NON-TESTABLE HYPOTHESES IN LINEAR MODELS

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Abstract

Non-testable hypotheses in linear models are formally defined and proof given that they cannot be tested. Consequences of carrying out calculations as if they were testable are considered.

1. Introduction

We represent the general linear model under normality as

$$\underline{y} \sim \mathcal{N}(\underline{X}\underline{b}, \sigma^2 \underline{I}_N) \quad (1)$$

where $\underline{y}$ is a vector of N observations with mean $\underline{X}\underline{b}$ and dispersion matrix $\sigma^2 \underline{I}_N$, where $\underline{I}_N$ is an identity matrix of order N . Estimation of the parameter vector $\underline{b}$ is often made by using the principle of least squares, or using maximum likelihood, both of which lead in the case of (1) to what are called the normal equations $\underline{X}'\underline{X}\underline{b}^0 = \underline{X}'\underline{y}$. A solution of them is taken as

$$\underline{b}^0 = \underline{G}\underline{X}'\underline{y} \quad (2)$$

for $\underline{G}$ being a generalized inverse of $\underline{X}'\underline{X}$, meaning that it satisfies

$$\underline{X}'\underline{X}\underline{G}\underline{X}'\underline{X} = \underline{X}'\underline{X}, \quad \text{from which } \underline{X} = \underline{X}\underline{G}\underline{X}'\underline{X}. \quad (3)$$

$\underline{G}'$ is also a generalized inverse of $\underline{X}'\underline{X}$. If $\underline{G}$ satisfies both (3) and

$$\underline{G}\underline{X}'\underline{X}\underline{G} = \underline{G} \quad (4)$$

it is said to be a reflexive generalized inverse; if $\underline{G}$ is not reflexive then

$$\underline{G}^* = \underline{G}\underline{X}'\underline{X}\underline{G} \quad (5)$$

is.

2. Defining Testable Hypotheses

A linear function $\underline{q}'\underline{b}$ of the elements of the parameter vector $\underline{b}$ is said to be an estimable function when $\underline{q}' = \underline{t}'\underline{X}$ for some $\underline{t}'$, i.e., when $\underline{q}'$ lies in the row space of $\underline{X}$. The best linear unbiased estimator

of the estimable function $\underline{q}'\underline{b}$ is $\underline{q}'\underline{b}^0$, a function of $\underline{b}^0$ that is invariant to whatever $\underline{G}$ is used for $\underline{b}^0$ in (2), i.e., to whatever particular solution, $\underline{b}^0$, of the normal equations is used.

A linear hypothesis concerning elements of $\underline{b}$ is defined as

$$H: \underline{K}'\underline{b} = \underline{m} \quad (6)$$

for known matrix $\underline{K}'$ and vector $\underline{m}$. It will be assumed throughout that $\underline{K}'$ is of full row rank. There is no loss of generality in making this assumption (Scheffé, 1959, p. 29) and it is also a practical requirement. $\underline{K}'$ is derived from the statistical problem at hand and redundant or inconsistent equations would not normally be specified as part of any hypothesis of interest.

When all elements of $\underline{K}'\underline{b}$ are estimable, $\underline{K}'\underline{b}$ is said to be estimable and

$$\underline{K}' = \underline{T}'\underline{X} \quad (7)$$

for some $\underline{T}'$. The hypothesis (6) is said to be testable when $\underline{K}'\underline{b}$ is estimable. The F-statistic for testing (6) is well known to be

$$F(H) = Q/r_K \hat{\sigma}^2 \quad (8)$$

where

$$Q = (\underline{K}'\underline{b}^0 - \underline{m})'(\underline{K}'\underline{G}\underline{K})^{-1}(\underline{K}'\underline{b}^0 - \underline{m}), \quad (9)$$

$$r_K = \text{rank of } \underline{K}' = \text{number of rows in } \underline{K}', \quad (10)$$

and

$$\hat{\sigma}^2 = \underline{y}'(\underline{I} - \underline{X}\underline{G}\underline{X}')\underline{y}/(N - r_X). \quad (11)$$

