. Going a step further, using an argument based on the Kurdyka-Łojasiewicz inequality, in the spirit of [9, 12, 74, 78], we show that bounded curves of near-maximal slope for semi-algebraic functions necessarily have finite length. Consequently, such curves defined on maximal domains must converge to a critical point of f. This, in turn, allows us to strengthen the obtained existence theory in the semi-algebraic setting. Namely, we will show that for semi-algebraic functions, there exist near-steepest descent curves emanating from any point that are either unbounded or converge to local minimizers (and not just to critical points). To illustrate the subtlety of this fact, we show in Example 4.6.1 that this can easily fail even for $\mathbf{C}^{\infty}$-smooth functions.

In our writing style, rather than striving for maximal generality, we have tried to make the basic ideas and the techniques as clear as possible. The outline of the chapter is as follows. Section 4.2 is a short self-contained treatment of absolutely continuous curves in metric spaces. In Section 4.3 we prove that curves of near-steepest descent exist under reasonable conditions. In Section 4.4 we analyze conditions under which curves of near-maximal slope are the same as solutions to subgradient dynamical systems. In Sections 4.5 and 4.6 we study descent curves in the setting of semi-algebraic geometry.

4.2 Absolute continuity, metric derivative, and steepest descent

In this section, we adhere closely to the notation and the development in [2].

Definition 4.2.1 (Absolutely continuous curves)**.** Consider a curve $\gamma\colon (a,b) \to \mathcal{X}$. We will say that γ is *absolutely continuous*, denoted $\gamma \in AC(a,b,\mathcal{X})$, provided that there exists an integrable function $m\colon (a,b) \to \mathbf{R}$ satisfying

$$d(\gamma(s), \gamma(t)) \leq \int_s^t m(\tau)\, d\tau, \quad \text{whenever } a < s \leq t < b. \tag{4.2}$$

Every curve $\gamma \in AC(a, b, \mathcal{X})$ is uniformly continuous. Moreover, the right and left limits of γ, denoted respectively by $\gamma(a)$ and $\gamma(b)$, exist. There is a canonical choice for the integrand appearing in the definition of absolute continuity, namely the "metric derivative".

Definition 4.2.2 (Metric derivative)**.** For any curve $\gamma\colon [a,b] \to \mathcal{X}$ and any $t \in (a,b)$, the quantity

$$\|\dot{\gamma}(t)\| := \lim_{s \to t} \frac{d(\gamma(s), \gamma(t))}{|s-t|},$$

if it exists, is the *metric derivative* of γ at t. If this limit does exist at t, then we will say that γ is *metrically differentiable* at t.

For any curve $\gamma \in AC(a, b, \mathcal{X})$, the metric derivative exists almost everywhere on (a,b). Moreover the function $t \mapsto \|\dot{\gamma}(t)\|$ is integrable on (a,b) and is an admissible integrand in inequality (4.2). In fact, as far as such integrands are concerned, the metric derivative is in a sense minimal. Namely, for any admissible integrand $m\colon (a,b) \to \mathbf{R}$ for the right-hand-side of (4.2), the inequality

$$\|\dot{\gamma}(t)\| \leq m(t) \quad \text{holds for a.e. } t \in (a,b).$$

See [2, Theorem 1.1.2] for more details. We can now define the *length* of any absolutely continuous curve $\gamma \in AC(a, b, \mathcal{X})$ by the formula

$$\text{length}\,(\gamma) := \int_a^b \|\dot{\gamma}(\tau)\|\, d\tau.$$

We adopt the following convention with respect to curve reparametrizations.

Definition 4.2.3 (Curve reparametrization)**.** Consider a curve $\gamma\colon [a,b] \to \mathcal{X}$. Then any curve $\omega\colon [c,d] \to \mathcal{X}$ is a *reparametrization* of γ whenever there exists a nondecreasing absolutely continuous function $s\colon [c,d] \to [a,b]$ with $s(c) = a$, $s(d) = b$, and satisfying $\omega = \gamma \circ s$.

Absolutely continuous curves can always be parametrized by arclength. See for example [2, Lemma 1.1.4].

Theorem 4.2.4 (Arclength parametrization)**.** *Consider an absolutely continuous curve $\gamma \in AC(a,b,\mathcal{X})$, and denote its length by $L = \operatorname{length}(\gamma)$. Then there exists a nondecreasing absolutely continuous map $s\colon [a,b] \to [0,L]$ with $s(a) = 0$ and $s(b) = L$, and a 1-Lipschitz curve $v\colon [0,L] \to \mathcal{X}$ satisfying*

$$\gamma = v \circ s \quad \textit{and} \quad \|\dot{v}\| = 1 \textit{ a.e. in } [0,L].$$

Consider a lsc function $f\colon \mathcal{X} \to \mathbf{R}$ and a 1-Lipschitz continuous curve $\gamma\colon [a,b] \to \mathcal{X}$. There are two intuitive requirements that we would like γ to satisfy in order to be called a steepest descent curve:

1. The composition $f \circ \gamma$ is non-increasing on a full-measure subset of $[a,b]$,
2. The instantaneous rate of decrease of $f \circ \gamma$ is almost always as great as possible.

To elaborate on the latter requirement, suppose that the composition $f \circ \gamma$ is indeed non-increasing on a full-measure subset of $[a,b]$. Then there exists a non-increasing function $\phi\colon [a,b] \to \overline{\mathbf{R}}$ coinciding almost everywhere with $f \circ \gamma$. Note that in particular, whenever $f \circ \gamma$ is continuous, we may simply take $\phi := f \circ \gamma$. Now taking into account that γ is 1-Lipschitz continuous and that monotone functions are differentiable a.e., one can readily verify

$$|\phi'(t)| \leq |\nabla f|(\gamma(t)) \qquad \text{for a.e. } t \in [a,b]. \tag{4.3}$$

Requiring the reverse inequality to hold amounts to forcing the curve to achieve fastest instantaneous rate of decrease. The discussion above motivates the following definition.

Definition 4.2.5 (Near-steepest descent curves)**.** Consider a lsc function $f\colon \mathcal{X} \to \overline{\mathbf{R}}$. Then a 1-Lipschitz curve $\gamma\colon [a,b] \to \mathcal{X}$ is a *steepest descent curve* if $f \circ \gamma$ coincides a.e. with some nonincreasing function $\phi\colon [a,b] \to \mathbf{R}$ and the inequality

$$|\phi'(t)| \geq |\nabla f|(\gamma(t)) \qquad \text{holds a.e. on } [a,b].$$

If instead the weaker inequality

$$|\phi'(t)| \geq \overline{|\nabla f|}(\gamma(t)) \qquad \text{holds a.e. on } [a,b],$$

then we will say that γ is a *near-steepest descent curve*.

Remark 4.2.6. It is easy to see that the defining inequalities of steepest and near-steepest descent curves are independent of the particular choice of the function $\phi\colon [a,b] \to \mathbf{R}$. Namely, if those inequalities hold for some nondecreasing function agreeing a.e. with $f \circ \gamma$, then they will hold for any other function agreeing a.e with $f \circ \gamma$.

In principle, near-steepest descent curves may fall short of achieving true "steepest descent", since the analogue of inequality (4.3) for the limiting slope may fail to hold in general. Our work, however, will revolve around near-steepest descent curves since the limiting slope is a much better behaved object, and anyway this is common practice in the literature (see for example [2,44,88]). The following example illustrates the difference between the two notions.

Example 4.2.7 (Steepest descent vs. near-steepest descent)**.** Consider the function $f\colon \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x,y) := -x+\min(y,0)$. Then the curve $x(t)=(t,0)$ is a near-steepest descent curve but is not a steepest descent curve, as one can easily verify.

It is often convenient to reparametrise near-steepest descent curves so that their speed is given by the slope. This motivates the following companion notion; related concepts appear in [2, Section 1.3], [35, 88].

Definition 4.2.8 (Curve of near-maximal slope)**.** Consider a lsc function $f\colon \mathcal{X} \to \overline{\mathbf{R}}$. A curve $\gamma\colon [a,b] \to \mathcal{X}$ is a *curve of near-maximal slope* if the following conditions holds:

(a) γ is absolutely continuous,

(b) $\|\dot{\gamma}(t)\| = \overline{|\nabla f|}(\gamma(t))$ a.e. on $[a,b]$,

(c) $f \circ \gamma$ coincides a.e. with some nonincreasing function $\phi\colon [a,b] \to \mathbf{R}$ and the inequality

$$\phi'(t) \leq -\big(\overline{|\nabla f|}(\gamma(t))\big)^2 \quad \text{holds a.e. on } [a,b].$$

The following proposition shows that, as alluded to above, under reasonable conditions near-steepest descent curves and curves of near-maximal slope are the same up to reparametrization. For a proof, see [49, Proposition 2.16].

Proposition 4.2.9 (Curves of near-steepest descent & near-maximal slope)**.** *Let $f\colon \mathcal{X} \to \overline{\mathbf{R}}$ be an lsc function.*

Consider a near-steepest descent curve $\gamma\colon [a,b] \to \mathcal{X}$ satisfying $\|\dot{x}\| = 1$ a.e. on $[a,b]$. If $(\overline{|\nabla f|} \circ \gamma)^{-1}$ is integrable, then there exists a reparametrization of γ that is a *curve of near-maximal slope.*

Conversely, consider a curve of near-maximal slope $\gamma\colon [a,b] \to \mathcal{X}$. If the composition $\overline{|\nabla f|} \circ \gamma$ is integrable, then there exists a reparametrization of γ that is a near-steepest descent curve satisfying $\|\dot{x}\| = 1$ a.e. on $[a,b]$.

4.3 Existence of descent curves

In this section, we provide a natural and transparent existence proof for near-steepest descent curves in complete locally convex metric spaces. We begin with a few relevant definitions, adhering closely to the notation of [77].

Definition 4.3.1 (Metric segments)**.** A subset S of a metric space $\mathcal{X}$ is a *metric segment* between two points x and y in $\mathcal{X}$ if there exists a closed interval $[a,b]$ and an isometry $\omega\colon [a,b] \to \mathcal{X}$ satisfying $\omega([a,b]) = S$, $\omega(a) = x$, and $\omega(b) = y$.

Definition 4.3.2 (Convex metric spaces)**.** We will say that $\mathcal{X}$ is a *convex metric space* if for any distinct points $x, y \in \mathcal{X}$ there exists a metric segment between them. We will call $\mathcal{X}$ a *locally convex metric space* if each point in $\mathcal{X}$ admits a neighborhood that is a convex metric space in the induced metric.

Some notable examples of locally convex metric spaces are complete Riemannian manifolds and, more generally, length spaces that are complete and locally compact. For more examples, we refer the reader to [64].

We now introduce the following very weak continuity condition, which has been essential in the study of descent curves in metric spaces. See for example [2, Theorem 2.3.1].

Definition 4.3.3 (Continuity on slope-bounded sets)**.** Consider a function $f: \mathcal{X} \to \overline{\mathbf{R}}$. We will say that f is *continuous on slope-bounded sets* provided that for any point $\bar{x} \in \operatorname{dom} f$ the implication

$$x_i \to \bar{x} \quad \text{with} \quad \sup_{i \in \mathbb{N}} \{|\nabla f|(x_i), f(x_i)\} < \infty \qquad \Longrightarrow \qquad f(x_i) \to f(\bar{x}),$$

holds.

We now arrive at the main result of this section. We should note that in the following theorem we will suppose strong compactness assumptions relative to the metric topology. As is now standard, such compactness assumptions can be sidestepped by instead introducing weaker topologies [2, Section 2.1]. On the other hand, following this route would take us far off field and would lead to technical details that may obscure the main proof ideas for the reader. Hence we do not dwell on this issue further. We have however designed our proof so as to make such an extension as easy as possible for interested readers.

An obvious implication of the following theorem is that for any function $f: \mathcal{X} \to \overline{\mathbf{R}}$, satisfying reasonable conditions, there exist near-steepest descent curves emanating from any point $\bar{x}$ in its domain. The theorem, however, does much more than that! To illustrate, observe that if $\bar{x}$ is a critical point of f, then the constant curve $x(t) \equiv \bar{x}$ is trivially a near-steepest descent curve. Theorem 4.3.4, on the other hand, shows that even if $\bar{x}$ is a critical point of f, provided that $\bar{x}$ is not a local minimizer and a certain condition that will come into focus in Section 4.4 holds, there will exist a *nontrivial* near-steepest descent curve $\gamma: [0, L] \to \mathcal{X}$ emanating from $\bar{x}$.

Theorem 4.3.4 (Existence of near-steepest descent curves)**.** *Consider a lsc function $f: \mathcal{X} \to \overline{\mathbf{R}}$ on a complete locally convex metric space $\mathcal{X}$, along with a point $\bar{x} \in \operatorname{dom} f$ which is not a local minimizer of f. Suppose that the following are true:*

1. *f is continuous on slope-bounded sets and bounded closed subsets of sublevel sets of f are compact.*

2. *There exist constants $\alpha < f(\bar{x})$ and $r, R > 0$ so that the implication*

$$\alpha < f(u) < f(\bar{x}) \quad \textit{and} \quad d(u, \bar{x}) < R \quad \Longrightarrow \quad |\nabla f|(u) \geq r,$$

holds for any point $u \in \mathcal{X}$.

Then there exists a curve $\gamma\colon [0, L] \to \mathcal{X}$ emanating from $\bar{x}$ and satisfying the following.

Decrease in value: *The composition $f \circ \gamma$ is strictly decreasing on a full-measure subset of $[0, L]$ and consequently the inclusion $\gamma(0, L] \in [f < f(\bar{x})]$ holds.*

Near-steepest descent: *γ is a near-steepest descent curve.*

Regularity: *For a.e. $t \in [0, L]$, the slope $\overline{|\nabla f|}(\gamma(t))$ is finite and we have $\|\dot{\gamma}(t)\| = 1$.*

Proof. First, by restricting attention to a sufficiently small neighborhood of $\bar{x}$ we can clearly assume that $\mathcal{X}$ is a convex metric space. Define $\eta := f(\bar{x}) - \alpha$ and $C > 0$ to be slightly smaller than R. Increasing α we may enforce the inequality $\eta < rC$. Let $0 = \tau_0 < \tau_1 < \ldots < \tau_k = \eta$ be a partition of $[0, \eta]$ into k equal parts. We will adopt the notation

$$\lambda := \frac{\tau_{i+1} - \tau_i}{\eta}, \qquad \alpha_i = f(\bar{x}) - \tau_i, \qquad L_i := [f \leq \alpha_i].$$

With this partition, we will associate a certain curve $u_k(\tau)$ for $\tau \in [0, \eta]$, naturally obtained by concatenating metric segments between points x_i, x_{i+1} lying on consecutive sublevel sets. See Figure 3.1 for an illustration. For notational convenience, we will often suppress the index k in $u_k(\tau)$. The construction is as follows. Set $x_0 := \bar{x}$ and let x_1 be any point of L_1 satisfying $d(x_1, \bar{x}) \leq d(\bar{x}, L_1) + \frac{1}{k}$. Observe that since $\bar{x}$ is not a local minimizer of f, the sublevel set L_1 is nonempty for all sufficiently large indices k, and hence the point x_1 is well-defined for such k. We will assume throughout the proof that k is chosen sufficiently large for this to be the case. We will now inductively define x_{j+1}. It is important, however, to keep in mind that the point x_1 is rather exceptional, since we assume nothing about the slope of f exactly at $\bar{x}$. Nevertheless, define $\rho := d(x_0, x_1)$ and observe that since $\bar{x}$ is not a local minimizer of f, the distance ρ tends to zero as k tends to infinity. We will assume throughout that k is sufficiently large to ensure $\rho < C - r^{-1}\eta$, $C + \rho < R$, and $\rho < (1-\lambda)C$.

We now proceed with the inductive definition of x_{j+1}. To this end, suppose that we have defined points x_i for $i = 1, \ldots, j$. Consider the quantity

$$r_j := \inf \{|\nabla f|(y) : \ \alpha_{j+1} < f(y) \leq f(x_j), \ d(y, x_j) < \lambda C\}.$$

and let x_{j+1} be any point satisfying

$$x_{j+1} \in L_{j+1} \qquad \text{and} \qquad d(x_{j+1}, x_j) \leq r_j^{-1}(f(x_j) - \alpha_{j+1})^+.$$

(In our setting, due to the compactness of bounded closed subsets of sublevel sets of f, we may simply define x_{j+1} to be any closest point of L_{j+1} to x_j.)

Claim 4.3.5 (Well-definedness)**.** *For all indices $i = 1, \ldots, k$, the points x_i are well defined and satisfy*

$$d(x_{i+1}, x_i) \leq r_i^{-1}(\tau_{i+1} - \tau_i), \tag{4.4}$$

and

$$r_i \geq r, \quad d(x_i, \bar{x}) \leq r^{-1}\tau_i + \rho. \tag{4.5}$$

Proof. The proof proceeds by induction. First, observe that due to our choice of how large to make k, the point x_1 is well-defined and inequalities (4.5) hold for $i = 1$. Proceeding to the inductive step, suppose that the points x_i are well defined for indices $i = 1, \ldots, j$, the inequalities (4.5) are valid for $i = 1, \ldots, j$, and the inequality (4.4) is valid for indices i satisfying $1 \leq i \leq j - 1$.

Observe if the inequality $f(x_j) \leq \alpha_{j+1}$ were true, then we may set $x_{j+1} := x_j$ and the inductive step would be true trivially. Hence suppose otherwise. We claim that the conditions of Lemma 2.2.3 are satisfied with $x = x_j$, $\alpha = \alpha_{j+1}$, $K = \lambda C$, and with r_j in place of r.

To this end, we show the following

- $f(x_j) - \alpha_{j+1} \leq \lambda r_j C$;
- $\alpha_{j+1} < f(y) \leq f(x_j)$ and $d(y, x_j) \leq \lambda C \quad \Longrightarrow \quad |\nabla f|(y) \geq r_j$.

Observe

$$f(x_j) - (f(\bar{x}) - \tau_{j+1}) \leq \tau_{j+1} - \tau_j \leq (\tau_{j+1} - \tau_j)\frac{rC}{\eta} = \lambda rC \leq \lambda r_j C,$$

which is the first of the desired relations. The second relation follows immediately from the definition of r_j.

Applying Lemma 2.2.3, we conclude that the point x_{j+1} is well-defined and the inequality $d(x_{j+1}, x_j) \leq r_j^{-1}(f(x_j) - \alpha_{j+1})^+ \leq r_j^{-1}(\tau_{j+1} - \tau_j)$ holds. Consequently, we obtain

$$d(x_{j+1}, \bar{x}) \leq d(x_{j+1}, x_j) + d(x_j, \bar{x}) \leq r_j^{-1}(\tau_{j+1} - \tau_j) + r^{-1}\tau_j + \rho \leq r^{-1}\tau_{j+1} + \rho.$$

Finally we claim that the inequality $r_{j+1} \geq r$ holds. To see this, consider a point y satisfying $f(\bar{x}) - \tau_{j+2} < f(y) \leq f(x_{j+1})$ and $d(y, x_{j+1}) < \lambda C$. Taking (4.5) into account, along with the inequality $r^{-1} \leq C/\eta$, we obtain

$$d(y, \bar{x}) \leq d(y, x_{j+1}) + d(x_{j+1}, \bar{x}) \leq \frac{\tau_{j+2} - \tau_{j+1}}{\eta}C + \frac{\tau_{j+1}}{r} + \rho < \frac{\tau_{j+2}}{\eta}C + \rho < R.$$

Combining this with the obvious inequality $f(\bar{x}) > f(y) > f(\bar{x}) - \eta$, we deduce $|\nabla f|(y) \geq r$ and consequently $r_{j+1} \geq r$. This completes the induction. $\square$

For each index $i = 0, \ldots, k-1$, let $\omega_i \colon [0, d(x_i, x_{i+1})] \to \mathcal{X}$ be the isometry parametrizing the metric segment between x_i and x_{i+1}. For reasons which will become apparent momentarily, we now rescale the domain of ω_i by instead declaring

$$\omega_i \colon [\tau_i, \tau_{i+1}] \to \mathcal{X} \quad \text{to be} \quad \omega_i(t) = \omega_i\Big(\frac{d(x_{i+1}, x_i)}{\tau_{i+1} - \tau_i}(t - \tau_i)\Big).$$

Observe now that for any index $i = 1, \ldots, k-1$ and any $s, t \in [\tau_i, \tau_{i+1}]$ with $s < t$, we have

$$d(\omega_i(t), \omega_i(s)) = \frac{d(x_{i+1}, x_i)}{\tau_{i+1} - \tau_i}(t - s) \leq r_i^{-1}(t - s). \tag{4.6}$$

It follows that all the curves ω_i, for $i = 1, \ldots, k-1$, are Lipschitz continuous with a uniform modulus r^{-1}. We may now define a curve $u_k \colon [0, \eta] \to \mathcal{X}$ by simply concatenating the domains of ω_i for each index $i = 0, \ldots, k-1$. Each such curve u_k is Lipschitz continuous on $[\rho_k, \eta]$ with modulus r^{-1} and the sequence of curves $\{u_k\}$ is uniformly bounded. Moreover, since f is lsc and bounded subsets of sub-level sets of f are compact, one can readily verify that the sequence $\{u_k\}$ is pointwise relatively compact. The well-known theorem of Arzelà and Ascoli ([76, Section 7]) then guarantees that a certain subsequence of u_k, which we continue to denote by u_k, converges uniformly on compact subsets of $(0, \eta]$ to some mapping $x \colon (0, \eta] \to \mathcal{X}$. We can then clearly extend the domain of x to the closed interval $[0, \eta]$ by declaring $x(0) := \bar{x}$. It follows that $x \colon [0, \eta] \to \mathcal{X}$ is Lipschitz continuous with modulus r^{-1}.

Observe that the metric derivative functions $\|\dot{u}_k(\cdot)\|$ are bounded in $L^2(a, \eta)$ for any $a \in (0, \eta)$. It follows that, up to a subsequence, the mappings $\|\dot{u}_k(\cdot)\|$

converge weakly in $L^2(0,\eta)$ to some integrable mapping $m\colon [0,\eta] \to \mathbf{R}$ satisfying

$$d(x(s), x(t)) \leq \int_s^t m(\tau)\, d\tau, \quad \text{whenever } 0 < s \leq t < \eta. \tag{4.7}$$

For what follows now, define the set of breakpoints

$$E := \bigcup_{k \in \mathbb{N}} \bigcup_{i \in \mathbb{N} \cap [0,k]} \left\{ \frac{i\lambda_k}{\eta} \right\}$$

and observe that it has zero measure in $[0,\eta]$. In addition, let D be the full-measure subset of $(0,\eta)$ on which all the curves u_k and x admit a metric derivative.

Claim 4.3.6. *For almost every $\tau \in [0,\eta]$ with $\|\dot{x}(\tau)\| \neq 0$, the following are true:*

- $f(x(\tau)) = f(\bar{x}) - \tau$,
- $\|\dot{x}(\tau)\| \leq \frac{1}{|\nabla f|(x(\tau))}$,

Proof. Fix a real $\tau \in D \setminus E$ with $\|\dot{x}(\tau)\| \neq 0$. Then using equation (4.6) we deduce that for sufficiently large k, we have

$$\|\dot{u}_k(\tau)\| \leq \frac{1}{r_{i_k}^{(k)}}, \tag{4.8}$$

for some $i_k \in \{0, \ldots, k\}$, where the superscript (k) refers to partition of the interval $[0,\eta]$ into k equal pieces. Noting that weak convergence does not increase the norm and using minimality of the metric derivative, we deduce

$$\liminf_{k \to \infty} \|\dot{u}_k(\tau)\| \geq m(\tau) \geq \|\dot{x}(\tau)\|, \quad \text{for a.e. } \tau \in [0,\eta]. \tag{4.9}$$

Consequently there exists a subsequence of $\|\dot{u}_k(\tau)\|$, which we continue to denote by $\|\dot{u}_k(\tau)\|$, satisfying $\lim_{k\to\infty} \|\dot{u}_k(\tau)\| \neq 0$. Taking into account (4.8), we deduce that $r_{i_k}^{(k)}$ remain bounded. We may then choose points $x_{i_k}^{(k)}$, y_k, and reals λ_k, $\tau_{i_k}^{(k)}$ with $\tau \in (\tau_{i_k}^{(k)}, \tau_{i_k+1}^{(k)})$ satisfying

$$d(y_k, x_{i_k}^{(k)}) < \lambda_k C, \qquad f(\bar{x}) - \tau_{i_k+1}^{(k)} < f(y_k) \leq f(x_{i_k}^{(k)}), \quad x_{i_k}^{(k)} \to x(\tau), \quad \tau_{i_k}^{(k)} \to \tau$$

and $|\nabla f|(y_k) \leq r_{i_k}^{(k)} + \frac{1}{k}$. Then since f is continuous on slope-bounded sets and the quantity $f(x_{i_k}^{(k)}) - (f(\bar{x}) - \tau_{i_k+1}^{(k)})$ tends to zero, we deduce

$$f(x(\tau)) = \lim_{k\to\infty} f(y_k) = \lim_{k\to\infty} f(\bar{x}) - \tau_{i_k+1}^{(k)} = f(\bar{x}) - \tau,$$

as claimed. Moreover

$$\liminf_{k\to\infty} r_{i_k}^{(k)} \geq \liminf_{k\to\infty} \{|\nabla f|(y_k) - \frac{1}{k}\} \geq \overline{|\nabla f|}(x(\tau)).$$

Combining this with (4.9) and taking the limit in (4.8), we obtain

$$\|\dot{x}(\tau)\| \leq \frac{1}{\overline{|\nabla f|}(x(\tau))},$$

as claimed. $\square$

Define now a strictly decreasing function $\phi\colon [0,\eta] \to \mathbf{R}$ by the formula $\phi(\tau) = f(\bar{x}) - \tau$. In particular, it follows from Claim 4.3.6 that ϕ coincides with $f \circ x$ almost everywhere on the set $\{\tau \in (0,\eta) : \|\dot{x}(\tau)\| \neq 0\}$. Moreover, for almost every $\tau \in (0,\eta)$ the implication

$$\|\dot{x}(\tau)\| \neq 0 \quad \Longrightarrow \quad \overline{|\nabla f|}(x(\tau)) < \infty \quad \text{and} \quad |\phi'(\tau)| \geq \overline{|\nabla f|}(x(\tau)) \cdot \|\dot{x}(\tau)\|,$$

holds.

Now in light of Theorem 4.2.4, there exists a nondecreasing absolutely continuous map $s\colon [a,b] \to [0,L]$ with $s(a) = 0$ and $s(b) = L$, and a 1-Lipschitz curve $\gamma\colon [0,L] \to \mathcal{X}$ satisfying

$$x(\tau) = (\gamma \circ s)(\tau) \quad \text{and} \quad \|\dot{\gamma}(t)\| = 1 \text{ for a.e. } t \in [0,L].$$

Define also the nondecreasing function $\tau\colon [0,L] \to [0,\eta]$ by setting

$$\tau(t) := \min\{\tau : s(\tau) = t\},$$

and observe the equality $(s \circ \tau)(t) = t$. Define now the nonincreasing function $\psi\colon [0,L] \to [0,\eta]$ by setting $\psi(t) := \phi(\tau(t))$.

It is easy to check that if s is differentiable at $\tau(t)$ with $s'(\tau(t)) \neq 0$, the function $\tau(\cdot)$ is continuous at t, and γ is metrically differentiable at t with $\|\dot{\gamma}(t)\| = 1$, then we have

$$\|\dot{x}(\tau(t))\| = s'(\tau(t)) \qquad \text{and} \qquad \tau'(t) = \frac{1}{s'(\tau(t))},$$

and consequently

$$\overline{|\nabla f|}(\gamma(t)) = \overline{|\nabla f|}(x(\tau(t))) \leq |\phi'(\tau(t))| \cdot \frac{1}{s'(\tau(t))} = |\psi'(t)|.$$

It easily follows (in part using [116, Fundamental Lemma]) that the collection of such real numbers t has full measure in $[0,L]$, that ψ and $f \circ \gamma$ coincide a.e. on $[0,L]$, and that $f \circ \gamma$ is strictly decreasing on a full-measure subset of $[0,L]$. This completes the proof. □

The following is now an easy consequence.

Corollary 4.3.7 (Existence of curves of near-maximal slope)**.** *Consider a lsc function $f\colon \mathcal{X} \to \overline{\mathbf{R}}$ on a complete locally convex metric space $\mathcal{X}$ and a point $\bar{x} \in \mathcal{X}$, with f finite at $\bar{x}$. Suppose that f is continuous on slope-bounded sets and that bounded closed subsets of sublevel sets of f are compact. Then there exists a curve of near-maximal slope $\gamma\colon [0,T] \to \mathcal{X}$ starting at $\bar{x}$.*

Proof. If $\bar{x}$ is a critical point of f then the constant curve $x(t) \equiv \bar{x}$ is a curve of near-maximal slope. Hence we may suppose that $\bar{x}$ is not a critical point of f. The result is now immediate from Proposition 4.2.9 and Theorem 4.3.4. □

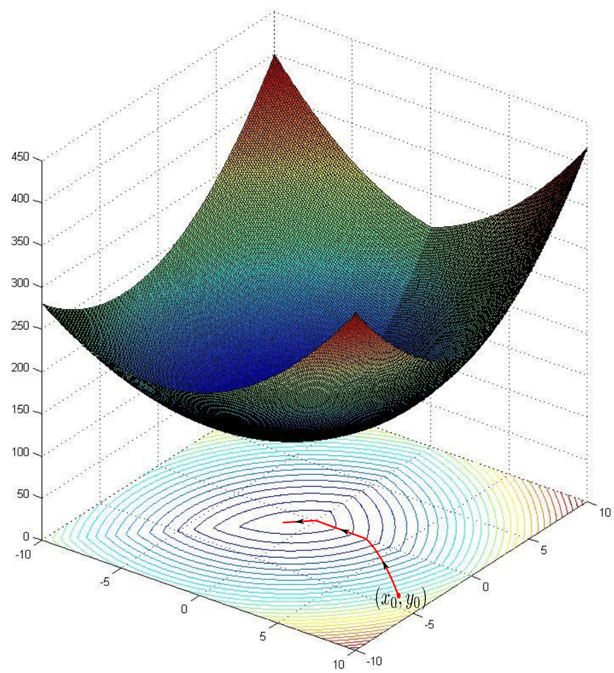


Figure 4.1: $f(x, y) = \max\{x + y, |x - y|\} + x(x + 1) + y(y + 1) + 100$

4.4 Descent curves and subgradient dynamical systems

In this subsection, we consider curves of near-maximal slope in Euclidean spaces. In this context, it is interesting to compare such curves to solutions $x\colon [0, \eta] \to \mathbf{R}^n$ of subgradient dynamical systems

$$\dot{x}(t) \in -\partial f(x(t)), \qquad \text{for a.e. } t \in [0, \eta].$$

It turns out that the same construction as in the proof of Theorem 4.3.4 shows that there exist near-steepest descent curves x so that essentially, up to rescaling, the vector $\dot{x}(t)$ lies in $-\partial f(x(t))$ for a.e. $t \in [0, \eta]$.

Theorem 4.4.1 (Existence of near-steepest descent curves)**.** *Consider a lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, along with a point $\bar{x}$ in the domain of f. Suppose that f is continuous on slope bounded sets. Then there exists a curve of near-maximal slope $x\colon [0, L] \to \mathcal{X}$*

emanating from $\bar{x}$ and satisfying

$$\dot{x}(t) \in -\mathrm{cl}\,\mathrm{cone}\,\partial_c f(x(t)), \quad \textit{for a.e. } t \in [0, L].$$

Proof. We can clearly assume that zero is not a subgradient of f at $\bar{x}$. We now specialize the construction of Theorem 4.3.4 to the Euclidean setting. Namely, let u_k be defined as in that theorem, except in this case we may more specifically define $u_k(0) = \bar{x}$, and inductively define $u_k(\tau_{i+1})$ to be any point belonging to the projection of $u_k(\tau_i)$ onto the lower level set $[f \leq f(\bar{x}) - \tau_{i+1}]$, provided that this set is nonempty. In particular, up to a subsequence, u_k converge uniformly on compact subsets to a Lipschitz continuous curve x.

Observe that in light of [108, Proposition 10.3], for any index k and any $\tau \in [\tau_i, \tau_{i+1}]$ (for $i = 1, \ldots, k$) we have $\dot{u}_k(\tau) \in -\big(\mathrm{cone}\,\partial f(u_k(\tau_{i+1}))\big) \cup \partial^{\infty} f(u_k(\tau_{i+1}))$. Furthermore, recall that restricting to a subsequence we may suppose that $\dot{u}_k$ converges weakly to $\dot{x}(\tau)$ in $L^2(0, \eta)$. Mazur's Lemma then implies that a sequence of convex combinations of the form $\sum_{n=k}^{N(k)} \alpha_n^k \dot{u}_n$ converges strongly to $\dot{x}$ as k tends to ∞. Since convergence in $L^2(0, \eta)$ implies almost everywhere pointwise convergence, we deduce that for almost every $\tau \in [0, \eta]$, we have

$$\Big\| \sum_{n=k}^{N(k)} \alpha_n^k \dot{u}_n(\tau) - \dot{x}(\tau) \Big\| \to 0.$$

Therefore if the inclusion

$$\dot{x}(\tau) \in -\mathrm{cl}\,\mathrm{conv}\Big[\big(\mathrm{cone}\ \partial f(x(\tau))\big) \cup \partial^{\infty} f(x(\tau))\Big]$$

did not hold, then we would deduce that there exists a subsequence of vectors $\dot{u}_{n_l}^{k_l}(\tau)$ with $\lim_{l\to\infty} \dot{u}_{n_l}^{k_l}(\tau)$ not lying in the set on the right-hand-side of the inclusion above. This immediately yields a contradiction. After the reparametrization performed in the proof of Theorem 4.3.4, the curve γ is subdifferentiable almost everywhere on $[0, L]$ and consequently satisfies

$$\dot{\gamma}(t) \in -\mathrm{cl}\,\mathrm{cone}\,\partial_c f(\gamma(t)), \quad \text{for a.e. } t \in [0, L],$$

as we needed to show. □

The above theorem motivates the question of when curves of near-maximal slope and solutions of subgradient dynamical systems are one and the same, that is when is the rescaling of the gradient $\dot{x}$ in the previous theorem not needed. The following property turns out to be crucial.

Definition 4.4.2 (Chain rule)**.** Consider an lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. We say that f has the *speed chain rule property* if for every curve $x \in AC(a, b, \mathbf{R}^n)$ such that f is subdifferentiable almost everywhere along x and so that $f \circ x$ coincides almost everywhere with some nonincreasing function $\theta\colon (a, b) \to \mathbf{R}$, the equation

$$\theta'(t) = \langle \partial f(x(t)), \dot{x}(t) \rangle \qquad \text{holds for a.e. } t \in (a, b).$$

The following simple proposition shows that whenever f has the speed chain rule property, solutions of subgradient dynamical systems and curves of near-maximal slope coincide.

Proposition 4.4.3 (Subgradient systems & curves of near-maximal slope)**.** *Consider a lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and suppose that f has the speed chain rule property. Then for any curve $x \in AC(a, b, \mathbf{R}^n)$ the following are equivalent.*

1. *x is a curve of near-maximal slope.*
2. *$f \circ x$ is nonincreasing on a full-measure subset of (a, b) and we have*

$$\dot{x}(t) \in -\partial f(x(t)), \quad \text{a.e. on } [a, b]. \tag{4.10}$$

3. *$f \circ x$ is nonincreasing on a full-measure subset of (a,b) and we have*

$$\dot{x}(t) \in -\partial f(x(t)), \quad \textit{and} \quad \|\dot{x}(t)\| = d(0, \partial f(x(t))), \quad \textit{a.e. on } [a,b]. \tag{4.11}$$

Proof. We first prove the implication $1 \Rightarrow 3$. To this end, suppose that x is a curve of near-maximal slope. Then $f \circ x$ coincides a.e. with some nonincreasing function $\theta\colon (a,b) \to \mathbf{R}$ and we have

$$\langle \partial f(x(t)), \dot{x}(t) \rangle = \theta'(t) \leq -\big(\overline{|\nabla f|}(x(t))\big)^2 \quad \text{a.e. on } [a,b].$$

Let $v(t) \in \partial f(x(t))$ be a vector of minimal norm. Then we have

$$\langle \partial f(x(t)), \dot{x}(t) \rangle \geq -\|v(t)\| \cdot \|\dot{x}(t)\| = -\big(\overline{|\nabla f|}(x(t))\big)^2,$$

with equality if and only if $\dot{x}(t)$ and $v(t)$ are collinear. We deduce $\dot{x}(t) = -v(t)$, as claimed.

The implication $3 \Rightarrow 2$ is trivial. Hence we focus now on $2 \Rightarrow 1$. To this end suppose that 2 holds and observe

$$\langle \partial f(x(t)), \dot{x}(t) \rangle = -\|\dot{x}(t)\|^2, \quad \text{for a.e. } t \in (a,b). \tag{4.12}$$

Given such t consider the subspace

$$V = \text{par}\,\partial f\big(x(t)\big).$$

Then we have

$$\text{aff}\,\partial f(x(t)) = -\dot{x}(t) + V.$$

We claim now that the inclusion $\dot{x}(t) \in V^{\perp}$ holds. To see this, observe that for any real λ_i and for vectors $v_i \in \partial f\big(x(t)\big)$, we have

$$\langle \dot{x}(t), \sum_{i=1}^{k} \lambda_i (v_i + \dot{x}(t)) \rangle = \sum_{i=1}^{k} \lambda_i \big[\langle \dot{x}(t), v_i \rangle + \|\dot{x}(t)\|^2\big] = 0,$$

where the latter equality follows from (4.12). Hence the inclusion

$$-\dot{x}(t) \in (-\dot{x}(t) + V) \cap V^{\perp},$$

holds. Consequently we deduce that $-\dot{x}(t)$ achieves the distance of the affine space, $\operatorname{aff} \partial f(x(t))$, to the origin. On the other hand, the inclusion $-\dot{x}(t) \in \partial f\big(x(t)\big)$ holds, and hence $-\dot{x}(t)$ actually achieves the distance of $\partial f(x(t))$ to the origin. The result follows. □

In light of the theorem above, it is interesting to understand which functions f have the speed chain rule property. Subdifferentially regular (in particular, all lsc convex) functions furnish a simple example. The convex case can be found in [22, Lemma 3.3, p 73](Chain rule).

Lemma 4.4.4 (Chain rule under subdifferential regularity)**.** *Consider a subdifferentially regular function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. Then f has the speed chain rule property.*

Proof. Consider a curve $x\colon (a,b) \to \mathbf{R}^n$ and a nonincreasing function $\theta\colon (a,b) \to \mathbf{R}$ coinciding with $f \circ \gamma$ almost everywhere. Suppose that for some real $t \in (a,b)$ both x and θ are differentiable at t and $\partial f(x(t))$ is nonempty. We then deduce

$$\theta'(t) = \lim_{i\to\infty} \frac{f(x(t_i)) - f(x(t))}{t_i - t}$$

where t_i is an arbitrary sequence such that $t_i \downarrow t$ and satisfying $(f \circ x)(t) = \theta(t)$. We deduce

$$\theta'(t) \geq \langle v, \dot{x}(t)\rangle, \quad \text{for any } v \in \hat{\partial} f(x(t).$$

Similarly we have

$$\theta'(t) = \lim_{i\to\infty} \frac{f(x(t_i) - f(x(t))}{t_i - t},$$

where t_i is an arbitrary sequence such that $t_i \uparrow t$ and satisfying $(f \circ x)(t_i) = \theta(t_i)$. We deduce

$$\theta'(t) \leq \langle v, \dot{x}(t) \rangle, \quad \text{for any } v \in \hat{\partial} f(x(t))$$

Hence the equation

$$\theta'(t) = \langle \hat{\partial} f(x(t)) \dot{x}(t) \rangle \quad \text{holds,}$$

and the result follows. □

Subdifferentially regular functions are very special, however. In particular, many nonpathological functions such as $-\|\cdot\|$ are not subdifferentially regular. So it is natural to consider prototypical nonpathological functions appearing often in practice — those that are semi-algebraic. This is the focus of the following section.

4.5 Semi-algebraic descent & subgradient dynamical systems

The main goal of this section is to establish an equivalence between curves of near-maximal slope and solution of subgradient dynamical systems for lsc semi-algebraic functions that are locally Lipschitz continuous on their domains. To this end, we analyze the speed chain rule property in the context of semi-algebraic functions. Before we proceed, we need to recall the notion of tangent cones.

Definition 4.5.1 (Tangent cone)**.** Consider a set $Q \subset \mathbf{R}^n$ and a point $\bar{x} \in Q$. Then the *tangent cone* to Q at $\bar{x}$, is simply the set

$$T_Q(\bar{x}) := \Big\{ \lim_{i \to \infty} \lambda_i (x_i - \bar{x}) : \lambda_i \uparrow \infty \text{ and } x_i \in Q \Big\}.$$

We now record the following simple lemma, whose importance in the context of semi-algebraic geometry will become apparent shortly. We omit the proof since it is rather standard.

Lemma 4.5.2 (Generic tangency). *Consider a set $M \subset \mathbf{R}^n$ and a path $x\colon [0,\eta] \to \mathbf{R}^n$ that is differentiable almost everywhere on $[0,\eta]$. Then for almost every $t \in [0,\eta]$, the implication*

$$x(t) \in M \quad \Longrightarrow \quad \dot{x}(t) \in T_M(x(t)), \quad \textit{holds}.$$

The following is a key property of semi-algebraic functions that we will exploit [11, Proposition 4].

Theorem 4.5.3 (Projection formula). *Consider a lsc semi-algebraic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. Then there exists a partition of* $\operatorname{dom} f$ *into finitely many $\mathbf{C}^1$-manifolds $\{M_i\}$ so that f restricted to each manifold M_i is $\mathbf{C}^1$-smooth. Moreover for any point x lying in a manifold M_i, the inclusion*

$$\partial_c f(x) \subset \nabla g(x) + N_{M_i}(x) \quad \textit{holds},$$

where $g\colon \mathbf{R}^n \to \mathbf{R}$ is any $\mathbf{C}^1$-smooth function agreeing with f on a neighborhood of x in M_i.

Theorem 4.5.4 (Semi-algebraic speed chain rule property). *Consider a lsc semi-algebraic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that is locally Lipschitz continuous on its domain. Consider also a curve $\gamma \in AC(a,b,\mathbf{R}^n)$ whose image is contained in the domain of f. Then equality*

$$(f \circ \gamma)'(t) = \langle \partial f(\gamma(t)), \dot{x}(t) \rangle = \langle \partial_c f(\gamma(t)), \dot{x}(t) \rangle$$

holds for almost every $t \in [a,b]$. In particular, the function f has the speed chain rule property.

Proof. Consider the partition of $\operatorname{dom} f$ into finitely many $\mathbf{C}^1$-manifolds $\{M_i\}$, guaranteed to exist by Theorem 4.5.3. We first record some preliminary observations. Clearly both x and $f \circ x$ are differentiable at a.e. $t \in (0, T)$. Furthermore, in light of Lemma 4.5.2, for any index i and for a.e. $t \in (0, \eta)$ the implication

$$x(t) \in M_i \quad \Longrightarrow \quad \dot{x}(t) \in T_{M_i}(x(t)), \quad \text{holds.}$$

Now suppose that for such t, the point $x(t)$ lies in a manifold M_i and let $g\colon \mathbf{R}^n \to \mathbf{R}$ be a $\mathbf{C}^1$-smooth function agreeing with f on a neighborhood of $x(t)$ in M_i. Lipschitzness of f on its domain then easily implies

$$\begin{aligned}
\frac{d(f \circ x)}{dt}(t) &= \lim_{\epsilon \downarrow 0} \frac{f(x(t+\epsilon)) - f(x(t))}{\epsilon} \\
&= \lim_{\epsilon \downarrow 0} \frac{f(P_{M_i}(x(t+\epsilon))) - f(x(t))}{\epsilon} \\
&= \lim_{\epsilon \downarrow 0} \frac{g(P_{M_i}(x(t+\epsilon))) - g(x(t))}{\epsilon} \\
&= \frac{d}{dt}\, g \circ P_{M_i} \circ x(t) = \langle \nabla g(x(t)), \dot{x}(t) \rangle \\
&= \langle \nabla g(x(t)) + N_{M_i}(x(t)), \dot{x}(t) \rangle.
\end{aligned}$$

The result follows. □

A noteworthy point about the theorem above is the appearance of the Clarke subdifferential in the chain rule. As a result, we can strengthen Theorem 4.4.3 in the context of lsc semi-algebraic functions $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that are locally Lipschitz continuous on their domains. The proof is analogous to that of Theorem 4.4.3.

Proposition 4.5.5 (Semi-algebraic equivalence)**.** *Consider a lsc semi-algebraic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that is locally Lipschitz continuous on its domain. Then for any curve $x \in AC(a, b, \mathbf{R}^n)$ the following are equivalent.*

1. *x is a curve of near-maximal slope.*

2. *$f \circ x$ is nonincreasing and we have*

$$\dot{x} \in -\partial f(x), \quad \textit{a.e. on } [a,b].$$

3. *$f \circ x$ is nonincreasing and we have*

$$\dot{x} \in -\partial f(x), \quad \|\dot{x}\| = d(0, \partial f(x)), \quad \textit{and} \quad \|\dot{x}\| = d(0, \partial_c f(x)) \quad \textit{a.e. on } [a,b].$$

4.6 Semi-algebraic descent: existence, length, and convergence

In this section, we study existence of near-steepest curves, along with lengths and convergence properties of curves of near-maximal slope. To motivate the discussion, consider a continuous function $f\colon \mathbf{R}^n \to \mathbf{R}$. Recall that Corollary 4.3.7 establishes existence of near-steepest descent curves emanating from any point $\bar{x} \in \mathbf{R}^n$. Suppose that $\bar{x}$ is a critical point of f, that is the equation $\overline{|\nabla f|}(\bar{x}) = 0$ holds. Then clearly the constant curve $\gamma \equiv \bar{x}$ is a near-steepest descent curve — a rather uninteresting one at that. It is then natural to ask whether in the case that $\bar{x}$ is *not* a local minimizer of f there exists a *nontrivial* near-steepest descent curve emanating from $\bar{x}$, that is a near-steepest descent curve $\gamma\colon [0,L] \to \mathbf{R}^n$ satisfying $\gamma(0) = \bar{x}$ and $f(\gamma(L)) < f(\bar{x})$.

This question is interesting even when f is $\mathbf{C}^\infty$-smooth. One special case stands out. If f is Morse at $\bar{x}$, that is the the Hessian $\nabla^2 f(\bar{x})$ is nonsingular, then it is easy to see that the answer is positive. Indeed, due to the Morse Lemma one can obtain nontrivial descent by following (in the local coordinate system) an eigenvector corresponding to a negative eigenvalue of the Hessian $\nabla^2 f(\bar{x})$. For general $\mathbf{C}^\infty$-smooth functions, with a possibly degenerate Hessian at the point of criticality, the answer is decisively false!

Example 4.6.1 (Nontrivial descent curves for smooth functions may fail to exist)**.** Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) := e^{-\frac{1}{|x|}} \sin(\frac{1}{x})$. It is easy to see that f is $\mathbf{C}^\infty$-smooth and that it does not admit any nontrivial near-steepest descent curve emanating from $\bar{x} = 0$, even though $\bar{x}$ is not a local minimizer.

The function f in the example above is not analytic, and this is no accident. We will see shortly that analytic functions $f\colon \mathbf{R}^n \to \mathbf{R}$ do admit non-trivial near-steepest descent curves emanating from any point $\bar{x}$ that is not a local minimizer. The key to our development is the Kurdyka-Łojasiewicz inequality. See Definition 2.4.7.

Theorem 4.6.2 (KŁ-inequality and existence of near-steepest descent curves)**.** *Consider a lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ satisfying the lower KŁ-inequality, along with a point $\bar{x} \in \operatorname{dom} f$ that is not a local minimizer of f. Suppose moreover that f is continuous on slope-bounded sets. Then there exists a curve $\gamma\colon [0, L] \to \mathbf{R}^n$ emanating from $\bar{x}$ and satisfying the following.*

Decrease in value: *The composition $f \circ \gamma$ is strictly decreasing on a full-measure subset of $[0, L]$ and consequently the inclusion $\gamma(0, L] \in [f < f(\bar{x})]$ holds.*

Near-steepest descent: *γ is a near-steepest descent curve.*

Regularity: *For a.e. $t \in [0, L]$, the slope $\overline{|\nabla f|}(\gamma(t))$ is finite and we have $\|\dot{\gamma}(t)\| = 1$.*

Proof. Let U be any bounded open neighborhood of $\bar{x}$. Since f satisfies the lower KŁ-inequality, we deduce that, there exists a real $\rho > 0$ and a non-negative continuous function $\psi\colon (f(\bar{x}) - \rho, f(\bar{x})] \to \mathbf{R}$, which is $\mathbf{C}^1$-smooth and strictly increasing on $(f(\bar{x}) - \rho, f(\bar{x}))$, and such that the inequality

$$|\nabla(\psi \circ f)|(x) \geq 1,$$

holds for all $x \in U$ with $f(\bar{x}) - \rho < f(x) < f(\bar{x})$. Since f is lsc, there exists $\epsilon > 0$ so that for each $x \in \overline{B}_\epsilon(\bar{x})$ we have $f(x) > f(\bar{x}) - \rho$. Define a function $h\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ by setting $h(x) = (\psi \circ f)(x) + \delta_{\overline{B}_\epsilon(\bar{x})}(x) + \delta_{[f \le f(\bar{x})]}(x)$. Theorem 4.3.4 then immediately implies that there exists a curve $\gamma\colon [0, L] \to \mathbf{R}^n$ emanating from $\bar{x}$ and satisfying the following.

- The composition $h \circ \gamma$ is strictly decreasing on a full-measure subset of $[0, L]$ and consequently the inclusion $\gamma(0, L] \in [h < h(\bar{x})]$ holds.
- γ is a near-steepest descent curve for h.
- For a.e. $t \in [0, L]$, the slope $\overline{|\nabla h|}(\gamma(t))$ is finite and we have $\|\dot{\gamma}(t)\| = 1$.

By decreasing L, we may assume without loss of generality that the image of γ is contained in $B_\epsilon(\bar{x})$. Let $\theta\colon [0, L] \to \mathbf{R}$ be an nonincreasing function coinciding a.e. with $h \circ \gamma$. We then deduce $\psi^{-1} \circ \theta$ is a nonincreasing function coinciding a.e. with $f \circ \gamma$. Moreover for a.e. $t \in [0, L]$ we have

$$|(\psi^{-1} \circ \theta)'(t)| = \frac{1}{\psi'(f(\gamma(t)))} |\theta'(t)| \ge \frac{1}{\psi'(f(\gamma(t)))} \overline{|\nabla(\psi \circ f)|}(\gamma(t)) = \overline{|\nabla f|}(\gamma(t)).$$

The result follows immediately. □

Shortly, we will investigate existence of near-steepest descent curves converging to local minimizers in the context of semi-algebraic geometry. Since we will be interested in asymptotic properties of descent curves, we will allow the curves to be defined on possibly infinite intervals. The entire theory developed trivially adapts to this setting.

We now show that bounded curves of near-maximal slope for semi-algebraic functions have finite length. The proof is almost identical to the proof of [74, Theorem 7.1]; hence we only provide a sketch. We will need the following nonsmooth Sard's theorem for semi-algebraic functions [11, Corollary 9].

Theorem 4.6.3 (Semi-algebraic Sard's theorem)**.** *Any lsc semi-algebraic function* $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ *has at most finitely many critical values.*

Theorem 4.6.4 (Lengths of curves of near-maximal slope)**.** *Consider a lsc, semi-algebraic function* $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$*, and let* U *be a bounded subset of* $\mathbf{R}^n$*. Then there exists a number* $N > 0$ *such that the length of any curve of near-maximal slope for* f *lying in* U *does not exceed* N*.*

Proof. Let $x\colon [0,T)$ be a curve of near-maximal slope for f and let ψ be any strictly increasing $\mathbf{C}^1$-smooth function on an interval containing the image of $f \circ x$. It is then easy to see then that, up to a reparametrization, x is a curve of near-maximal slope for the composite function $\psi \circ f$. In particular, we may assume that f is bounded on U, since otherwise we may for example replace f by $\psi \circ f$ where $\psi(t) = \frac{t}{\sqrt{1+t^2}}$.

Define the function

$$\xi(s) = \inf\{|\nabla f|(y) : y \in U,\ f(y) = s\}.$$

Standard arguments show that ξ is semi-algebraic. Consequently, with the exception of finitely many points, the domain of ξ is a union of finitely many open intervals (α_i, β_i), with ξ continuous and either strictly monotone or constant on each such interval. Define for each index i, the quantity

$$c_i = \inf\{\xi(s) : s \in (\alpha_i, \beta_i)\}.$$

We first claim that ξ is strictly positive on each interval (α_i, β_i). This is clear for indices i with $c_i > 0$. On the other hand if we have $c_i = 0$, then by Sard's theorem 4.6.3 the function ξ is strictly positive on (α_i, β_i) as well.

Define ζ_i and η_i by

$$\zeta = \inf\{t : f(x(t)) = \alpha_i\} \quad \text{and} \quad \eta = \sup\{t : f(x(t)) = \beta_i\},$$

and let l_i be the length of $x(t)$ between ζ_i and η_i.

Then we have

$$l_i = \int_{\zeta_i}^{\eta_i} \|\dot{x}(t)\| dt = \int_{\zeta_i}^{\eta_i} \overline{|\nabla f|}(x(t)) dt \leq \Big((\eta_i - \zeta_i) \int_{\zeta_i}^{\eta_i} \overline{|\nabla f|}(x(t))^2 dt \Big)^{\frac{1}{2}}.$$

On the other hand, observe

$$\int_{\zeta_i}^{\eta_i} \overline{|\nabla f|}(x(t))^2 dt \leq f(x(\eta_i)) - f(x(\zeta_i)) = \beta_i - \alpha_i.$$

Finally in the case $c_i > 0$ we have $l_i \geq c_i(\eta_i - \zeta_i)$, which combined with the two equations above yields the bound

$$l_i \leq \frac{\beta_i - \alpha_i}{c_i}.$$

If the equation $c_i = 0$ holds, then by the upper Kurdyka-Łojasiewicz inequality we can find a continuous function $\theta_i \colon [\alpha_i, \alpha_i + \rho) \to \mathbf{R}$, for some $\rho > 0$, where θ_i is strictly positive and $\mathbf{C}^1$-smooth on $(\alpha_i, \alpha_i + \rho)$ and satisfying $|\nabla(\theta_i \circ f)|(y) \geq 1$ for any $y \in U$ with $\alpha_i < f(y) < \alpha_i + \rho$. Since θ_i is strictly increasing on $(\alpha_i, \alpha_i + \rho)$, it is not difficult to check that we may extend θ_i to a continuous function on $[\alpha_i, \beta_i]$ and so that this extension is $\mathbf{C}^1$-smooth and strictly increasing on (α_i, β_i) with the inequality $|\nabla(\theta_i \circ f)|(y) \geq 1$ being valid for any $y \in U$ with $\alpha_i < f(y) < \beta_i$.

Then as we have seen before, up to a reparametrization, the curve $x(t)$ for $t \in [\zeta_i, \eta_i]$ is a curve of near maximal slope for the function $\theta_i \circ f$. Then as above, we obtain the bound $l_i \leq \theta_i(\beta_i) - \theta_i(\alpha_i)$. We conclude that the length of the curve $x(t)$ is bounded by a constant that depends only on f and on U, thereby completing the proof. □

The following consequence is now immediate.

Corollary 4.6.5 (Convergence of curves of near-maximal slope)**.** *Consider a lsc, semi-algebraic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. Then any curve of near-maximal slope for f that is bounded and has a maximal domain of definition converges to a critical point of f.*

We finally arrive at the following existence result.

Theorem 4.6.6 (Near-steepest descent curves and local minimizers)**.** *Consider an lsc, semi-algebraic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a point $\bar{x} \in \operatorname{dom} f$ that is not a local minimizer of f. Suppose moreover that f is continuous on slope-bounded sets. Then there exists a curve $\gamma\colon [0, L) \to \mathbf{R}^n$ emanating from $\bar{x}$ and satisfying the following.*

Decrease in value: *The composition $f \circ \gamma$ is strictly decreasing on a full-measure subset of $[0, L)$ and consequently the inclusion $\gamma(0, L) \in [f < f(\bar{x})]$ holds.*

Near-steepest descent: *γ is a near-steepest descent curve.*

Regularity: *For a.e. $t \in [0, L)$, the slope $\overline{|\nabla f|}(\gamma(t))$ is finite and we have $\|\dot{\gamma}(t)\| = 1$.*

Asymptotics: *Either γ is unbounded or γ converges to a local minimizer of f.*

Proof. This follows immediately by combining Theorem 4.6.2, Theorem 4.6.3, and Corollary 4.6.5. □

Theorem 4.6.7 (Smooth near-steepest descent curves)**.** *Consider a smooth semi-algebraic function $f\colon \mathbf{R}^n \to \mathbf{R}$ and a point $\bar{x}$. Then the curve guaranteed to exist by Theorem 4.6.6 can be assumed to be piecewise $\mathbf{C}^\infty$-smooth.*

Proof. Consider the curve $\gamma\colon [0,T) \to \mathbf{R}^n$ emanating from $\bar{x}$ guaranteed to exist by Theorem 4.6.6. Fix any real numbers $a, b \in [0,T)$ with $a < b$ and having the property that the interval $[f(b), f(a)]$ contains no critical values. Since γ is absolutely continuous, there exists an integrable function $g\colon [a,b] \to \mathbf{R}$ satisfying

$$\gamma(b) - \gamma(a) = \int_a^b g(t)\, dt, \tag{4.13}$$

Observe that g coincides a.e. on $[a,b]$ with $\dot{\gamma}$. Moreover by Theorem 4.5.5, we have $\dot{\gamma}(t) = \frac{\nabla f(\gamma(t))}{\|\nabla f(\gamma(t))\|}$ a.e. on $[a,b]$. Consequently equation (4.13) would remain true if we replace g with the continuous mapping $\frac{\nabla f(\gamma(t))}{\|\nabla f(\gamma(t))\|}$ on $[a,b]$. This immediately implies that γ is $\mathbf{C}^\infty$-smooth on $[a,b]$. Combining this with the fact that f has only finitely many critical values, we arrive at the result. $\square$

CHAPTER 5

GLOBAL ANALYSIS OF SEMI-ALGEBRAIC SUBDIFFERENTIAL GRAPHS

5.1 Introduction

A principal goal of variational analysis is the search for generalized critical points of nonsmooth functions $f\colon \mathbf{R}^n \to \mathbf{R}$. For example, given a locally Lipschitz function f, we might be interested in points $x \in \mathbf{R}^n$ having zero in the Clarke subdifferential $\partial_c f(x)$. Adding a linear perturbation, we might seek Clarke critical points of the function $x \mapsto f(x) - \langle v, x\rangle$ for a given vector $v \in \mathbf{R}^m$, or, phrased in terms of the graph of the subdifferential mapping $\partial_c f$, solutions to the inclusion

$$(x, v) \in \operatorname{gph} \partial_c f.$$

More generally, given a smooth function $G\colon \mathbf{R}^m \to \mathbf{R}^n$, we might be interested in solutions $(x, y) \in \mathbf{R}^m \times \mathbf{R}^n$ to the system

$$(G(x), y) \in \operatorname{gph} \partial_c f \ \text{ and } \ \nabla G(x)^* y = v \tag{5.1}$$

(where $*$ denotes the adjoint). Such systems arise naturally when we seek Clarke critical points of the composite function $x \mapsto f(G(x)) - \langle v, x\rangle$.

Generalized critical points of *smooth* functions f are, of course, simply the critical points in the classical sense. However, the more general theory is particularly interesting to optimization specialists, because critical points of continuous convex functions are just minimizers [108, Proposition 8.12], and more generally, for a broader class of functions (for instance, those that are Clarke regular [31]), a point is critical exactly when the directional derivative is nonnegative in every direction.

The system (5.1) could, in principle, be uninformative if the graph gph $\partial_c f$ is large. In particular, if the dimension (appropriately defined) of the graph is larger than n, then we could not typically expect the system to be a very definitive tool, since it involves $m + n$ variables constrained by only m linear equations and the inclusion. Such examples are not hard to construct: indeed, there exists a function $f\colon \mathbf{R} \to \mathbf{R}$ with Lipschitz constant one and with the property that its Clarke subdifferential is the interval $[-1, 1]$ at every point [105]. Alarmingly, in a precise mathematical sense, this property is actually typical for such functions [19].

Optimization theorists often consider subdifferentials that are smaller than Clarke's, the "limiting" subdifferential ∂f being a popular choice [20,32,93,108]. However, the Clarke subdifferential can be easier to approximate numerically (see [24]), and in any case the potential difficulty posed by functions with large subdifferential graphs persists with the limiting subdifferential [15].

Notwithstanding this pathology, concrete functions $f\colon \mathbf{R}^n \to \mathbf{R}$ encountered in practice have subdifferentials $\partial_c f$ whose graphs are, in some sense, small and this property can be useful, practically. For instance, Robinson [102] considers algorithmic aspects of functions whose subdifferential graphs are everywhere locally Lipschitz homeomorphic to an open subset of $\mathbf{R}^n$. As above, dimensional considerations suggest reassuringly that this property should help the definitive power of critical point systems like (5.1), and Robinson furthermore argues that it carries powerful computational promise. An example of the applicability of Robinson's techniques is provided by Minty's theorem, which states that the graph of the subdifferential of a proper, lower semicontinuous, convex function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is Lipschitz homeomorphic to $\mathbf{R}^n$ [92].

When can we be confident that a function has a subdifferential graph that is, by some definition, small? The study of classes of functions that are favorable for subdifferential analysis, in particular excluding the pathological examples above, is well-developed. The usual starting point is a unification of smooth and convex analysis, arriving at such properties as amenability [108, Chapter 10.F.], prox-regularity [98], and cone-reducibility [13, Section 3.4.4]. Using Minty's theorem, Poliquin and Rockafellar [98] showed that prox-regular functions, in particular, have small subdifferentials in the sense of Robinson. Aiming precisely at a class of functions with small subdifferentials (in fact minimal in the class of upper semicontinuous mappings with nonempty compact convex images), [17] considers "essential strict differentiability".

In this work we take a different, very concrete approach. We focus on the dimension of the subdifferential graph, unlike the abstract minimality results of [17], but we consider the class of *semi-algebraic* functions—those functions whose graphs are semi-algebraic, meaning composed of finitely-many sets, each defined by finitely-many polynomial inequalities—and prove that such functions have small subdifferentials in the sense of dimension: the Clarke subdifferential has n-dimensional graph. This result subsumes neither the simple case of a smooth function, nor the case of a convex function, neither of which is necessarily semi-algebraic. Nonetheless, it has a certain appeal: semi-algebraic functions are common, they serve as an excellent model for "concrete" functions in variational analysis [74], and in marked contrast with many other classes of favorable functions, such as amenable functions, they may not even be Clarke regular. Furthermore, semi-algebraic functions are easy to recognize (as a consequence of the Tarski-Seidenberg theorem on preservation of semi-algebraicity under projection). For instance, observe that the spectral radius function on $n \times n$ matrices is neither Lipschitz nor convex, but it is easy to see that it is semi-algebraic.

To illustrate our results, consider the critical points of the function $x \mapsto f(x) - \langle v, x\rangle$ for a semi-algebraic function $f\colon \mathbf{R}^n \to [-\infty, +\infty]$. As a consequence of the subdifferential graph being small, we show that for a *generic* choice of the vector v, the number of critical points is finite. More precisely, there exists a number N, and a semi-algebraic set $S \subset \mathbf{R}^n$ of dimension strictly less than n, such that for all vectors v outside S, there exist at most N critical points. A result of a similar flavor can be found in [75], where criticality of so called "constraint systems" is considered. Specifically, [75] shows that if a semi-algebraic constrained minimization problem is "normal", then it has only finitely many critical points. Furthermore, it is shown that normality is a generic property. To contrast their approach to ours, we should note that [75] focuses on perturbations to the constraint structure, whereas we address linear perturbations to the function itself.

5.2 Dimension of subdifferential graphs

In this section, we study the dimension of semi-algebraic subdifferential graphs. The following notion will be key for our development.

Definition 5.2.1 (Normal bundle of a stratification)**.** Consider a Whitney (a)-regular stratification $\mathcal{A}$ of a semi-algebraic set $Q \subset \mathbf{R}^n$. We define the *normal* *bundle* $N_{\mathcal{A}}$ associated with the stratification $\mathcal{A}$ to be the union of the normal bundles of each stratum, that is

$$N_{\mathcal{A}} = \bigcup_{M \in \mathcal{A}} \operatorname{gph} N_M = \bigcup_{M \in \mathcal{A}} \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n : x \in M,\ y \in N_M(x)\}.$$

In the definition above, since there are finitely many strata and for each stratum $M \in \mathcal{A}$, the semi-algebraic set $\operatorname{gph} N_M$ is n-dimensional, we deduce that the normal bundle $N_{\mathcal{A}}$ is a semi-algebraic set of dimension n.

Proposition 5.2.2. *Consider a semi-algebraic set $Q \subset \mathbf{R}^n$ and suppose it admits a Whitney (a)-regular stratification $\mathcal{A} = \{M_i\}$. Then for any stratum M_i and any point $\bar{x} \in M_i$, the Clarke normal cone, $N_Q^c(\bar{x})$, is contained in the normal space, $N_{M_i}(\bar{x})$. Consequently, the inclusion* $\operatorname{gph} N_Q^c \subset N_{\mathcal{A}}$ *holds and so the graph of the Clarke normal cone has dimension no greater than n.*

Proof Observe that for any stratum M_j, we have the inclusion $M_j \subset Q$. Hence for any point $x \in M_j$, the inclusion

$$\hat{N}_Q(x) \subset \hat{N}_{M_j}(x) = N_{M_j}(x) \tag{5.2}$$

holds. Now fix some stratum M_i and a point $\bar{x} \in M_i$. We claim that the limiting normal cone $N_Q(\bar{x})$ is contained in $N_{M_i}(\bar{x})$. To see this, consider a vector $v \in N_Q(\bar{x})$. By definition of the limiting normal cone, there exist sequences (x_r) and (v_r) such that $x_r \xrightarrow{Q} \bar{x}$ and $v_r \to v$ with $v_r \in \hat{N}_Q(x_r)$. Since there are finitely many strata, we can assume that there is some stratum M_j such that the entire sequence (x_r) is contained in M_j. From (5.2), we deduce $\hat{N}_Q(x_r) \subset N_{M_j}(x_r)$, and hence $v_r \in N_{M_j}(x_r)$. Therefore by Whitney condition (a), we have $v \in N_{M_i}(\bar{x})$. Since v was arbitrarily chosen from $N_Q(\bar{x})$, we deduce $N_Q(\bar{x}) \subset N_{M_i}(\bar{x})$ and thus $N_Q^c(\bar{x}) = \operatorname{cl}\operatorname{conv} N_Q(\bar{x}) \subset N_{M_i}(\bar{x})$, as we needed to show. $\square$

Applying the previous proposition to the epigraph of a function, we obtain the following.

Theorem 5.2.3. *Let $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ be a semi-algebraic function. Then the graph of the Clarke subdifferential,* $\operatorname{gph} \partial_c f$*, has dimension no greater than n.*

Proof Let $F := \operatorname{epi} f$ and

$$A := \{(x,r,y) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^{n+1} : ((x,r),y) \in \operatorname{gph} N_F^c,\ r = f(x),\ y_{n+1} < 0\}.$$

Using Proposition 5.2.2, we see

$$\dim A \leq \dim \operatorname{gph} N_F^c \leq n+1. \tag{5.3}$$

Consider the continuous semi-algebraic map

$$\phi\colon A \to \mathbf{R}^n \times \mathbf{R}^n$$
$$(x, f(x), y) \mapsto (x, \pi\Big(\frac{y}{|y_{n+1}|}\Big)),$$

where $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ is the canonical projection onto the first n coordinates. Observe that the image of ϕ is exactly the graph of the Clarke subdifferential $\partial_c f$. Furthermore, for any pair $(x,v) \in \operatorname{gph} \partial_c f$, we have

$$\phi^{-1}(x,v) = \{x\} \times \{f(x)\} \times \mathbf{R}_+(v,-1),$$

and hence $\dim \phi^{-1}(c) = 1$ for any point c in the image of ϕ. By [52, Proposition 3.3], we deduce

$$\dim \operatorname{gph} \partial_c f + 1 = \dim A \leq n+1,$$

where the last inequality follows from (5.3). Hence, we obtain $\dim \operatorname{gph} \partial_c f \leq n$, as we needed to show. □

Shortly we will show that for a proper semi-algebraic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, both $\operatorname{gph} \partial_c f$ and $\operatorname{gph} \partial f$ have dimension exactly equal to n. In the case that the domain of f is full-dimensional, this fact is easy to show. The argument is as follows. By Theorem 2.4.3, the domain of f can be partitioned into semi-algebraic manifolds $\{X_i\}$ such that $f|_{X_i}$ is smooth. Let X_i be a manifold of maximal dimension. Observe that for $x \in X_i$, we have $\partial f(x) = \{\nabla f(x)\}$ and it easily follows that $\dim \operatorname{gph} \partial f|_{X_i} = n$. Thus we have

$$n \leq \dim \operatorname{gph} \partial f \leq \dim \operatorname{gph} \partial_c f \leq n,$$

where the last inequality follows from Theorem 5.2.3, and hence there is equality throughout. The argument just presented no longer works when the domain of f is not full-dimensional. A slightly more involved argument is required.

Theorem 5.2.4. *Let $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ be a proper semi-algebraic function. Then the graphs of the regular, limiting, and Clarke subdifferentials have dimension exactly n.*

Proof We know

$$\dim \operatorname{gph} \hat{\partial} f \leq \dim \operatorname{gph} \partial f \leq \dim \operatorname{gph} \partial_c f \leq n,$$

where the last inequality follows from Theorem 5.2.3. Thus if we show that the dimension of $\operatorname{gph} \hat{\partial} f$ is no less than n, we will be done. With that aim, applying Theorem 2.4.3 to the function f, we obtain a Whitney (a)-regular stratification $\{M_i\}$ of the domain of f such that for every stratum M_i, the restriction $f|_{M_i}$ is smooth. Let M_j be a stratum of $\operatorname{dom} f$ of maximal dimension, and let $\bar{x}$ be an arbitrary point of M_j.

Now consider the function $h\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, which agrees with f on M_j and is plus infinity elsewhere. Observe that the functions h and f coincide on a neighbourhood of $\bar{x}$. Applying [52, Proposition 3.6], we deduce that f is subdifferentially regular at $\bar{x}$ and $\partial f(\bar{x})$ is nonempty with dimension $n - \dim M_j$. Since the point $\bar{x}$ was arbitrarily chosen from M_j, we deduce $\dim \hat{\partial} f(x) = n - \dim M_j$ for any point $x \in M_j$. The result follows. □

5.3 Consequences

Definition 5.3.1. Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. We say that a point $x \in \mathbf{R}^n$ is *Clarke-critical* for the function f if $0 \in \partial_c f(x)$, and we call such a critical point x *nondegenerate* if the stronger property $0 \in \mathrm{ri}\, \partial_c f(x)$ holds.

Recall that for a proper convex function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a point $\bar{x} \in \mathrm{dom}\, f$, the subdifferentials $\hat{\partial} f(\bar{x})$, $\partial f(\bar{x})$, and $\partial_c f(\bar{x})$ all coincide and are equal to the convex subdifferential of f at $\bar{x}$. So in this case, the notions of Clarke-criticality and Clarke-nondegeneracy reduce to more familiar notions from Convex Analysis. The importance of nondegeneracy for the sensitivity analysis of convex functions is well known: in [80], for example, it is an underlying assumption for a pioneering conceptual approach to superlinearly convergent convex minimization algorithms. Consider the following largely classical theorem (see [10, Proposition 1] and [51]).

Theorem 5.3.2 (Generic uniqueness of minimizers for convex functions)**.** *Let $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ be a proper convex function. Consider the collection of perturbed functions $h_v(x) = f(x) - \langle v, x\rangle$, parametrized by vectors $v \in \mathbf{R}^n$. Then for a full measure set of vectors $v \in \mathbf{R}^n$, the function h_v has at most one minimizer, which furthermore is nondegenerate.*

Shortly, we will prove that a natural analogue of Theorem 5.3.2 holds for arbitrary semi-algebraic functions, with no assumption of convexity. We will then reference an example of a locally Lipschitz function that is not semi-algebraic, and for which the conclusion of our analogous result fails, thus showing that the assumption of semi-algebraicity is not superfluous. In what follows, for a set S, the number of elements in S will be denoted by $S^{\#}$. We begin with the following simple proposition, whose proof we omit.

Proposition 5.3.3. *Let $F\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ be a semi-algebraic set-valued mapping whose graph has dimension no greater than n. Then there exists $\beta \in \mathbb{N}$ such that for a generic set of points $c \in \mathbf{R}^n$, we have $F(c)^{\#} \leq \beta$.*

Corollary 5.3.4. *Let $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ be a semi-algebraic function and consider the collection of perturbed functions $h_v(x) = f(x) - \langle v, x\rangle$, parametrized by vectors $v \in \mathbf{R}^n$. Then there exists a positive integer β, such that for generic $v \in \mathbf{R}^n$, the number of Clarke-critical points of the perturbed function h_v is no greater than β.*

Proof Observe

$$0 \in \partial_c h_v(x) \Leftrightarrow v \in \partial_c f(x) \Leftrightarrow x \in (\partial_c f)^{-1}(v).$$

Thus the set $(\partial_c f)^{-1}(v)$ is equal to the set of Clarke-critical points of the function h_v. By Theorem 5.2.3, we have $\dim \operatorname{gph} \partial_c f \leq n$, hence $\dim \operatorname{gph} (\partial_c f)^{-1} \leq n$. Applying Theorem 5.3.3 to $(\partial_c f)^{-1}$, we deduce that there exists a positive integer β, such that for generic v, we have $((\partial_c f)^{-1}(v))^{\#} \leq \beta$. The result follows. □

Corollary 5.3.5. *Let $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ be a semi-algebraic function and consider the collection of perturbed functions $h_v(x) = f(x) - \langle v, x\rangle$, parametrized by vectors $v \in \mathbf{R}^n$. Then for generic $v \in \mathbf{R}^n$, every Clarke-critical point of the function h_v is nondegenerate.*

Corollary 5.3.5 follows immediately from the observation

$$0 \in \operatorname{ri} \partial_c h_v(x) \Leftrightarrow v \in \operatorname{ri} \partial_c f(x),$$

and the following result.

Corollary 5.3.6. *Let $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ be a semi-algebraic function. Then for generic $v \in \mathbf{R}^n$, we have that*

$$x \in (\partial_c f)^{-1}(v) \Longrightarrow v \in \mathbf{ri}\, \partial_c f(x).$$

Proof Let $D = \operatorname{dom} \partial_c f$. Consider the semi-algebraic set-valued mapping

$$\widetilde{F}\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^n, \;\; x \mapsto \text{rb}\; \partial_c f(x).$$

Our immediate goal is to show that the dimension of $\operatorname{gph} \widetilde{F}$ is no greater than $n-1$. Observe that for each $x \in \mathbf{R}^n$, we have $\widetilde{F}(x) \subset \partial_c f(x)$. Applying [52, Corollary 2.27] to the mapping $\partial_c f$, we get a finite partition of D into semi-algebraic sets $\{X_i\}$, such that

$$\operatorname{gph} \partial_c f|_{X_i} \cong X_i \times \partial_c f(x)$$

and

$$\operatorname{gph} \widetilde{F}|_{X_i} \cong X_i \times \widetilde{F}(x)$$

for any $x \in X_i$ (for each i). By Theorem 5.2.3, we have that

$$n \geq \dim \operatorname{gph} \partial_c f|_{X_i} = \dim X_i + \dim \partial_c f(x).$$

Since $\widetilde{F}(x) = \text{rb}\; \partial_c f(x)$, it follows that

$$\dim \widetilde{F}(x) \leq \dim \partial_c f(x) - 1.$$

Therefore

$$\dim \operatorname{gph} \widetilde{F}|_{X_i} = \dim X_i + \dim \widetilde{F}(x) \leq \dim X_i + \dim \partial_c f(x) - 1 \leq n - 1.$$

Thus

$$\dim \operatorname{gph} \widetilde{F} = \dim \Big(\bigcup_i \operatorname{gph} \widetilde{F}|_{X_i} \Big) \leq n - 1.$$

And so if we let

$$\pi\colon \text{gph}\, \widetilde{F} \to \mathbf{R}^n$$

be the projection onto the last n coordinates, we deduce that $\dim \pi(\text{gph}\, \widetilde{F}) \leq n-1$. Finally, observe

$$\pi(\text{gph}\, \widetilde{F}) \;=\; \Big\{ v \in \mathbf{R}^n : v \in \text{rb}\, \partial_c f(x) \text{ for some } x \in \mathbf{R}^n \Big\},$$

and so the result follows. □

Remark 5.3.7. Observe that if a convex function has finitely many minimizers then, in fact, it has a unique minimizer. Thus, for a proper convex semi-algebraic function, Corollaries 5.3.4 and 5.3.5 reduce to Theorem 5.3.2.

Remark 5.3.8. In Corollaries 5.3.4 and 5.3.5, if the function f is not semi-algebraic, then the results of these corollaries can fail. In fact, these results can fail even if the function f is locally Lipschitz continuous. For instance, there is a locally Lipschitz function $f\colon \mathbf{R} \to \mathbf{R}$ such that $\partial_c f(x) = [-x, x]$ for every $x \in \mathbf{R}$. See the article of Borwein-Moors-Wang [18]. For all $v \in \mathbf{R}$, the perturbed function h_v has infinitely many critical points, and for all $v \in \mathbf{R} \setminus \{0\}$, the function h_v has critical points that are degenerate.

5.4 Composite optimality conditions

Consider a composite optimization problem $\min_x g(F(x))$. It is often computationally more convenient to replace the criticality condition $0 \in \partial(g \circ F)(x)$ with the potentially different condition $0 \in \nabla F(x)^* \partial g(F(x))$, related to the former condition by an appropriate chain rule. See for example the discussion of Lagrange multipliers in [106]. Thus it is interesting to study the graph of the set-valued mapping $x \mapsto \nabla F(x)^* \partial g(F(x))$.

5.4.1 Dimensional analysis of the chain rule.

The following is a standard result in subdifferential calculus.

Theorem 5.4.1. *[108, Theorem 10.6] Consider a function* $g\colon \mathbf{R}^m \to \overline{\mathbf{R}}$ *and a smooth mapping* $F\colon \mathbf{R}^n \to \mathbf{R}^m$*. Then at any point* $\bar{x} \in \mathbf{R}^n$*, one has*

$$\hat{\partial}(g \circ F)(\bar{x}) \supset \nabla F(\bar{x})^* \hat{\partial} g(F(\bar{x})).$$

Now assuming that the functions g and F in the theorem above are semi-algebraic, we immediately deduce, using Theorem 5.2.3, that the dimension of the graph of the mapping $x \mapsto \nabla F(\bar{x})^* \hat{\partial} g(F(\bar{x}))$ is at most n. One can ask what happens more generally in the case of the limiting and Clarke subdifferentials. It is well known that the inclusion

$$\partial(g \circ F)(\bar{x}) \supset \nabla F(\bar{x})^* \partial g(F(\bar{x}))$$

is only guaranteed to hold under certain conditions [108, Theorem 10.6]. The Clarke case is similar [31, Theorem 2.3.10]. Hence, a priori, the dimension of the graph of the set-valued mapping $x \mapsto \nabla F(x)^* \partial g(F(x))$ is unclear. In this section, we will show that if g is lower semicontinuous, then this dimension is no greater than n and we will derive some consequences.

The proofs of Proposition 5.2.2 and Theorem 5.2.3 are self-contained and purely geometric. There is, however, an alternative approach using [11, Proposition 4], which will be useful for us. We state this proposition now. We denote the linear subspace of $\mathbf{R}^n$ parallel to a nonempty convex set $S \subset \mathbf{R}^n$ by par S.