All of the preceding development is well known. What is not so well known is the treatment of the hypothesis $H: \underline{\underline{K}}' \underline{\underline{b}} = \underline{\underline{m}}$ when not all elements of $\underline{\underline{K}}' \underline{\underline{b}}$ are estimable, i.e., when $\underline{\underline{K}}' \underline{\underline{b}}$ is not estimable. Generally speaking we simply say that such a hypothesis is not testable and leave it at that. The purpose of this paper is twofold. First, non-testable and partially-testable hypotheses are formally defined and proof given that a non-testable hypothesis cannot be tested. Second, in cases where $\underline{\underline{K}}'$ and $\underline{\underline{G}}$ are such that $(\underline{\underline{K}}' \underline{\underline{G}} \underline{\underline{K}})^{-1}$ exists, thus enabling Q of (9) to be calculated and used in $F(H)$ in (8), it is shown what is being tested.

3. The General Procedure for Hypothesis Testing

The residual sum of squares after fitting the model (1) is well known to be

$$SSE = (\underline{\underline{y}} - \underline{\underline{Xb}}^0)' (\underline{\underline{y}} - \underline{\underline{Xb}}^0) . \quad (12)$$

To derive the residual sum of squares under the hypothesis we fit the model

$$\underline{\underline{y}} \sim \mathcal{N}(\underline{\underline{Xb}}, \sigma^2 \underline{\underline{I}}) \quad \text{and} \quad \underline{\underline{K}}' \underline{\underline{b}} = \underline{\underline{m}} . \quad (13)$$

Using Lagrange multipliers 2θ to account for $\underline{\underline{K}}' \underline{\underline{b}} = \underline{\underline{m}}$ in minimizing $(\underline{\underline{y}} - \underline{\underline{Xb}})' (\underline{\underline{y}} - \underline{\underline{Xb}})$ subject to $\underline{\underline{K}}' \underline{\underline{b}} = \underline{\underline{m}}$, leads to the well known equations

$$\underline{\underline{X}}' \underline{\underline{Xb}}_H^0 + \underline{\underline{K}} \theta = \underline{\underline{X}}' \underline{\underline{y}} \quad (14)$$

and

$$\underline{\underline{K}}' \underline{\underline{b}}_H^0 = \underline{\underline{m}} \quad (15)$$

for $\underline{\underline{b}}_H^0$ and $\underline{\underline{\theta}}$. The error sum of squares under the hypothesis is then

$$SSE_H = (\underline{\underline{y}} - \underline{\underline{X}}\underline{\underline{b}}_H^0)'(\underline{\underline{y}} - \underline{\underline{X}}\underline{\underline{b}}_H^0) , \quad (16)$$

and the F-statistic for testing H is

$$F = (SSE_H - SSE) / r_K \hat{\sigma}^2 . \quad (17)$$

4. Testable Hypotheses

$H: \underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ is defined as testable when $\underline{\underline{K}}'\underline{\underline{b}}$ is estimable. A solution to the minimizing equations (14) and (15) is then (see, e.g., Searle, 1971, p. 191)

$$\underline{\underline{\theta}} = (\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}(\underline{\underline{K}}'\underline{\underline{b}}^0 - \underline{\underline{m}}) \quad (18)$$

and

$$\underline{\underline{b}}_H^0 = \underline{\underline{G}}\underline{\underline{X}}'\underline{\underline{y}} - \underline{\underline{G}}\underline{\underline{K}}(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}(\underline{\underline{K}}'\underline{\underline{b}}^0 - \underline{\underline{m}}) \quad (19)$$

where, when $\underline{\underline{K}}'$ has full row rank and $\underline{\underline{K}}'\underline{\underline{b}}$ is estimable, $(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}$ always exists, ensuring that $\underline{\underline{b}}_H^0$ of (19) and Q of (9) can be calculated. On substituting $\underline{\underline{b}}_H^0$ into (16) it will be found that

$$SSE_H = SSE + (\underline{\underline{K}}'\underline{\underline{b}}^0 - \underline{\underline{m}})'(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}(\underline{\underline{K}}'\underline{\underline{b}}^0 - \underline{\underline{m}}) .$$