Proposition 5.4.2. *[11, Proposition 4] Consider a proper, lower semicontinuous, semi-algebraic function* $g\colon \mathbf{R}^m \to \overline{\mathbf{R}}$*. Then there exists a Whitney (a)-regular stratification*

$\{M_i\}$ *of the domain of* g *such that for each stratum* M_i *and for any point* $x \in M_i$*, the inclusion* $\text{par}\, \partial_c g(x) \subset N_{M_i}(x)$ *holds.*

Before proceeding, we record the following special case of Theorem 5.4.1. Consider a smooth function $F\colon \mathbf{R}^n \to \mathbf{R}^m$ and a nonempty set $Q \subset \mathbf{R}^m$. Consider any point $\bar{x} \in \mathbf{R}^n$. Applying Theorem 5.4.1 to the indicator function of Q, we deduce

$$\hat{N}_{F^{-1}(Q)}(\bar{x}) \supset \nabla F(\bar{x})^* \hat{N}_Q(F(\bar{x})).$$

If we let $Q = F(X)$, for some set $X \subset \mathbf{R}^n$, then we obtain

$$\hat{N}_{F^{-1}(F(X))}(\bar{x}) \supset \nabla F(\bar{x})^* \hat{N}_{F(X)}(F(\bar{x})). \tag{5.4}$$

Theorem 5.4.3. *Consider a proper, lower semicontinuous, semi-algebraic function* $g\colon \mathbf{R}^m \to \overline{\mathbf{R}}$ *and a smooth semi-algebraic mapping* $F\colon \mathbf{R}^n \to \mathbf{R}^m$*. Then the graph of the semi-algebraic set-valued mapping* $x \mapsto \nabla F(x)^* \partial_c g(F(x))$ *has dimension no greater than* n*.*

Proof Consider the Whitney (a)-regular stratification $\{M_i\}$ of $\text{dom}\, g$ that is guaranteed to exist by applying Proposition 5.4.2 to the function g. Now applying Theorem 2.4.3 to the mapping F, we obtain a Whitney (a)-regular stratification $\{X_i\}$ of $\mathbf{R}^n$ and a Whitney (a)-regular stratification $\{K_j\}$ of $\mathbf{R}^m$ compatible with $\{M_i\}$ such that for each index i, we have $F(X_i) = K_j$ for some index j. Fix some stratum X and a point $\bar{x} \in X$. If $F(X)$ is not a subset of the domain of g, then clearly $\nabla F(\cdot)^* \partial_c g(F(\cdot))|_X \equiv \emptyset$. Hence, we only consider X such that $F(X) \subset \text{dom}\, g$. Let M be the stratum satisfying $F(X) \subset M$. Observe by our choice of the stratification $\{M_i\}$, we have

$$\nabla F(\bar{x})^* \partial_c g(F(\bar{x})) \subset \nabla F(\bar{x})^* v + \nabla F(\bar{x})^* N_M(F(\bar{x})),$$

for some vector $v \in \mathbf{R}^m$. Hence we have the inclusions

$$\text{par}\, \nabla F(\bar{x})^* \partial_c g(F(\bar{x})) \subset \nabla F(\bar{x})^* N_M(F(\bar{x})) \subset \nabla F(\bar{x})^* N_{F(X)}(F(\bar{x})), \tag{5.5}$$

where the last inclusion follows since the manifold $F(X)$ is a subset of M. Combining (5.4) and (5.5), we obtain

$$\text{par}\, \nabla F(\bar{x})^* \partial_c g(F(\bar{x})) \subset \hat{N}_{F^{-1}(F(X))}(\bar{x}) \subset N_X(\bar{x}),$$

where the last inclusion follows since the manifold X is a subset of $F^{-1}(F(X))$. So we deduce

$$\dim \nabla F(\bar{x})^* \partial_c g(F(\bar{x})) \leq n - \dim X.$$

Since the point $\bar{x}$ was arbitrarily chosen from X, we conclude (using [52, Corollary 3.4]) the inequality $\dim \text{gph}\, \nabla F(\cdot)^* \partial_c g(F(\cdot))|_X \leq n$. Taking the union over the strata $\{X_i\}$ yields

$$\dim \text{gph}\, \nabla F(\cdot)^* \partial_c g(F(\cdot)) \leq n,$$

as we claimed. □

Observe that Theorem 5.4.3 is a generalization of Theorem 5.2.3. This can easily be seen by taking F to be the identity map in Theorem 5.4.3.

Remark 5.4.4. Consider a proper, lower semicontinuous, semi-algebraic function $g\colon \mathbf{R}^m \to \overline{\mathbf{R}}$ and a smooth semi-algebraic mapping $F\colon \mathbf{R}^n \to \mathbf{R}^m$ satisfying $\text{dom}\, g \circ F = F^{-1}(\text{dom}\, g) \neq \emptyset$. A natural question, in line with Theorem 5.2.4, is whether the graph of the mapping $x \mapsto \nabla F(x)^* \partial_c g(F(x))$ has dimension exactly n. In fact, there is no hope for that to hold generally. For instance, it is possible to have $\partial_c g(y) = \emptyset$ for every point y in the image of F. This example, however motivates the following easy proposition, whose proof we omit.

Proposition 5.4.5. *Consider a proper, lower semicontinuous, semi-algebraic function $g\colon \mathbf{R}^m \to \overline{\mathbf{R}}$ and a smooth semi-algebraic mapping $F\colon \mathbf{R}^n \to \mathbf{R}^m$. Assume that the set $F^{-1}(\mathrm{dom}\,\hat{\partial} g)$ has a nonempty interior. Then the graph of the set-valued mapping $\nabla F(\cdot)^*\hat{\partial} g(F(\cdot))$ has dimension exactly n. Analogous results hold in the limiting and Clarke cases.*

5.4.2 Consequences

Let $F\colon \mathbf{R}^n \to \mathbf{R}^m$ be a smooth mapping and $g\colon \mathbf{R}^m \to \overline{\mathbf{R}}$ a proper lower semicontinuous function. (For simplicity, here we assume that the mapping F is defined on all of $\mathbf{R}^n$. However the whole section extends immediately to a mapping F defined only on an open subset $U \subset \mathbf{R}^n$.) Consider the following collection of composite minimization problems, parametrized by vectors $v \in \mathbf{R}^n$.

$$(P(v)) \qquad \min_{x\in\mathbf{R}^n} g(F(x)) - \langle v, x\rangle$$

For a point $\bar{x}$ to be a minimizer for $P(v)$, the inclusion $v \in \partial(g \circ F)(\bar{x})$ must necessarily hold. As discussed in the beginning of the section, it is often more convenient to replace this condition with the potentially different condition $v \in \nabla F(\bar{x})^*\partial g(F(\bar{x}))$. This motivates the following definition.

Definition 5.4.6. We say that a point x is *Clarke critical* for the problem $(P(v))$ if the inclusion $v \in \nabla F(x)^*\partial_c g(F(x))$ holds, and we call such a critical point x *nondegenerate* for the problem $(P(v))$ if the stronger property $v \in \mathrm{ri}\,\nabla F(x)^*\partial_c g(F(x))$ holds.

We are now in position to state a natural generalization of Corollaries 5.3.4 and 5.3.5.

Corollary 5.4.7. *Let $F\colon \mathbf{R}^n \to \mathbf{R}^m$ be a semi-algebraic smooth function and $g\colon \mathbf{R}^m \to \overline{\mathbf{R}}$ a proper lower semicontinuous semi-algebraic function. Consider the following collection of optimization problems, parametrized by vectors $v \in \mathbf{R}^n$.*

$$(P(v)) \qquad \min_{x\in\mathbf{R}^n} g(F(x)) - \langle v, x\rangle$$

Then there exists a positive integer β, such that for a generic vector $v \in \mathbf{R}^n$, the number of Clarke-critical points for the problem $(P(v))$ is no greater than β. Furthermore, for a generic vector $v \in \mathbf{R}^n$, every Clarke-critical point for the problem $(P(v))$ is nondegenerate.

Proof Observe that by Theorem 5.4.3, the graph of the mapping $x \mapsto \nabla F(x)^*\partial_c g(F(x))$ has dimension no greater than n. The proof now proceeds along the same lines as the proofs of Corollaries 5.3.4 and 5.3.5. □

Observe that Corollaries 5.3.4 and 5.3.5 can be considered as special cases of Corollary 5.4.7, in which the map F is the identity map.

A noteworthy illustration of Corollary 5.4.7 is the problem of constrained minimization, which we discuss now. Let $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ be a semi-algebraic function and $D \subset \mathbf{R}^n$ a closed semi-algebraic set. Consider the following collection of constrained minimization problems, parametrized by vectors $v \in \mathbf{R}^n$.

$$\begin{aligned} (P'(v)) \qquad & \min \quad f(x) - \langle v, x\rangle \\ & \text{s.t.} \quad x \in D \end{aligned}$$

Observe that $(P'(v))$ is equivalent to the problem $\min\limits_{x\in\mathbf{R}^n} g(F(x)) - \langle v, x\rangle$, where we define $F(x) = (x, x)$ and $g(x, y) = f(x) + \delta_D(y)$.

Hence, in the sense of composite minimization, it is easy to check that a point $x \in D$ is *Clarke critical* for the problem $(P'(v))$ if $v \in \partial_c f(x) + N^c_D(x)$, and such a critical point x is *nondegenerate* for the problem $(P'(v))$ if the stronger property $v \in \mathrm{ri}\, \partial_c f(x) + \mathrm{ri}\, N_D^c(x)$ holds.

Corollary 5.4.8. *Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a semi-algebraic function and let D be a closed, semi-algebraic set. Consider the following collection of optimization problems, parametrized by vectors $v \in \mathbf{R}^n$.*

$$
\begin{array}{lll}
(P(v)) \qquad & \min & f(x) - \langle v, x \rangle \\
& \text{s.t.} & x \in D
\end{array}
$$

Then there exists a positive integer β, such that for a generic vector $v \in \mathbf{R}^n$, the number of Clarke-critical points for the problem $(P(v))$ is no greater than β. Furthermore, for a generic vector $v \in \mathbf{R}^n$, every Clarke-critical point for the problem $(P(v))$ is nondegenerate.

Proof This follows directly from Corollary 5.4.7. □

CHAPTER 6

LOCAL ANALYSIS OF SEMI-ALGEBRAIC SUBDIFFERENTIAL GRAPHS

6.1 Introduction.

Variational analysis, a subject that has been vigorously developing for the past 40 years, has proven itself to be extremely effective at describing nonsmooth phenomena. The Clarke subdifferential (or generalized gradient) and the limiting subdifferential of a function are the earliest and most widely used constructions of the subject. A key distinction between these two notions is that, in contrast to the limiting subdifferential, the Clarke subdifferential is always convex. From a computational point of view, convexity of the Clarke subdifferential is a great virtue. To illustrate, by the classical Rademacher theorem, a *locally Lipschitz continuous* function f on an open subset U of $\mathbf{R}^n$ is differentiable almost everywhere on U, in the sense of Lebesgue measure. Clarke, in [30], showed that for such functions, the Clarke subdifferential admits the simple presentation

$$\partial_c f(\bar{x}) = \text{conv}\{\lim_{i\to\infty} \nabla f(x_i) : x_i \xrightarrow{\Omega} \bar{x}\}, \tag{LipR}$$

where $\bar{x}$ is any point of U and Ω is any full measure subset of U. Such a formula holds great computational promise since gradients are often cheap to compute. For example, utilizing (LipR), Burke, Lewis, and Overton developed an effective computational scheme for approximating the Clarke subdifferential by sampling gradients [24], and, motivated by this idea, developed a robust optimization algorithm [25].

The authors of [24] further extended Clarke's result to the class of *finite-valued, continuous* functions $f\colon U \to \mathbf{R}$, defined on an open subset U of $\mathbf{R}^n$, that are *absolutely continuous on lines*, and are *directionally Lipschitzian*; the latter means that the Clarke normal cone to the epigraph of f is pointed. Under these assumptions on f, the authors derived the representation

$$\partial_c f(\bar{x}) = \bigcap_{\delta>0} \operatorname{cl}\operatorname{conv}\Big(\nabla f\big(\Omega \cap B_\delta(\bar{x})\big)\Big), \tag{ACLR}$$

where $B_\delta(\bar{x})$ is an open ball of radius δ around $\bar{x}$ and Ω is any full measure subset of U, and they extended their computational scheme to this more general setting. One can easily see that this formula generalizes Clarke's result, since locally Lipschitz functions are absolutely continuous on lines, and for such functions (ACLR) reduces to (LipR). Pointedness of the Clarke normal cone is a common theoretical assumption. For instance, closed convex sets with nonempty interior have this property. Some results related to (ACLR) appear in [61].

In optimization theory, one is often interested in *extended real-valued* functions (functions that are allowed to take on the value $+\infty$), so as to model constraints, for instance. The results above are not applicable in such instances. An early predecessor of (LipR) and (ACLR) does rectify this problem, at least when convexity is present. Rockafellar [103, Theorem 25.6] showed that for any closed *convex* function $f\colon \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, whose domain dom f has a nonempty interior, the convex subdifferential has the form

$$\partial f(\bar{x}) = \operatorname{conv}\{\lim_{i\to\infty} \nabla f(x_i) : x_i \to \bar{x}\} + N_{\operatorname{dom} f}(\bar{x}), \tag{CoR}$$

where $\bar{x}$ is any point in the domain of f and $N_{\operatorname{dom} f}(\bar{x})$ is the normal cone to the domain of f at $\bar{x}$.

Our goal is to provide an intuitive and geometric proof of a representation formula unifying (LipR), (ACLR), and (CoR). To do so, we will impose a certain structural assumption on the functions f that we consider. Namely, we will assume that the domain of f can be locally "stratified" into a finite collection of smooth manifolds, so that f is smooth on each such manifold. Many functions of practical importance in optimization and in nonsmooth analysis possess this property. All semi-algebraic functions (those functions whose graphs can be described as a union of finitely many sets, each defined by finitely many polynomial inequalities), and more generally, tame functions fall within this class [74]. We will show (Theorem 6.3.10) that for a *directionally Lipschitzian*, *stratifiable* function $f\colon \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, that is continuous on its domain (for simplicity), the Clarke subdifferential admits the intuitive form

$$\partial_c f(\bar{x}) = \text{conv}\{\lim_{i\to\infty} \nabla f(x_i) : x_i \xrightarrow{\Omega} \bar{x}\} + \text{cone}\,\{\lim_{\substack{i\to\infty \\ t_i \downarrow 0}} t_i \nabla f(x_i) : x_i \xrightarrow{\Omega} \bar{x}\} + N^c_{\text{dom}\, f}(\bar{x}), \tag{6.1}$$

or equivalently,

$$\partial_c f(\bar{x}) = \bigcap_{\delta>0} \text{cl}\,\text{conv}\Big(\nabla f\big(\Omega \cap B_\delta(\bar{x})\big)\Big) + N^c_{\text{dom}\, f}(\bar{x}),$$

where Ω is any dense subset of $\text{dom}\, f$ and cone denotes the convex conical hull. (In contrast to the aforementioned results, we do not require Ω to have full-measure).

This is significant both from theoretical and computational perspectives. Proofs of (LipR) and (ACLR) are based largely on Fubini's theorem and analysis of directional derivatives, and though the arguments are elegant, they do not shed light on the geometry driving such representations to hold. Similarly, Rockafellar's argument of (CoR) relies heavily on the well-oiled machinery of convex analysis. Consequently, a simple unified geometric argument is extremely desirable. From a practical point of view, representation (6.1) decouples the behavior of the function from the geometry of the domain; consequently, when the domain is a simple set (polyhedral perhaps) and the behavior of the function on the interior of the domain is complex, our result provides a convenient method of calculating the Clarke subdifferential purely in terms of limits of gradients and the normal cone to the domain — information that is often readily available. Furthermore, using (6.1), the functions we consider in the current chapter become amenable to the techniques developed in [24].

Whereas (6.1) deals with pointwise estimation of the Clarke subdifferential, our second result addresses the geometry of subdifferential graphs, as a whole. In particular, we consider the size of subdifferential graphs, a feature that may have important algorithmic applications. For instance, Robinson [101, 102] shows computational promise for functions defined on $\mathbf{R}^n$ whose subdifferential graphs are locally homeomorphic to an open subset of $\mathbf{R}^n$. Due to the results of Minty [92] and Poliquin-Rockafellar [98], Robinson's techniques are applicable for convex, and more generally, for "prox-regular" functions. Trying to understand the size of subdifferential graphs in the absence of convexity (or monotonicity), the authors in [45] were led to consider the semi-algebraic setting. The authors proved that the *limiting subdifferential graph* of a closed, proper, *semi-algebraic* function on $\mathbf{R}^n$ has *uniform local dimension* n. Applications to sensitivity analysis were also discussed. We show how the techniques developed in the current chapter drastically simplify the proof of this striking fact. Remarkably, this dimensional uniformity does not hold for the Clarke subdifferential graph.

The rest of the chapter is organized as follows. In Section 6.2, we establish notation and recall some basic facts from variational analysis. In Section 6.3, we derive a characterization formula for the Clarke subdifferential of a directionally Lipschitzian, stratifiable function that possesses a certain continuity property on its domain. In Section 6.4, we prove the theorem concerning the local dimension of semi-algebraic subdifferential graphs. We have designed this last section to be entirely independent from the previous ones, since it does require a short foray into semi-algebraic geometry.

6.2 Preliminary results.

We begin with the following standard result in smooth manifold theory.

Theorem 6.2.1 (Prox-normal neighborhood)**.** *Consider a $\mathbf{C}^2$-manifold $M \subset \mathbf{R}^n$ and a point $\bar{x} \in M$. Then there exists an open neighborhood U of $\bar{x}$, such that*

1. *the projection map P_M is single-valued on U,*
2. *for any two points $x \in M \cap U$ and $v \in U$, the equivalence,*

$$v \in x + N_M(x) \Leftrightarrow x = P_M(v),$$

holds.

Following the notation of [67], we call the set U that is guaranteed to exist by Theorem 6.2.1, a *prox-normal neighborhood* of M at $\bar{x}$. For more details about Theorem 6.2.1, see [108, Exercise 13.38], [32, Proposition 1.9]. We should note that the theorem above holds for all "prox-regular" sets M [98].

Often, we will work with discontinuous functions $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. For such functions, it is useful to consider *f-attentive* convergence of a sequence x_i to a point $\bar{x}$, denoted $x_i \underset{f}{\to} \bar{x}$. In this notation we have

$$x_i \underset{f}{\to} \bar{x} \quad \Longleftrightarrow \quad x_i \to \bar{x} \text{ and } f(x_i) \to f(\bar{x}).$$

If in addition we have a set $Q \subset \mathbf{R}^n$, then $x_i \xrightarrow[f]{Q} \bar{x}$ will mean that x_i converges f-attentively to $\bar{x}$ and the points x_i all lie in Q.

Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that is locally lower semi-continuous at a point $\bar{x}$, with $f(\bar{x})$ finite. Then f is locally Lipschitz continuous around $\bar{x}$ if and only if the horizon subdifferential is trivial, that is the condition $\partial^\infty f(\bar{x}) = \{0\}$ holds [108, Theorem 9.13]. Weakening the latter condition to requiring $\partial^\infty f(\bar{x})$ to simply be pointed, we arrive at the following central notion [108, Exercise 9.42].

Definition 6.2.2 (epi-Lipschitzian sets and directionally Lipschitzian functions)**.**

1. A set $Q \subset \mathbf{R}^n$ is *epi-Lipschitzian* at one of its points $\bar{x}$ if Q is locally closed at $\bar{x}$ and the normal cone $N_Q(\bar{x})$ is pointed.
2. A function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, that is finite at $\bar{x}$, is *directionally Lipschitzian* at $\bar{x}$ if f is locally lower-semicontinuous at $\bar{x}$ and the cone $\partial^\infty f(\bar{x})$ is pointed.

Rockafellar [104, Section 4] proved that an epi-Lipschitzian set in $\mathbf{R}^n$, up to a rotation, locally coincides with an epigraph of a Lipschitz continuous function defined on $\mathbf{R}^{n-1}$. We should further note that the Clarke normal cone mapping of an epi-Lipschitzian set is outer-semicontinuous [32, Proposition 6.8].

It is easy to see that a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is directionally Lipschitzian at $\bar{x}$ if and only if the epigraph epi f is epi-Lipschitzian at $(\bar{x}, f(\bar{x}))$. Furthermore, for a set Q that is locally closed at $\bar{x}$, the limiting normal cone $N_Q(\bar{x})$ is pointed if and only if the Clarke normal cone $N^c_Q(\bar{x})$ is pointed [108, Exercise 9.42].

Consider the two functions

$$f_1(x) = x \quad \text{and} \quad f_2(x) = \begin{cases} x & \text{if } x \le 0 \\ x+1 & \text{if } x > 0 \end{cases}$$

defined on the real line. Clearly both f_1 and f_2 are directionally Lipschitzian, and have the same derivatives at each point of differentiability. However $\partial_c f_1(0) \neq \partial_c f_2(0)$. Roughly speaking, this situation arises because some normal cones to the epigraph of a function f, namely at points (x, r) with $r > f(x)$, may not correspond to any subdifferential. Consequently, if we have any hope of deriving a characterization of the Clarke subdifferential purely in terms of gradients and the normal cone to the domain, we must eliminate the situation above. Evidently, an assumption of continuity of the function on the domain would do the trick. However, such an assumption would immediately eliminate some interesting convex functions from consideration. Rather than doing so, we identify a new condition, which arises naturally as a byproduct of our arguments. At the risk of sounding extravagant, we give this property a name.

Definition 6.2.3 (Vertical continuity)**.** We say that a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is *vertically continuous* at a point $\bar{x} \in \operatorname{dom} f$ if the equation

$$\limsup_{\substack{x_i \to \bar{x},\ r \to f(\bar{x}) \\ r > f(\bar{x})}} N_{\operatorname{epi} f}(x, r) = N_{\operatorname{dom} f}(\bar{x}) \times \{0\}, \tag{6.2}$$

holds.

To put this condition in perspective, we record the following observations. For a proof, see [48, Proposition 2.14].

Proposition 6.2.4 (Properties of vertically continuous functions)**.** *Consider a proper function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that is lsc at a point $\bar{x}$, with $f(\bar{x})$ finite.*

1. *Suppose that whenever a pair* $(x,r) \in \operatorname{epi} f$*, with* $r > f(x)$*, is near* $(\bar{x}, f(\bar{x}))$ *we have*

$$N_{\operatorname{epi} f}(x,r) = N_{\operatorname{dom} f}(x) \times \{0\}.$$

Then f *is vertically continuous at* $\bar{x}$*.*

2. *Suppose that* $\bar{x}$ *lies in the interior of* $\operatorname{dom} f$ *and that* f *is vertically continuous at* $\bar{x}$*. Then* f *is continuous at* $\bar{x}$*, in the usual sense.*

3. *Suppose that* f *is continuous on a neighborhood of* $\bar{x}$*, relative to the domain of* f*. Then* f *is vertically continuous at all points of* $\operatorname{dom} f$ *near* $\bar{x}$*.*

4. *If* f *is convex, then* f *is vertically continuous at every point* $\bar{x}$ *in* $\operatorname{dom} f$*.*

5. *Suppose that* f *is "amenable" at* $\bar{x}$ *in the sense of [97]; that is,* f *is finite at* $\bar{x}$ *and there exists a neighborhood* V *of* $\bar{x}$ *so that* f *can be written as a composition* $f = g \circ F$*, for a* $\mathbf{C}^1$ *mapping* $F \colon V \to \mathbf{R}^m$ *and a proper, lsc, convex function* $g \colon \mathbf{R}^m \to \overline{\mathbf{R}}$*, so that the qualification condition*

$$N_{\operatorname{dom} g}(F(\bar{x})) \cap \ker \nabla F(\bar{x})^* = \{0\},$$

is satisfied. Then f *is vertically continuous at* $\bar{x}$*.*

As can be seen from the proposition above, vertical continuity bridges the gap between continuity of the function on the interior of the domain and continuity on the whole domain, and hence the name. In summary, all convex and amenable functions have this property, as do functions that are continuous on their domains. An illustrative example is provided by the proper, lower semicontinuous, convex (directionally Lipschitzian) function f on $\mathbf{R}^2$, defined by

$$f(x,y) = \begin{cases} y^2/2x & \text{if } x > 0 \\ 0 & \text{if } x = 0, y = 0 \\ \infty & \text{otherwise} \end{cases}$$

This function is discontinuous at the origin, despite being vertically continuous there.

6.3 Characterization of the Clarke Subdifferential.

As was mentioned in the introduction, a key feature of Clarke's construction is that the Clarke subdifferential of a locally Lipschitz function f, on $\mathbf{R}^n$, can be described purely in terms of gradient information. It is then reasonable to hope that the same property holds for continuous directionally Lipschitzian functions, but this is too good to be true. Though such functions are differentiable almost everywhere [14], their gradients may fail to generate the entire Clarke subdifferential. A simple example is furnished by the classical ternary Cantor function — a nondecreasing, continuous, and therefore directionally Lipschitzian function, with zero derivative at each point of differentiability. The Clarke subdifferential of this function does not identically consist of the zero vector [20, Exercise 3.5.5], and consequently cannot be recovered from classical derivatives. This example notwithstanding, one does not expect the Cantor function to arise often in practice.

Nonsmoothness arises naturally in many applications, but not pathologically so. On the contrary, nonsmoothness is usually highly structured. Often such structure manifests itself through existence of a *stratification*. In the current work, we consider so-called *stratifiable* functions. Roughly speaking, domains of such function can be decomposed into smooth manifolds (called strata), which fit together in a "regular" way, and so that the function is smooth on each such stratum. In particular, this rich class of functions includes all semi-algebraic, and more generally, all o-minimally defined functions. See for example [115]. We now make this notion precise.

Definition 6.3.1 (Locally finite stratifications)**.** Consider a set Q in $\mathbf{R}^n$. A *locally finite stratification* of Q is a partition of Q into disjoint manifolds M_i (called strata) satisfying

- **(frontier condition)** for each index i, the the closure of M_i in Q is the union of some M_j's, and
- **(local finiteness)** each point $x \in Q$ has a neighborhood that intersects only finitely many strata.

We say that a set $Q \subset \mathbf{R}^n$ is *stratifiable* if it admits a locally finite stratification.

Observe that due to the frontier condition, a stratum M_i intersects the closure of another stratum M_j if and only if the inclusion $M_i \subset \operatorname{cl} M_j$ holds. Consequently, given a locally finite stratification of a set Q into manifolds $\{M_i\}$, we can impose a natural partial order on the strata, namely

$$M_i \preceq M_j \Leftrightarrow M_i \subset \operatorname{cl} M_j.$$

A good example to keep in mind is the partition of a convex polyhedron into its open faces.

Definition 6.3.2 (Stratifiable functions)**.** A function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is *stratifiable* if there exists a locally finite stratification of $\operatorname{dom} f$ so that f is smooth on each stratum.

The following result nicely illustrates the geometric insight one obtains by working with stratifications explicitly.

Proposition 6.3.3 (Dense differentiability)**.** *Consider a proper stratifiable function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that is directionally Lipschitzian at all points of* $\operatorname{dom} f$ *near $\bar{x}$, and let Ω be any dense subset of* $\operatorname{dom} f$*. Then the set $\Omega \cap \operatorname{dom} \nabla f$ is dense in the domain of f, in the f-attentive sense, locally near $\bar{x}$.*

Proof. Consider a locally finite stratification of $\operatorname{dom} f$ into manifolds M_i so that f is smooth on each stratum. Suppose for the sake of contradiction that there exists a point $x \in \operatorname{dom} f$ arbitrarily close to $\bar{x}$ and an f-attentive neighborhood $V = \{y \in \mathbf{R}^n : |y - x| < \epsilon, |f(y) - f(x)| < \delta\}$ so that $V \cap \Omega$ does not intersect any strata of dimension n. Shrinking V, we may assume that V intersects only finitely many strata, say $\{M_j\}$ for $j \in J := \{1, \ldots, k\}$, and that the inclusion $x \in \operatorname{cl} M_j$ holds for each index $j \in J$. Notice that since f is continuous on each stratum, the set V is a union of open subsets of the strata M_j for $j \in J$.

Now among the strata M_j with $j \in J$, choose a stratum M that is maximal with respect to the partial order $\preceq$. Clearly, we have

$$M \cap \operatorname{cl} M_j = \emptyset, \text{ for each } j \in J \text{ with } M_j \neq M.$$

Now let y be any point of $V \cap M$ and observe that there exists a neighborhood Y of y so that the functions f and $f + \delta_M$ coincide on $Y \cap M$. We deduce that $\partial f(y)$ is a nontrivial affine subspace. Since f is directionally Lipschitzian at all points in $\operatorname{dom} f$ near $\bar{x}$, and in particular at y, we have arrived at a contradiction. Thus $\Omega \cap \operatorname{dom} \nabla f$ is dense (in the f-attentive sense) in $\operatorname{dom} f$, locally near $\bar{x}$. $\square$

In this section, we will derive a characterization formula for the Clarke subdifferential of a stratifiable, vertically continuous, directionally Lipschitzian function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. This formula will only depend on the gradients of f and on the normal cone to the domain. It is important to note that the characterization formula, we obtain, is independent of any particular stratification of dom f; one only needs to know that f is stratifiable in order to apply our result. The argument we present is entirely constructive and is motivated by the following fact.

Proposition 6.3.4. *Consider a closed, convex cone $Q \subset \mathbf{R}^n$, which is neither $\mathbf{R}^n$ nor a half-space. Then the equality,*

$$Q = \text{cone}\,(\text{bd}\,Q).$$

holds.

Hence in light of Proposition 6.3.4, in order to obtain a representation formula for the Clarke subdifferential, it is sufficient to study the boundary structure of the Clarke normal cone. This is precisely the route we take. [1]

Lemma 6.3.5 (Frechét accessibility)**.** *Consider a closed set $Q \subset \mathbf{R}^n$, a $\mathbf{C}^2$-manifold $M \subset Q$, and a point $\bar{x} \in M$. Recall that the inclusion $\hat{N}_Q(\bar{x}) \subset N_M(\bar{x})$ holds. Suppose that a vector $\bar{v} \in \hat{N}_Q(\bar{x})$ lies in the boundary of $\hat{N}_Q(\bar{x})$, relative to the linear space $N_M(\bar{x})$. Then there exists a sequence $(x_i, v_i) \to (\bar{x}, \bar{v})$, with $v_i \in N^P_Q(x_i)$, and so that all points x_i lie outside of M.*

Proof. Choose a vector $\bar{w} \in N_M(\bar{x})$ in such a way so as to guarantee

$$\bar{v} + t\bar{w} \notin \hat{N}_Q(\bar{x}), \quad \text{for all } t > 0.$$

[1]The idea to study the boundary structure of the Clarke normal cone in order to establish a convenient representation for the Clarke subdifferential is by no means new. For instance the same idea was used by Rockafellar to establish the representation formula for the convex subdifferential [103, Theorem 25.6]. While working on the manuscript forming the basis of this chapter, the authors became aware that the same strategy was also used to prove a representation formula for the subdifferential of finite-valued, continuous functions whose epigraph has positive reach [60, Theorem 4.9], [61, Theorem 2]. In particular, Proposition 6.3.4 also appears as [61, Proposition 3].

Consider the vectors

$$y(t) := \bar{x} + t(\bar{v} + t\bar{w}), \tag{6.3}$$

and observe $y(t) \notin \bar{x} + \hat{N}_Q(\bar{x})$ for every $t > 0$. Consider a selection of the projection operator,

$$x(t) \in P_Q(y(t)).$$

Clearly, $y(t) \to \bar{x}$ and $x(t) \to \bar{x}$, as $t \to 0$. Observe

$$\frac{y(t) - x(t)}{t} \in N_Q^P(x(t)),$$
$$x(t) \neq \bar{x}, \tag{6.4}$$

for every t.

We claim that the points $x(t)$ all lie outside of M for all sufficiently small $t > 0$. Indeed, if this were not the case, then for sufficiently small t, the points $x(t)$ and $y(t)$ would lie in the prox-normal neighborhood of M near $\bar{x}$, and we would deduce

$$x(t) = P_M(y(t)) = \bar{x},$$

contradicting (6.4).

Thus all that is left is to show the convergence, $\frac{y(t)-x(t)}{t} \to \bar{v}$. To this end, observe that from (6.3), we have

$$\frac{y(t) - \bar{x}}{t} \to \bar{v}. \tag{6.5}$$

Hence it suffices to argue $\frac{x(t)-\bar{x}}{t} \to 0$. By definition of $x(t)$, we have

$$|y(t) - \bar{x}| \geq |(y(t) - \bar{x}) + (\bar{x} - x(t))|. \tag{6.6}$$

Squaring and simplifying the inequality above, we obtain

$$\left\langle \frac{y(t) - \bar{x}}{t}, \frac{x(t) - \bar{x}}{t} \right\rangle \geq \frac{1}{2}\left|\frac{x(t) - \bar{x}}{t}\right|^2. \tag{6.7}$$

From (6.5) and (6.6), we deduce that the vectors $\frac{x(t)-\bar{x}}{t}$ are bounded as $t \to 0$. Consider any limit point $\gamma \in \mathbf{R}^n$. Taking the limit in (6.7), we obtain

$$\langle \bar{v}, \gamma \rangle \geq \frac{1}{2}|\gamma|^2. \tag{6.8}$$

Since $\bar{v}$ is a Frechét normal, we deduce

$$\langle \bar{v}, x(t) - \bar{x} \rangle \leq o(|x(t) - \bar{x}|).$$

It immediately follows that

$$\langle \bar{v}, \gamma \rangle \leq 0,$$

and in light of (6.8), we obtain $\gamma = 0$. Hence

$$\frac{y(t) - x(t)}{t} \to \bar{v},$$

as we claimed. □

Remark 6.3.6. We note, in passing, that an analogue of Lemma 6.3.5 (with an identical proof) holds when M is simply "prox-regular", in the sense of [98], around $\bar{x}$. In particular, the lemma is valid when M is a convex set.

The combination of Lemma 6.3.5 and Proposition 6.3.4 yields dividends even in the simplest case when the manifold M of Lemma 6.3.5 is a singleton set. We should emphasize that in the following proposition, we do not even assume that the function in question is directionally Lipschitzian or stratifiable.

Proposition 6.3.7 (Isolated singularity)**.** *Consider a continuous function $f\colon U \to \mathbf{R}$, defined on an open set $U \subset \mathbf{R}^n$. Suppose that f is differentiable on $U \setminus \{\bar{x}\}$ for some point $\bar{x} \in U$, and that $\partial f(\bar{x}) \neq \emptyset$. Then*

$$\partial_c f(\bar{x}) = \mathrm{cl}\Big(\mathrm{conv}\{\lim_{i\to\infty} \nabla f(x_i) : x_i \to \bar{x}\} + \mathrm{cone}\,\{\lim_{\substack{i\to\infty \\ t_i \downarrow 0}} t_i \nabla f(x_i) : x_i \to \bar{x}\}\Big),$$

under the convention that $\mathrm{conv}\emptyset = \{0\}$.

Proof. Define the two sets

$$E := \{\lim_{i\to\infty} \nabla f(x_i) : x_i \to \bar{x}\}, \quad H := \{\lim_{\substack{i\to\infty \\ t_i\downarrow 0}} t_i \nabla f(x_i) : x_i \to \bar{x}\},$$

and consider the epigraph $Q :=$ epi f and the singleton set $M := \{(\bar{x}, f(\bar{x}))\}$.

By Lemma 6.3.5 and continuity of f, we have

$$\text{bd}\, \hat{N}_Q(\bar{x}, f(\bar{x})) \subset \text{cone}\,\big(E \times \{-1\}\big) \cup \big(H \times \{0\}\big). \tag{6.9}$$

CASE 1. Suppose $\hat{N}_Q(\bar{x}, f(\bar{x}))$ is not equal to $\mathbf{R}^n \times [0, -\infty)$. Then from Proposition 6.3.4 and (6.9), we deduce

$$N^c_Q(\bar{x}, f(\bar{x})) = \text{cl}\,\text{cone}\,\big(E \times \{-1\}\big) \cup \big(H \times \{0\}\big). \tag{6.10}$$

From (6.10), we see that an inclusion $(v, -1) \in N^c_Q(\bar{x}, f(\bar{x}))$ holds if and only if for every $\epsilon > 0$, there exist vectors $y_i \in E \cup H$, and real numbers $\lambda_i > 0$, for $1 \le i \le n+1$, satisfying

$$\Big| v - \Big(\sum_{i:y_i\in E} \lambda_i y_i + \sum_{i:y_i\in H} \lambda_i y_i \Big) \Big| < \epsilon,$$
$$1 = \sum_{i:y_i\in E} \lambda_i$$

Thus $\partial_c f(\bar{x}) = \text{cl}\,(\text{conv}E + \text{cone}\,H)$, as we claimed.

CASE 2. Now suppose $\hat{N}_Q(\bar{x}, f(\bar{x})) = \mathbf{R}^n \times [0, -\infty)$. Then from (6.9), we deduce $H = \mathbf{R}^n$ and $\partial_c f(\bar{x}) = \mathbf{R}^n = \text{conv}E + \text{cone}\,H$, under the convention $\text{conv}\emptyset = \{0\}$. □

As an illustration, consider the following simple example.

Example 6.3.8. Consider the function $f(x,y) := \sqrt[4]{x^4 + y^2}$ on $\mathbf{R}^2$. Clearly f is differentiable on $\mathbf{R}^2 \setminus \{(0,0)\}$. The gradient has the form

$$\nabla f(x,y) = \frac{1}{(x^4+y^2)^{3/4}} \begin{pmatrix} x^3 \\ \frac{1}{2}y \end{pmatrix}.$$

From Proposition 6.3.7, we obtain

$$\partial_c f(\bar{x}) = \text{cl}\left(\text{conv}\{\lim_{i\to\infty} \nabla f(x_i) : x_i \to \bar{x}\} + \text{cone}\,\{\lim_{\substack{i\to\infty \\ t_i\downarrow 0}} t_i \nabla f(x_i) : x_i \to \bar{x}\}\right),$$

Observe that the vectors $\nabla f(x,0)$ are equal to $(\pm 1, 0)$, and the vectors $2\sqrt{|y|}\nabla f(0,\pm y)$ are equal to $(0,\pm 1)$, whenever $x \neq 0 \neq y$. Thus we obtain

$$[-1,1] \times \{0\} \subset \text{conv}\{\lim_{i\to\infty} \nabla f(x_i) : x_i \to \bar{x}\}.$$

$$\{0\} \times \mathbf{R} \subset \text{cone}\,\{\lim_{\substack{i\to\infty \\ t_i\downarrow 0}} t_i \nabla f(x_i) : x_i \to \bar{x}\}.$$

Consequently, the inclusion

$$[-1,1] \times \mathbf{R} \subset \partial_c f(0,0)$$

holds. The absolute value of the first coordinate of $\nabla f(x,y)$ is always bounded by 1, which implies the reverse inclusion above. Thus we have exact equality $\partial_c f(0,0) = [-1,1] \times \mathbf{R}$.

We record the following observation for ease of reference.

Corollary 6.3.9. *Consider a closed, convex cone $Q \subset \mathbf{R}^n$ with nonempty interior. Suppose that* $\text{bd}\, Q$ *is contained in a proper linear subspace. Then Q is either all of $\mathbf{R}^n$ or a half-space.*

Proof. Clearly if Q were neither $\mathbf{R}^n$ or a half-space, then by Proposition 6.3.4, we would deduce that $Q = \text{cone}\,(\text{bd}\, Q)$ has empty interior, which is a contradiction. □

Armed with Proposition 6.3.4 and Lemma 6.3.5, we can now prove the main result of this section with ease.

Theorem 6.3.10 (Characterization)**.** *Consider a proper, stratifiable function* $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ *that is finite at* $\bar{x}$*. Suppose that* f *is vertically continuous and directionally Lipschitzian at all points of* dom f *near* $\bar{x}$*. Then for any dense subset* $\Omega \subset$ dom f*, we have*

$$N^c_{\mathrm{epi}\, f}(\bar{x}, f(\bar{x})) = \mathrm{cone}\,\Big\{ \lim_{i\to\infty} \frac{(\nabla f(x_i), -1)}{\sqrt{1+|\nabla f(x_i)|}} : x_i \xrightarrow[f]{\Omega} \bar{x}\Big\} + \big(N^c_{\mathrm{dom}\, f}(\bar{x}) \times \{0\}\big). \quad (6.11)$$

Consequently, the Clarke subdifferential admits the presentation

$$\partial_c f(\bar{x}) = \mathrm{conv}\Big\{ \lim_{i\to\infty} \nabla f(x_i) : x_i \xrightarrow[f]{\Omega} \bar{x}\Big\} + \mathrm{cone}\,\Big\{ \lim_{\substack{i\to\infty \\ t_i \downarrow 0}} t_i \nabla f(x_i) : x_i \xrightarrow[f]{\Omega} \bar{x}\Big\} + N^c_{\mathrm{dom}\, f}(\bar{x}).$$

Proof. We first prove (6.11). Observe that since f is vertically continuous at $\bar{x}$, we have $N^c_{\mathrm{dom}\, f}(\bar{x}) \times \{0\} \subset N^c_{\mathrm{epi}\, f}(\bar{x}, f(\bar{x}))$, and hence the inclusion "$\supset$" holds. Therefore we must establish the reverse inclusion. To this effect, intersecting the domain of f with a small open ball around $\bar{x}$, we may assume that f is directionally Lipschitzian and vertically continuous at each point $x \in$ dom f. For notational convenience, for a vector $v \in \mathbf{R}^n$, let $\overline{v} := \frac{(v,-1)}{\sqrt{1+|v|^2}}$. Define the set-valued mapping

$$F(x) := \mathrm{cone}\,\Big(\big\{ \lim_{i\to\infty} \overline{\nabla f(x_i)} : x_i \xrightarrow[f]{\Omega} x\big\} \cup (N_{\mathrm{dom}\, f}(x) \cap \mathbf{B}) \times \{0\}\Big), \quad (6.12)$$

By Proposition 6.3.3, the set $\{\lim_{i\to\infty} \overline{\nabla f(x_i)} : x_i \xrightarrow[f]{\Omega} x\}$ is nonempty. Furthermore, from the established inclusion "$\supset$", we see that $N_{\mathrm{dom}\, f}(x)$ is pointed and hence the set cone $N_{\mathrm{dom}\, f}(x)$ is closed for all $x \in$ dom f. Consequently, we deduce

$$F(x) = \mathrm{cone}\,\big\{ \lim_{i\to\infty} \overline{\nabla f(x_i)} : x_i \xrightarrow[f]{\Omega} x\big\} + \big(N^c_{\mathrm{dom}\, f}(x) \times \{0\}\big),$$

Combining (6.12) with [48, Lemma 2.2], we see that F is outer-semicontinuous with respect to f-attentive convergence.

Now consider a stratification of dom f into manifolds $\{M_i\}$ having the property that f is smooth on each stratum M_i. Restricting the domain of f, we may assume that the stratification $\{M_i\}$ consists of only finitely many sets. We prove the theorem by induction on the dimension of the strata M_i in which the point $\bar{x}$ lies.

Clearly, the result holds for all strata of dimension n, since f is smooth on such strata and Ω is dense in dom f. As an inductive hypothesis, suppose that the claim holds for all strata that are strictly greater in the partial order $\preceq$ than a certain stratum M and let $\bar{x}$ be an arbitrary point of M.

Since f is smooth on M, we deduce that gph $f\big|_M$ is a smooth manifold. Then by Lemma 6.3.5, for every vector $0 \neq v \in \text{rb}\, \hat{N}_{\text{epi}\, f}(\bar{x}, f(\bar{x}))$, there exists a sequence $(x_l, r_l, v_l) \to (\bar{x}, f(\bar{x}), v)$, with $v_l \in \hat{N}_{\text{epi}\, f}(x_l, r_l)$ and $(x_l, r_l) \notin$ gph $f\big|_M$. Suppose that there exists a subsequence satisfying $r_l \neq f(x_l)$ for each index l. Then since f is vertically continuous at $\bar{x}$, we obtain

$$v = \lim_{l\to\infty} v_l \subset \limsup_{l\to\infty} N_{\text{epi}\, f}(x_l, r_l) \subset N_{\text{dom}\, f}(\bar{x}) \times \{0\} \subset F(\bar{x}).$$

On the other hand, if $r_l = f(x_l)$ for all large indices i, then restricting to a subsequence, we may assume that all the points x_l lie in a a stratum M' with $M' \succ M$. The inductive hypothesis and f-attentive outer-semicontinuity of F yield the inclusion $v \in F(\bar{x})$.

Thus we have established the inclusion,

$$\text{rb}\, \hat{N}_{\text{epi}\, f}(\bar{x}, f(\bar{x})) \subset F(\bar{x}).$$

Since $\partial^\infty f(\bar{x})$ is pointed, we deduce that the cone $\hat{N}_{\text{epi}\, f}(\bar{x})$ is neither a linear subspace nor a half-subspace. Consequently by [48, Lemma 2.2], we deduce

$$\hat{N}_{\text{epi}\, f}(\bar{x}, f(\bar{x})) = \text{cone}\,\text{rb}\, \hat{N}_{\text{epi}\, f}(\bar{x}, f(\bar{x})) \subset F(\bar{x}). \tag{6.13}$$

In fact, we have shown that (6.13) holds for all points $\bar{x} \in M$.

Finally consider a limiting normal $v \in N_{\text{epi}\, f}(\bar{x})$. Then there exists a sequence $(x_l, r_l, v_l) \to (\bar{x}, f(\bar{x}), v)$, with $v_l \in \hat{N}_{\text{epi}\, f}(x_l, r_l)$. It follows from (6.13), the inductive hypothesis, and f-attentive outer-semicontinuity of F that the inclusion $v \in F(\bar{x})$ holds. Thus the induction is complete, as is the proof of (6.11).

To finish the proof of the theorem, define the two sets

$$E := \{\lim_{i\to\infty} \nabla f(x_i) : x_i \xrightarrow[f]{\Omega} \bar{x}\}, \quad H := \{\lim_{\substack{i\to\infty \\ t_i \downarrow 0}} t_i \nabla f(x_i) : x_i \xrightarrow[f]{\Omega} \bar{x}\}. \tag{6.14}$$

Observe

$$\text{cone}\,\{\lim_{i\to\infty} \overrightarrow{\nabla f(x_i)} : x_i \xrightarrow[f]{\Omega} \bar{x}\} = \text{cone}\,\big(E \times \{-1\}\big) \cup \big(H \times \{0\}\big).$$

Thus an inclusion $(v, -1) \in N^c_Q(\bar{x}, f(\bar{x}))$ holds if and only if there exist vectors $y_i \in E \cup H$ and $y \in N^c_{\text{dom}\, f}(\bar{x})$, and real numbers $\lambda_i > 0$, for $1 \le i \le n+1$, satisfying

$$\begin{aligned} v &= \sum_{i: y_i \in E} \lambda_i y_i + \sum_{i: y_i \in H} \lambda_i y_i + y, \\ 1 &= \sum_{i: y_i \in E} \lambda_i \end{aligned}$$

The result follows. □

Recovering representation (ACLR) of the introduction, in the setting of stratifiable functions, is now an easy task.

Corollary 6.3.11. *Consider a proper, stratifiable function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, that is finite at $\bar{x}$. Suppose that f is directionally Lipschitzian at all points of* $\operatorname{dom} f$ *near $\bar{x}$, and is continuous near $\bar{x}$ relative to the domain of f. Then we have*

$$\partial_c f(\bar{x}) = \bigcap_{\delta>0} \operatorname{cl}\operatorname{conv}\Big(\nabla f\big(\Omega \cap B_\delta(\bar{x})\big)\Big) + N^c_{\operatorname{dom} f}(\bar{x}),$$

where Ω is any dense subset of $\operatorname{dom} f$.

Proof. Since the cone $N^c_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))$ is pointed, one can easily verify, much along the lines of [48, Lemma 2.2], that the equation

$$\bigcap_{\delta>0} \operatorname{cl}\operatorname{cone}\Big\{\frac{(\nabla f(x), -1)}{\sqrt{1+|\nabla f(x)|}} : x \in \Omega \cap B_\delta(\bar{x})\Big\} = \operatorname{cone}\Big\{\lim_{i\to\infty} \frac{(\nabla f(x_i), -1)}{\sqrt{1+|\nabla f(x_i)|}} : x_i \xrightarrow{\Omega} \bar{x}\Big\},$$

holds. The result follows by an application of Theorem 6.3.10. We leave the details to the reader. □

Our next goal is to recover the representation of the convex subdifferential (CoR) of the introduction in the stratifiable setting. In fact, we will consider the more general case of amenable functions. Before proceeding, recall that a proper, lsc, convex function is directionally Lipschitzian at some point if and only if its domain has nonempty interior. A completely analogous situation occurs for amenable functions.

Lemma 6.3.12. *Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that is amenable at $\bar{x}$. Let V be a neighborhood of $\bar{x}$ so that f can be written as a composition $f = g \circ F$, for a $\mathbf{C}^1$ mapping $F\colon V \to \mathbf{R}^m$ and a proper, lsc, convex function $g\colon \mathbf{R}^m \to \overline{\mathbf{R}}$, so that the qualification condition*

$$N_{\operatorname{dom} g}(F(\bar{x})) \cap \ker \nabla F(\bar{x})^* = \{0\},$$

is satisfied. Then there exists a neighborhood U of $\bar{x}$ so that

1. $F(U \cap \operatorname{int} \operatorname{dom} f) \subset \operatorname{int} \operatorname{dom} g$,
2. $U \cap F^{-1}(\operatorname{int} \operatorname{dom} g) \subset \operatorname{int} \operatorname{dom} f$.

Furthermore f is directionally Lipschitzian at $\bar{x}$ if and only if $\bar{x}$ lies in $\operatorname{cl}(\operatorname{int} \operatorname{dom} f)$.

Proof. Let us first recall a few useful formulas. To this end, [97, Theorem 3.3] shows that there exists a neighborhood U of $\bar{x}$ so that for all points $x \in U \cap \operatorname{dom} f$, we have

$$\{0\} = N_{\operatorname{dom} g}(F(x)) \cap \ker \nabla F(x)^*, \tag{6.15}$$

$$\partial f(x) = \nabla F(x)^* \partial g(F(x)), \tag{6.16}$$

$$N_{\operatorname{dom} f}(x) = \nabla F(x)^* N_{\operatorname{dom} g}(F(x)). \tag{6.17}$$

Furthermore a computation in the proof of Proposition 6.2.4 (item 5) shows that for any $x \in U \cap \operatorname{dom} f$ and any $r > f(x)$, we have

$$N_{\operatorname{epi} f}(x, r) = \nabla F(x)^* N_{\operatorname{dom} g}(F(x)) \times \{0\}. \tag{6.18}$$

Observe for any $x \in U \cap \operatorname{int} \operatorname{dom} f$, we have

$$0 = N_{\operatorname{dom} f}(x) = \nabla F(x)^* N_{\operatorname{dom} g}(F(x)),$$

and consequently $N_{\operatorname{dom} g}(F(x)) = 0$. We conclude $F(x) \in \operatorname{int} \operatorname{dom} g$, thus establishing 1.

Now consider a point $x \in U \cap F^{-1}(\operatorname{int} \operatorname{dom} g)$. Using (6.18), we deduce $N_{\operatorname{epi} f}(x, r) = 0$ for any $r > f(x)$. Hence by [108, Exercise 6.19], we conclude $(x, r) \in \operatorname{int} \operatorname{epi} f$ and consequently $x \in \operatorname{int} \operatorname{dom} f$, thus establishing 2.

By [108, Exercise 10.25 (a)], we have $\partial^\infty f(\bar{x}) = N_{\operatorname{dom} f}(\bar{x})$, and in light of (6.15) and (6.17) one can readily verify that the cone $N_{\operatorname{dom} f}(\bar{x})$ is pointed if and only if $N_{\operatorname{dom} g}(F(\bar{x}))$ is pointed.

Now suppose that $\bar{x}$ lies in cl (int dom f). Then by 1 the domain dom g has nonempty interior and consequently $N_{\text{dom}\, g}(F(\bar{x}))$ is pointed, as is $N_{\text{dom}\, f}(\bar{x})$.

Conversely suppose that f is directionally Lipschitzian at $\bar{x}$. Then $N_{\text{dom}\, g}(F(\bar{x}))$ is pointed, and consequently dom g has nonempty interior. Observe since F is continuous, the set $F^{-1}(\text{int dom}\, g) \subset$ dom f is open. Hence it is sufficient to argue that this set contains $\bar{x}$ in its closure. Suppose this is not the case. Then there exists a neighborhood U of $\bar{x}$ so that the image $F(U)$ does not intersect int dom g. It follows that the range of the linearised mapping $w \mapsto F(\bar{x}) + \nabla F(\bar{x})w$ can be separated from dom g, thus contradicting (6.15). See [108, Theorem 10.6] for a more detailed explanation of this latter assertion. □

We can now easily recover, in the stratifiable setting, representation (CoR) of the introduction. In fact, an entirely analogous formula holds more generally for amenable functions.

Corollary 6.3.13. *Consider a proper stratifiable function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$, that is amenable at a point $\bar{x}$, and so that $\bar{x}$ lies in the closure of the interior of* dom f. *Let Ω be any dense subset of* dom f. *Then the subdifferential admits the presentation*

$$\partial f(\bar{x}) = \text{conv}\big\{ \lim_{i\to\infty} \nabla f(x_i) : x_i \xrightarrow{\Omega} \bar{x} \big\} + N_{\text{dom}\, f}(\bar{x}).$$

Proof. By [108, Exercise 10.25], we have $\partial^{\infty} f(\bar{x}) = N_{\text{dom}\, f}(\bar{x})$. Thus we have

$$\text{cone}\,\big\{ \lim_{\substack{i\to\infty \\ t_i \downarrow 0}} t_i \nabla f(x_i) : x_i \xrightarrow[f]{\Omega} \bar{x} \big\} \subset N_{\text{dom}\, f}(\bar{x}).$$

Observe f is amenable, directionally Lipschitzian (Lemma 6.3.12), and vertically continuous (Proposition 6.2.4) at each point of dom f near $\bar{x}$. Applying

Theorem 6.3.10, we deduce

$$\partial f(\bar{x}) = \text{conv}\{\lim_{i\to\infty} \nabla f(x_i) : x_i \xrightarrow[f]{\Omega} \bar{x}\} + N_{\text{dom}\, f}(\bar{x}).$$

Noting that the subdifferential map ∂f of an amenable function is outer-semicontinuous, the result follows. □

A natural question arises. Does the corollary above hold more generally without the stratifiability assumption? The answer turns out to be yes. This is immediate, in light of (CoR), for the subclass of *lower*-$\mathbf{C}^2$ functions (those functions that are locally representable as a difference of convex functions and convex quadratics). A first attempt at a proof for general amenable functions might be to consider the representation $f = g \circ F$ and the chain rule

$$\partial f(x) = \nabla F(\bar{x})^* \partial g(F(\bar{x})).$$

One may then try to naively use Rockafellar's representation formula (CoR) for the convex subdifferential

$$\partial g(F(\bar{x})) = \text{conv}\{\lim_{i\to\infty} \nabla g(y_i) : y_i \to F(\bar{x})\}$$

to deduce the result. However, we immediately run into trouble since F may easily fail to be surjective onto a neighborhood of $F(\bar{x})$ in dom g. Hence a different more sophisticated proof technique is required. For completeness, we present an argument below, which is a natural extension of the proof of [103, Theorem 25.6]. It is furthermore instructive to emphasize how the stratifiability assumption allowed us in Corollary 6.3.13 to bypass essentially all the technical details of the argument below.

Theorem 6.3.14. *Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that is amenable at a point $\bar{x}$ lying in* cl (int dom f)*. Then the subdifferential admits the presentation*

$$\partial f(\bar{x}) = \text{conv}\{\lim_{i\to\infty} \nabla f(x_i) : x_i \xrightarrow{\Omega} \bar{x}\} + N_{\text{dom}\, f}(\bar{x}),$$

where Ω *is any full-measure subset of* $\operatorname{dom} f$.

Proof. Recall that f is Clarke regular at $\bar{x}$, and therefore $\partial^{\infty} f(\bar{x}) = N_{\operatorname{dom} f}(\bar{x})$ is the recession cone of $\partial f(\bar{x})$. Combining this with the fact that the map ∂f is outer-semicontinuous at $\bar{x}$, we immediately deduce the inclusion "$\supset$".

We now argue the reverse inclusion. To this end, let V be a neighborhood of $\bar{x}$ so that f can be written as a composition $f = g \circ F$, for a $\mathbf{C}^1$ mapping $F \colon V \to \mathbf{R}^m$ and a proper, lsc, convex function $g \colon \mathbf{R}^m \to \overline{\mathbf{R}}$, so that the qualification condition

$$N_{\operatorname{dom} g}(F(\bar{x})) \cap \ker \nabla F(\bar{x})^* = \{0\},$$

is satisfied. Since f is directionally Lipschitzian at $\bar{x}$, the subdifferential $\partial f(x)$ is the sum of the convex hull of its extreme points and the recession cone $N_{\operatorname{dom} f}(\bar{x})$. Furthermore every extreme point is a limit of exposed points. Thus

$$\partial f(\bar{x}) = \operatorname{conv}(\operatorname{cl} E) + N_{\operatorname{dom} f}(\bar{x}),$$

where E is the set of all exposed point of $\partial f(\bar{x})$.

Hence to prove the needed inclusion, it suffices to argue the inclusion

$$E \subset \operatorname{conv}\big\{ \lim_{i\to\infty} \nabla f(x_i) : x_i \xrightarrow{\Omega} \bar{x} \big\}.$$

Here, we should note that since f is directionally Lipschitzian at $\bar{x}$, the set on the right hand side is closed.

To this end, let $\bar{v}$ be an arbitrary exposed point of $\partial f(\bar{x})$. By definition, there exists a vector $\bar{a} \in \mathbf{R}^n$ with $|\bar{a}| = 1$ and satisfying

$$\langle \bar{a}, \bar{v} \rangle > \langle \bar{a}, v \rangle \text{ for all } v \in \partial f(\bar{x}) \text{ with } v \neq \bar{v}.$$

Since $N_{\text{dom}\, f}(\bar{x})$ is the recession cone of $\partial f(\bar{x})$, from above we deduce

$$\langle \bar{a}, z \rangle < 0 \text{ for all } 0 \neq z \in N_{\text{dom}\, f}(\bar{x}),$$

and consequently

$$\langle \nabla F(\bar{x})\bar{a}, w \rangle < 0 \text{ for all } 0 \neq w \in N_{\text{dom}\, g}(F(\bar{x})).$$

Consider the half-line $\{F(\bar{x}) + t\nabla F(\bar{x})\bar{a} : t \geq 0\}$. We claim that this half-line cannot be separated from $\text{dom}\, g$. Indeed, otherwise there would exist a nonzero vector $\bar{w} \in N_{\text{dom}\, g}(\bar{x})$ so that for all $t > 0$ and all $x \in \text{dom}\, g$ we have

$$\langle x, \bar{w} \rangle \leq \langle F(\bar{x}) + t\nabla F(\bar{x})\bar{a}, \bar{w} \rangle < \langle F(\bar{x}), \bar{w} \rangle,$$

which is a contradiction. Hence by [103, Theorem 11.3], this half-line must meet the interior of $\text{dom}\, g$. By convexity then there exists a real number $\alpha > 0$ satisfying

$$\{F(\bar{x}) + t\nabla F(\bar{x})\bar{a} : 0 < t \leq \alpha\} \subset \text{int}\,(\text{dom}\, g).$$

Consequently the points $F(\bar{x} + t\bar{a})$ lie in $\text{int}\,\text{dom}\, g$ for all sufficiently small $t > 0$. By Lemma 6.3.12, we deduce that there exists a real number $\beta > 0$ so that

$$\{\bar{x} + t\bar{a} : 0 < t \leq \beta\} \subset \text{int}\,(\text{dom}\, f).$$

Hence f is Lipschitz continuous at each point $\bar{x} + t\bar{a}$ (for $0 < t \leq \beta$), and so from (LipR) we obtain

$$\partial f(\bar{x} + t\bar{a}) = \text{conv}\{\lim_{j\to\infty} \nabla f(x_j) : x_j \xrightarrow{\Omega} \bar{x} + t\bar{a}\}. \tag{6.19}$$

Now choose a sequence $t_i \to 0$ and observe that by [103, Theorem 24.6], for any $\epsilon > 0$ we have

$$\begin{aligned}
\partial g(F(\bar{x} + t_i\bar{a})) &\subset \operatorname*{argmax}_{v \in \partial g(F(\bar{x}))} \langle \nabla F(\bar{x})\bar{a}, v \rangle + \epsilon \mathbf{B}, \\
&= \operatorname*{argmax}_{v \in \partial g(F(\bar{x}))} \langle \bar{a}, \nabla F(\bar{x})^* v \rangle + \epsilon \mathbf{B},
\end{aligned}$$

for all large i. We deduce,

$$\nabla F(\bar{x})^* \partial g(F(\bar{x} + t_i \bar{a})) \subset \underset{w \in \partial f(\bar{x})}{\operatorname{argmax}} \langle \bar{a}, w \rangle + \epsilon \mathbf{B} = \bar{v} + \epsilon \mathbf{B}.$$

Thus there exists a sequence $w_i \in \partial g(F(\bar{x} + t_i \bar{a}))$ with $\nabla F(\bar{x})^* w_i \to \bar{v}$. Consequently the vectors $\nabla F(\bar{x} + t_i \bar{a})^* w_i \in \partial f(\bar{x} + t_i \bar{a})$ converge to $\bar{v}$. The result now follows from (6.19) and the fact that f is directionally Lipschitzian at $\bar{x}$. □

The following is a further illustration of the applicability of our results to a wide variety of situations.

Corollary 6.3.15. *Consider a proper stratifiable function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that is locally Lipschitz continuous at a point $\bar{x}$, relative to* dom f. *Suppose furthermore that* dom f *is an epi-Lipschitzian set at $\bar{x}$. Then the formula*

$$\partial f(\bar{x}) = \text{conv}\{ \lim_{i \to \infty} \nabla f(x_i) : x_i \xrightarrow{\Omega} \bar{x} \} + N^c_{\text{dom}\, f}(\bar{x}),$$

holds, where Ω is any dense subset of dom f.

Proof. Since f is locally Lipschitz near $\bar{x}$ relative to dom f, there exists a globally Lipschitz function $\tilde{f} \colon \mathbf{R}^n \to \mathbf{R}$, agreeing with f on dom f near $\bar{x}$. Hence, we have

$$f(x) = \tilde{f}(x) + \delta_{\text{dom}\, f}(x), \text{ locally near } \bar{x}.$$

Combining this with [108, Exercise 10.10], we deduce

$$\partial^\infty f(x) \subset N_{\text{dom}\, f}(x), \text{ for } x \text{ near } \bar{x}.$$

We conclude that f is directionally Lipschitzian at all points of dom f near $\bar{x}$, and furthermore since the gradients of $\tilde{f}$ are bounded near $\bar{x}$ so are the gradients of f. The result follows immediately by an application of Proposition 6.2.4 and Theorem 6.3.10. □

6.4 Local dimension of semi-algebraic subdifferential graphs.

We begin with a definition.

Definition 6.4.1 (Subjets)**.** For a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, the *limiting subjet* is given by

$$[\partial f] := \{(x, f(x), v) : v \in \partial f(x)\}.$$

Subjets corresponding to the other subdifferentials are defined analogously.

Much like f-attentive convergence, subjets are useful for keeping track of variational information in absence of continuity. In this section, we build on the following theorem. This result and its consequences for generic semi-algebraic optimization problems are discussed extensively in [52].

Theorem 6.4.2. *[52, Theorem 3.6] Let $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ be a proper semi-algebraic function. Then the subjets $[\partial_P f]$, $[\hat{\partial} f]$, $[\partial f]$ and $[\partial_c f]$ have dimension exactly n.*

An immediate question arises: Can the four subjets associated to a semi-algebraic function have local dimension smaller than n at some of their points? In a recent paper [45], the authors showed that this indeed may easily happen for $[\partial_c f]$. Remarkably the authors showed that the subjets $[\partial_P f]$, $[\hat{\partial} f]$, and $[\partial f]$ of a lsc, semi-algebraic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ do have uniform local dimension n. The significance of this result and the relation to Minty's theorem were also discussed. In this section, we provide a much simplified proof of this rather striking fact (Theorem 6.4.5). The main tool we use is the following accessibility lemma, which is a special case of Lemma 6.3.5. Since the proof is much simpler than that of Lemma 6.3.5, we include the full argument below.

Lemma 6.4.3 (Accessibility). *Consider a closed set $Q \subset \mathbf{R}^n$, a manifold $M \subset Q$, and a point $\bar{x} \in M$. Recall that the inclusion $N_Q^P(\bar{x}) \subset N_M(\bar{x})$ holds. Suppose that a proximal normal vector $\bar{v} \in N_Q^P(\bar{x})$ lies in the boundary of $N_Q^P(\bar{x})$, relative to the linear space $N_M(\bar{x})$. Then there exist sequences $x_i \to \bar{x}$ and $v_i \to \bar{v}$, with $v_i \in N_Q^P(x_i)$, and so that all the points x_i lie outside of M.*

Proof. There exists a real number $\lambda > 0$ so that $\bar{x} + \lambda\bar{v}$ lies in the prox-normal neighborhood W of M at $\bar{x}$ and such that the equality $P_Q(\bar{x} + \lambda\bar{v}) = \bar{x}$ holds. Consider any sequence $v_i \in \mathbf{R}^n$ satisfying

$$v_i \to \bar{v}, v_i \in N_M(\bar{x}), v_i \notin N_Q^P(\bar{x}).$$

Choose arbitrary points $x_i \in P_Q(\bar{x} + \lambda v_i)$. We have

$$(\bar{x} - x_i) + \lambda v_i \in N_Q^P(x_i).$$

We deduce $x_i \neq \bar{x}$. Clearly, the sequence x_i converges to $\bar{x}$. We claim $x_i \notin M$ for all sufficiently large indices i. Indeed, if it were otherwise, then for large i, the points $\bar{x} + \lambda v_i$ would lie in W and we would have $x_i \in P_M(\bar{x} + \lambda v_i) = \bar{x}$, which is a contradiction. Thus we have obtained a sequence $(x_i, \frac{1}{\lambda}(\bar{x} - x_i) + v_i) \in \operatorname{gph} N_Q^P$, with $x_i \notin M$, and satisfying $(x_i, \frac{1}{\lambda}(\bar{x} - x_i) + v_i)) \to (\bar{x}, \bar{v})$. □

The following is now immediate.

Corollary 6.4.4. *Consider a lower semicontinuous function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$, a manifold $M \subset \mathbf{R}^n$, and a point $\bar{x} \in M$. Suppose that f is smooth on M and the strict inequality $\dim \partial_P f(\bar{x}) < \dim N_M(\bar{x})$ holds. Then for every vector $\bar{v} \in \partial_P f(\bar{x})$, there exist sequences $(x_i, f(x_i), v_i) \to (\bar{x}, f(\bar{x}), \bar{v})$, with $v_i \in \partial_P f(x_i)$, and so that all the points x_i lie outside of M.*

Proof. From the strict inequality, one can easily see that the normal cone $N^P_{\text{epi} f}(\bar{x}, f(\bar{x}))$ has empty interior relative to the normal space $N_{\text{gph}}(\bar{x}, f(\bar{x}))$. An application of Lemma 6.4.3 completes the proof. □

We can now prove the main result of this section.

Theorem 6.4.5. *Consider a lsc, semi-algebraic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. Then the subjets $[\partial_P f]$, $[\hat{\partial} f]$, and $[\partial f]$ have constant local dimension n around each of their points.*

Proof. We first prove the theorem for the subjet $[\partial_P f]$. Consider the semi-algebraic set-valued mapping

$$F(x) := \{f(x)\} \times \partial_P f(x),$$

whose graph is precisely $[\partial_P f]$. We may stratify the domain of F into finitely many semi-algebraic manifolds $\{M_i\}$, so that on each stratum M_i, the mapping F is inner-semicontinuous, the images $F(x)$ have constant dimension, and f is smooth. Consider a triple $(x, f(x), v) \in [\partial_P f]$. We prove the theorem by induction on the dimension of the strata M in which the point x lies. Clearly the result holds for the strata of dimension n, if there are any. As an inductive hypothesis, assume that the theorem holds for all points $(x, f(x), v) \in [\partial_P f]$ with x lying in strata of dimension at least k, for some integer $k \geq 1$.

Now consider a stratum M of dimension $k-1$ and a point $x \in M$. If $\dim F(x) = n - \dim M$, then recalling that F is inner-semicontinuous on M and applying [48, Proposition 5.11], we see that the set $\text{gph}\, F\Big|_M$ has local dimension n around $(x, f(x), v)$ for any $v \in \partial_P f(x)$. The result follows in this case.

Now suppose $\dim F(x) < n - \dim M$. Then, by Corollary 6.4.4, for such a vector v, there exists a sequence $(x_i, f(x_i), v_i) \to (x, f(x), v)$ satisfying

$(x_i, f(x_i), v_i) \in [\partial_P f]$ and $x_i \notin M$ for each index i. Restricting to a subsequence, we may assume that all the points x_i lie in a stratum K satisfying $\dim K \geq k$. By the inductive hypothesis, we deduce

$$\dim_{[\partial_P f]}(x, f(x), v) \geq \limsup_{i \to \infty} \dim_{[\partial_P f]}(x_i, f(x_i), v_i) = n.$$

This completes the proof of the inductive step and of the theorem for the subjet $[\partial_P f]$.

Now observe that $[\partial_P f]$ is dense in $[\hat{\partial} f]$ and in $[\partial f]$. It follows that $[\hat{\partial} f]$ and $[\partial f]$ also have local dimension n around each of their points. $\square$

Surprisingly Theorem 6.4.5 may fail in the Clarke case, even for Lipschitz continuous functions.

Example 6.4.6. Consider the function $f \colon \mathbf{R}^3 \to \mathbf{R}$, defined by

$$f(x, y, z) = \begin{cases} \min\{x, y, z^2\} & \text{if } (x, y, z) \in \mathbf{R}^3_+ \\ \min\{-x, -y, z^2\} & \text{if } (x, y, z) \in \mathbf{R}^3_- \\ 0 & \text{otherwise.} \end{cases}$$

Let $\bar{x} \in \mathbf{R}^n$ be the origin and let $\Gamma := \text{conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 0)\}$. One can check that the local dimension of $\text{gph}\, \partial_c f$ at $(\bar{x}, \bar{v})$ is two for any vector $\bar{v} \in \big(\text{conv}(\Gamma \cup -\Gamma)\big) \setminus (\Gamma \cup -\Gamma)$. For more details see [45, Example 3.11].

CHAPTER 7

ACTIVE SETS IN OPTIMIZATION

7.1 Introduction

Active set ideas permeate traditional nonlinear optimization. Classical problems involve a list of smooth nonlinear constraints: the active set for a particular feasible solution — the collection of binding constraints at that point — is crucial in first and second order optimality conditions, in sensitivity analysis, and for certain algorithms. Contemporary interest in more general constraints (such as semidefiniteness) suggests a reappraisal. A very thorough modern study of sensitivity analysis in its full generality appears in [13]. Approaches more variational-analytic in flavor appear in texts such as [107]. Our aim here is rather different: to present a simple fresh approach, combining wide generality with mathematical elegance.

Our approach has its roots in the notion of an "identifiable surface" [119], and its precursors [1, 23, 26, 27, 54, 57, 59]. In essence, the idea is extremely simple: given a critical point x for a function f, a set M is *identifiable* if any sequence of points approaching x that is approximately critical (meaning corresponding subgradients approach zero) must eventually lie in M. The terminology comes from the idea that an iterative algorithm that approximates x along with an approximate criticality certificate must "identify" M. To take the classical example where f is a pointwise maximum of smooth functions, around any critical point x, assuming a natural constraint qualification, we can define M as those points with the same corresponding "active set" of functions attaining the maximum.

Identifiable sets M are useful computationally because the problem of minimizing the function f near the critical point x is equivalent to minimizing the restriction of f to M, which may be an easier problem, and because the identifiability property allows convergent algorithms to find M — the motivation for active-set methods. We show moreover how M is a natural tool for optimality conditions: under reasonable conditions, quadratic growth of f around x is equivalent to quadratic growth on M — a potentially easier condition to check.

Clearly the smaller the identifiable set M, the more informative it is. Ideal would be a "locally minimal identifiable set". We note that such sets may fail to exist, even for finite convex functions f. However, when a minimal identifiable set M does exist, we show that it is both unique (locally), and central to sensitivity analysis: it consists locally of all critical points of small linear perturbations to f. We show furthermore that, under reasonable conditions, variational analysis of f simplifies because, locally, the graph of its subdifferential mapping is influenced only by the restriction of f to M. One appealing consequence is a close relationship between minimal identifiable sets and critical cones appearing in the study of variational inequalities.

The case when an identifiable set M is in fact a manifold around the point x (as in the classical example above) is particularly interesting. In particular, the manuscript [87] performs (a much simplified) sensitivity analysis in this setting. Remarkably, this case is equivalent to a powerful but seemingly stringent list of properties known as "partial smoothness" [83], nondegeneracy and prox-regularity — related work on "$\mathcal{VU}$ algorithms" and "the fast track" appears in [89–91] and [65,66]. By contrast, our approach here is to offer a concise mathematical development emphasizing how this important scenario is in fact very natural indeed.

The outline of the chapter is as follows. In Section 7.2, we introduce the notion of identifiability for arbitrary set-valued mappings. Then in Section 7.3, we specialize this idea to subdifferential mappings, laying the foundation for the rest of the chapter. Section 7.4 contains basic examples of identifiable sets, while Section 7.5 establishes a calculus of identifiability, which could be skipped at first reading. Arriving at our main results in Section 7.6, we study variational geometry of identifiable sets; this in particular allows us to establish a strong relationship between identifiable sets and critical cones in Section 7.7. Finally in Section 7.8, we consider optimality conditions in the context of identifiable sets, while in Section 7.9 we establish a relationship between identifiable manifolds and partial smoothness — one of our central original goals.

7.2 Identifiability in set-valued analysis

The key property that we explore in this work is that of finite identification. For two sets M and Q in $\mathbf{R}^n$, we will say that the inclusion $M \subset Q$ *holds locally around a point* $\bar{x}$, if there exists a neighborhood U of $\bar{x}$ satisfying $M \cap U \subset Q \cap U$.

Definition 7.2.1 (Identifiable sets)**.** Consider a mapping $G\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$. We say that a subset $M \subset \mathbf{R}^n$ is *identifiable* at $\bar{x}$ for $\bar{v}$, where $\bar{v} \in G(\bar{x})$, if the inclusion

$$\operatorname{gph} G \subset M \times \mathbf{R}^m \text{ holds locally around } (\bar{x}, \bar{v}).$$

Equivalently, a set M is identifiable at $\bar{x}$ for $\bar{v} \in G(\bar{x})$ if for any sequence $(x_i, v_i) \to (\bar{x}, \bar{v})$ in $\operatorname{gph} G$, the points x_i must lie in M for all sufficiently large indices i. Clearly M is identifiable at $\bar{x}$ for $\bar{v}$ if and only if the same can be said of $M \cap \operatorname{dom} G$. Hence we will make light of the distinction between two such sets.

Clearly $\operatorname{dom} G$ is identifiable at $\bar{x}$ for $\bar{v} \in G(\bar{x})$. More generally, if $\bar{v}$ lies in the interior of some set U, then $G^{-1}(U)$ is identifiable at $\bar{x}$ for $\bar{v}$. Sometimes *all* identifiable subsets of $\operatorname{dom} G$ arise locally in this way. In particular, one can readily check that this is the case for any set-valued mapping G satisfying $G^{-1} \circ G = \mathrm{Id}$, an important example being the inverse of the projection map $G = P_Q^{-1}$ onto a nonempty, closed, convex set Q.

The "smaller" the set M is, the more interesting and the more useful it becomes. Hence an immediate question arises. When is the identifiable set M *locally minimal*, in the sense that for any other identifiable set M' at $\bar{x}$ for $\bar{v}$, the inclusion $M \subset M'$ holds locally around $\bar{x}$? The following notion will be instrumental in addressing this question.

Definition 7.2.2 (Necessary sets)**.** Consider a set-valued mapping $G\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$, a point $\bar{x} \in \mathbf{R}^n$, and a vector $\bar{v} \in G(\bar{x})$. We say that a subset $M \subset \mathbf{R}^n$, containing $\bar{x}$, is *necessary* at $\bar{x}$ for $\bar{v}$ if the function

$$x \mapsto d(\bar{v}, G(x)),$$

restricted to M, is continuous at $\bar{x}$.

Thus M is necessary at $\bar{x}$ for $\bar{v} \in G(\bar{x})$ if for any sequence $x_i \to \bar{x}$ in M, there exists a sequence $v_i \in G(x_i)$ with $v_i \to \bar{v}$. The name "necessary" arises from the following simple observation.

Lemma 7.2.3. *Consider a set-valued mapping $G\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$, a point $\bar{x} \in \mathbf{R}^n$, and a* *vector $\bar{v} \in G(\bar{x})$. Let M and M' be two subsets of $\mathbf{R}^n$. Then the implication*

$$\left.\begin{array}{c} M \text{ is identifiable at } \bar{x} \text{ for } \bar{v} \\ M' \text{ is necessary at } \bar{x} \text{ for } \bar{v} \end{array}\right\} \Rightarrow M' \subset M \text{ locally around } \bar{x},$$

holds.

The following elementary characterization of locally minimal identifiable sets will be used extensively in the sequel, often without an explicit reference. For a proof, see [53, Proposition 2.4].

Proposition 7.2.4 (Characterizing locally minimal identifiability)**.**
Consider a set-valued mapping $G\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ and a pair $(\bar{x}, \bar{v}) \in \operatorname{gph} G$. The following are equivalent.

1. *M is a locally minimal identifiable set at $\bar{x}$ for $\bar{v}$.*
2. *There exists neighborhood V of $\bar{v}$ such that for any subneighborhood $W \subset V$ of $\bar{v}$, the representation*

   $$M = G^{-1}(W) \text{ holds locally around } \bar{x}.$$

3. *M is a locally maximal necessary set at $\bar{x}$ for $\bar{v}$.*
4. *M is identifiable and necessary at $\bar{x}$ for $\bar{v}$.*

Remark 7.2.5. It is clear from Proposition 7.2.4 that whenever locally minimal identifiable sets exist, they are locally unique. That is, if M_1 and M_2 are both locally minimal identifiable sets at $\bar{x}$ for $\bar{v} \in G(\bar{x})$, then we have $M_1 = M_2$ locally around $\bar{x}$.

The central goal in sensitivity analysis is to understand the behavior of solutions x, around $\bar{x}$, to the inclusion

$$v \in G(x),$$

as v varies near $\bar{v}$. Characterization 2 of Proposition 7.2.4 shows that a locally minimal identifiable set at $\bar{x}$ for $\bar{v}$ is a locally minimal set that captures *all* the sensitivity information about the inclusion above.

This characterization yields a constructive approach to finding locally minimal identifiable sets. Consider any open neighborhoods $V_1 \supset V_2 \supset V_3 \supset \ldots,$ around $\bar{v}$ with the diameters of V_i tending to zero. If the chain

$$G^{-1}(V_1) \supset G^{-1}(V_2) \supset G^{-1}(V_3) \supset \ldots,$$

stabilizes, in the sense that for all large indices i and j, we have $G^{-1}(V_i) = G^{-1}(V_j)$ locally around $\bar{x}$, then $G^{-1}(V_i)$ is a locally minimal identifiable set at $\bar{x}$ for $\bar{v}$, whenever i is sufficiently large. Moreover, the locally minimal identifiable set at $\bar{x}$ for $\bar{v}$, if it exists, must arise in this way.

The following example shows that indeed a set-valued mapping can easily fail to admit a locally minimal identifiable set.

Example 7.2.6 (Failure of existence)**.** Consider the mapping $G\colon \mathbf{R}^2 \rightrightarrows \mathbf{R}$, defined in polar coordinates, by

$$G(r,\theta) = \begin{cases} |\theta| & \text{if } r \neq 0,\ \theta \in [-\pi,\pi]\,, \\ [-1,1] & \text{if } r = 0. \end{cases}$$

Let $\bar{x}$ be the origin in $\mathbf{R}^2$ and $\bar{v} := 0 \in G(\bar{x})$. Observe that for $\epsilon \to 0$, the preimages

$$G^{-1}(-\epsilon,\epsilon) = \{(r,\theta) \in \mathbf{R}^2 : G(r,\theta) \cap (-\epsilon,\epsilon) \neq \emptyset\} = \{(r,\theta) : |\theta| < \epsilon\},$$

never coincide around $\bar{x}$. Consequently, there is no locally minimal identifiable set at $\bar{x}$ for $\bar{v}$.

Notwithstanding the previous example, locally minimal identifiable sets do often exist. In particular, clearly inner-semicontinuous mappings always admit locally minimal identifiable sets.

Proposition 7.2.7 (Identifiability under continuity)**.**
Consider a set-valued mapping $G\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ that is inner semicontinuous, relative to $\operatorname{dom} G$*, at a point $\bar{x} \in \operatorname{dom} G$. Then $\operatorname{dom} G$ is a locally minimal identifiable set at $\bar{x}$ for any vector $\bar{v} \in G(\bar{x})$.*

More interesting examples can be constructed by taking pointwise unions of maps admitting locally minimal identifiable sets.