Hence for (17)

$$SSE_H - SSE = (\underline{\underline{K}}'\underline{\underline{b}}^0 - \underline{\underline{m}})'(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}(\underline{\underline{K}}'\underline{\underline{b}}^0 - \underline{\underline{m}}) ,$$

i.e.,

$$SSE_H - SSE = Q \quad (20)$$

for Q of (9). Thus, when $H: \underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ is testable, F of (17) is F(H) introduced in (8).

5. Non-testable Hypotheses

a. Definition

We define a linear hypothesis $H: \underline{\underline{K}}' \underline{\underline{b}} = \underline{\underline{m}}$ as being non-testable when every element, and every linear combination of elements, of $\underline{\underline{K}}' \underline{\underline{b}}$ is non-estimable. For example, in the case where $\underline{\underline{b}} = [\mu \quad \alpha_1 \quad \alpha_2 \quad \alpha_3]'$ as in the subsequent example (Section 6), this precludes describing

$$H: \begin{cases} \alpha_1 = 7 \\ \alpha_2 = 4 \end{cases} \quad (21)$$

as non-testable in a situation where α_1 and α_2 are each non-estimable, but $\alpha_1 - \alpha_2$ is estimable.

In our definition of a non-testable hypothesis the non-estimability of elements of $\underline{\underline{K}}' \underline{\underline{b}}$ means that $\underline{\underline{K}}'$ must be such that there is no $\underline{\underline{T}}'$ such that $\underline{\underline{K}}' = \underline{\underline{T}}' \underline{\underline{X}}$; and, more generally, non-estimability of linear combinations of elements of $\underline{\underline{K}}' \underline{\underline{b}}$ means that there must be no $\underline{\underline{R}}$ and $\underline{\underline{T}}'$ other than both null such that

$$\underline{\underline{R}} \underline{\underline{K}}' = \underline{\underline{T}}' \underline{\underline{X}}. \quad (22)$$

The reason for defining a non-testable hypothesis this way is that otherwise a hypothesis such as (21), written in terms of non-estimable functions, could be rewritten as

$$H: \begin{cases} \alpha_1 = 7 \\ \alpha_1 - \alpha_2 = 3 \end{cases}$$

where it is now in terms of a mixture of non-estimable and estimable functions. We call such a mixture a partially-testable hypothesis and formally define it in Section 5f.

b. A Non-testable Hypothesis Cannot be Tested

Equations (14), (15), and (16) apply whether or not $H: \underline{K}'\underline{b} = \underline{m}$ is testable; when it is, (18) and (19) are a solution and (20) follows.

But when $H: \underline{K}'\underline{b} = \underline{m}$ is non-testable, we find the solution differs from (18) and (19). To see this consider the following equations in elements of a vector $\underline{u}$:

$$\underline{K}'(\underline{I} - \underline{H})\underline{u} = \underline{m} - \underline{K}'\underline{G}\underline{X}'\underline{y}, \quad (23)$$

where $\underline{H}$ is defined as $\underline{H} = \underline{G}\underline{X}'\underline{X}$, with $\underline{X}\underline{H} = \underline{X}$ from (3). First note that when $\underline{K}'\underline{b}$ is estimable, $\underline{K}' = \underline{T}'\underline{X}$ and $\underline{K}'(\underline{I} - \underline{H}) = \underline{T}'(\underline{X} - \underline{X}\underline{H}) = \underline{0}$ and equations (23) are inconsistent. More generally, when $\underline{K}'$ is such that (22) is true, pre-multiplication of (23) by $\underline{R}$ leads to $\underline{0} = \underline{R}\underline{m} - \underline{R}\underline{K}'\underline{G}\underline{X}'\underline{y}$ and again the equations are inconsistent. Indeed, $\underline{R}\underline{K}'(\underline{I} - \underline{H}) = \underline{0}$, if and only if (22) holds. Since (22) is assumed to hold, we proceed to consider (23).