Proposition 7.2.8 (Pointwise union)**.** *Consider a finite collection of outer-semicontinuous mappings, $G_i\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$, for $i = 1, \ldots, k$. Define the pointwise union mapping $G\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ to be*

$$G(x) = \bigcup_{i=1}^{m} G_i(x).$$

Fix a point $\bar{x} \in \mathbf{R}^n$ and a vector $\bar{v} \in G(\bar{x})$, and suppose that for each index i, satisfying $\bar{v} \in G_i(\bar{x})$, there exists a locally minimal identifiable set M_i (with respect to G_i) at $\bar{x}$ for $\bar{v}$. Then the set

$$M := \bigcup_{i : \bar{v} \in G_i(\bar{x})} M_i,$$

is a locally minimal identifiable set (with respect to G) at $\bar{x}$ for $\bar{v}$.

Proof. This readily follows from Proposition 7.2.4. □

In particular, locally minimal identifiable sets exist for *piecewise polyhedral* mappings. These are those mappings whose graphs can be decomposed into a union of finitely many convex polyhedra.

Example 7.2.9 (Piecewise polyhedral mappings)**.**

Consider a piecewise polyhedral mapping $G\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$, where $\operatorname{gph} G = \bigcup_{i=1}^k V_i$ and $V_i \subset \mathbf{R}^n$ are convex polyhedral sets. It is easy to check that set-valued mappings whose graphs are convex polyhedral are inner-semicontinuous on their domains. Fix a point $\bar{x} \in \mathbf{R}^n$ and a vector $\bar{v} \in G(\bar{x})$, and let $\pi\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ be the canonical projection onto $\mathbf{R}^n$. Consequently, by Propositions 7.2.7 and 7.2.8, the set

$$\bigcup_{i:(\bar{x},\bar{v})\in V_i} \pi(V_i)$$

is a locally minimal identifiable set at $\bar{x}$ for $\bar{v}$.

For the remainder of the current work, we will be investigating the notion of identifiability in the context of the workhorse of variational analysis, the subdifferential set-valued mapping.

7.3 Identifiability in variational analysis

We are now ready to define the appropriate notion of identifiability in the context of optimization.

Definition 7.3.1 (Identifiability for functions)**.**

Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, a point $\bar{x} \in \mathbf{R}^n$, and a subgradient $\bar{v} \in \partial f(\bar{x})$. A set $M \subset \mathbf{R}^n$ is *identifiable at $\bar{x}$ for $\bar{v}$* if for any sequences $(x_i, f(x_i), v_i) \to (\bar{x}, f(\bar{x}), \bar{v})$, with $v_i \in \partial f(x_i)$, the points x_i must all lie in M for all sufficiently large indices i.

The definition above can be interpreted in the sense of Section 7.2. Indeed, consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a subgradient $\bar{v} \in \partial f(\bar{x})$, for some point $\bar{x} \in \mathbf{R}^n$. Define the set-valued mapping

$$G\colon \mathbf{R}^n \rightrightarrows \mathbf{R} \times \mathbf{R}^n,$$

$$x \mapsto \{f(x)\} \times \partial f(x).$$

Then M is identifiable (relative to f) at $\bar{x}$ for $\bar{v}$ if and only if it is identifiable (relative to G) at $\bar{x}$ for the vector $(f(\bar{x}), \bar{v})$. Here, we have to work with the mapping G, rather than the subdifferential mapping ∂f directly, so as to facilitate coherence between normal cone mappings and subdifferential mappings via epigraphical geometry. (See Proposition 7.3.7.) This slight annoyance can be avoided whenever f is subdifferentially continuous at $\bar{x}$ for $\bar{v}$.

Definition 7.3.2 (Subdifferential continuity)**.** A function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is *subdifferentially continuous at* $\bar{x}$ *for* $\bar{v} \in \partial f(\bar{x})$ if for any sequences $x_i \to \bar{x}$ and $v_i \to \bar{v}$, with $v_i \in \partial f(x_i)$, it must be the case that $f(x_i) \to f(\bar{x})$.

Subdifferential continuity of a function f at $\bar{x}$ for $\bar{v}$ was introduced in [98, Definition 1.14], and it amounts to requiring the function $(x, v) \mapsto f(x)$, restricted to $\operatorname{gph} \partial f$, to be continuous in the usual sense at the point $(\bar{x}, \bar{v})$. In particular, any lsc convex function is subdifferentially continuous [108, Example 13.30].

Similarly, we define necessary sets as follows.

Definition 7.3.3 (Necessity for functions)**.** Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. A set $M \subset \mathbf{R}^n$ is *necessary at* $\bar{x}$ *for* $\bar{v} \in \partial f(\bar{x})$ if both the function f and the mapping

$$x \mapsto d(\bar{v}, \partial f(x)),$$

restricted to M, are continuous at $\bar{x}$.

Specializing the characterization in Proposition 7.2.4 to this setting, we obtain the following.

Proposition 7.3.4 (Characterizing locally minimal identifiability)**.**
Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, a point $\bar{x} \in \mathbf{R}^n$, and a subgradient $\bar{v} \in \partial f(\bar{x})$. Then the following are equivalent.

1. *M is a locally minimal identifiable set at $\bar{x}$ for $\bar{v}$,*
2. *There exists a neighborhood V of $\bar{v}$ and a real number $\epsilon > 0$ such that for any subneighborhood $W \subset V$ of $\bar{v}$ and a real number $0 < \epsilon' < \epsilon$, the presentation*

   $$M = (\partial f)^{-1}(W) \cap \{x \in \mathbf{R}^n : |f(x) - f(\bar{x})| < \epsilon'\} \text{ holds locally around } \bar{x}.$$

3. *M is a locally maximal necessary set at $\bar{x}$ for $\bar{v}$.*
4. *M is identifiable and necessary at $\bar{x}$ for $\bar{v}$*

Definition 7.3.5 (Identifiability for sets)**.** Given a set $Q \subset \mathbf{R}^n$, we will say that a subset $M \subset Q$ is *identifiable* (relative to Q) at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$ if M is identifiable (relative to δ_Q) at $\bar{x}$ for $\bar{v} \in \partial\delta_Q(\bar{x})$. Analogous conventions will hold for necessary sets and locally minimal identifiable sets.

It is instructive to observe the relationship between identifiability and the metric projection in presence of convexity.

Proposition 7.3.6 (Identifiability for convex sets)**.**
Consider a closed, convex set Q and a subset $M \subset Q$. Let $\bar{x} \in M$ and $\bar{v} \in N_Q(\bar{x})$. Then the following are equivalent.

1. *M is identifiable (relative to Q) at $\bar{x}$ for $\bar{v}$.*
2. *M is identifiable (relative to P_Q^{-1}) at $\bar{x}$ for $\bar{x} + \bar{v}$.*

Analogous equivalence holds for necessary sets.

Proof. Suppose that M is identifiable (relative to Q) at $\bar{x}$ for $\bar{v}$. Consider a sequence $(x_i, y_i) \to (\bar{x}, \bar{x} + \bar{v})$ in gph P_Q^{-1}. Observe $x_i = P_Q(y_i)$ and the sequence $y_i - x_i \in N_Q(x_i)$ converges to $\bar{v}$. Consequently, the points x_i all eventually lie in M.

Conversely suppose that M is identifiable (relative to P_Q^{-1}) at $\bar{x}$ for $\bar{x} + \bar{v}$. Consider a sequence $(x_i, v_i) \to (\bar{x}, \bar{v})$ in gph N_Q. Then the sequence $(x_i, x_i + v_i) \in$ gph P_Q^{-1} converges to $(\bar{x}, \bar{x} + \bar{v})$. Consequently, we have $x_i \in M$ for all large i.

We leave the verification of the analogous equivalence for necessary sets to the reader. □

Thus a subset M of a closed, convex set Q is identifiable at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$ if and only if the equality, $P_Q = P_M$, holds locally around $\bar{x} + \bar{v}$.

The following simple proposition establishes epigraphical coherence, alluded to above, between normal cone mappings and subdifferential mappings in the context of identifiability. The proof is immediate.

Proposition 7.3.7 (Epigraphical coherence)**.**
Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a subgradient $\bar{v} \in \partial f(\bar{x})$, for some point $\bar{x} \in \mathbf{R}^n$. Then $M \subset$ dom f is an identifiable set (relative to f) at $\bar{x}$ for $\bar{v}$ if and only if gph $f\big|_M$ *is an identifiable set (relative to* epi f*) at $(\bar{x}, f(\bar{x}))$ for $(\bar{v}, -1)$. Analogous statements hold for necessary, and consequently for locally minimal identifiable sets.*

7.4 Basic examples

In this section, we present some basic examples of identifiable sets of functions.

Example 7.4.1 (Smooth functions)**.** If U is an open set containing $\bar{x}$ and $f: U \to \mathbf{R}$ is $\mathbf{C}^1$-smooth. Then U is a locally minimal identifiable set at $\bar{x}$ for $\nabla f(\bar{x})$.

Example 7.4.2 (Smooth manifolds)**.** If M is a $\mathbf{C}^1$ manifold, then M is a locally minimal identifiable set at any $x \in M$ for any $v \in N_M(x)$. This follows immediately by observing that the normal cone mapping $x \mapsto N_M(x)$ is inner-semicontinuous on M.

We define the *support* of any vector $v \in \mathbf{R}^n$, denoted $\operatorname{supp} v$, to be the set consisting of all indices $i \in \{1, \ldots, n\}$ such that $v_i \neq 0$. The *rank* of v, denoted $\operatorname{rank} v$, is then the size of the support $\operatorname{supp} v$.

Example 7.4.3 (Convex polyhedra)**.** Let $Q \subset \mathbf{R}^n$ be a convex polyhedron. Example 7.2.9 shows that $M := N_Q^{-1}(\bar{v})$ (equivalently, $M := \operatorname{argmax}_{x \in Q} \langle \bar{v}, x \rangle$) is a locally minimal identifiable set at $\bar{x}$ for $\bar{v}$.

More concretely, suppose that Q has the representation

$$Q = \{x \in \mathbf{R}^n : \langle a_i, x \rangle \leq b_i \text{ for all } i \in I\}, \tag{7.1}$$

for some index set $I = \{1, \ldots, m\}$ and vectors $a_1, \ldots, a_m \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$. For any point $x \in \mathbf{R}^n$, define the active index set

$$I(x) := \{i \in I : \langle a_i, x \rangle = b_i\}.$$

Then we have $N_Q(x) = \operatorname{cone} \{a_i : i \in I(x)\}$ and consequently there exist multipliers $\bar{\lambda} \in \mathbf{R}^m_+$ with $\operatorname{supp} \bar{\lambda} \subset I(\bar{x})$ satisfying $\bar{v} = \sum_{i \in I(\bar{x})} \bar{\lambda}_i a_i$. Hence for any point $y \in Q$, the equivalence

$$y \in M \iff \Big\langle \sum_{i \in I(\bar{x})} \bar{\lambda}_i a_i, y \Big\rangle = \Big\langle \sum_{i \in I(\bar{x})} \bar{\lambda}_i a_i, \bar{x} \Big\rangle$$
$$\iff \sum_{i \in I(\bar{x})} \bar{\lambda}_i [\langle a_i, y \rangle - b_i] = 0,$$

holds. We deduce that M has the alternate description

$$M = \{x \in Q : \text{supp}\, \bar{\lambda} \subset I(x)\}.$$

We should note that under a strict complementarity condition, $\bar{v} \in \text{ri}\, N_Q(\bar{x})$, we may choose $\bar{\lambda}$ with supp $\bar{\lambda} = I(\bar{x})$. Then M would consist of all points $x \in Q$, whose active index set $I(x)$ coincides with $I(\bar{x})$. It is then standard to check that M coincides with an affine subspace locally around $\bar{x}$.

Example 7.4.4 (Polyhedral functions)**.** Analogously, we may analyse a convex polyhedral function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$, a function whose epigraph is a convex polyhedron. To be more precise, we may express f as

$$f(x) = \begin{cases} \max_{i \in I} \{\langle a_i, x \rangle + b_i\} & \text{whenever } \langle c_j, x \rangle \le d_j \text{ for all } j \in J, \\ \infty & \text{otherwise,} \end{cases}$$

for some index sets $I = \{1, \ldots, m\}$ and $J = \{1, \ldots, k\}$, vectors $a_i, c_j \in \mathbf{R}^n$, and real numbers b_i, d_j for $i \in I$ and $j \in J$. For any point $x \in \mathbf{R}^n$, define the active index sets

$$I(x) = \{i \in I : \langle a_i, x \rangle + b_i = f(x)\},$$
$$J(x) = \{j \in J : \langle c_j, x \rangle = d_j\}.$$

A straightforward computation shows

$$\partial f(x) = \text{conv}\{a_i : i \in I(x)\} + \text{cone}\,\{c_j : j \in J(x)\}.$$

Consider a pair $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$. Then there exist multipliers $(\bar{\lambda}, \bar{\mu}) \in \mathbf{R}^m_+ \times \mathbf{R}^k_+$ satisfying

$$\bar{v} = \sum_{i \in I} \bar{\lambda}_i a_i + \sum_{j \in J} \bar{\mu}_j c_j,$$

with $\sum_{i \in I} \bar{\lambda}_i = 1$, $\operatorname{supp} \bar{\lambda} \subset I(\bar{x})$, and $\operatorname{supp} \bar{\mu} \subset J(\bar{x})$. Applying the same argument as in Example 7.4.3 to epi f, we deduce that the set

$$M = \{x \in \operatorname{dom} f : \operatorname{supp} \bar{\lambda} \subset I(x),\ \operatorname{supp} \bar{\mu} \subset J(x)\},$$

is a locally minimal identifiable set at $\bar{x}$ for $\bar{v}$. Again we should note that a particularly nice situation occurs under a strict complementarity condition, $\bar{v} \in \operatorname{ri} \partial f(\bar{x})$. In this case there exist multipliers $(\bar{\lambda}, \bar{\mu})$ so that $\operatorname{supp} \bar{\lambda} = I(\bar{x})$ and $\operatorname{supp} \bar{\mu} = J(\bar{x})$, and then M coincides with an affine subspace locally around $\bar{x}$.

Example 7.4.5 (Maximum function)**.** The example above, in particular, applies to the maximum function mx: $\mathbf{R}^n \to \mathbf{R}$, defined by

$$\text{mx}(x) := \max\{x_1, \ldots, x_n\}.$$

Given a point $\bar{x}$ and a vector $\bar{v} \in \partial(\text{mx})(\bar{x})$, the set $M = \{x \in \mathbf{R}^n : \operatorname{supp} \bar{v} \subset I(x)\}$, where

$$I(x) := \{i : x_i = \text{mx}(x)\},$$

is a locally minimal identifiable set at $\bar{x}$ for $\bar{v}$. Alternatively, M admits the presentation

$$M = \{x \in \mathbf{R}^n : \operatorname{mult} \text{mx}(x) \geq \operatorname{rank} \bar{v}\} \quad \text{locally around } \bar{x},$$

where $\operatorname{mult} \text{mx}(x)$ simply denotes the size of the set $I(x)$.

Generalizing beyond polyhedrality, we now consider the so-called *piecewise linear-quadratic functions*; these are those functions whose domain can be represented as the union of finitely many convex polyhedra, so that the function is linear or quadratic on each such set. Convex piecewise linear-quadratic functions are precisely the convex functions whose subdifferential mappings are piecewise polyhedral [100].

Proposition 7.4.6 (Piecewise linear-quadratic functions)**.**
Consider a convex, piecewise linear-quadratic function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$*. Then there exists a locally minimal identifiable set at any point* $x \in \operatorname{dom} f$ *for any vector* $y \in \partial f(x)$*.*

Proof. Convex piecewise linear-quadratic functions have piecewise polyhedral subdifferential mappings [100]. Consequently, Example 7.2.9 shows that the mapping $x \mapsto \partial f(x)$ admits a locally minimal identifiable set at any point $x \in \mathbf{R}^n$ for any vector $v \in \partial f(x)$. Since piecewise linear-quadratic functions are lower-semicontinuous [108, Proposition 10.21], and lower-semicontinuous convex functions are subdifferentially continuous [108, Example 13.30], the result follows. □

We now briefly consider the three standard convex cones of mathematical programming.

Example 7.4.7 (Non-negative Orthant)**.** Consider a point $\bar{x} \in \mathbf{R}^n_+$ and a vector $\bar{v} \in N_{\mathbf{R}^n_+}(\bar{x})$. Then $M := \{x \in \mathbf{R}^n_+ : x_i = 0 \text{ for each } i \in \operatorname{supp} \bar{v}\}$ is a locally minimal identifiable set at $\bar{x}$ for $\bar{v}$. Observe that M also admits the presentation

$$M = \{x \in \mathbf{R}^n_+ : \operatorname{rank} x + \operatorname{rank} \bar{v} \leq n\} \quad \text{locally around } \bar{x}.$$

Example 7.4.8 (Lorentz cone)**.** Consider the Lorentz cone

$$\mathcal{L}^n := \{(x, r) \in \mathbf{R}^n \times \mathbf{R} : r \geq |x|\}.$$

Observe that $\mathcal{L}^n$ coincides with the epigraph $\operatorname{epi} |\cdot|$. Let $\bar{x} = 0$ and consider any $v \in \partial|\cdot|(0)$ with $|v| = 1$. Then for any real $\epsilon > 0$, the set $M_\epsilon := \{x \in \mathbf{R}^n : \langle \frac{x}{|x|}, \bar{v}\rangle \leq$

$\epsilon\}$ is identifiable at $\bar{x}$ for $\bar{v}$. In particular, for $n \geq 2$ and $\epsilon \neq \epsilon'$ the sets M_ϵ and $M_{\epsilon'}$ do not coincide on any neighborhood of $\bar{x}$, and consequently there is no locally minimal identifiable set at $\bar{x}$ for $\bar{v}$.

In what follows $\mathbf{S}^n$ will denote the space of $n \times n$ real symmetric matrices with the trace inner product while $\mathbf{S}^n_+$ will denote the convex cone of symmetric positive semi-definite matrices. With every matrix $X \in \mathbf{S}^n$ we will associate its largest eigenvalue, denoted by $\lambda_1(X)$. The multiplicity of $\lambda_1(X)$ as an eigenvalue of X will be written as $\operatorname{mult} \lambda_1(X)$. Finally $\mathbf{M}^{n\times m}$ will denote the space of $n \times m$ matrices with real entries. We defer the verification of the following two examples to a forthcoming paper [36]. We should also emphasize the intriguing parallel between these two examples and Examples 7.4.5 and 7.4.7.

Example 7.4.9 (Positive semi-definite cone)**.** Consider a matrix $\bar{X} \in \mathbf{S}^n_+$ and a normal $\bar{V} \in N_{\mathbf{S}^n_+}(\bar{X})$. Then

$$M = \{X \in \mathbf{S}^n_+ : \operatorname{rank} X + \operatorname{rank} \bar{V} \leq n\},$$

is an identifiable set at $\bar{X}$ for $\bar{V}$. It is interesting to note that M may fail to be locally minimal in general. Indeed, it is possible that $\mathbf{S}^n_+$ admits no locally minimal identifiable set at $\bar{X}$ for $\bar{V}$. This can easily be seen from the previous example and the fact that $\mathbf{S}^2_+$ and $\mathcal{L}^2$ are isometrically isomorphic.

However, under the strict complementarity condition $\bar{V} \in \operatorname{ri} N_{\mathbf{S}^n_+}(\bar{X})$, we have $\operatorname{rank} \bar{X} + \operatorname{rank} \bar{V} = n$, and consequently M coincides with $\{X \in \mathbf{S}^n_+ : \operatorname{rank} X = \operatorname{rank} \bar{X}\}$ around $\bar{X}$. It is then standard that M is an analytic manifold around $\bar{X}$, and furthermore one can show that M is indeed a locally minimal identifiable set at $\bar{X}$ for $\bar{V}$. For more details see [36].

Example 7.4.10 (Maximum eigenvalue)**.** Consider a matrix $\bar{X}$ and a subgradient

$\bar{V} \in \partial\lambda_1(\bar{X})$, where $\lambda_1 \colon \mathbf{S}^n \to \mathbf{R}$ is the maximum eigenvalue function. Then

$$M := \{X \in \mathbf{S}^n : \text{mult}\,\lambda_1(X) \geq \text{rank}\,\bar{V}\},$$

is an identifiable set at $\bar{X}$ for $\bar{V}$. Again under a strict complementarity condition $\bar{V} \in \text{ri}\,\partial\lambda_1(\bar{X})$, we have $\text{rank}\,\bar{V} = \text{mult}\,\lambda_1(\bar{X})$, and consequently M coincides with the manifold $\{X \in \mathbf{S}^n : \text{mult}\,\lambda_1(X) = \text{mult}\,\lambda_1(\bar{X})\}$ locally around $\bar{X}$. Furthermore under this strict complementarity condition, M is locally minimal. For more details see [36].

Example 7.4.11 (The rank function)**.** Consider the rank function, denoted $\text{rank} : \mathbf{M}^{n\times m} \to \mathbf{R}$. Then

$$M := \{X \in \mathbf{M}^{n\times m} : \text{rank}\,X = \text{rank}\,\bar{X}\}$$

is a locally minimal identifiable set at $\bar{X}$ for any $\bar{V} \in \partial(\text{rank}\,)(\bar{X})$. To see this, observe that the equality

$$\text{epi rank } = \text{epi}\,(\text{rank}\,\bar{X} + \delta_M) \text{ holds locally around } (\bar{X}, \text{rank}\,\bar{X}).$$

Combining this with the standard fact that M is an analytic manifold verifies the claim.

In Examples 7.4.8 and 7.4.9, we already saw that there are simple functions $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that do not admit a locally minimal identifiable set at some point $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$. However in those examples $\bar{v}$ was degenerate in the sense that $\bar{v}$ was contained in the relative boundary of $\partial f(\bar{x})$. We end this section by demonstrating that locally minimal identifiable sets may, in general, fail to exist even for subgradients $\bar{v}$ lying in the relative interior of the convex subdifferential $\partial f(\bar{x})$.

Example 7.4.12 (Failure of existence)**.**

Consider the convex function $f\colon \mathbf{R}^2 \to \mathbf{R}$, given by

$$f(x,y) = \sqrt{x^4 + y^2}.$$

Observe that f is continuously differentiable on $\mathbf{R}^2 \setminus \{(0,0)\}$, with

$$|\nabla f(x,y)|^2 = \frac{4x^6 + y^2}{x^4 + y^2},$$

and

$$\partial f(0,0) = \{0\} \times [-1,1].$$

We claim that f does not admit a locally minimal identifiable set at $(0,0)$ for the vector $(0,0) \in \partial f(0,0)$. To see this, suppose otherwise and let M be such a set.

Consider the curves

$$L_n := \{(x,y) \in \mathbf{R}^2 : y = \frac{1}{n}x^2\},$$

parametrized by integers n. For a fixed integer n, consider a sequence of points $(x_i, y_i) \to (0,0)$ in L_n. Then

$$\lim_{i\to\infty} |\nabla f(x_i, y_i)| = \frac{n^2}{n^4+1}.$$

Since M is necessary at $(0,0)$ for $(0,0)$, we deduce that for each integer n, there exists a real number $\epsilon_n > 0$ such that

$$\mathbf{B}_{\epsilon_n} \cap L_n \cap M = \{(0,0)\}.$$

However observe $\lim_{n\to\infty} \frac{n^2}{n^4+1} = 0$. Therefore we can choose a sequence $(x_n, y_n) \in \mathbf{B}_{\epsilon_n} \cap L_n$, with $(x_n, y_n) \neq (0,0)$, $(x_n, y_n) \to (0,0)$, and the gradients $\nabla f(x_n, y_n)$ tending to $(0,0)$. Since M is identifiable at $(0,0)$ for $(0,0)$, the points (x_n, y_n) lie in M for all large indices n, which is a contradiction.

7.5 Calculus of identifiability

To build more sophisticated examples, it is necessary to develop some calculus rules. Our starting point is the following intuitive chain rule.

Proposition 7.5.1 (Chain Rule). *Consider a function $f(x) := g(F(x))$ defined on an open neighborhood $V \subset \mathbf{R}^n$, where $F\colon V \to \mathbf{R}^m$ is a $\mathbf{C}^1$-smooth mapping and $g\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is a lsc function. Suppose that at some point $\bar{x} \in \operatorname{dom} f$, the qualification condition*

$$\ker \nabla F(\bar{x})^* \cap \partial^\infty g(F(\bar{x})) = \{0\}, \tag{7.2}$$

is valid, and hence the inclusion

$$\partial f(\bar{x}) \subset \nabla F(\bar{x})^* \partial g(F(\bar{x})) \text{ holds.}$$

Consider a vector $\bar{v} \in \partial f(\bar{x})$ and the corresponding multipliers

$$\Lambda := \{y \in \partial g(F(\bar{x})) : \bar{v} = \nabla F(\bar{x})^* y\}.$$

Suppose that for each vector $y \in \Lambda$, there exists an identifiable set M_y (with respect to g) at $F(\bar{x})$ for y. Then the set

$$M := \bigcup_{y \in \Lambda} F^{-1}(M_y),$$

is identifiable (with respect to f) at $\bar{x}$ for $\bar{v}$.

If, in addition,

- *g is Clarke regular at all points in $\operatorname{dom} g$ around $F(\bar{x})$,*
- *the collection $\{M_y\}_{y \in S}$ is finite, and*
- *each set M_y is a locally minimal identifiable set (with respect to g) at $F(\bar{x})$ for y,*

then M is a locally minimal identifiable set (with respect to f) at $\bar{x}$ for $\bar{v}$.

Proof. We first argue the identifiability of M. To this effect, consider any sequence $(x_i, f(x_i), v_i) \to (\bar{x}, f(\bar{x}), \bar{v})$, with $v_i \in \partial f(x_i)$. It is easy to see that the transversality condition

$$\ker \nabla F(x_i)^* \cap \partial^\infty g(F(x_i)) = \{0\}, \tag{7.3}$$

holds for all sufficiently large indices i. Then by [108, Theorem 10.6], we have

$$v_i \in \partial f(x_i) \subset \nabla F(x_i)^* \partial g(F(x_i)).$$

Choose a sequence $y_i \in \partial g(F(x_i))$ satisfying $v_i = \nabla F(x_i)^* y_i$. We claim that the sequence y_i is bounded. Indeed suppose otherwise. Then restricting to a subsequence, we can assume $|y_i| \to \infty$ and $\frac{y_i}{|y_i|} \to \tilde{y}$, for some nonzero vector $\tilde{y} \in \partial^\infty g(F(\bar{x}))$. Consequently

$$\nabla F(\bar{x})^* \tilde{y} = \lim_{i\to\infty} \nabla F(x_i)^* \frac{y_i}{|y_i|} = \lim_{i\to\infty} \frac{v_i}{|y_i|} = 0,$$

thus contradicting (7.3).

Now restricting to a subsequence, we may suppose that the vectors $y_i \in \partial g(F(x_i))$ converge to $\bar{y}$ for some vector $\bar{y} \in \partial g(F(\bar{x}))$. Furthermore, observe $\bar{y} \in \Lambda$. So for all sufficiently large indices i, the points $F(x_i)$ all lie in $M_{\bar{y}}$. Consequently the points x_i lie in M for all large indices i, and we conclude that M is identifiable (with respect to f) at $\bar{x}$ for $\bar{v}$.

Now suppose that g is Clarke regular at all points of $\operatorname{dom} g$ near $F(\bar{x})$, the collection $\{M_y\}_{y\in S}$ is finite, and each set M_y is a locally minimal identifiable set (with respect to g) at $F(\bar{x})$ for y. We now show that M is necessary (with respect to f) at $\bar{x}$ for $\bar{v}$. To this effect, consider a sequence $x_i \to \bar{x}$ in M. Then restricting to a subsequence, we may suppose that the points $F(x_i)$ all lie in $M_{\bar{y}}$ for some $\bar{y} \in \Lambda$. Consequently there exists a sequence $y_i \in \partial g(F(x_i))$ converging to $\bar{y}$. Hence we deduce

$$v_i := \nabla F(x_i)^* y_i \to \nabla F(\bar{x})^* \bar{y} = \bar{v}.$$

Since g is Clarke regular at all points of dom g near $F(\bar{x})$, by [108, Theorem 10.6], the inclusion $v_i \in \partial f(x_i)$ holds for all large i. Hence M is necessary (with respect to f) at $\bar{x}$ for $\bar{v}$. □

Our goal now is to obtain a sum rule. The passage to this result though the chain rule is fairly standard. The first step is to deal with separable functions.

Proposition 7.5.2 (Separable functions)**.**
Consider proper, lsc functions $f_i \colon \mathbf{R}^{n_i} \to \overline{\mathbf{R}}$, for $i = 1, \ldots, k$, and define

$$f(x_1, \ldots, x_k) = \sum_{i=1}^{k} f_i(x_i).$$

Suppose that $M_{\bar{v}_i} \subset \mathbf{R}^{n_i}$ is an identifiable set (with respect to f_i) at $\bar{x}_i$ for $\bar{v}_i \in \partial f_i(\bar{x}_i)$, for each $i = 1, \ldots, k$. Then the set

$$M := M_{\bar{v}_1} \times \ldots \times M_{\bar{v}_k},$$

is identifiable (with respect to f) at $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_k)$ for $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_k)$. An analogous result holds for necessary sets.

Proof. Clearly $M := M_{\bar{v}_1} \times \ldots \times M_{\bar{v}_k}$ is identifiable for the set-valued mapping

$$(x_1, \ldots, x_k) \mapsto \prod_{i=1}^{k} \{f_i(x_i)\} \times \partial f_i(x_i),$$

at $\bar{x}$ for $\prod_{i=1}^{k} (f_i(\bar{x}_i), \bar{v}_i)$. Furthermore lower-semicontinuity of the functions f_i readily implies that M is also identifiable for

$$(x_1, \ldots, x_k) \mapsto \{f(\bar{x})\} \times \prod_{i=1}^{k} \partial f_i(x_i),$$

at $\bar{x}$ for $(f(\bar{x}), \bar{v})$. Using the identity $\partial f(x_1, \ldots, x_k) = \prod_{i=1}^k \partial f_i(x_i)$, we deduce the result. The argument in the context of necessary sets is similar. □

Corollary 7.5.3 (Sum Rule). *Consider proper, lsc functions $f_i \colon \mathbf{R}^n \to \overline{\mathbf{R}}$, for $i = 1, \ldots, k$, and define the sum $f(x) = \sum_{i=1}^k f_i(x)$. Assume at some point $\bar{x} \in \operatorname{dom} f$, the qualification condition*

$$\sum_{i=1}^{k} v_i = 0 \text{ and } v_i \in \partial f_i^{\infty}(\bar{x}) \text{ for each } i \implies v_i = 0 \text{ for each } i.$$

Consider a vector $\bar{v} \in \partial f(\bar{x})$ and define the set

$$\Lambda = \{(v_1, \ldots, v_k) \in \prod_{i=1}^{k} \partial f_i(\bar{x}) : \bar{v} = \sum_{i=1}^{k} v_i\}.$$

For each $(v_1, \ldots, v_k) \in \Lambda$, let M_{v_i} be an identifiable set (with respect to f_i) at $\bar{x}$ for v_i. Then

$$M := \bigcup_{(v_1, \ldots, v_k) \in \Lambda} M_{v_1} \cap \ldots \cap M_{v_k},$$

is identifiable (with respect to f) at $\bar{x}$ for $\bar{v}$.

If, in addition,

- *each f_i is Clarke regular at all points in $\operatorname{dom} f_i$ around $\bar{x}$,*
- *the collection $\{M_{v_1} \times \ldots \times M_{v_k}\}_{(v_1, \ldots, v_k) \in \Lambda}$ is finite, and*
- *for each $(v_1, \ldots, v_k) \in \Lambda$, the set M_{v_i} is a locally minimal identifiable set (with respect to f_i) at $\bar{x}$ for v_i,*

then M is a locally minimal identifiable set (with respect to f) at $\bar{x}$ for $\bar{v}$.

Proof. We may rewrite f in the composite form $g \circ F$, where $F(x) := (x, \ldots, x)$ and $g(x_1, \ldots, x_k) := \sum_{i=1}^k f_i(x_i)$ is separable. Then applying Proposition 7.5.1 and Proposition 7.5.2 we obtain the result. □

In particular, we now obtain the following geometric version of the chain rule.

Proposition 7.5.4 (Sets with constraint structure)**.**
Consider closed sets $Q \in \mathbf{R}^n$ *and* $K \in \mathbf{R}^m$*, and a* $\mathbf{C}^1$ *smooth mapping* $F\colon \mathbf{R}^n \to \mathbf{R}^m$*. Define the set*

$$L = \{x \in Q : F(x) \in K\}.$$

Consider a pair $(\bar{x}, \bar{v}) \in \operatorname{gph} N_L$ *and suppose that the constraint qualification*

$$\left.\begin{array}{c} y \in N_Q(\bar{x}), w \in N_K(F(\bar{x})) \\ y + \nabla F(\bar{x})^* w = 0 \end{array}\right\} \Longrightarrow (y, w) = (0, 0),$$

holds. Define the set

$$\Lambda = \{(y, w) \in N_Q(\bar{x}) \times N_K(F(\bar{x})) : y + \nabla F(\bar{x})^* w = \bar{v}\},$$

and for each pair $(y, w) \in \Lambda$*, let* M_y *be an identifiable set (relative to* Q*) at* $\bar{x}$ *for* y *and let* K_w *be an identifiable set (relative to* K*) at* $F(\bar{x})$ *for* w*. Then*

$$M := \bigcup_{(v,w) \in \Lambda} M_v \cap F^{-1}(K_w),$$

is identifiable (relative to L*) at* $\bar{x}$ *for* $\bar{v}$*.*

If, in addition,

- Q *(respectively* K*) is Clarke regular at each of its point near* $\bar{x}$ *(respectively* $F(\bar{x})$*),*
- *the collection* $\{M_y \times K_w\}_{(y,w) \in \Lambda}$ *is finite,*
- *for each* $(y, w) \in \Lambda$*, the set* M_y *(respectively* K_w*) is a locally minimal identifiable set with respect to* Q *(respectively* K*) at* $\bar{x}$ *for* y *(respectively at* $F(\bar{x})$ *for* w*),*

then M *is a locally minimal identifiable set (relative to* L*) at* $\bar{x}$ *for* $\bar{v}$*.*

Proof. Observe $\delta_L = \delta_Q + \delta_{F^{-1}(K)}$. Combining Proposition 7.5.1 and Corollary 7.5.3, we obtain the result. □

Corollary 7.5.5 (Max-type functions)**.**

Consider $\mathbf{C}^1$*-smooth functions* $f_i \colon \mathbf{R}^n \to \overline{\mathbf{R}}$*, for* $i \in I := \{1, \ldots, m\}$*, and let* $f(x) := \max\{f_1(x), f_2(x), \ldots, f_m(x)\}$*. For any* $x \in \mathbf{R}^n$*, define the active set*

$$I(x) = \{i \in I : f(x) = f_i(x)\}.$$

Consider a pair $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$*, and the corresponding set of multipliers*

$$\Lambda = \{\lambda \in \mathbf{R}^m : \bar{v} = \sum_{i \in I(\bar{x})} \lambda_i \nabla f_i(\bar{x}),\ \operatorname{supp} \lambda \subset I(\bar{x})\}.$$

Then

$$M = \bigcup_{\lambda \in \Lambda} \{x \in \mathbf{R}^n : \operatorname{supp} \lambda \subset I(x)\},$$

is a locally minimal identifiable set (relative to f*) at* $\bar{x}$ *for* $\bar{v}$*.*

Proof. This follows directly from Proposition 7.5.1 and Example 7.4.4 by writing f as the composition $\mathrm{mx} \circ F$, where $F(x) = (f_1(x), \ldots, f_m(x))$. □

Corollary 7.5.6 (Smooth constraints)**.**

Consider $\mathbf{C}^1$*-smooth functions* $g_i \colon \mathbf{R}^n \to \overline{\mathbf{R}}$*, for* $i \in I := \{1, \ldots, m\}$*, and define the set*

$$Q = \{x \in \mathbf{R}^n : g_i(x) \leq 0 \text{ for each } i \in I\}.$$

For any $x \in \mathbf{R}^n$*, define the active set*

$$I(x) = \{i \in I : g_i(x) = 0\}.$$

and suppose that for a certain pair $(\bar{x}, \bar{v}) \in \operatorname{gph} N_Q$*, the constraint qualification*

$$\sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0 \text{ and } \lambda_i \geq 0 \text{ for all } i \in I(\bar{x}) \implies \lambda_i = 0 \text{ for all } i \in I(\bar{x}),$$

holds. Then in terms of the Lagrange multipliers

$$\Lambda := \{\lambda \in \mathbf{R}^m : \bar{v} = \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}),\ \operatorname{supp} \lambda \subset I(\bar{x})\},$$

the set

$$M = \bigcup_{\lambda \in \Lambda} \{x \in Q : g_j(x) = 0 \text{ for each } j \in \operatorname{supp} \lambda\},$$

is a locally minimal identifiable set (relative to Q) at $\bar{x}$ for $\bar{v}$.

Proof. This follows immediately from Proposition 7.5.1 and Example 7.4.7. □

We end the section by observing that, in particular, the chain rule, established in Proposition 7.5.1, allows us to consider the rich class of fully amenable functions, introduced in [97].

Definition 7.5.7 (Fully amenable functions)**.**
A function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is *fully amenable* at $\bar{x}$ if f is finite at $\bar{x}$, and there is an open neighborhood U of $\bar{x}$ on which f can be represented as $f = g \circ F$ for a $\mathbf{C}^2$-smooth mapping $F \colon V \to \mathbf{R}^m$ and a convex, piecewise linear-quadratic function $g \colon \mathbf{R}^m \to \overline{\mathbf{R}}$, and such that the qualification condition

$$\ker \nabla F(\bar{x})^* \cap \partial g^\infty(F(\bar{x})) = \{0\},$$

holds.

The qualification condition endows the class of fully amenable functions with exact calculus rules. Such functions are indispensable in nonsmooth second order theory. For more details, see [97].

Proposition 7.5.8 (Identifiable sets for fully amenable functions)**.**
A function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that is fully amenable at a point $\bar{x} \in \mathbf{R}^n$ admits a locally minimal identifiable set at $\bar{x}$ for any vector $\bar{v} \in \partial f(\bar{x})$.

Proof. This follows immediately from Propositions 7.4.6 and 7.5.1. □

7.6 Variational geometry of identifiable sets

In the previous sections, we have introduced the notions of identifiability, analyzed when locally minimal identifiable sets exist, developed calculus rules, and provided important examples. In this section, we consider the interplay between variational geometry of a set Q and its identifiable subsets M. Considering sets rather than functions has the advantage of making our arguments entirely geometric. We begin with the simple observation that locally minimal identifiable sets are locally closed.

Proposition 7.6.1. *Consider a closed set $Q \subset \mathbf{R}^n$ and a subset $M \subset Q$ that is a locally minimal identifiable set at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$. Then M is locally closed at $\bar{x}$*

Proof. Suppose not. Then there exists a sequence $x_i \in (\mathrm{bd}\, M) \setminus M$ with $x_i \to \bar{x}$. Since M is identifiable at $\bar{x}$ for $\bar{v}$, there exists a neighborhood V of $\bar{v}$ satisfying $\mathrm{cl}\, V \cap N_Q(x_i) = \emptyset$ for all large indices i. Observe that for each index i, every point y sufficiently close to x_i satisfies $V \cap N_Q(y) = \emptyset$. Consequently, there exists a sequence $y_i \in Q$ converging to $\bar{x}$ with $V \cap N_Q(y) = \emptyset$, which contradicts the necessity of M at $\bar{x}$ for $\bar{v}$. □

Recall that for a set $Q \subset \mathbf{R}^n$ and a subset $M \subset Q$, the inclusion $\hat{N}_Q(x) \subset$

$N_M(x)$ holds for each point $x \in M$, while the analogous inclusion for the limiting normal cone may fail. This pathology does not occur for identifiable sets.

Proposition 7.6.2. *Consider a closed set Q and a set M that is identifiable at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$. Then the equation*

$$\text{gph}\, N_Q \subset \text{gph}\, N_M \quad \textit{holds locally around } (\bar{x}, \bar{v}).$$

Proof. Consider a sequence $(x_i, v_i) \in \text{gph}\, N_Q$ converging to $(\bar{x}, \bar{v})$. Then for each i, there exists a sequence $(x_i^j, v_i^j) \in \text{gph}\, \hat{N}_Q$ converging to (x_i, v_i). For sufficiently large indices i, the points x_i^j lie in M for all large j. For such indices we have $v_i^j \in \hat{N}_Q(x_i^j) \subset \hat{N}_M(x_i^j)$, and consequently $v_i \in N_M(x_i)$. This verifies the inclusion $\text{gph}\, N_Q \subset \text{gph}\, N_M$ locally around $(\bar{x}, \bar{v})$. □

The following characterization [98, Corollary 3.4] will be of some use for us.

Proposition 7.6.3 (Prox-regularity and monotonicity)**.**
For a set $Q \subset \mathbf{R}^n$ and $\bar{x} \in Q$, with Q locally closed at $\bar{x}$, the following are equivalent.

1. *Q is prox-regular at $\bar{x}$ for $\bar{v}$.*
2. *The vector $\bar{v}$ is a proximal normal to Q at $\bar{x}$, and there exists a real number $r > 0$ satisfying*

$$\langle v_1 - v_0, x_1 - x_0 \rangle \geq -r|x_1 - x_0|^2,$$

for any pairs $(x_i, v_i) \in \text{gph}\, N_Q$ (for $i = 0, 1$) near $(\bar{x}, \bar{v})$.

So Q is prox-regular at $\bar{x}$ for a proximal normal $\bar{v} \in N_Q^P(\bar{x})$ as long as $N_Q + rI$ has a monotone localization around $(\bar{x}, \bar{v} + r\bar{x})$, for some real number $r > 0$.

The following proposition shows that prox-regularity of an identifiable subset M of a set Q implies that Q itself is prox-regular.

Proposition 7.6.4 (Prox-regularity of identifiable sets)**.**
Consider a closed set Q and a subset $M \subset Q$ that is identifiable at $\bar{x}$ for $\bar{v} \in \hat{N}_Q(\bar{x})$. In addition, suppose that M is prox-regular at $\bar{x}$ for $\bar{v}$. Then Q is prox-regular at $\bar{x}$ for $\bar{v}$.

Proof. To show that Q is prox-regular at $\bar{x}$ for $\bar{v}$, we will utilize Proposition 7.6.3. To this end, we first claim that the inclusion $\bar{v} \in N_Q^P(\bar{x})$ holds. To see this, choose a sequence of real numbers $r_i \to \infty$ and a sequence of points

$$x_i \in P_Q(\bar{x} + r_i^{-1}\bar{v}).$$

We have

$$v_i := r_i(\bar{x} - x_i) + \bar{v} \in N_Q^P(x_i).$$

Clearly $x_i \to \bar{x}$. We now claim that the sequence v_i converges to $\bar{v}$. Observe by definition of x_i, we have

$$|(\bar{x} - x_i) + r_i^{-1}\bar{v}| \leq |r_i^{-1}\bar{v}|.$$

Squaring and cancelling terms, we obtain

$$2\langle \bar{v}, \bar{x} - x_i \rangle \leq -r_i|\bar{x} - x_i|^2.$$

Combining this with the inclusion $\bar{v} \in \hat{N}_Q(\bar{x})$, we deduce

$$\frac{o(\bar{x} - x_i)}{|\bar{x} - x_i|} \leq 2\langle \bar{v}, \frac{\bar{x} - x_i}{|\bar{x} - x_i|} \rangle \leq -r_i|\bar{x} - x_i|$$

We conclude $r_i(\bar{x} - x_i) \to 0$ and consequently $v_i \to \bar{v}$. Since M is identifiable at $\bar{x}$ for $\bar{v}$, we deduce $x_i \in M$ for all large indices i. In addition, since M is prox-regular at $\bar{x}$ for $\bar{v}$, we have

$$x_i = P_{M \cap B_\epsilon(\bar{x})}(\bar{x} + r_i^{-1}\bar{v}) = \bar{x},$$

for some $\epsilon > 0$ and for sufficiently large indices i. Hence the inclusion $\bar{v} \in N_Q^P(\bar{x})$ holds.

Now since M is prox-regular at $\bar{x}$ for $\bar{v}$ we deduce, using Proposition 7.6.3, that there exists a real number $r > 0$ satisfying

$$\langle v_1 - v_0, x_1 - x_0 \rangle \geq -r|x_1 - x_0|^2,$$

for any pairs $(x_i, v_i) \in \operatorname{gph} N_M$ (for $i = 0, 1$) near $(\bar{x}, \bar{v})$.

By Proposition 7.6.2, we have

$$\operatorname{gph} N_Q \subset \operatorname{gph} N_M \text{ locally around } (\bar{x}, \bar{v}).$$

Recalling that $\bar{v}$ is a proximal normal to Q at $\bar{x}$ and again appealing to Proposition 7.6.3, we deduce that Q is prox-regular at $\bar{x}$ for $\bar{v}$. □

Remark 7.6.5. In Proposition 7.6.4, we assumed that the vector $\bar{v}$ is a Frechét normal. Without this assumption, the analogous result fails. For instance, consider the sets

$$Q := \{(x, y) \in \mathbf{R}^2 : xy = 0\},$$

$$M := \{(x, y) \in \mathbf{R}^2 : y = 0\}.$$

Then clearly M is identifiable at $\bar{x} := (0, 0)$ for the normal vector $\bar{v} := (0, 1) \in N_Q(\bar{x})$. However, $\bar{v}$ is not a proximal normal.

The following result brings to the fore the insight one obtains by combining the notions of identifiability and prox-regularity. It asserts that given a prox-regular identifiable set M at $\bar{x}$ for $\bar{v} \in \hat{N}_Q(\bar{x})$, not only does the inclusion $\operatorname{gph} N_Q \subset \operatorname{gph} N_M$ hold locally around $(\bar{x}, \bar{v})$, but rather the two sets $\operatorname{gph} N_Q$ and $\operatorname{gph} N_M$ coincide around $(\bar{x}, \bar{v})$.

Proposition 7.6.6 (Reduction I)**.** *Consider a closed set Q and let $M \subset Q$ be a set that is prox-regular at a point $\bar{x}$ for $\bar{v} \in \hat{N}_Q(\bar{x})$. Then M is identifiable at $\bar{x}$ for $\bar{v}$ if and only* *if*

$$\operatorname{gph} N_Q = \operatorname{gph} N_M \ \textit{locally around } (\bar{x}, \bar{v}).$$

Proof. We must show that locally around $(\bar{x}, \bar{v})$, we have the equivalence

$$\operatorname{gph} N_Q \subset M \times \mathbf{R}^n \Leftrightarrow \operatorname{gph} N_Q = \operatorname{gph} N_M.$$

The implication "$\Leftarrow$" is clear. Now assume $\operatorname{gph} N_Q \subset M \times \mathbf{R}^n$ locally around $(\bar{x}, \bar{v})$. By prox-regularity, there exist real numbers $r, \epsilon > 0$ so that $P_Q(\bar{x} + r^{-1}\bar{v}) = \bar{x}$ (Proposition 7.6.4) and so that the implication

$$\left.\begin{array}{ll} x \in M, & v \in N_M(x) \\ |x - \bar{x}| < \epsilon, & |v - \bar{v}| < \epsilon \end{array}\right\} \Rightarrow P_{M \cap B_\epsilon(\bar{x})}(x + r^{-1}v) = x,$$

holds. By Proposition 7.6.2, it is sufficient to argue that the inclusion

$$\operatorname{gph} N_M \subset \operatorname{gph} N_Q \text{ holds locally around } (\bar{x}, \bar{v}).$$

Suppose this is not the case. Then there exists a sequence $(x_i, v_i) \to (\bar{x}, \bar{v})$, with $(x_i, v_i) \in \operatorname{gph} N_M$ and $(x_i, v_i) \notin \operatorname{gph} N_Q$. Let $z_i \in P_Q(x_i + r^{-1}v_i)$. We have

$$\begin{aligned} (x_i - z_i) + r^{-1}v_i &\in N_Q^P(z_i), \\ x_i &\neq z_i. \end{aligned} \tag{7.4}$$

Observe $z_i \to \bar{x}$ by the continuity of the projection map. Consequently, by the finite identification property, for large indices i, we have $z_i \in M$ and

$$x_i + r^{-1}v_i \in z_i + N_Q^P(z_i) \subset z_i + N_M(z_i).$$

Hence $z_i = P_{M \cap B_\epsilon(\bar{x})}(x_i + r^{-1}v_i) = x_i$, for large i, thus contradicting (7.4). □

Recall that Proposition 7.6.4 shows that prox-regularity of an identifiable subset $M \subset Q$ is inherited by Q. It is then natural to consider to what extent the converse holds. It clearly cannot hold in full generality, since identifiable sets may contain many extraneous pieces. However we will see shortly that the converse does hold for a large class of identifiable sets M, and in particular for ones that are locally minimal. The key tool is the following lemma, which may be of independent interest.

Lemma 7.6.7 (Accessibility). *Consider a closed set $Q \subset \mathbf{R}^n$ and a subset $M \subset Q$ containing a point $\bar{x}$. Suppose that for some vector $\bar{v} \in \hat{N}_Q(\bar{x})$, there exists a sequence*

$$y_i \in N^P_M(\bar{x}) \setminus N^P_Q(\bar{x}) \text{ with } y_i \to \bar{v}.$$

Then there exists a sequence $(x_i, v_i) \in \operatorname{gph} N^P_Q$ converging to $(\bar{x}, \bar{v})$ with $x_i \notin M$ for each index i.

Proof. For each index i, there exists a real $r_i > 0$ satisfying $P_M(\bar{x} + r_i^{-1} y_i) = \{\bar{x}\}$. Furthermore we can clearly assume $r_i \to \infty$. Define a sequence $(x_i, v_i) \in \operatorname{gph} N^P_Q$ by

$$x_i \in P_Q(\bar{x} + r_i^{-1} y_i) \quad \text{and} \quad v_i := r_i(\bar{x} - x_i) + y_i.$$

Observe $x_i \notin M$ since otherwise we would have $x_i = \bar{x}$ and $y_i = r_i v_i \in N^P_Q(\bar{x})$, a contradiction. By continuity of the projection P_Q, clearly we have $x_i \to \bar{x}$. Now observe

$$|(\bar{x} - x_i) + r_i^{-1} y_i| \leq r_i^{-1} |y_i|.$$

Squaring and simplifying we obtain

$$r_i |\bar{x} - x_i| + 2\Big\langle \frac{\bar{x} - x_i}{|\bar{x} - x_i|}, y_i \Big\rangle \leq 0.$$

Since $\bar{v}$ is a Frechét normal, we deduce

$$\liminf_{i \to \infty} \Big\langle \frac{\bar{x} - x_i}{|\bar{x} - x_i|}, y_i \Big\rangle = \liminf_{i \to \infty} \Big\langle \frac{\bar{x} - x_i}{|\bar{x} - x_i|}, \bar{v} \Big\rangle \geq 0.$$

Consequently we obtain $r_i|\bar{x} - x_i| \to 0$ and $v_i \to \bar{v}$, as claimed. $\square$

Proposition 7.6.8 (Reduction II)**.** *Consider a closed set $Q \subset \mathbf{R}^n$, a point $\bar{x}$, and a normal $\bar{v} \in N_Q(\bar{x})$. Suppose* $\mathrm{gph}\, N_Q^P = \mathrm{gph}\, N_Q$ *locally around $(\bar{x}, \bar{v})$, and consider a set $M := N_Q^{-1}(V)$, where V is a convex, open neighborhood of $\bar{v}$. Then the equation*

$$\mathrm{gph}\, N_Q = \mathrm{gph}\, N_M \text{ holds locally around } (\bar{x}, \bar{v}).$$

Proof. First observe that since M is identifiable at $\bar{x}$ for $\bar{v}$, applying Proposition 7.6.2, we deduce that the inclusion $\mathrm{gph}\, N_Q \subset \mathrm{gph}\, N_M$ holds locally around $(\bar{x}, \bar{v})$. To see the reverse inclusion, suppose that there exists a pair $(x, v) \in \mathrm{gph}\, N_M^P$, arbitrarily close to $(\bar{x}, \bar{v})$, with $v \in V$ and $v \notin N_Q^P(x)$. By definition of M, we have $N_Q^P(x) \cap V \neq \emptyset$. Let z be a vector in this intersection, and consider the line segment γ joining z and v. Clearly the inclusion $\gamma \subset V \cap N_M^P(x)$ holds. Observe that the line segment $\gamma \cap N_Q^P(x)$ is strictly contained in γ, with z being one of its endpoints. Let w be the other endpoint of $\gamma \cap N_Q^P(x)$. Then we immediately deduce that there exists a sequence

$$y_i \in N_M^P(x) \setminus N_Q^P(x) \text{ with } y_i \to w.$$

Applying Lemma 7.6.7, we obtain a contradiction. Therefore the inclusion $\mathrm{gph}\, N_Q \supset \mathrm{gph}\, N_M^P$ holds locally around $(\bar{x}, \bar{v})$. Taking the closure the result follows. $\square$

In particular, we obtain the following essential converse of Proposition 7.6.4.

Proposition 7.6.9 (Prox-regularity under local minimality)**.**
Consider a closed set Q and a subset $M \subset Q$ that is a locally minimal identifiable set at $\bar{x}$ for $\bar{v} \in \hat{N}_Q(\bar{x})$. Then Q is prox-regular at $\bar{x}$ for $\bar{v}$ if and only if M is prox-regular at $\bar{x}$ for $\bar{v}$.

Proof. The implication $\Leftarrow$ was proven in Proposition 7.6.4. To see the reverse implication, first recall that M is locally closed by Proposition 7.6.1. Furthermore Propositions 7.3.4 shows that there exists an open convex neighborhood V of $\bar{v}$ so that M coincides locally with $N_Q^{-1}(V)$. In turn, applying Proposition 7.6.8 we deduce that the equation

$$\text{gph}\, N_Q = \text{gph}\, N_M \text{ holds locally around } (\bar{x}, \bar{v}).$$

Finally by Proposition 7.6.3, prox-regularity of Q at $\bar{x}$ for $\bar{v}$ immediately implies that M is prox-regular at $\bar{x}$ for $\bar{v}$. □

We end this section by exploring the strong relationship between identifiable sets and the metric projection map. We begin with the following proposition.

Proposition 7.6.10 (Identifiability and the metric projection)**.**
Consider a closed set Q and a subset $M \subset Q$. Let $\bar{x} \in M$ and $\bar{v} \in N_Q^P(\bar{x})$. Consider the following conditions.

1. *M is identifiable (relative to Q) at $\bar{x}$ for $\bar{v}$.*
2. *For all sufficiently small $\lambda > 0$, the set M is identifiable (relative to P_Q^{-1}) at $\bar{x}$ for $\bar{x} + \lambda\bar{v}$.*

Then the implication $1 \Rightarrow 2$ holds. If in addition Q is prox-regular at $\bar{x}$ for $\bar{v}$, then the equivalence $1 \Leftrightarrow 2$ holds.

Proof. $1 \Rightarrow 2$: Recall that for all small $\lambda > 0$, we have $P_Q(\bar{x} + \lambda\bar{v}) = \bar{x}$. Fix such a real number λ and consider a sequence $(x_i, y_i) \to (\bar{x}, \bar{x} + \lambda\bar{v})$ in gph P_Q^{-1}. Observe $x_i \in P_Q(y_i)$ and the sequence $\lambda^{-1}(y_i - x_i) \in N_Q(x_i)$ converges to $\bar{v}$. Consequently, the points x_i all eventually lie in M.

Suppose now that Q is prox-regular at $\bar{x}$ for $\bar{v}$.

$2 \Rightarrow 1$: We may choose $\lambda, \epsilon > 0$ satisfying

$$\left.\begin{array}{rr} x \in Q, & v \in N_Q(x) \\ |x - \bar{x}| < \epsilon, & |v - \bar{v}| < \epsilon \end{array}\right\} \Rightarrow P_{Q \cap B_\epsilon(\bar{x})}(x + \lambda v) = x,$$

Consider a sequence $(x_i, v_i) \to (\bar{x}, \bar{v})$ in gph N_Q. Then the sequence $(x_i, x_i + \lambda v_i)$ converges to $(\bar{x}, \bar{x} + \lambda \bar{v})$ and lies in gph P_Q^{-1} for all sufficiently large i. Consequently, we have $x_i \in M$ for all large i. $\square$

Assuming prox-regularity, a simple way to generate identifiable subsets $M \subset Q$ is by projecting open sets onto Q.

Proposition 7.6.11 (Projections of neighborhoods are identifiable).
Consider a set $Q \subset \mathbf{R}^n$ that is prox-regular at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$. If for all sufficiently small $\lambda > 0$, the inclusion $\bar{x} + \lambda \bar{v} \in \operatorname{int} U$ holds for some set U, then $P_Q(U)$ is identifiable at $\bar{x}$ for $\bar{v}$.

Proof. Suppose $(x_i, v_i) \to (\bar{x}, \bar{v})$ in gph N_Q. Then by prox-regularity for all sufficiently small $\lambda > 0$, we have $P_Q(x_i + \lambda v_i) = x_i$ and $x_i + \lambda v_i \in U$ for all large i. We deduce $x_i \in P_Q(U)$ for all large i, as we needed to show. $\square$

In fact, we will see shortly that under the prox-regularity assumption, *all* locally minimal identifiable sets arise in this way.

Lemma 7.6.12. *Consider a closed set $Q \subset \mathbf{R}^n$ and a subset M that is identifiable at $\bar{x}$ for $\bar{v} \in N_Q^P(\bar{x})$. Then for all sufficiently small $\lambda > 0$ and all $\epsilon > 0$, the inclusion*

$$\bar{x} + \lambda \bar{v} \in \operatorname{int} \{x + \lambda v : x \in M, v \in N_Q(x), |x - \bar{x}| < \epsilon, |v - \bar{v}| < \epsilon\},$$

holds.

Proof. For all sufficiently small $\lambda > 0$, we have $P_Q(\bar{x} + \lambda\bar{v}) = \bar{x}$. Consider a sequence $z_i \to \bar{x} + \lambda\bar{v}$ and choose points

$$x_i \in P_Q(z_i).$$

Observe $x_i \to \bar{x}$ and the vectors $\lambda^{-1}(z_i - x_i) \in N_Q(x_i)$ converge to $\bar{v}$. Consequently, the points x_i lie in M for all large i, and hence $z_i = x_i + \lambda(\lambda^{-1}(z_i - x_i))$ lie in the desired set eventually. □

Proposition 7.6.13 (Representing locally minimal identifiable sets)**.**
Consider a set $Q \subset \mathbf{R}^n$ that is prox-regular at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$ and let M be a locally minimal identifiable set at $\bar{x}$ for $\bar{v}$. For $\lambda, \epsilon > 0$, define

$$U := \{x + \lambda v : x \in M, v \in N_Q(x), |x - \bar{x}| < \epsilon, |v - \bar{v}| < \epsilon\}.$$

Then for all sufficiently small $\lambda, \epsilon > 0$, we have $\bar{x} + \lambda\bar{v} \in \operatorname{int} U$ and M admits the presentation

$$M = P_Q(U) \text{ locally around } \bar{x}.$$

Proof. Using prox-regularity and Lemma 7.6.7, we deduce that for all sufficiently small $\lambda, \epsilon > 0$ we have

$$\left.\begin{array}{cc} x \in Q, & v \in N_Q(x) \\ |x - \bar{x}| < \epsilon, & |v - \bar{v}| < \epsilon \end{array}\right\} \Rightarrow P_Q(x + \lambda v) = x,$$

and $\bar{x} + \lambda\bar{v} \in \operatorname{int} U$. Using the fact that M is locally minimal at $\bar{x}$ for $\bar{v}$, it is easy to verify that M and $P_Q(U)$ coincide locally around $\bar{x}$. □

The following characterization is now immediate.

Proposition 7.6.14 (Characterization of identifiable sets)**.**
Consider a closed set $Q \subset \mathbf{R}^n$, a subset $M \subset Q$, a point $\bar{x} \in M$, and a normal vector $\bar{v} \in \hat{N}_Q(\bar{x})$. Consider the properties:

1. *M is identifiable at $\bar{x}$ for $\bar{v}$.*

2. *Q is prox-regular at $\bar{x}$ for $\bar{v}$ and for all sufficiently small $\lambda > 0$ and all $\epsilon > 0$ the inclusion*

$$\bar{x} + \lambda\bar{v} \in \operatorname{int}\{x + \lambda v : x \in M, v \in N_Q(x), |x - \bar{x}| < \epsilon, |v - \bar{v}| < \epsilon\},$$

   holds.

Then the implication $(2) \Rightarrow (1)$ holds. If M is prox-regular at $\bar{x}$ for $\bar{v}$, then we have the equivalence $(1) \Leftrightarrow (2)$.

7.7 Identifiable sets and critical cones

In this section, we consider *critical cones*, a notion that has been instrumental in sensitivity analysis, particularly in connection with polyhedral variational inequalities. See [107, Section 2E] for example. We will see that there is a strong relationship between these objects and locally minimal identifiable sets. We begin with the notion of tangency.

Definition 7.7.1 (Tangent cones)**.** Consider a set $Q \subset \mathbf{R}^n$ and a point $\bar{x} \in Q$. The *tangent cone* to Q at $\bar{x}$, written $T_Q(\bar{x})$, consists of all vectors w such that

$$w = \lim_{i\to\infty} \frac{x_i - \bar{x}}{\tau_i}, \text{ for some } x_i \xrightarrow{Q} \bar{x},\ \tau_i \downarrow 0.$$

The tangent cone is always closed but may easily fail to be convex. For any cone $K \in \mathbf{R}^n$, we consider the polar cone

$$K^* := \{y : \langle y, v\rangle \leq 0 \text{ for all } v \in K\}.$$

It turns out that the sets $\operatorname{cl}\operatorname{conv} T_Q(\bar{x})$ and $\hat{N}_Q(\bar{x})$ are dual to each other, that is the equation

$$\hat{N}_Q(\bar{x}) = T_Q(\bar{x})^*,$$

holds [108, Theorem 6.28]. Consequently if Q is locally closed at $\bar{x}$, then Q is Clarke regular at $\bar{x}$ if and only if the equation $N_Q(\bar{x}) = T_Q(\bar{x})^*$ holds.

A companion notion to tangency is smooth derivability.

Definition 7.7.2 (smooth derivability)**.** Consider a set Q and a point $\bar{x} \in Q$. Then a tangent vector $w \in T_Q(\bar{x})$ is *smoothly derivable* if there exists a $\mathbf{C}^1$-smooth path $\gamma\colon [0, \epsilon) \to Q$ satisfying

$$w = \lim_{t \downarrow 0} \frac{\gamma(t) - \bar{x}}{t},$$

where $\epsilon > 0$ is a real number and $\gamma(0) = \bar{x}$. We will say that Q is *smoothly derivable* at $\bar{x}$ if every tangent vector $w \in T_Q(\bar{x})$ is smoothly derivable.

We should note that there is a related weaker notion of geometric derivability, where the path γ is not required to be $\mathbf{C}^1$-smooth. For more details see [108, Definition 6.1].

Most sets that occur in practice are smoothly derivable. In particular, any smooth manifold is smoothly derivable at each of its point, as is any semi-algebraic set $Q \subset \mathbf{R}^n$. We omit the proof of the latter claim, since it is a straightforward consequence of the curve selection lemma [115, Property 4.6] and the details needed for the argument would take us far off field. For a nice survey on semi-algebraic geometry, see [34].

We now arrive at the following central notion.

Definition 7.7.3 (Critical cones)**.** For a set $Q \subset \mathbf{R}^n$ that is Clarke regular at a point $\bar{x} \in Q$, the *critical cone* to Q at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$ is the set

$$K_Q(\bar{x}, \bar{v}) := N_{N_Q(\bar{x})}(\bar{v}).$$

Because of the polarity relationship between normals and tangents, the critical cone $K_Q(\bar{x}, \bar{v})$ can be equivalently described as

$$K_Q(\bar{x}, \bar{v}) = T_Q(\bar{x}) \cap \bar{v}^{\perp},$$

where $\bar{v}^{\perp}$ is the subspace perpendicular to $\bar{v}$. For more information about critical cones and their use in variational inequalities and complementarity problems, see [55].

Connecting the classical theory of critical cones to our current work, we will now see that critical cones provide tangential approximations to locally minimal identifiable sets. In what follows, we denote the closed convex hull of any set $Q \subset \mathbf{R}^n$ by $\overline{\text{co}}\, Q$.

Proposition 7.7.4 (Critical cones as tangential approximations)**.**
Consider a set Q that is Clarke regular at a point $\bar{x}$ and a locally minimal identifiable set M at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$. Suppose furthermore that M is prox-regular at $\bar{x}$ for $\bar{v}$ and is smoothly derivable at $\bar{x}$. Then the equation

$$\overline{\text{co}}\, T_M(\bar{x}) = K_Q(\bar{x}, \bar{v}),$$

holds.

Proof. Observe

$$K_Q(\bar{x}, \bar{v}) = N_{N_Q(\bar{x})}(\bar{v}) = N_{N_M(\bar{x})}(\bar{v}) = N_{\hat{N}_M(\bar{x})}(\bar{v}) = \overline{\text{co}}\, T_M(\bar{x}) \cap \bar{v}^{\perp},$$

where the second equality follows from Proposition 7.6.6 and the last equality follows from polarity of $\text{cl conv}\, T_M(\bar{x})$ and $\hat{N}_M(\bar{x})$. Hence to establish the claim, it is sufficient to argue that every tangent vector $w \in T_M(\bar{x})$ is orthogonal to $\bar{v}$.

To this end, fix a vector $w \in T_M(\bar{x})$ and a $\mathbf{C}^1$-smooth path $\gamma\colon [0, \epsilon) \to Q$ satisfying

$$w = \lim_{t \downarrow 0} \frac{\gamma(t) - \bar{x}}{t},$$

where $\epsilon > 0$ is a real number and $\gamma(0) = \bar{x}$.

Let $t_i \in (0, \epsilon)$ be a sequence converging to 0 and define $x_i := \gamma(t_i)$. Observe that for each index i, the tangent cone $T_M(x_i)$ contains the line $\{\lambda\dot{\gamma}(t_i) : \lambda \in \mathbf{R}\}$. Since M is necessary at $\bar{x}$ for $\bar{v}$, there exist vectors $v_i \in N_Q(\gamma(t_i))$ with $v_i \to \bar{v}$. By Proposition 7.6.6, we have $v_i \in \hat{N}_M(\gamma(t_i))$ for all large i. For such indices, we have $\langle v_i, \dot{\gamma}(t_i)\rangle = 0$. Letting i tend to ∞, we deduce $\langle \bar{v}, \bar{w}\rangle = 0$, as we needed to show. □

Classically, the main use of critical cones has been in studying polyhedral variational inequalities. Their usefulness in that regard is due to Proposition 7.7.5, stated below. We provide a simple proof of this proposition that makes it evident that this result is simply a special case of Proposition 7.6.6. This further reinforces the theory developed in our current work. For an earlier proof that utilizes representations of polyhedral sets, see for example [107, Lemma 2E.4].

Proposition 7.7.5 (Polyhedral reduction). *Consider a polyhedron $Q \subset \mathbf{R}^n$ and a normal vector $\bar{v} \in N_Q(\bar{x})$, for some point $\bar{x} \in Q$. Let $K := K_Q(\bar{x}, \bar{v})$. Then we have*

$$\text{gph}\, N_Q - (\bar{x}, \bar{v}) = \text{gph}\, N_K \textit{ locally around } (0, 0).$$

Proof. By Example 7.4.4, the set $M := \operatorname{argmax}_{x \in Q} \langle x, \bar{v} \rangle$ is the locally minimal identifiable set at $\bar{x}$ for $\bar{v}$. Being polyhedral, M is smoothly derivable and it satisfies

$$\bar{x} + T_M(\bar{x}) = M \text{ locally around } \bar{x}.$$

In light of Proposition 7.7.4, we deduce $M - \bar{x} = K$ locally around 0.

Thus for all (u, w) sufficiently near $(0, 0)$ we have

$$\begin{aligned} \bar{v} + u \in N_Q(\bar{x} + w) &\iff \bar{v} + u \in N_M(\bar{x} + w) \\ &\iff \bar{v} + u \in N_K(w) \\ &\iff u \in N_K(w) \end{aligned}$$

where the first equivalence follows from Proposition 7.6.6, and the last equivalence follows from the fact that $K \subset \bar{v}^{\perp}$ and so for all $w \in K$, the cone $N_K(w)$ contains the line spanned by $\bar{v}$. $\square$

Proposition 7.7.5 easily fails for nonpolyhedral sets. Indeed, in light of Proposition 7.7.4, this is to be expected since critical cones provide only tangential approximations to locally minimal identifiable sets. Such an approximation is exact only for polyhedral sets. Hence the theory of locally minimal identifiable sets (in particular, Proposition 7.6.6) extends Proposition 7.7.5 far beyond polyhedrality.

We end this section by showing how Proposition 7.7.4 can be extended even further to the situation when locally minimal identifiable sets do not even exist. Indeed, consider a set Q that is Clarke regular at a point $\bar{x}$, and let $\bar{v} \in N_Q(\bar{x})$. Consider a nested sequence of open neighborhoods V_i of $\bar{v}$ satisfying $\bigcap_{i=1}^{\infty} V_i =$

$\{\bar{v}\}$. One would then expect that, under reasonable conditions, the equality

$$K_Q(\bar{x},\bar{v}) = \overline{\text{co}} \bigcap_{i=1}^{\infty} T_{N_Q^{-1}(V_i)}(\bar{x}),$$

holds. To put this in perspective, observe that if there exists a locally minimal identifiable set M at $\bar{x}$ for $\bar{v}$, then the sets $T_{N_Q^{-1}(V_i)}(\bar{x})$ are equal to $T_M(\bar{x})$ for all large i, and the equation above reduces to Proposition 7.7.4. More generally, the following is true.

Proposition 7.7.6 (Critical cones more generally)**.** *Consider a set Q that is Clarke regular at a point $\bar{x}$, and let $\bar{v} \in N_Q(\bar{x})$. Consider a nested sequence of open neighborhoods V_i of $\bar{v}$ satisfying $\bigcap_{i=1}^{\infty} V_i = \{\bar{v}\}$ and the corresponding preimages $M_i := \hat{N}_Q^{-1}(V_i)$. Assume that each M_i is smoothly derivable at $\bar{x}$. Then the inclusion*

$$K_Q(\bar{x},\bar{v}) \supset \overline{\text{co}} \bigcap_{i=1}^{\infty} T_{M_i}(\bar{x}), \tag{7.5}$$

holds. Assume in addition that each M_i is prox-regular at $\bar{x}$ for $\bar{v}$ and that the formula

$$\overline{\text{co}} \bigcap_{i=1}^{\infty} T_{M_i}(\bar{x}) = \bigcap_{i=1}^{\infty} \overline{\text{co}}\, T_{M_i}(\bar{x}), \tag{7.6}$$

holds. Then each M_i is an identifiable set at $\bar{x}$ for $\bar{v}$ and we have

$$K_Q(\bar{x},\bar{v}) = \overline{\text{co}} \bigcap_{i=1}^{\infty} T_{M_i}(\bar{x}).$$

We omit the proof of the proposition above since it follows along the same lines as the proof of Proposition 7.7.4. In particular, let us note that (7.6) holds whenever the tangent spaces $T_{M_i}(\bar{x})$ all coincide for sufficiently large indices i or whenever all M_i are Clarke regular at $\bar{x}$.

7.8 Optimality conditions

In this section, we will see that the order of growth of a function f around a critical point (a point satisfying $0 \in \partial f(x)$) is dictated entirely by its order of growth around this point on a corresponding identifiable set. Here is a preliminary geometric result.

Proposition 7.8.1 (Restricted optimality). *Consider a closed set Q and a subset $M \subset Q$ that is identifiable at $\bar{x}$ for a proximal normal $\bar{v} \in N_Q^P(\bar{x})$. Then $\bar{x}$ is a (strict) local maximizer of the linear function $\langle \bar{v}, \cdot \rangle$ on M if and only if $\bar{x}$ is a (strict) local maximizer of $\langle \bar{v}, \cdot \rangle$ on Q.*

Proof. One implication is clear. To establish the converse, suppose that $\bar{x}$ is a local maximizer of the linear function $\langle \bar{v}, \cdot \rangle$ on M. We will show that the inequality, $\langle \bar{v}, \bar{x} \rangle > \langle \bar{v}, x \rangle$, holds for all points $x \in Q \setminus M$ near $\bar{x}$. Indeed, suppose this is not the case. Then there exists a sequence $x_i \to \bar{x}$ in $Q \setminus M$ satisfying

$$\langle \bar{v}, \bar{x} \rangle \leq \langle \bar{v}, x_i \rangle. \tag{7.7}$$

Since $\bar{v}$ is a proximal normal, we deduce that there exists a real number $r > 0$ satisfying $P_Q(\bar{x} + r^{-1}\bar{v}) = \{\bar{x}\}$. Consider any points z_i with

$$z_i \in P_Q(x_i + r^{-1}\bar{v}).$$

Clearly we have $z_i \to \bar{x}$ and

$$(x_i - z_i) + r^{-1}\bar{v} \in N_Q(z_i).$$

Since M is identifiable at $\bar{x}$ for $\bar{v}$, we deduce $z_i \in M$ for all large indices i. Consequently, for such indices i, we have $x_i \neq z_i$.

Observe

$$|(x_i - z_i) + r^{-1}\bar{v}| \leq r^{-1}|\bar{v}|.$$

Squaring and cancelling terms, we obtain

$$\langle \bar{v}, z_i - x_i \rangle \geq \frac{r}{2}|z_i - x_i|^2. \tag{7.8}$$

Consequently,

$$\frac{r}{2}|z_i - x_i|^2 \leq \langle \bar{v}, \bar{x} - x_i \rangle \leq 0,$$

which is a contradiction. Claim (1) now follows. □

Recall that a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is said to *grow quadratically* around $\bar{x}$ provided that the inequality

$$\liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{|x - \bar{x}|^2} > 0,$$

holds. We now arrive at the main result of this section.

Proposition 7.8.2 (Order of growth)**.** *Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a set $M \subset \mathbf{R}^n$. Suppose that M is identifiable at $\bar{x}$ for $\bar{v} = 0 \in \partial_P f(\bar{x})$. Then the following are true.*

1. *$\bar{x}$ is a (strict) local minimizer of f restricted to M $\Leftrightarrow$ $\bar{x}$ is a (strict) local minimizer of the unrestricted function f.*
2. *More generally, consider a growth function $g\colon U \to \mathbf{R}$, defined on an open neighborhood U of 0, that is $\mathbf{C}^1$-smooth and satisfies*

$$f(\bar{x}) < f(x) - g(x - \bar{x}) \text{ for all } x \in M \text{ near } \bar{x},$$
$$g(0) = 0,\ \nabla g(0) = 0,$$

Then the above inequality, in fact, holds for all points $x \in \mathbf{R}^n$ near $\bar{x}$.

In particular, the function f, restricted to M, grows quadratically near $\bar{x}$ if and only if the unrestricted function f grows quadratically near $\bar{x}$.

Proof. We first prove claim (1). By Proposition 7.3.7, gph $f\big|_M$ is identifiable, with respect to epi f, at $(\bar{x}, f(\bar{x}))$ for $(0, -1)$. Now observe that $\bar{x}$ is a (strict) local minimizer of $f\big|_M$ if and only of $(\bar{x}, f(\bar{x}))$ is a (strict) local maximizer of the linear function, $(x, r) \mapsto -r$, on gph $f\big|_M$. Similarly $\bar{x}$ is a (strict) local minimizer of f if and only of $(\bar{x}, f(\bar{x}))$ is a (strict) local maximizer of the linear function, $(x, r) \mapsto -r$, on epi f. Combining these equivalences with Proposition 7.8.1 establishes the claim.

We now prove claim (2). Suppose that the growth condition is satisfied. Let $h := f - g(x - \bar{x})$. Since f is $\mathbf{C}^1$-smooth, $g(0) = 0$, and $\nabla g(0) = 0$, it easily follows that M is identifiable, now with respect to h, at $\bar{x}$ for $0 \in \partial_P h(\bar{x})$. Furthermore, the point $\bar{x}$ is a strict local minimizer of $h\big|_M$. Applying claim (1) of the current proposition, we deduce that $\bar{x}$ is a strict local minimizer of the unrestricted function h, that is

$$f(\bar{x}) = h(\bar{x}) < h(x) = f(x) - g(x - \bar{x}), \text{ for all } x \text{ near } \bar{x},$$

as we needed to show. □

In particular, we obtain the following curious characterization of quadratic growth.

Corollary 7.8.3 (Refined optimality)**.** *Consider a function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a point $\bar{x}$ with $0 \in \partial_P f(\bar{x})$. Then f grows quadratically around $\bar{x}$ if and only if*

$$\liminf_{\substack{(x, f(x), v) \to (\bar{x}, f(\bar{x}), 0) \\ v \in \partial f(x)}} \frac{f(x) - f(\bar{x})}{|x - \bar{x}|^2} > 0. \tag{7.9}$$

Proof. Clearly if f grows quadratically around $\bar{x}$, then (7.9) holds. Conversely, assume (7.9) holds and let V_i be a sequence of neighborhoods of 0 shrinking to 0 and let $\epsilon_i > 0$ be real number tending to 0. Then the sets

$$M_i := (\partial f)^{-1}(V_i) \cap \{x \in \mathbf{R}^n : |f(x) - f(\bar{x})| < \epsilon_i\},$$

are identifiable at $\bar{x}$ for 0. Furthermore, f restricted to M_i must grow quadratically around $\bar{x}$, for all sufficiently large indices i, since the alternative would contradict (7.9). Applying Proposition 7.8.2, we obtain the result. □

7.9 Identifiable manifolds

Consider a closed set Q and a normal vector $\bar{v} \in \hat{N}_Q(\bar{x})$, for a point $\bar{x} \in Q$. The inherent difficulty in analyzing properties of the optimization problem,

$$\begin{aligned} P(v): \quad & \max \ \langle v, x\rangle, \\ & \text{s.t.} \ x \in Q, \end{aligned}$$

such as dependence of the local maximizers of $P(v)$ on v or the order of growth of the function $x \mapsto \langle x, \bar{v}\rangle$ on Q near $\bar{x}$, stem entirely from the potential nonsmoothness of Q. However, as we have seen in Proposition 7.6.6, the local geometry of $\operatorname{gph} N_Q$ is entirely the same as that of a prox-regular identifiable set M at $\bar{x}$ for $\bar{v}$. Thus, for instance, existence of an *identifiable manifold* M at $\bar{x}$ for $\bar{v}$ shows that the nonsmoothness of Q is not intrinsic to the problem at all. Our goal in this section is to investigate this setting. We begin with the following easy consequence of Proposition 7.6.6.

Proposition 7.9.1. *Consider a closed set $Q \subset \mathbf{R}^n$ and suppose that a subset $M \subset Q$ is a $\mathbf{C}^2$ identifiable manifold at $\bar{x}$ for $\bar{v} \in \hat{N}_Q(\bar{x})$. Then the following properties hold.*

1. *$\bar{v}$ lies in the interior of the cone $N_Q^P(\bar{x})$, relative to its linear span $N_M(\bar{x})$.*

2. *There exists an open neighborhood U of $\bar{x}$ and V of $\bar{v}$ such that the mapping $x \mapsto V \cap N_Q(x)$, restricted to M, is inner-semicontinuous at each $x \in U \cap M$.*

Proof. To see the validity of the first claim, observe that if it did not hold, then we could choose a sequence of vectors v_i satisfying

$$v_i \to \bar{v}, \;\; v_i \in N_M(\bar{x}), \;\; v_i \notin N_Q(\bar{x}),$$

thus contradicting Proposition 7.6.6.

The second claim now easily follows from Proposition 7.6.6. □

Consider a locally minimal identifiable subset $M \subset Q$ at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$. Then M remains identifiable at x for $v \in N_Q(x)$, whenever the pair (x, v) is sufficiently close to $(\bar{x}, \bar{v})$. However under such perturbations, M might cease to be locally minimal, as one can see even from polyhedral examples. (Indeed when Q is a convex polyhedron, this instability occurs whenever the inclusion $\bar{v} \in \text{rb}\, N_Q(\bar{x})$ holds.)

In the case of identifiable manifolds, the situation simplifies. Identifiable manifolds at $\bar{x}$ for $\bar{v} \in \hat{N}_Q(\bar{x})$ are automatically locally minimal, and furthermore they remain locally minimal under small perturbations to $(\bar{x}, \bar{v})$ in gph N_Q.

This important observation is summarized below.

Proposition 7.9.2. *Consider a closed set Q and a $\mathbf{C}^2$ identifiable manifold $M \subset Q$ at $\bar{x}$ for $\bar{v} \in \hat{N}_Q(\bar{x})$. Then M is automatically a locally minimal identifiable set at $x \in M$ for $v \in N_Q(x)$ whenever the pair (x, v) is near $(\bar{x}, \bar{v})$.*

Proof. This follows directly from Proposition 7.2.4 and Proposition 7.9.1. □

In particular, identifiable manifolds at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$ are locally unique. We now relate identifiable manifolds to the notion of partial smoothness, introduced in [83]. The motivation behind partial smoothness is two-fold. On one hand, it is an attempt to model an intuitive idea of a "stable active set". On the other hand, partial smoothness, along with certain nondegeneracy and growth conditions, provides checkable sufficient conditions for optimization problems to possess good sensitivity properties. Evidently, partial smoothness imposes conditions that are unnecessarily strong. We now describe a variant of partial smoothness that is localized in a directional sense. This subtle distinction, however, will be important for us.