For $\underline{X}$ having order $N \times p$ and rank r , the matrices $\underline{H}$ and $\underline{I} - \underline{H}$ are idempotent of order p , and $\underline{I} - \underline{H}$ has rank $p - r$. Therefore $(\underline{I} - \underline{H})\underline{u}$ in (23) has $p - r$ arbitrary elements. Furthermore, $\underline{X}$ has rows of order p and its rank is r ; and $\underline{K}'$, through having full row rank, has $r_{\underline{K}}$ rows of order p which, by (22), have no linear dependencies with the rows of $\underline{X}$. Therefore $r_{\underline{K}} \leq p - r$. Hence (23) consists of no more than $p - r$ equations in $p - r$ unknowns, the $p - r$ arbitrary elements of $(\underline{I} - \underline{H})\underline{u}$. Thus (23) has a solution when $H: \underline{K}'\underline{b} = \underline{m}$ is non-testable.

Now consider equations (14) and (15) for determining $\underline{b}_{\underline{H}}^0$, when $H: \underline{K}'\underline{b} = \underline{m}$ is non-testable. It is easily shown that they are satisfied by

$$\underline{\underline{\theta}} = \underline{\underline{0}} \quad (24)$$

and

$$\underline{\underline{b}}_H^0 = \underline{\underline{GX}}'\underline{\underline{y}} + (\underline{\underline{I}} - \underline{\underline{H}})\underline{\underline{u}} \quad (25)$$

for $\underline{\underline{u}}$ being a solution of (23), i.e.,

$$\underline{\underline{X}}'\underline{\underline{X}}\underline{\underline{b}}_H^0 + \underline{\underline{K}}\underline{\underline{\theta}} = \underline{\underline{X}}'\underline{\underline{X}}\underline{\underline{GX}}'\underline{\underline{y}} + \underline{\underline{X}}'\underline{\underline{X}}(\underline{\underline{I}} - \underline{\underline{H}})\underline{\underline{u}} + \underline{\underline{K}}\underline{\underline{0}} = \underline{\underline{X}}'\underline{\underline{y}} + \underline{\underline{0}} + \underline{\underline{0}} = \underline{\underline{X}}'\underline{\underline{y}},$$

thus satisfying (14), and, using (23),

$$\underline{\underline{K}}'\underline{\underline{b}}_H^0 = \underline{\underline{K}}'\underline{\underline{GX}}'\underline{\underline{y}} + \underline{\underline{K}}'(\underline{\underline{I}} - \underline{\underline{H}})\underline{\underline{u}} = \underline{\underline{K}}'\underline{\underline{GX}}'\underline{\underline{y}} + \underline{\underline{m}} - \underline{\underline{K}}'\underline{\underline{GX}}'\underline{\underline{y}} = \underline{\underline{m}},$$

which satisfies (15).

We now use (25) in (16); because $\underline{\underline{X}} = \underline{\underline{XH}}$ we have $\underline{\underline{Xb}}_H^0 = \underline{\underline{XGX}}'\underline{\underline{y}} = \underline{\underline{Xb}}^0$ and so in (16)

$$SSE_H = (\underline{\underline{y}} - \underline{\underline{Xb}}^0)'(\underline{\underline{y}} - \underline{\underline{Xb}}^0) = \text{SSE of (12)}.$$

Therefore

$$SSE_H - SSE = 0$$

and so F of (17) is zero, i.e., there is no test of $H: \underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ for non-testable hypotheses.