Definition 7.9.3 (Directional Partial Smoothness)**.** Consider a closed set $Q \subset \mathbf{R}^n$ and a $\mathbf{C}^2$-manifold $M \subset Q$. Then Q is *partly smooth with respect to* M *at* $\bar{x} \in M$ *for* $\bar{v} \in N_Q(\bar{x})$ if

1. **(prox-regularity)** Q is prox-regular at $\bar{x}$ for $\bar{v}$.
2. **(sharpness)** span $\hat{N}_Q(\bar{x}) = N_M(\bar{x})$.
3. **(continuity)** There exists a neighborhood V of $\bar{v}$, such that the mapping, $x \mapsto V \cap N_Q(x)$, when restricted to M, is inner-semicontinuous at $\bar{x}$.

We arrive at the main result of this subsection.

Proposition 7.9.4 (Identifiable manifolds and partial smoothness)**.**
Consider a closed set $Q \subset \mathbf{R}^n$ and a subset $M \subset Q$ that is a $\mathbf{C}^2$ manifold around a point $\bar{x} \in Q$. Let $\bar{v} \in \hat{N}_Q(\bar{x})$. Then the following are equivalent.

1. *M is an identifiable manifold at $\bar{x}$ for $\bar{v}$.*

2. *We have*

$$\operatorname{gph} N_Q = \operatorname{gph} N_M \;\textit{ locally around } (\bar{x}, \bar{v}).$$

3. • *Q is partly smooth with respect to M at $\bar{x}$ for $\bar{v}$.*

   • *the strong inclusion $\bar{v} \in \operatorname{ri} \hat{N}_Q(\bar{x})$ holds.*

4. *The set Q is prox-regular at $\bar{x}$ for $\bar{v}$, and for all sufficiently small real numbers $\lambda, \epsilon > 0$, the inclusion*

$$\bar{x} + \lambda\bar{v} \in \operatorname{int}\Big(\bigcup_{x \in M \cap B_\epsilon(\bar{x})} \big(x + N_Q(x)\big)\Big),$$

   holds.

Proof. The equivalence $(1) \Leftrightarrow (2)$ has been established in Proposition 7.6.6. The implication $(1) \Rightarrow (3)$ follows trivially from Propositions 7.6.4 and 7.9.1.

$(3) \Rightarrow (4)$: There exist real numbers $r, \epsilon > 0$ so that the implication

$$\left.\begin{array}{cc} x \in Q, & v \in N_Q(x) \\ |x - \bar{x}| < \epsilon, & |v - \bar{v}| < \epsilon \end{array}\right\} \Rightarrow P_{Q \cap B_\epsilon(\bar{x})}(x + r^{-1}v) = x,$$

holds.

For the sake of contradiction, suppose

$$\bar{x} + r^{-1}\bar{v} \in \operatorname{bd}\Big(\bigcup_{x \in M \cap B_\epsilon(\bar{x})} \big(x + N_Q(x)\big)\Big).$$

Then there exists a sequence of points $z_i \to \bar{x} + r^{-1}\bar{v}$ with

$$z_i \notin \bigcup_{x \in M \cap B_\epsilon(\bar{x})} \big(x + N_Q(x)\big),$$

for each index i. Let $x_i \in P_M(z_i)$. Observe

$$x_i \to \bar{x}, \;\; z_i - x_i \in N_M(x_i).$$

Clearly,

$$z_i - x_i \notin N_Q^P(x_i) \subset N_M(x_i),$$

for large indices i. Hence, there exist separating vectors $a_i \in N_M(x_i)$ with $|a_i| = 1$ satisfying

$$\sup_{v \in N_Q^P(x_i)} \langle a_i, v \rangle \leq \langle a_i, z_i - x_i \rangle = \langle a_i, z_i - \bar{x} \rangle + \langle a_i, \bar{x} - x_i \rangle.$$

We deduce,

$$\sup_{v \in N_Q^P(x_i)} \langle a_i, v \rangle \leq \langle a_i, r(z_i - \bar{x}) \rangle + \langle a_i, r(\bar{x} - x_i) \rangle.$$

Passing to a subsequence, we may assume $a_i \to a$ for some nonzero vector $a \in N_M(\bar{x})$. Observe $\bar{v} + \delta a \in N_Q(x_i)$ for all small $\delta > 0$. Consequently for all sufficiently small $\delta > 0$, there exist vectors $v_i \in N_Q(x_i)$ with $v_i \to \bar{v} + \delta a$. Observe

$$\langle a_i, v_i \rangle \leq \langle a_i, r(z_i - \bar{x}) \rangle + \langle a_i, r(\bar{x} - x_i) \rangle.$$

Letting i tend to ∞, we obtain

$$\langle a, \bar{v} + \delta a \rangle \leq \langle a, \bar{v} \rangle,$$

which is a contradiction.

$(4) \Rightarrow (1)$: Choose $r, \epsilon > 0$ so as to ensure

$$\bar{x} + r^{-1}\bar{v} \in \text{int}\Big(\bigcup_{x \in M \cap B_\epsilon(\bar{x})} \big(x + N_Q(x)\big) \Big).$$

Consider any sequence of points $x_i \in \mathbf{R}^n$ and vectors $v_i \in N_Q(x_i)$, with $x_i \to \bar{x}$ and $v_i \to \bar{v}$. Then for all large indices i, the inclusion

$$x_i + r^{-1}v_i \in \bigcup_{x \in M \cap B_\epsilon(\bar{x})} \big(x + N_Q(x)\big),$$

holds. Shrinking r and ϵ, from prox-regularity of Q, we deduce $x_i \in M$ for all large indices i. Hence M is identifiable at $\bar{x}$ for $\bar{v}$. $\square$

Some comments concerning characterization (4) of the previous proposition are in order. Consider a convex set Q containing a point $\bar{x}$, and let $\bar{v} \in N_Q(\bar{x})$ be a normal vector. Then the arguments $(3) \Rightarrow (4)$ and $(4) \Rightarrow (1)$ show that a manifold $M \subset Q$ is identifiable at $\bar{x}$ for $\bar{v}$ if and only if the inclusion

$$\bar{x} + \bar{v} \in \operatorname{int}\Big(\bigcup_{x \in M} \big(x + N_Q(x)\big)\Big),$$

holds. The region $\bigcup_{x \in M} \big(x + N_Q(x)\big)$ is formed by attaching cones $N_Q(x)$ to each point $x \in M$. This set is precisely the set of points in $\mathbf{R}^n$ whose projections onto Q lie in M. Thus a manifold M is identifiable at $\bar{x}$ for $\bar{v}$ whenever the region $\bigcup_{x \in M} \big(x + N_Q(x)\big)$ is "valley-like" around $\bar{x} + \bar{v}$. We end the section with an observation relating identifiable manifolds to critical cones.

Corollary 7.9.5. *Consider a closed set $Q \subset \mathbf{R}^n$ that is Clarke regular at a point $\bar{x} \in Q$. Suppose that $M \subset Q$ is a $\mathbf{C}^2$ identifiable manifold at $\bar{x}$ for $\bar{v}$. Then the critical cone $K_Q(\bar{x}, \bar{v})$ coincides with the tangent space $T_M(\bar{x})$.*

Proof. This is an immediate consequence of Proposition 7.7.4. □

CHAPTER 8

ORTHOGONAL INVARIANCE AND IDENTIFIABILITY

8.1 Introduction

Nonsmoothness is inherently present throughout even classical mathematics and engineering - the spectrum of a symmetric matrix variable is a good example. The nonsmooth behavior is not, however, typically pathological, but on the contrary is highly structured. The theory of *identifiability* (or its synonym, *partial smoothness*) [53, 67, 83, 119] models this idea by positing existence of smooth manifolds capturing the full "activity" of the problem. Such manifolds, when they exist, are simply composed of approximate critical points of the minimized function. In the classical case of nonlinear programming, this theory reduces to the active-set philosophy. Illustrating the ubiquity of the notion, the authors of [10] prove that identifiable manifolds exist generically for convex semi-algebraic optimization problems.

Identifiable manifolds are particularly prevalent in the context of eigenvalue optimization. One of our goals is to shed new light on this phenomenon. To this end, we will consider so-called *spectral functions*. These are functions F, defined on the space of symmetric matrices $\mathbf{S}^n$, that depend on matrices only through their eigenvalues, that is, functions that are invariant under the action of the orthogonal group by conjugation. Spectral functions can always be written as the composition $F = f \circ \lambda$ where f is a permutation-invariant function on $\mathbf{R}^n$ and λ is the mapping assigning to each matrix $X \in \mathbf{S}$ the vector of its eigenvalues $(\lambda_1(X), \ldots, \lambda_n(X))$ in non-increasing order, see [16, Section 5.2]. Notable examples of functions fitting in this category are $X \mapsto \lambda_1(X)$ and $X \mapsto \sum_{i=1}^n |\lambda_i(X)|$.

Though the spectral mapping λ is very badly behaved, as far as say differentiability is concerned, the symmetry of f makes up for the fact, allowing powerful analytic results to become available.

In particular, the *Transfer Principle* asserts that F inherits many geometric (more generally variational analytic) properties of f, or equivalently, F inherits many properties of its restriction to diagonal matrices. For example, when f is a permutation-invariant norm, then F is an orthogonally invariant norm on the space of symmetric matrices — a special case of von Neumann's theorem on unitarily invariant matrix norms [117]. The collection of properties known to satisfy this principle is impressive: convexity [41,106], prox-regularity [37], Clarke-regularity [86,106], smoothness [38,81,106,109,111,112], algebraicity [38], and stratifiability [50, Theorem 4.8]. In this work, we add identifiability (and partial smoothness) to the list (Theorems 8.3.11 and 8.3.15). In particular, many common spectral functions (like the two examples above) can be written in the composite form $f \circ \lambda$, where f is a permutation-invariant convex *polyhedral* function. As a direct corollary of our results, we conclude that such functions always admit partly smooth structure! Furthermore, a "polyhedral-like" duality theory of partly smooth manifolds becomes available.

One of our intermediary theorems is of particular interest. We will give an elementary argument showing that a permutation-invariant set M is a $\mathbf{C}^\infty$ manifold if and only if the spectral set $\lambda^{-1}(M)$ is a $\mathbf{C}^\infty$ manifold (Theorem 8.2.6). The converse implication of our result is apparently new. On the other hand, the authors of [38] proved the forward implication even for $\mathbf{C}^k$ manifolds (for $k = 2, \ldots, \infty$). This being said, their proof is rather long and dense, whereas the proof of our result is very accessible. The key idea of our approach is to consider the metric projection onto M.

The outline of the chapter is as follows. In Section 8.2 we establish some basic notation and give an elementary proof of the spectral lifting property for $\mathbf{C}^\infty$ manifolds. In Section 8.3 we prove the lifting property for identifiable sets and partly smooth manifolds, while in Section 8.4 we explore duality theory of partly smooth manifolds. Section 8.5 illustrates how our results have natural analogues in the world of nonsymmetric matrices.

8.2 Spectral functions and lifts of manifolds

8.2.1 Notation

Throughout, the symbol $\mathbf{E}$ will denote a Euclidean space (by which we mean a finite-dimensional real inner-product space). Two particular realizations of $\mathbf{E}$ will be important for us, namely $\mathbf{R}^n$ and the space $\mathbf{S}^n$ of $n \times n$-symmetric matrices.

Throughout, we will fix an orthogonal basis of $\mathbf{R}^n$, along with an inner product $\langle \cdot, \cdot \rangle$. The corresponding norm will be written as $\| \cdot \|$. The group of permutations of coordinates of $\mathbf{R}^n$ will be denoted by Σ^n, while an application of a permutation $\sigma \in \Sigma^n$ to a point $x \in \mathbf{R}^n$ will simply be written as σx. We denote by $\mathbf{R}^n_{\geq}$ the set of all points $x \in \mathbf{R}^n$ with $x_1 \geq x_2 \geq \ldots \geq x_n$. A function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is said to be *symmetric* if we have $f(x) = f(\sigma x)$ for every $x \in \mathbf{R}^n$ and every $\sigma \in \Sigma^n$.

The vector space of real $n \times n$ symmetric matrices $\mathbf{S}^n$ will always be endowed with the trace inner product $\langle X, Y\rangle = \text{tr}\,(XY)$, while the associated norm (Frobenius norm) will be denoted by $\|\cdot\|_F$. The group of orthogonal $n \times n$ matrices will be denoted by $\mathbf{O}^n$. Note that the group of permutations Σ^n naturally embeds in $\mathbf{O}^n$. The action of $\mathbf{O}^n$ by conjugation on $\mathbf{S}^n$ will be written as $U.X := U^T XU$, for matrices $U \in \mathbf{O}^n$ and $X \in \mathbf{S}^n$. A function $h\colon \mathbf{S}^n \to \overline{\mathbf{R}}$ is said to be *spectral* if we have $h(X) = h(U.X)$ for every $X \in \mathbf{S}^n$ and every $U \in \mathbf{O}^n$.

8.2.2 Spectral functions and the transfer principle

We can now consider the spectral mapping $\lambda\colon \mathbf{S}^n \to \mathbf{R}^n$ which simply maps symmetric matrices to the vector of its eigenvalues in nonincreasing order. Then a function on $\mathbf{S}^n$ is spectral if and only if it can be written as a composition $f \circ \lambda$, for some symmetric function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$. (See for example [106, Proposition 4].) As was mentioned in the introduction, the *Transfer Principle* asserts that a number of variational-analytic properties hold for the spectral function $f \circ \lambda$ if and only if they hold for f. We will encounter a number of such properties in the current work. Evidently, analogous results hold even when f is only *locally symmetric* (to be defined below). The proofs follow by a reduction to the symmetric case by simple symmetrization arguments, and hence we will omit details in the current chapter.

For each point $x \in \mathbf{R}^n$, we consider the *stabilizer*

$$\text{Fix}(x) := \{\sigma \in \Sigma^n : \sigma x = x\}.$$

Definition 8.2.1 (Local symmetry)**.** A function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is *locally symmetric* at a point $\bar{x} \in \mathbf{R}^n$ if we have $f(x) = f(\sigma x)$ for all points x near $\bar{x}$ and all permutations $\sigma \in \mathrm{Fix}(\bar{x})$.

A set $Q \subset \mathbf{R}^n$ is symmetric (respectively locally symmetric) if the indicator function δ_Q is symmetric (respectively locally symmetric). The following shows that smoothness satisfies the Transfer Principle [111, 112].

Theorem 8.2.2 (Lifts of smoothness)**.** *Consider a function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a matrix $\overline{X} \in \mathbf{S}^n$. Suppose that f is locally symmetric around $\bar{x} := \lambda(\overline{X})$. Then f is $\mathbf{C}^p$-smooth $(p = 1, \ldots, \infty)$ around $\bar{x}$ if and only if the spectral function $f \circ \lambda$ is $\mathbf{C}^p$-smooth around $\overline{X}$.*

It will be important for us to relate properties of a set Q with those of the metric projection P_Q. It is reassuring then that prox-regularity also satisfies the transfer principle [37, Proposition 2.3, Theorem 2.4].

Theorem 8.2.3 (Lifts of prox-regularity)**.** *Consider a matrix $\overline{X} \in \mathbf{S}^n$ and a set $Q \subset \mathbf{R}^n$ that is locally symmetric around the point $\bar{x} := \lambda(\overline{X})$. Then the function d_Q is locally symmetric near $\bar{x}$ and the distance to the spectral set $\lambda^{-1}(Q)$ satisfies*

$$d_{\lambda^{-1}(Q)} = d_Q \circ \lambda, \text{ locally around } \overline{X}.$$

Furthermore, Q is prox-regular at $\bar{x}$ if and only if $\lambda^{-1}(Q)$ is prox-regular at $\overline{X}$.

If a set $Q \subset \mathbf{E}$ is prox-regular at $\bar{x}$, then the *proximal normal cone*

$$N_Q(\bar{x}) := \mathbf{R}_+\{v \in \mathbf{E} : \bar{x} \in P_Q(\bar{x} + v)\},$$

and the *tangent cone*

$$T_Q(\bar{x}) := \Big\{ \lim_{i\to\infty} \lambda_i(x_i - \bar{x}) : \lambda_i \uparrow \infty \text{ and } x_i \in Q \Big\}.$$

are closed convex cones and are polar to each other [108, Corollary 6.29]. Here, we mean polarity in the standard sense of convex analysis, namely for any closed convex cone $K \subset \mathbf{E}$, the polar of K is another closed convex cone defined by

$$K^o := \{v \in \mathbf{E} : \langle v, w\rangle \leq 0 \text{ for all } w \in K\}.$$

8.2.3 Lifts of symmetric manifolds

It turns out (not surprisingly) that smoothness of the projection P_Q is inherently tied to smoothness of Q itself, which is the content of the following lemma.

For any mapping $F\colon \mathbf{E} \to \mathbf{E}$, the directional derivative of F at $\bar{x}$ in direction w (if it exists) will be denoted by

$$DF(\bar{x})(w) := \lim_{t\downarrow 0} \frac{F(\bar{x}+tw) - F(\bar{x})}{t},$$

while the Gâteaux derivative of F at $\bar{x}$ (if it exists) will be denoted by $DF(\bar{x})$.

Lemma 8.2.4 (Smoothness of the metric projection)**.** *Consider a set $Q \subset \mathbf{E}$ that is prox-regular at a point $\bar{x} \in Q$. Then*

$$DP_Q(\bar{x})(v) = 0, \quad \text{for any } v \in N_Q(\bar{x}). \tag{8.1}$$

If P_Q is directionally differentiable at $\bar{x}$, then we also have

$$DP_Q(\bar{x})(w) = w, \quad \text{for any } w \in T_Q(\bar{x}). \tag{8.2}$$

In particular, if P_Q is Gâteaux differentiable at $\bar{x}$, then $N_Q(\bar{x})$ and $T_Q(\bar{x})$ are orthogonal subspaces and $DP_Q(\bar{x}) = P_{T_Q(\bar{x})}$. If P_Q is $\mathbf{C}^k$ ($k = 1,\ldots,\infty$) smooth near $\bar{x}$, then P_Q automatically has constant rank near $\bar{x}$ and consequently Q is a $\mathbf{C}^k$ manifold around $\bar{x}$.

Proof. Observe that for any normal vector $\bar{v} \in N_Q(\bar{x})$ there exists $\epsilon > 0$ so that $P_Q(\bar{x} + \epsilon'\bar{v}) = \bar{x}$ for all nonnegative $\epsilon' < \epsilon$. Equation (8.1) is now immediate.

Suppose now that P_Q is directionally differentiable at $\bar{x}$ and consider a vector $w \in T_Q(\bar{x})$ with $\|w\| = 1$. Then there exists a sequence $x_i \in Q$ converging to $\bar{x}$ and satisfying $w = \lim_{i\to\infty} \frac{x_i - \bar{x}}{\|x_i - \bar{x}\|}$. Define $t_i := \|x_i - \bar{x}\|$ and observe that since P_Q is Lipschitz continuous, for some constant L we have

$$\frac{\|P_Q(\bar{x} + t_i w) - P_Q(x_i)\|}{t_i} \leq L \Big\| w - \frac{x_i - \bar{x}}{t_i} \Big\|,$$

and consequently this quantity converges to zero. We obtain

$$DP_Q(\bar{x})(w) = \lim_{i\to\infty} \frac{P_Q(\bar{x} + t_i w) - \bar{x}}{t_i} = \lim_{i\to\infty} \frac{P_Q(x_i) - \bar{x}}{t_i} = w,$$

as claimed.

Suppose now that P_Q is Gâteaux differentiable at $\bar{x}$. Then clearly from (8.1) we have $N_Q(\bar{x}) \subset \ker DP_Q(\bar{x})$. If $N_Q(\bar{x})$ were a proper convex subset of $\ker DP_Q(\bar{x})$, then we would deduce

$$T_Q(\bar{x}) \cap \ker DP_Q(\bar{x}) = [N_Q(\bar{x})]^{\circ} \cap \ker DP_Q(\bar{x}) \neq \{0\},$$

thereby contradicting equation (8.2). Hence $N_Q(\bar{x})$ and $T_Q(\bar{x})$ are orthogonal subspaces and the equation $DP_Q(\bar{x}) = P_{T_Q(\bar{x})}$ readily follows from (8.1) and (8.2).

Suppose now that P_Q is $\mathbf{C}^k$-smooth (for $k = 1, \ldots, \infty$) around $\bar{x}$. Then clearly we have

$$\text{rank}\, DP_Q(x) \geq \text{rank}\, DP_Q(\bar{x}), \text{ for all } x \text{ near } \bar{x}.$$

Towards establishing equality above, we now claim that the set-valued mapping T_Q is outer-semicontinuous at $\bar{x}$. To see this, consider sequences $x_i \to \bar{x}$ and $w_i \in T_Q(x_i)$, with w_i converging to some vector $\bar{w} \in \mathbf{E}$. From equation (8.2), we deduce $w_i = DP_Q(x_i)(w_i)$. Passing to the limit, while taking into account the continuity of DP_Q, we obtain $\bar{w} = DP_Q(\bar{x})(\bar{w})$. On the other hand, since $DP_Q(\bar{x})$

is simply the linear projection onto $T_Q(\bar{x})$, we deduce the inclusion $\bar{w} \in T_Q(\bar{x})$, thereby establishing outer-semicontinuity of T_Q at $\bar{x}$. It immediately follows that the inequality, $\dim T_Q(x) \leq \dim T_Q(\bar{x})$, holds for all $x \in Q$ near $\bar{x}$.

One can easily verify that for any point x near $\bar{x}$, the inclusion $N_Q(P_Q(x)) \subset \ker DP_Q(x)$ holds. Consequently we deduce

$$\text{rank}\, DP_Q(x) \leq \dim T_Q(P_Q(x)) \leq \dim T_Q(\bar{x}) = \text{rank}\, DP_Q(\bar{x}),$$

for all $x \in \mathbf{E}$ sufficiently close to $\bar{x}$. as claimed. Hence P_Q has constant rank near $\bar{x}$. By the constant rank theorem, for all sufficiently small $\epsilon > 0$, the set $P_Q(B_\epsilon(\bar{x}))$ is a $\mathbf{C}^k$ manifold. Observing that the set $P_Q(B_\epsilon(\bar{x}))$ coincides with Q near $\bar{x}$ completes the proof. □

The following observation will be key. It shows that the metric projection map onto a prox-regular set is itself a gradient of a $\mathbf{C}^1$-smooth function. This easily follows from [99, Proposition 3.1]. In the convex case, this observation has been recorded and used explicitly for example in [58, Proposition 2.2] and [70, Preliminaries], and even earlier in [3] and [121].

Lemma 8.2.5 (Projection as a derivative)**.** *Consider a set $Q \subset \mathbf{E}$ that is prox-regular at $\bar{x}$. Then the function*

$$h(x) := \frac{1}{2}\|x\|^2 - \frac{1}{2}d_Q^2(x),$$

is $\mathbf{C}^1$-smooth on a neighborhood of $\bar{x}$, with $\nabla h(x) = P_Q(x)$ for all x near $\bar{x}$.

We are now ready to state and prove the main result of this section.

Theorem 8.2.6 (Spectral lifts of manifolds)**.** *Consider a matrix $\overline{X} \in \mathbf{S}^n$ and a set $M \subset \mathbf{R}^n$ that is locally symmetric around $\bar{x} := \lambda(\overline{X})$. Then M is a $\mathbf{C}^\infty$ manifold around $\bar{x}$ if and only if the spectral set $\lambda^{-1}(M)$ is a $\mathbf{C}^\infty$ manifold around $\overline{X}$.*

Proof. Consider the function

$$h(x) := \frac{1}{2}\|x\|^2 - \frac{1}{2}d_M^2(x).$$

Suppose that M is a $\mathbf{C}^\infty$ manifold around $\bar{x}$. In particular M is prox-regular, see [108, Example 13.30]. Then using Theorem 8.2.3 we deduce that h is locally symmetric around $\bar{x}$. In turn, Lemma 8.2.5 implies the equality $\nabla h = P_M$ near $\bar{x}$. Since M is a $\mathbf{C}^\infty$ manifold, the projection mapping P_M is $\mathbf{C}^\infty$-smooth near $\bar{x}$. Combining this with Theorem 8.2.2, we deduce that the spectral function $h \circ \lambda$ is $\mathbf{C}^\infty$-smooth near $\overline{X}$. Observe

$$\begin{aligned}(h \circ \lambda)(X) &= \frac{1}{2}\|\lambda(X)\|^2 - \frac{1}{2}d_M^2(\lambda(X)) \\ &= \frac{1}{2}\|X\|_F^2 - \frac{1}{2}d_{\lambda^{-1}(M)}^2(X),\end{aligned}$$

where the latter equality follows from Theorem 8.2.3. Applying Theorem 8.2.3, we deduce that $\lambda^{-1}(M)$ is prox-regular at $\overline{X}$. Combining this with Lemma 8.2.5, we obtain equality $\nabla(h\circ\lambda)(X) = P_{\lambda^{-1}(M)}(X)$ for all X near $\overline{X}$. Consequently the mapping $X \mapsto P_{\lambda^{-1}(M)}(X)$ is $\mathbf{C}^\infty$-smooth near $\overline{X}$. Appealing to Lemma 8.2.4, we conclude that $\lambda^{-1}(M)$ is a $\mathbf{C}^\infty$ manifold. The proof of the converse implication is analogous. □

Remark 8.2.7. The proof of Theorem 8.2.6 falls short of establishing the lifting property for $\mathbf{C}^k$ manifolds, when k is finite, but not by much. The reason for that is that $\mathbf{C}^k$ manifolds yield projections that are only $\mathbf{C}^{k-1}$ smooth. Nevertheless, the same proof shows that $\mathbf{C}^k$ manifolds do lift to $\mathbf{C}^{k-1}$ manifolds, and conversely $\mathbf{C}^k$ manifolds project down by λ to $\mathbf{C}^{k-1}$ manifolds.

8.2.4 Dimension of the lifted manifold

The proof of Theorem 8.2.6 is relatively simple and short, unlike the involved proof of [38]. One shortcoming however is that it does not a priori yield information about the dimension of the lifted manifold $\lambda^{-1}(M)$. In this section, we outline how we can use the fact that $\lambda^{-1}(M)$ is a manifold to establish a formula between the dimensions of M and $\lambda^{-1}(M)$. This section can safely be skipped upon first reading.

We adhere closely to the notation and some of the combinatorial arguments of [38] and [39]. With any point $x \in \mathbf{R}^n$ we associate a partition $\mathcal{P}_x = \{I_1, \ldots, I_\rho\}$ of the set $\{1, \ldots, n\}$, whose elements are defined as follows:

$$i, j \in I_\ell \Longleftrightarrow x_i = x_j.$$

It follows readily that for $x \in \mathbf{R}^n_{\geq}$ there exists a sequence

$$1 = i_0 \leq i_1 < \ldots < i_\rho = n$$

such that

$$I_\ell = \{i_{\ell-1}, \ldots, i_\ell\}, \quad \text{for each } \ell \in \{1, \ldots, \rho\}.$$

For any such partition $\mathcal{P}$ we set

$$\Delta_{\mathcal{P}} := \{x \in \mathbf{R}^n_{\geq} : \mathcal{P}_x = \mathcal{P}\}.$$

As explained in [38, Section 2.2], the set of all such $\Delta_{\mathcal{P}}$'s defines an affine stratification of $\mathbf{R}^n_{\geq}$. Observe further that for every point $x \in \mathbf{R}^n_{\geq}$ we have

$$\lambda^{-1}(x) = \{U^T X U : U \in \mathbf{O}^n\}.$$

Let $\mathbf{O}^n_X := \{U \in \mathbf{O}^n : U^T X U = X\}$ denote the stabilizer of X, which is a $\mathbf{C}^\infty$ manifold of dimension

$$\dim \mathbf{O}^n_X \;=\; \dim \left(\prod_{1\le \ell \le \rho} \mathbf{O}^{|I_\ell|} \right) = \sum_{1\le \ell\le\rho} \frac{|I_\ell|\,(|I_\ell|-1)}{2},$$

as one can easily check. Since the orbit $\lambda^{-1}(x)$ is isomorphic to $\mathbf{O}^n/\mathbf{O}^n_X$, it follows that it is a submanifold of $\mathbf{S}^n$. A computation, which can be found in [38], then yields the equation

$$\dim\, \lambda^{-1}(x) = \dim \mathbf{O}^n - \dim \mathbf{O}^n_X = \sum_{1\le i<j\le\rho} |I_i|\,|I_j|.$$

Consider now any locally symmetric manifold M of dimension d. There is no loss of generality to assume that M is connected and has nonempty intersection with $\mathbf{R}^n_\geq$. Let us further denote by Δ_* an affine stratum of the aforementioned stratification of $\mathbf{R}^n_\geq$ with the property that its dimension is maximal among all of the strata Δ enjoying a nonempty intersection with M. It follows that there exists a point $\bar{x} \in M \cap \Delta_*$ and $\delta > 0$ satisfying $M \cap B(\bar{x},\delta) \subset \Delta_*$ (see [38, Section 3] for details). Since $\dim\, \lambda^{-1}(M) = \dim\, \lambda^{-1}(M \cap B(\bar{x},\delta))$ and since $\lambda^{-1}(M \cap B(\bar{x},\delta))$ is a fibration we obtain

$$\dim\, \lambda^{-1}(M) \;=\; \dim\, M \;+ \sum_{1\le i<j\le\rho_*} |I_i^*|\;|I_j^*|, \tag{8.3}$$

where $\mathcal{P}_* = \{I_1^*, \ldots, I_\rho^*\}$ is the partition associated to $\bar{x}$ (or equivalently, to any $x \in \Delta_*$).

Remark 8.2.8. It's worth to point out that it is possible to have strata $\Delta_1 \neq \Delta_2$ of $\mathbf{R}^n_\geq$ of the same dimension, but giving rise to stabilizers of different dimension for their elements. The argument above shows that a connected locally symmetric manifold cannot intersect simultaneously these strata. This also follows implicitly from the forthcoming Lemma 8.4.4, asserting the connectedness of $\lambda^{-1}(M)$, whenever M is connected.

8.3 Spectral lifts of identifiable sets and partly smooth manifolds

We begin this section by summarizing some of the basic tools used in variational analysis of spectral functions.

8.3.1 Variational analysis of spectral functions

Recall that by Theorem 8.2.2, smoothness of functions satisfies the Transfer Principle. Shortly, we will need a slightly strengthened version of this result, where smoothness is considered only relative to a certain locally symmetric subset. We record it now.

Corollary 8.3.1 (Lifts of restricted smoothness)**.** *Consider a function* $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$*, a matrix* $\overline{X} \in \mathbf{S}^n$*, and a set* $M \subset \mathbf{R}^n$ *containing* $\bar{x} := \lambda(\bar{X})$*. Suppose that* f *and* M *are locally symmetric around* $\bar{x}$*. Then the restriction of* f *to* M *is* $\mathbf{C}^p$*-smooth* ($p = 1, \ldots, \infty$) *around* $\bar{x}$ *if and only if the restriction of* $f \circ \lambda$ *to* $\lambda^{-1}(M)$ *is* $\mathbf{C}^p$*-smooth around* $\overline{X}$*.*

Proof. Suppose that the restriction of f to M is $\mathbf{C}^p$-smooth around $\bar{x}$. Then there exists a $\mathbf{C}^p$-smooth function $\tilde{f}$, defined on $\mathbf{R}^n$, and agreeing with f on M near $\bar{x}$. Consider then the symmetrized function

$$\tilde{f}_{\mathrm{sym}}(x) := \frac{1}{|\mathrm{Fix}(\bar{x})|} \sum_{\sigma \in \mathrm{Fix}(\bar{x})} \tilde{f}(\sigma x),$$

where $|\mathrm{Fix}(\bar{x})|$ denotes the cardinality of the set $\mathrm{Fix}(\bar{x})$. Clearly $\tilde{f}_{\mathrm{sym}}$ is $\mathbf{C}^p$-smooth, locally symmetric around $\bar{x}$, and moreover it agrees with f on M near

$\bar{x}$. Finally, using Theorem 8.2.2, we deduce that the spectral function $\tilde{f}_{\mathrm{sym}} \circ \lambda$ is $\mathbf{C}^p$-smooth around $\overline{X}$ and it agrees with $f \circ \lambda$ on $\lambda^{-1}(M)$ near $\overline{X}$. This proves the forward implication of the corollary.

To see the converse, define $F := f \circ \lambda$, and suppose that the restriction of F to $\lambda^{-1}(M)$ is $\mathbf{C}^p$-smooth around $\overline{X}$. Then there exists a $\mathbf{C}^p$-smooth function $\widetilde{F}$, defined on $\mathbf{S}^n$, and agreeing with F on $\lambda^{-1}(M)$ near $\overline{X}$. Consider then the function

$$\widetilde{F}_{\mathrm{sym}}(X) := \frac{1}{|\mathbf{O}^n|} \sum_{U \in \mathbf{O}^n} \widetilde{F}(U.X),$$

where $|\mathbf{O}^n|$ denotes the cardinality of the set $\mathbf{O}^n$. Clearly $\widetilde{F}_{\mathrm{sym}}$ is $\mathbf{C}^p$-smooth, spectral, and it agrees with F on $\lambda^{-1}(M)$ near $\overline{X}$. Since $\widetilde{F}_{\mathrm{sym}}$ is spectral, we deduce that there is a symmetric function $\tilde{f}$ on $\mathbf{R}^n$ satisfying $\widetilde{F}_{\mathrm{sym}} = \tilde{f} \circ \lambda$. Theorem 8.2.2 then implies that $\tilde{f}$ is $\mathbf{C}^p$-smooth. Hence to complete the proof, all we have to do is verify that $\tilde{f}$ agrees with f on M near $\bar{x}$. To this end consider a point $x \in M$ near $\bar{x}$ and choose a permutation $\sigma \in \mathrm{Fix}(\bar{x})$ satisfying $\sigma x \in \mathbf{R}^n_{\geq}$. Let $U \in \mathbf{O}^n$ be such that $\overline{X} = U^T(\mathrm{Diag}\, \bar{x})U$. Then we have

$$\tilde{f}(x) = \tilde{f}(\sigma x) = \widetilde{F}_{\mathrm{sym}}\big(U^T(\mathrm{Diag}\, x)U\big) = F\big(U^T(\mathrm{Diag}\, x)U\big) = f(\sigma x) = f(x),$$

as claimed. □

We now recall from [106, Proposition 2] the following lemma, which shows that subdifferentials behave as one would expect in presence of symmetry.

Lemma 8.3.2 (Subdifferentials under symmetry)**.** *Consider a function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ that is locally symmetric at $\bar{x}$. Then the equation*

$$\partial f(\sigma x) = \sigma \partial f(x), \quad \textit{holds for any } \sigma \in \mathrm{Fix}(\bar{x}) \textit{ and all } x \textit{ near } \bar{x}.$$

Similarly, in terms of the spectral function $F := f \circ \lambda$, we have

$$\partial F(U.X) = U.(\partial F(X)), \quad \textit{for any } U \in \mathbf{O}^n.$$

Remark 8.3.3. In particular, if $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is locally symmetric around $\bar{x}$, then the sets $\hat{\partial} f(\bar{x})$, $\operatorname{ri} \hat{\partial} f(\bar{x})$, $\operatorname{rb} \hat{\partial} f(\bar{x})$, $\operatorname{aff} \hat{\partial} f(\bar{x})$, and $\operatorname{par} \hat{\partial} f(\bar{x})$ are invariant under the action of the group $\operatorname{Fix}(\bar{x})$.

The following result is the cornerstone for the variational theory of spectral mappings [106, Theorem 6].

Theorem 8.3.4 (Subdifferential under local symmetry)**.** *Consider a lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a symmetric matrix $X \in \mathbf{S}^n$, and suppose that f is locally symmetric at $\lambda(X)$. Then we have*

$$\partial(f \circ \lambda)(X) = \{U^T(\operatorname{Diag} v)U : v \in \partial f(\lambda(X)) \text{ and } U \in \mathbf{O}^n_X\},$$

where

$$\mathbf{O}^n_X = \{U \in \mathbf{O}^n : X = U^T(\operatorname{Diag} \lambda(X))U\}.$$

The following theorem shows that directional prox-regularity also satisfies the Transfer Principle [37, Theorem 4.2].

Theorem 8.3.5 (Directional prox-regularity under spectral lifts)**.**
Consider a lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a symmetric matrix $\bar{X}$. Suppose that f is locally symmetric around $\bar{x} := \lambda(\overline{X})$. Then f is prox-regular at $\bar{x}$ if and only if $f \circ \lambda$ is prox-regular at $\overline{X}$.

The following two standard results of Linear Algebra will be important for us [106, Proposition 3].

Lemma 8.3.6 (Simultaneous Conjugacy)**.** *Consider vectors $x, y, u, v \in \mathbf{R}^n$. Then there exists an orthogonal matrix $U \in \mathbf{O}^n$ with*

$$\operatorname{Diag} x = U^T(\operatorname{Diag} u)U \quad \text{and} \quad \operatorname{Diag} y = U^T(\operatorname{Diag} v)U,$$

if and only if there exists a permutation $\sigma \in \Sigma^n$ with $x = \sigma u$ and $y = \sigma v$.

Corollary 8.3.7 (Conjugations and permutations)**.** *Consider vectors $v_1, v_2 \in \mathbf{R}^n$ and a matrix $X \in \mathbf{S}^n$. Suppose that for some $U_1, U_2 \in \mathbf{O}^n_X$ we have*

$$U_1^T(\text{Diag}\, v_1)U_1 = U_2^T(\text{Diag}\, v_2)U_2.$$

Then there exists a permutation $\sigma \in \text{Fix}(\lambda(X))$ satisfying $\sigma v_1 = v_2$.

Proof. Observe

$$(U_1U_2^T)^T\text{Diag}\, v_1(U_1U_2^T) = \text{Diag}\, v_2,$$

$$(U_1U_2^T)^T\text{Diag}\, \lambda(X)(U_1U_2^T) = \text{Diag}\, \lambda(X).$$

The result follows by an application of Lemma 8.3.6. □

8.3.2 Main results

In this section, we consider partly-smooth sets of functions. The directional version of partial smoothness in the context of sets was discussed in Section 7.9.

Definition 8.3.8 (Partial Smoothness)**.** Consider a function $f\colon \mathbf{E} \to \overline{\mathbf{R}}$ and a set $M \subset \mathbf{E}$ containing a point $\bar{x}$. Then f is $\mathbf{C}^p$*-partly smooth* ($p = 2, \ldots, \infty$) at $\bar{x}$ relative to M if

(i) **(Smoothness)** M is a $\mathbf{C}^p$ manifold around $\bar{x}$ and f restricted to M is $\mathbf{C}^p$-smooth near $\bar{x}$,

(ii) **(Regularity)** f is prox-regular at $\bar{x}$,

(iii) **(Sharpness)** the affine span of ∂f is a translate of $N_M(x)$,

(iv) **(Continuity)** ∂f restricted to M is continuous at $\bar{x}$.

If the above properties hold, then we will refer to M as the *partly smooth manifold* of f at $\bar{x}$.

It is reassuring to know that partly smooth manifolds are locally unique. This is the content of the following theorem [67, Corollary 4.2].

Theorem 8.3.9 (Local uniqueness of partly smooth manifolds)**.** *Consider a function $f\colon \mathbf{E} \to \overline{\mathbf{R}}$ that is $\mathbf{C}^p$-partly smooth ($p \geq 2$) at $\bar{x}$ relative to two manifolds M_1 and M_2. Then there exists a neighborhood U of $\bar{x}$ satisfying $U \cap M_1 = U \cap M_2$.*

Our goal in this section is to prove that partly smooth manifolds satisfy the Transfer Principle. However, proving this directly is rather difficult. This is in large part because the continuity of the subdifferential mapping $\partial(f \circ \lambda)$ seems to be intrinsically tied to continuity properties of the mapping

$$X \mapsto \mathbf{O}^n_X = \{U \in \mathbf{O}^n : X = U^T(\mathrm{Diag}\,\lambda(X))U\},$$

which are rather difficult to understand.

We however will side-step this problem entirely by instead focusing on *finite identification*. The relationship between finite identification and partial smoothness, which follows directly from Subsection 7.9.4 is summarized below.

Proposition 8.3.10 (Partial smoothness and identifiability)**.** *Consider a lsc function $f\colon \mathbf{E} \to \overline{\mathbf{R}}$ that is prox-regular at a point $\bar{x}$. Let $M \subset \mathrm{dom}\, f$ be a $\mathbf{C}^p$ manifold ($p = 2, \ldots, \infty$) containing $\bar{x}$, with the restriction $f\big|_M$ being $\mathbf{C}^p$-smooth near $\bar{x}$. Then the following are equivalent*

1. *f is $\mathbf{C}^p$-partly smooth at $\bar{x}$ relative to M*
2. *M is an identifiable set (relative to f) at $\bar{x}$ for every subgradient $\bar{v} \in \mathrm{ri}\, \partial f(\bar{x})$.*

In light of the theorem above, our strategy for proving the Transfer Principle for partly smooth sets is two-fold: first prove the analogous result for identifiable sets and then gain a better understanding of the relationship between the sets $\mathrm{ri}\, \partial f(\lambda(X))$ and $\mathrm{ri}\, \partial (f \circ \lambda)(X)$.

Proposition 8.3.11 (Spectral lifts of Identifiable sets)**.** *Consider a lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a symmetric matrix $\overline{X} \in \mathbf{S}^n$. Suppose that f is locally symmetric around $\bar{x} := \lambda(\overline{X})$ and consider a subset $M \subset \mathbf{R}^n$ that is locally symmetric around $\bar{x}$. Then M is identifiable (relative to f) at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$, if and only if $\lambda^{-1}(M)$ is identifiable (relative to $f \circ \lambda$) at $\overline{X}$ for $U^T(\mathrm{Diag}\, \bar{v})U \in \partial (f \circ \lambda)(\overline{X})$, where $U \in \mathbf{O}^n_{\overline{X}}$ is arbitrary.*

Proof. We first prove the forward implication. Fix a subgradient

$$\overline{V} := \overline{U}^T(\mathrm{Diag}\, \bar{v})\overline{U} \in \partial (f \circ \lambda)(\overline{X}),$$

for an arbitrary transformation $\overline{U} \in \mathbf{O}^n_{\overline{X}}$. For convenience, let $F := f \circ \lambda$ and consider a sequence $(X_i, F(X_i), V_i) \to (\overline{X}, F(\overline{X}), \overline{V})$. Our goal is to show that for all large indices i, the inclusion $\lambda(X_i) \in M$ holds. To this end, there exist matrices $U_i \in \mathbf{O}^n_{X_i}$ and subgradients $v_i \in \partial f(\lambda(X_i))$ with

$$U_i^T(\mathrm{Diag}\, \lambda(X_i))U_i = X_i \quad \text{and} \quad U_i^T(\mathrm{Diag}\, v_i)U_i = V_i.$$

Restricting to a subsequence, we may assume that there exists a matrix $\widetilde{U} \in \mathbf{O}^n_{\overline{X}}$ satisfying $U_i \to \widetilde{U}$, and consequently there exists a subgradient $\tilde{v} \in \partial f(\lambda(\overline{X}))$ satisfying $v_i \to \tilde{v}$. Hence we obtain

$$\widetilde{U}^T(\mathrm{Diag}\, \lambda(\overline{X}))\widetilde{U} = \overline{X} \quad \text{and} \quad \widetilde{U}^T(\mathrm{Diag}\, \tilde{v})\widetilde{U} = \overline{V} = \overline{U}^T(\mathrm{Diag}\, \bar{v})\overline{U}.$$

By Corollary 8.3.7, there exists a permutation $\sigma \in \mathrm{Fix}(\bar{x})$ with $\sigma\tilde{v} = \bar{v}$. Observe $(\lambda(X_i), f(\lambda(X_i)), v_i) \to (\bar{x}, f(\bar{x}), \tilde{v})$. Observe that the set $\sigma^{-1}M$ is identifiable (relative to f) at $\bar{x}$ for $\tilde{v}$. Consequently for all large indices i, the inclusion

$\lambda(X_i) \in \sigma^{-1}M$ holds. Since M is locally symmetric at $\bar{x}$, we deduce that all the points $\lambda(X_i)$ eventually lie in M.

To see the reverse implication, fix an orthogonal matrix $\overline{U} \in \mathbf{O}^n_{\overline{X}}$ and define $\overline{V} := \overline{U}^T(\text{Diag}\,\bar{v})\overline{U}$. Consider a sequence $(x_i, f(x_i), v_i) \to (\bar{x}, f(\bar{x}), \bar{v})$ with $v_i \in \partial f(x_i)$. It is not difficult to see then that there exist permutations $\sigma_i \in \text{Fix}(\bar{x})$ satisfying $\sigma_i x_i \in \mathbf{R}_{\geq}$. Restricting to a subsequence, we may suppose that σ_i are equal to a fixed $\sigma \in \text{Fix}(\bar{x})$. Define

$$X_i := \overline{U}^T(\text{Diag}\,\sigma x_i)\overline{U} \quad \text{and} \quad V_i := \overline{U}^T(\text{Diag}\,\sigma v_i)\overline{U}.$$

Letting $A_{\sigma^{-1}} \in \mathbf{O}^n$ denote the matrix representing the permutation σ^{-1}, we have

$$\begin{aligned} X_i &:= (\overline{U}^T A_{\sigma^{-1}}\overline{U})^T\big[\overline{U}^T(\text{Diag}\,x_i)\overline{U}\big]\overline{U}^T A_{\sigma^{-1}}\overline{U} \quad \text{and} \\ V_i &:= (\overline{U}^T A_{\sigma^{-1}}\overline{U})^T[\overline{U}^T(\text{Diag}\,v_i)\overline{U}]\overline{U}^T A_{\sigma^{-1}}\overline{U}. \end{aligned}$$

We deduce $X_i \to (\overline{U}^T A_{\sigma^{-1}}\overline{U})^T\overline{X}(\overline{U}^T A_{\sigma^{-1}}\overline{U})$ and $V_i \to (\overline{U}^T A_{\sigma^{-1}}\overline{U})^T\overline{V}(\overline{U}^T A_{\sigma^{-1}}\overline{U})$. On the other hand, observe $\overline{X} = (\overline{U}^T A_{\sigma^{-1}}\overline{U})^T\overline{X}(\overline{U}^T A_{\sigma^{-1}}\overline{U})$. Since $\lambda^{-1}(M)$ is identifiable (relative to F) at $\overline{X}$ for $(\overline{U}^T A_{\sigma^{-1}}\overline{U})^T\overline{V}(\overline{U}^T A_{\sigma^{-1}}\overline{U})$, we deduce that the matrices X_i lie in $\lambda^{-1}(M)$ for all sufficiently large indices i. Since M is locally symmetric around $\bar{x}$, the proof is complete. $\square$

Using the results of Section 8.2, we can now describe in a natural way the affine span, relative interior, and relative boundary of the Fréchet subdifferential. We begin with a lemma.

Lemma 8.3.12 (Affine generation)**.** *Consider a matrix $X \in \mathbf{S}^n$ and suppose that the point $x := \lambda(X)$ lies in an affine subspace $\mathcal{V} \subset \mathbf{R}^n$ that is invariant under the action of* Fix(x)*. Then the set*

$$\{U^T(\text{Diag}\,v)U : v \in \mathcal{V} \text{ and } U \in \mathbf{O}^n_X\},$$

is an affine subspace of $\mathbf{S}^n$*.*

Proof. Define the set $L := (\operatorname{par} \mathcal{V})^{\perp}$. Observe that the set $L \cap \mathcal{V}$ consists of a single vector; call this vector w. Since both L and $\mathcal{V}$ are invariant under the action of $\operatorname{Fix}(x)$, we deduce $\sigma w = w$ for all $\sigma \in \operatorname{Fix}(x)$.

Now define a function $g \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ by declaring

$$g(y) = \langle w, y \rangle + \delta_{x+L}(y),$$

and note that the equation

$$\hat{\partial} g(x) := w + N_{x+L}(x) = \mathcal{V}, \quad \text{holds.}$$

Observe that for any permutation $\sigma \in \operatorname{Fix}(x)$, we have

$$g(\sigma y) = \langle w, \sigma y \rangle + \delta_{x+L}(\sigma y) = \langle \sigma^{-1} w, y \rangle + \delta_{x+\sigma^{-1}L}(y) = g(y).$$

Consequently g is locally symmetric at x. Observe

$$(g \circ \lambda)(Y) = \langle w, \lambda(Y) \rangle + \delta_{\lambda^{-1}(x+L)} Y.$$

It is immediate from Theorems 8.2.2 and 8.2.6, that the function $Y \mapsto \langle w, \lambda(Y) \rangle$ is $\mathbf{C}^{\infty}$-smooth around X and that $\lambda^{-1}(x+L)$ is a $\mathbf{C}^{\infty}$ manifold around X. Consequently $\hat{\partial}(g \circ \lambda)(X)$ is an affine subspace of $\mathbf{S}^n$. On the other hand, we have

$$\hat{\partial}(g \circ \lambda)(X) = \{U^T (\operatorname{Diag} v) U : v \in \mathcal{V} \text{ and } U \in \mathbf{O}^n_X\},$$

thereby completing the proof. □

Proposition 8.3.13 (Affine span of the spectral Fréchet subdifferential)**.** *Consider a function* $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ *and a matrix* $X \in \mathbf{S}^n$*. Suppose that* f *is locally symmetric at*

$\lambda(X)$. *Then we have*

$$\text{aff}\,\hat{\partial}(f \circ \lambda)(X) = \{U^T(\text{Diag}\,v)U : v \in \text{aff}\,\hat{\partial} f(\lambda(X)) \textit{ and } U \in \mathbf{O}^n_X\}, \tag{8.4}$$

$$\text{rb}\,\hat{\partial}(f \circ \lambda)(X) = \{U^T(\text{Diag}\,v)U : v \in \text{rb}\,\hat{\partial} f(\lambda(X)) \textit{ and } U \in \mathbf{O}^n_X\}. \tag{8.5}$$

$$\text{ri}\,\hat{\partial}(f \circ \lambda)(X) = \{U^T(\text{Diag}\,v)U : v \in \text{ri}\,\hat{\partial} f(\lambda(X)) \textit{ and } U \in \mathbf{O}^n_X\}. \tag{8.6}$$

Proof. Throughout the proof, let $x := \lambda(X)$. We prove the formulas in the order that they are stated. To this end, observe that the inclusion $\supset$ in (8.4) is immediate. Furthermore, the inclusion

$$\hat{\partial}(f \circ \lambda)(X) \subset \{U^T(\text{Diag}\,v)U : v \in \text{aff}\,\hat{\partial} f(\lambda(X)) \text{ and } U \in \mathbf{O}^n_X\}.$$

clearly holds. Hence to establish the reverse inclusion in (8.4), it is sufficient to show that the set

$$\{U^T(\text{Diag}\,v)U : v \in \text{aff}\,\hat{\partial} f(\lambda(X)) \text{ and } U \in \mathbf{O}^n_X\},$$

is an affine subspace; but this is immediate from Remark 8.3.3 and Lemma 8.3.12. Hence (8.4) holds.

We now prove (8.5). Consider a matrix $U^T(\text{Diag}\,v)U \in \text{rb}\,\hat{\partial}(f \circ \lambda)(X)$ with $U \in \mathbf{O}^n_X$ and $v \in \hat{\partial} f(\lambda(X))$. Our goal is to show the stronger inclusion $v \in \text{rb}\,\hat{\partial} f(x)$. Observe from (8.4), there exists a sequence $U_i^T(\text{Diag}\,v_i)U_i \to U^T(\text{Diag}\,v)U$ with $U_i \in \mathbf{O}^n_X$, $v_i \in \text{aff}\,\hat{\partial} f(x)$, and $v_i \notin \hat{\partial} f(x)$. Restricting to a subsequence, we may assume that there exists a matrix $\widetilde{U} \in \mathbf{O}^n_X$ with $U_i \to \widetilde{U}$ and a vector $\tilde{v} \in \text{aff}\,\hat{\partial} f(x)$ with $v_i \to \tilde{v}$. Hence the equation

$$\widetilde{U}^T(\text{Diag}\,\tilde{v})\widetilde{U} = U^T(\text{Diag}\,v)U, \quad \text{holds.}$$

Consequently, by Corollary 8.3.7, there exists a permutation $\sigma \in \text{Fix}(x)$ satisfying $\sigma\tilde{v} = v$. Since $\hat{\partial} f(x)$ is invariant under the action of Fix(x), it follows that

$\tilde{v}$ lies in $\mathrm{rb}\,\hat{\partial} f(x)$, and consequently from Remark 8.3.3 we deduce $v \in \mathrm{rb}\,\hat{\partial} f(x)$. This establishes the inclusion $\subset$ of (8.5). To see the reverse inclusion, consider a sequence $v_i \in \mathrm{aff}\,\hat{\partial} f(x)$ converging to $v \in \hat{\partial} f(x)$ with $v_i \notin \hat{\partial} f(x)$ for each index i. Fix an arbitrary matrix $U \in \mathbf{O}^n_X$ and observe that the matrices $U^T(\mathrm{Diag}\, v_i)U$ lie in $\mathrm{aff}\,\hat{\partial}(f \circ \lambda)(x)$ and converge to $U^T(\mathrm{Diag}\, v)U$. We now claim that the matrices $U^T(\mathrm{Diag}\, v_i)U$ all lie outside of $\hat{\partial}(f \circ \lambda)(x)$. Indeed suppose this is not the case. Then there exist matrices $\widetilde{U}_i \in \mathbf{O}^n_X$ and subgradients $v_i \in \hat{\partial} f(x)$ satisfying

$$U^T(\mathrm{Diag}\, v_i)U = \widetilde{U}_i^T(\mathrm{Diag}\, \tilde{v}_i)\widetilde{U}_i.$$

An application of Corollary 8.3.7 and Remark 8.3.3 then yields a contradiction. Therefore the inclusion $U^T(\mathrm{Diag}\, v)U \in \mathrm{rb}\,\hat{\partial}(f \circ \lambda)(X)$ holds, and the validity of (8.5) follows.

Finally, we aim to prove (8.6). Observe that the inclusion $\subset$ of (8.6) is immediate from equation (8.5). To see the reverse inclusion, consider a matrix $U^T(\mathrm{Diag}\, v)U$, for some $U \in \mathbf{O}^n_X$ and $v \in \mathrm{ri}\,\hat{\partial} f(x)$. Again, an easy application of Corollary 8.3.7 and Remark 8.3.3 yields the inclusion $U^T(\mathrm{Diag}\, v)U \in \mathrm{ri}\,\hat{\partial}(f \circ \lambda)(X)$. We conclude that (8.6) holds. □

Lemma 8.3.14 (Symmetry of partly smooth manifolds)**.** *Consider a lsc function* $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ *that is locally symmetric at* $\bar{x}$*. Suppose that* f *is* $\mathbf{C}^p$*-partly smooth at* $\bar{x}$ *relative to* M*. Then* M *is locally symmetric around* $\bar{x}$*.*

Proof. Consider a permutation $\sigma \in \mathrm{Fix}(\bar{x})$. Then the function f is partly smooth at $\bar{x}$ relative to σM. On the other hand, partly smooth manifolds are locally unique by Theorem 8.3.9. Consequently we deduce equality $M = \sigma M$ locally around $\bar{x}$. The claim follows. □

The main result of this section is now immediate.

Theorem 8.3.15 (Lifts of $\mathbf{C}^\infty$-partly smooth functions)**.** *Consider a lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a matrix $\overline{X} \in \mathbf{S}^n$. Suppose that f is locally symmetric around $\bar{x} := \lambda(\overline{X})$. Then f is $\mathbf{C}^\infty$-partly smooth at $\bar{x}$ relative to M if and only if $f \circ \lambda$ is $\mathbf{C}^\infty$-partly smooth at $\overline{X}$ relative to $\lambda^{-1}(M)$.*

Proof. Suppose that f is $\mathbf{C}^\infty$-partly smooth at $\bar{x}$ relative to M. In light of Lemma 8.3.14, we deduce that M is locally symmetric at $\bar{x}$. Consequently, Theorem 8.2.6 implies that the set $\lambda^{-1}(M)$ is a $\mathbf{C}^\infty$ manifold, while Corollary 8.3.1 implies that $f \circ \lambda$ is $\mathbf{C}^\infty$-smooth on $\lambda^{-1}(M)$ near $\overline{X}$. Applying Theorem 8.3.5, we conclude that $f \circ \lambda$ is prox-regular at $\bar{X}$. Consider now a subgradient $V \in \operatorname{ri} \partial(f \circ \lambda)(\overline{X})$. Then by Proposition 8.3.13, there exists a vector $v \in \operatorname{ri} \partial f(\bar{x})$ and a matrix $U \in \mathbf{O}^n_{\overline{X}}$ satisfying

$$V = U^T(\operatorname{Diag} v)U \quad \text{and} \quad \overline{X} = U^T(\operatorname{Diag} \bar{x})U.$$

Observe by Proposition 8.3.10, the set M is identifiable at $\bar{x}$ for $\bar{v}$. Then applying Proposition 8.3.11, we deduce that $\lambda^{-1}(M)$ is identifiable (relative to $f \circ \lambda$) at $\overline{X}$ relative to V. Since V is an arbitrary element of $\operatorname{ri} \partial(f \circ \lambda)(\overline{X})$, applying Proposition 8.3.10, we deduce that $f \circ \lambda$ is $\mathbf{C}^\infty$-partly smooth at $\overline{X}$ relative to $\lambda^{-1}(M)$, as claimed. The converse follows along the same lines. □

The forward implication of Theorem 8.3.15 holds in the case of $\mathbf{C}^p$-partly smooth functions (for $p = 2, \ldots, \infty$). The proof is identical except one needs to use [38, Theorem 4.21] instead of Theorem 8.2.6. We record this result for ease of reference in future works.

Theorem 8.3.16 (Lifts of $\mathbf{C}^p$-partly smooth functions)**.** *Consider a lsc function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and a matrix $\overline{X} \in \mathbf{S}^n$. Suppose that f is locally symmetric around*

$\bar{x} := \lambda(\overline{X})$. If f is $\mathbf{C}^p$-partly smooth (for $p = 2, \dots, \infty$) at $\bar{x}$ relative to M, then $f \circ \lambda$ is $\mathbf{C}^\infty$-partly smooth at $\overline{X}$ relative to $\lambda^{-1}(M)$.

8.4 Partly smooth duality for polyhedrally generated spectral functions

Consider a lsc, convex function $f\colon \mathbf{E} \to \overline{\mathbf{R}}$. Then the *Fenchel conjugate* $f^*\colon \mathbf{E} \to \overline{\mathbf{R}}$ is defined by setting

$$f^*(y) = \sup_{x \in \mathbf{R}^n} \{\langle x, y\rangle - f(x)\}.$$

Moreover, in terms of the powerset of $\mathbf{E}$, denoted $\mathbb{P}(\mathbf{E})$, we define a correspondence $\mathcal{J}_f\colon \mathbb{P}(\mathbf{E}) \to \mathbb{P}(\mathbf{E})$ by setting

$$\mathcal{J}_f(Q) := \bigcup_{x \in Q} \operatorname{ri} \partial f(x).$$

The significance of this map will become apparent shortly. Before proceeding, we recall some basic properties of the conjugation operation:

Biconjugation: $f^{**} = f$,

Subgradient inversion formula: $\partial f^* = (\partial f)^{-1}$,

Fenchel-Young inequality: $\langle x, y\rangle \le f(x) + f^*(y)$ for every $x, y \in \mathbf{R}^n$.

Moreover, convexity and conjugation behave well under spectral lifts. See for example [16, Section 5.2].

Theorem 8.4.1 (Lifts of convex sets and conjugation)**.** *If $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is a symmetric function, then f^* is also symmetric and the formula*

$$(f \circ \lambda)^* = f^* \circ \lambda, \quad \textit{holds}.$$

Furthermore f is convex if and only if the spectral function $f \circ \lambda$ is convex.

The following definition is standard.

Definition 8.4.2 (Stratification)**.** A finite partition $\mathcal{A}$ of a set $Q \subset \mathbf{E}$ is a *stratification* provided that for any partitioning sets (called *strata*) M_1 and M_2 in $\mathcal{A}$, the implication

$$M_1 \cap \operatorname{cl} M_2 \neq \emptyset \quad \Longrightarrow \quad M_1 \subset \operatorname{cl} M_2, \quad \text{holds.}$$

If the strata are open polyhedra, then $\mathcal{A}$ is a *polyhedral stratification*. If the strata are $\mathbf{C}^k$ manifolds, then $\mathcal{A}$ is a $\mathbf{C}^k$*-stratification*.

Stratification duality for convex polyhedral functions. We now establish the setting and notation for the rest of the section. Suppose that $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ is a convex *polyhedral* function (epigraph of f is a closed convex polyhedron). Then f induces a finite polyhedral stratification $\mathcal{A}_f$ of dom f in a natural way. Namely, consider the partition of epi f into open faces $\{F_i\}$. Projecting all faces F_i, with $\dim F_i \leq n$, onto the first n-coordinates we obtain a stratification of the domain dom f of f that we denote by $\mathcal{A}_f$. In fact, one can easily see that f is $\mathbf{C}^\infty$-partly smooth relative to each polyhedron $M \in \mathcal{A}_f$.

A key observation for us will be that the correspondence $f \overset{*}{\longleftrightarrow} f^*$ is not only a pairing of functions, but it also induces a duality pairing between $\mathcal{A}_f$ and $\mathcal{A}_{f^*}$. Namely, one can easily check that the mapping $\mathcal{J}_f$ restricts to an invertible mapping $\mathcal{J}_f\colon \mathcal{A}_f \to \mathcal{A}_{f^*}$ with inverse given by $\mathcal{J}_{f^*}$.

Limitations of stratification duality. It is natural to ask whether for general (nonpolyhedral) lsc, convex functions $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, the correspondence $f \overset{*}{\longleftrightarrow} f^*$, along with the mapping $\mathcal{J}$, induces a pairing between *partly smooth manifolds* of

f and f^*. Little thought, however shows an immediate obstruction: images of $\mathbf{C}^\infty$-smooth manifolds under the map $\mathcal{J}_f$ may fail to be even $\mathbf{C}^2$-smooth.

Example 8.4.3 (Failure of smoothness)**.** Consider the conjugate pair

$$f(x,y) = \frac{1}{4}(x^4 + y^4) \qquad \text{and} \qquad f^*(x,y) = \frac{3}{4}(|x|^{\frac{4}{3}} + |y|^{\frac{4}{3}}).$$

Clearly f is partly smooth relative to $\mathbf{R}^2$, whereas any possible partition of $\mathbf{R}^2$ into partly smooth manifolds relative to f^* must consist of at least three manifolds (one manifold in each dimension: one, two, and three). Hence no duality pairing between partly smooth manifolds is possible. See the Figures 8.1 and 8.2 for an illustration.

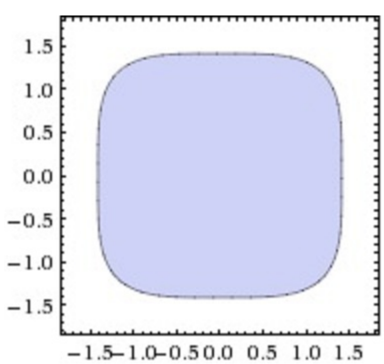


Figure 8.1: $\{(x,y) : x^4 + y^4 \leq 4\}$

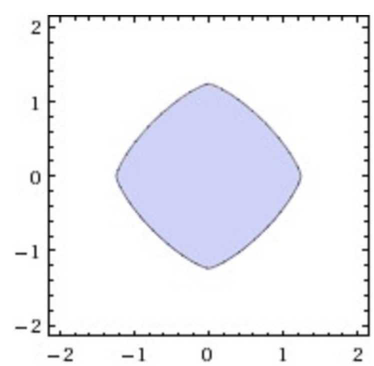


Figure 8.2: $\{(x,y) : |x|^{\frac{4}{3}} + |y|^{\frac{4}{3}} \leq \frac{4}{3}\}$

Indeed, this is not very surprising, since the convex duality is really a duality between smoothness and *strict* convexity. See for example [96, Section 4] or [108, Theorem 11.13]. Hence in general, one needs to impose tough strict convexity conditions in order to hope for this type of duality to hold. Rather than doing so, and more in line with the current work, we consider the spectral setting. Namely, we will show that in the case of spectral functions $F := f \circ \lambda$, with f symmetric and polyhedral — functions of utmost importance in eigenvalue optimization — the mapping $\mathcal{J}$ does induce a duality correspondence between partly smooth manifolds of F and F^*.

In the sequel, let us denote by

$$M^{\text{sym}} := \bigcup_{\sigma \in \Sigma} \sigma M$$

the symmetrization of any subset $M \subset \mathbf{R}^n$. Before we proceed, we will need the following result.

Lemma 8.4.4 (Path-connected lifts)**.** *Let $M \subseteq \mathbf{R}^n$ be a path-connected set and assume that for any permutation $\sigma \in \Sigma$, we either have $\sigma M = M$ or $\sigma M \cap M = \emptyset$. Then $\lambda^{-1}(M^{\text{sym}})$ is a path-connected subset of $\mathbf{S}^n$.*

Proof. Let X_1, X_2 be in $\lambda^{-1}(M^{\text{sym}})$, and set $x_i = \lambda(X_i) \in M^{\text{sym}} \cap \mathbf{R}^n_{\geq}$, for $i \in \{1, 2\}$. It is standard to check that the sets $\lambda^{-1}(x_i)$ are path-connected manifolds for $i = 1, 2$. Consequently the matrices X_i and $\text{Diag}(x_i)$ can be joined via a path lying in $\lambda^{-1}(x_i)$. Thus in order to construct a path joining X_1 to X_2 and lying in $\lambda^{-1}(M^{\text{sym}})$ it would be sufficient to join x_1 to x_2 inside M^{sym}. This in turn will follow immediately if both $\sigma x_1, \sigma x_2$ belong in M for some $\sigma \in \Sigma$. To establish this, we will assume without loss of generality that x_1 lies in M. In particular, we have $M \cap \mathbf{R}^n_{\geq} \neq \emptyset$ and we will establish the inclusion $x_2 \in M$.

To this end, consider a permutation $\sigma \in \Sigma$ satisfying $x_2 \in \sigma M \cap \mathbf{R}^n_{\geq}$. Our immediate goal is to establish $\sigma M \cap M \neq \emptyset$, and thus $\sigma M = M$ thanks to our assumption. To this end, consider the point $y \in M$ satisfying $x_2 = \sigma y$. If y lies in $\mathbf{R}^n_{\geq}$, then we deduce $y = x_2$ and we are done. Therefore, we can assume $y \notin \mathbf{R}^n_{\geq}$. We can then consider the decomposition $\sigma = \sigma_k \cdots \sigma_1$ of the permutation σ into 2-cycles σ_i each of which permutes exactly two coordinates of y that are not in the right (decreasing) order. For the sake of brevity, we omit details of the construction of such a decomposition; besides, it is rather standard. We claim now $\sigma_1 M = M$. To see this, suppose that σ_1 permutes the i and j coordinates of

y where $y_i < y_j$ and $i > j$. Since x_1 lies in $\mathbf{R}^n_{\geq}$ and M is path-connected, there exists a point $z \in M$ satisfying $z_i = z_j$. Then $\sigma_1 z = z$, whence $\sigma_1 M = M$ and $\sigma_1 y \in M$. Applying the same argument to $\sigma_1 y$ and $\sigma_1 M$ with the 2-cycle σ_2 we obtain $\sigma_2\sigma_1 M = M$ and $\sigma_2\sigma_1 y \in M$. By induction, $\sigma M = M$. Thus $x_2 \in M$ and the assertion follows. □

Stratification duality for spectral lifts. Consider a symmetric, convex polyhedral function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ together with its induced stratification $\mathcal{A}_f$ of dom f. Then with each polyhedron $M \in \mathcal{A}_f$, we may associate the symmetric set M^{sym}. We record some properties of such sets in the following lemma.

Lemma 8.4.5 (Properties of $\mathcal{A}_f$)**.** *Consider a symmetric, convex polyhedral function $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and the induced stratification $\mathcal{A}_f$ of* dom f*. Then the following are true.*

(i) *For any set $M_1, M_2 \in \mathcal{A}_f$ and any permutation $\sigma \in \Sigma$, the sets σM_1 and M_2 either coincide or are disjoint.*

(ii) *The action of Σ on $\mathbf{R}^n$ induces an action of Σ on*

$$\mathcal{A}_f^k := \{M \in \mathcal{A}_f : \dim M = k\}$$

for each $k = 0, \ldots, n$. In particular, the set M^{sym} is simply the union of all polyhedra belonging to the orbit of M under this action.

(iii) *For any polyhedron $M \in \mathcal{A}_f$, and every point $x \in M$, there exists a neighborhood U of x satisfying $U \cap M^{\text{sym}} = U \cap M$. Consequently, M^{sym} is a $\mathbf{C}^\infty$ manifold of the same dimension as M.*

Moreover, $\lambda^{-1}(M^{\text{sym}})$ is connected, whenever M is.

The last assertion follows from Lemma 8.4.4. The remaining assertions are straightforward and hence we omit their proof.

Notice that the strata of the stratification $\mathcal{A}_f$ are connected $\mathbf{C}^\infty$ manifolds, which fail to be symmetric in general. In light of Lemma 8.4.5, the set M^{sym} is a $\mathbf{C}^\infty$ manifold and a disjoint union of open polyhedra. Thus the collection

$$\mathcal{A}_f^{\mathrm{sym}} := \{M^{\mathrm{sym}} : M \in \mathcal{A}_f\},$$

is a stratification of $\operatorname{dom} f$, whose strata are now symmetric manifolds. Even though the new strata are disconnected, they give rise to connected lifts $\lambda^{-1}(M^{\mathrm{sym}})$. One can easily verify that, as before, $\mathcal{J}_f$ restricts to an invertible mapping $\mathcal{J}_f \colon \mathcal{A}_f^{\mathrm{sym}} \to \mathcal{A}_{f^*}^{\mathrm{sym}}$ with inverse given by the restriction of $\mathcal{J}_{f^*}$.

We now arrive at the main result of the section.

Theorem 8.4.6 (Lift of the duality map)**.** *Consider a symmetric, convex polyhedral function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and define the spectral function $F := f \circ \lambda$. Let $\mathcal{A}_f$ be the finite polyhedral partition of* $\operatorname{dom} f$ *induced by f, and define the collection*

$$\mathcal{A}_F := \Big\{\lambda^{-1}(M^{\mathrm{sym}}) : M \in \mathcal{A}_f\Big\}.$$

Then the following properties hold:

(i) *$\mathcal{A}_F$ is a $\mathbf{C}^\infty$-stratification of* $\operatorname{dom} F$ *comprised of connected manifolds,*

(ii) *F is $\mathbf{C}^\infty$-partly smooth relative to each set $\lambda^{-1}(M^{\mathrm{sym}}) \in \mathcal{A}_F$.*

(iii) *The assignment $\mathcal{J}_F \colon \mathbb{P}(\mathbf{S}^n) \to \mathbb{P}(\mathbf{S}^n)$ restricts to an invertible mapping $\mathcal{J}_F \colon \mathcal{A}_F \to \mathcal{A}_{F^*}$ with inverse given by the restriction of $\mathcal{J}_{F^*}$.*

(iv) *The following diagram commutes:*

That is, the equation $(\lambda^{-1} \circ \mathcal{J}_f)(M^{\mathrm{sym}}) = (\mathcal{J}_F \circ \lambda^{-1})(M^{\mathrm{sym}})$ holds for every set $M^{\mathrm{sym}} \in \mathcal{A}_f^{\mathrm{sym}}$.

$$\begin{array}{ccc} \mathcal{A}_F & \xrightarrow{\mathcal{J}_F} & \mathcal{A}_{F^*} \\ \lambda^{-1}\Big\uparrow & & \Big\uparrow\lambda^{-1} \\ \mathcal{A}_f^{\mathrm{sym}} & \xrightarrow[\mathcal{J}_f]{} & \mathcal{A}_{f^*}^{\mathrm{sym}} \end{array}$$

Proof. In light of Lemma 8.4.5, each set $M^{\mathrm{sym}} \in \mathcal{A}_f^{\mathrm{sym}}$ is a symmetric $\mathbf{C}^\infty$ manifold. The fact that $\mathcal{A}_F$ is a $\mathbf{C}^\infty$-stratification of $\operatorname{dom} F$ now follows from the transfer principle for stratifications [50, Theorem 4.8], while the fact that each manifold $\lambda^{-1}(M^{\mathrm{sym}})$ is connected follows immediately from Lemma 8.4.5. Moreover, from Theorem 8.3.15, we deduce that F is $\mathbf{C}^\infty$-partly smooth relative to each set in $\mathcal{A}_F$.

Consider now a set $M^{\mathrm{sym}} \in \mathcal{A}_f^{\mathrm{sym}}$ for some $M \in \mathcal{A}_f$. Then we have:

$$\begin{aligned} \mathcal{J}_F(\lambda^{-1}(M^{\mathrm{sym}})) &= \bigcup_{X \in \lambda^{-1}(M^{\mathrm{sym}})} \operatorname{ri} \partial F(X) \\ &= \bigcup_{X \in \lambda^{-1}(M^{\mathrm{sym}})} \{U^T (\operatorname{Diag} v) U : v \in \operatorname{ri} \partial f(\lambda(X)) \text{ and } U \in \mathbf{O}^n_X\}, \end{aligned}$$

and concurrently,

$$\lambda^{-1}(\mathcal{J}_f(M^{\mathrm{sym}})) = \lambda^{-1}\Big(\bigcup_{x \in M^{\mathrm{sym}}} \operatorname{ri} \partial f(x)\Big) = \bigcup_{x \in M^{\mathrm{sym}},\ v \in \operatorname{ri} \partial f(x)} \mathbf{O}^n.(\operatorname{Diag} v).$$

We claim that the equality $\lambda^{-1}(\mathcal{J}_f(M^{\mathrm{sym}})) = \mathcal{J}_F(\lambda^{-1}(M^{\mathrm{sym}}))$ holds. The inclusion "$\supset$" is immediate. To see the converse, fix a point $x \in M^{\mathrm{sym}}$, a vector $v \in \operatorname{ri} \partial f(x)$, and a matrix $U \in \mathbf{O}^n$. We must show $V := U^T (\operatorname{Diag} v) U \in \mathcal{J}_F(\lambda^{-1}(M^{\mathrm{sym}}))$. To see this, fix a permutation $\sigma \in \Sigma$ with $\sigma x \in \mathbf{R}^n_{\geq}$, and observe

$$U^T (\operatorname{Diag} v) U = (A_\sigma U)^T (\operatorname{Diag} \sigma v) A_\sigma U,$$

where A_σ denotes the matrix representing the permutation σ. Define a matrix $X := (A_\sigma U)^T (\operatorname{Diag} \sigma x) A_\sigma U$. Clearly, we have $V \in \operatorname{ri} \partial F(X)$ and $X \in \lambda^{-1}(M^{\mathrm{sym}})$.

This proves the claimed equality. Consequently, we deduce that the assignment $\mathcal{J}_F\colon \mathbb{P}(\mathbf{S}^n) \to \mathbb{P}(\mathbf{S}^n)$ restricts to a mapping $\mathcal{J}_F\colon \mathcal{A}_F \to \mathcal{A}_{F^*}$, and that the diagram commutes. Commutativity of the diagram along with the fact that $\mathcal{J}_{f^*}$ restricts to be the inverse of $\mathcal{J}_f\colon \mathcal{A}_f^{\mathrm{sym}} \to \mathcal{A}_{f^*}^{\mathrm{sym}}$ implies that $\mathcal{J}_{F^*}$ restricts to be the inverse of $\mathcal{J}_F\colon \mathcal{A}_F \to \mathcal{A}_{F^*}$. □

Example 8.4.7 (Constant rank manifolds)**.** Consider the closed convex cones of positive (respectively negative) semi-definite matrices $\mathbf{S}^n_+$ (respectively $\mathbf{S}^n_-$). Clearly, we have equality $\mathbf{S}^n_\pm = \lambda^{-1}(\mathbf{R}^n_\pm)$. Define the constant rank manifolds

$$M_k^\pm := \{X \in \mathbf{S}^n_\pm : \operatorname{rank} X = k\}, \quad \text{for } k = 0, \ldots, n.$$

Then using Theorem 8.4.6 one can easily check that the manifolds $M_k^\pm$ and $M_{n-k}^\mp$ are dual to each other under the conjugacy correspondence $\delta_{\mathbf{S}^n_+} \overset{*}{\longleftrightarrow} \delta_{\mathbf{S}^n_-}$.

8.5 Extensions to nonsymmetric matrices

Consider the space of $n \times m$ real matrices $\mathbf{M}^{n\times m}$, endowed with the trace inner-product $\langle X, Y\rangle = \mathrm{tr}\,(X^TY)$, and the corresponding Frobenius norm. We will let the group $\mathbf{O}^{n,m} := \mathbf{O}^n \times \mathbf{O}^m$ act on $\mathbf{M}^{n\times m}$ simply by defining

$$(U,V).X = U^TXV \text{ for all } (U,V) \in \mathbf{O}^{n,m} \text{ and } X \in \mathbf{M}^{n\times m}.$$

Recall that *singular values* of a matrix $A \in \mathbf{M}^{n\times m}$ are defined to be the square roots of the eigenvalues of the matrix A^TA. The *singular value mapping* $\sigma : \mathbf{M}^{n\times m} \to \mathbf{R}^m$ is simply the mapping taking each matrix X to its vector $(\sigma_1(X), \ldots, \sigma_m(X))$ of singular values in non-increasing order. We will be interested in functions $F\colon \mathbf{M}^{n\times m} \to \overline{\mathbf{R}}$ that are invariant under the action of $\mathbf{O}^{n,m}$. Such functions F can necessarily be represented as a composition $F = f \circ \sigma$, where the outer-function $f\colon \mathbf{R}^m \to \overline{\mathbf{R}}$ is *absolutely permutation-invariant*, meaning invariant under all signed permutations of coordinates. As in the symmetric case, it is useful to localize this notion. Namely, we will say that a function f is *locally absolutely permutation-invariant* around a point $\bar{x}$ provided that for each signed permutation σ fixing $\bar{x}$, we have $f(\sigma x) = f(x)$ for all x near $\bar{x}$. Then essentially all of the results presented in the symmetric case have natural analogues in this setting (with nearly identical proofs).

Theorem 8.5.1 (The nonsymmetric case: lifts of manifolds)**.** *Consider a matrix $\bar{X} \in \mathbf{M}^{n\times m}$ and a set $M \subset \mathbf{R}^m$ that is locally absolutely permutation-invariant around $\bar{x} := \sigma(\bar{X})$. Then M is a $\mathbf{C}^\infty$ manifold around $\bar{x}$ if and only if the set $\sigma^{-1}(M)$ is a $\mathbf{C}^\infty$ manifold around $\bar{X}$.*

Proposition 8.5.2 (The nonsymmetric case: lifts of identifiable sets)**.** *Consider a lsc $f\colon \mathbf{R}^m \to \overline{\mathbf{R}}$ and a matrix $\bar{X} \in \mathbf{M}^{n\times m}$. Suppose that f is locally absolutely permutation-invariant around $\bar{x} := \sigma(\bar{X})$ and consider a subset $M \subset \mathbf{R}^m$ that is locally absolutely permutation-invariant around $\bar{x}$. Then M is identifiable (relative to f) at $\bar{x}$ for $\bar{v} \in \partial f(\bar{x})$, if and only if $\sigma^{-1}(M)$ is identifiable (relative to $f \circ \sigma$) at $\bar{X}$ for $U^T(\mathrm{Diag}\,\bar{v})V \in \partial(f\circ\sigma)(\bar{X})$, where $(U,V) \in \mathbf{O}^{n,m}$ is any pair satisfying $\bar{X} = U^T(\mathrm{Diag}\,\sigma(\bar{X}))V$.*

Theorem 8.5.3 (The nonsymmetric case: lifts of partly smooth manifolds)**.** *Consider a lsc function $f\colon \mathbf{R}^m \to \overline{\mathbf{R}}$ and a matrix $X \in \mathbf{M}^{n\times m}$. Suppose that f is locally absolutely permutation-invariant around $\bar{x} := \sigma(\bar{X})$. Then f is $\mathbf{C}^\infty$-partly smooth at $\bar{x}$ relative to M if and only if $f \circ \sigma$ is $\mathbf{C}^\infty$-partly smooth at $\bar{X}$ relative to $\sigma^{-1}(M)$.*

It is unknown whether the analogue of the latter theorem holds in the case of $\mathbf{C}^p$ partial smoothness, where $p < \infty$. This is so because it is unknown whether a nonsymmetric analogue of [38, Theorem 4.21] holds in case of functions that are differentiable only finitely many times.

Finally, we should note that Section 8.4 also has a natural analogue in the nonsymmetric setting. For the sake of brevity, we do not record it here.

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