c. Alternative Forms for $\underline{\underline{b}}_H^0$

To eliminate $\underline{\underline{u}}$ from $\underline{\underline{b}}_H^0$ of (25) we solve (23) as

$$\underline{\underline{u}} = [\underline{\underline{K}}'(\underline{\underline{I}} - \underline{\underline{H}})]^-(\underline{\underline{m}} - \underline{\underline{K}}'\underline{\underline{GX}}'\underline{\underline{y}})$$

and get

$$\underline{\underline{b}}_H^0 = \underline{\underline{GX}}'\underline{\underline{y}} + (\underline{\underline{I}} - \underline{\underline{H}})[\underline{\underline{K}}'(\underline{\underline{I}} - \underline{\underline{H}})]^-(\underline{\underline{m}} - \underline{\underline{K}}'\underline{\underline{GX}}'\underline{\underline{y}}) \quad (26)$$

where, in general, $\underline{\underline{A}}^-$ is a generalized inverse of $\underline{\underline{A}}$. Rearranging (26) gives

$$\underline{b}_H^0 = \{ \underline{G} - (\underline{I} - \underline{H})[\underline{K}'(\underline{I} - \underline{H})]^{-1}\underline{K}'\underline{G} \} \underline{X}'\underline{y} + (\underline{I} - \underline{H})[\underline{K}'(\underline{I} - \underline{H})]^{-1}\underline{m}. \quad (27)$$

Define the coefficient of $\underline{X}'\underline{y}$ in (27) as $\underline{G}_r$:

$$\underline{G}_r = \underline{G} - (\underline{I} - \underline{H})[\underline{K}'(\underline{I} - \underline{H})]^{-1}\underline{K}'\underline{G}. \quad (28)$$

Then, because $\underline{X}(\underline{I} - \underline{H}) = \underline{0}$, it is easily seen that $\underline{G}_r$ is a generalized inverse of $\underline{X}'\underline{X}$. Furthermore, for

$$\underline{I} - \underline{H}_r = \underline{I} - \underline{G}_r \underline{X}'\underline{X},$$

$$(\underline{I} - \underline{H}_r)(\underline{I} - \underline{H}) = \underline{I} - \underline{H} - \underline{G}_r \underline{X}'\underline{X} + \underline{G}_r \underline{X}'\underline{X}\underline{G}_r \underline{X}'\underline{X} = \underline{I} - \underline{H}$$

so that in (27)

$$\begin{aligned} \underline{b}_H^0 &= \underline{G}_r \underline{X}'\underline{y} + (\underline{I} - \underline{H}_r) \{ (\underline{I} - \underline{H})[\underline{K}'(\underline{I} - \underline{H})]^{-1} \} \underline{m} \\ &= \underline{G}_r \underline{X}'\underline{y} + (\underline{I} - \underline{H}_r) \underline{w} \end{aligned} \quad (29)$$

for

$$\underline{w} = (\underline{I} - \underline{H})[\underline{K}'(\underline{I} - \underline{H})]^{-1}\underline{m}. \quad (30)$$

Thus in (29) we have $\underline{b}_H^0$ expressed in the general form of any solution to equations $\underline{X}'\underline{X}\underline{b}^0 = \underline{X}'\underline{y}$; and so we know at once that $\underline{X}\underline{b}_H^0 = \underline{X}\underline{b}^0$, and hence $SSE_{\underline{H}} = SSE$ and so $F = 0$ as earlier determined.

Another form for $\underline{b}_H^0$ is obtained by dealing with equations (14) and (15) directly, after writing them as

$$\begin{bmatrix} \underline{X}'\underline{X} & \underline{K} \\ \underline{K}' & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{b}_H^0 \\ \underline{\theta} \end{bmatrix} = \begin{bmatrix} \underline{X}'\underline{y} \\ \underline{m} \end{bmatrix}. \quad (31)$$

Then, on defining

$$\underline{L} = (\underline{X}'\underline{X} + \underline{K}\underline{K}')^{-1}, \quad (32)$$

which is a generalized inverse of $\underline{\underline{X}}'\underline{\underline{X}}$ (e.g., Searle, 1971, p. 23), it is easily verified that

$$\begin{bmatrix} \underline{\underline{X}}'\underline{\underline{X}} & \underline{\underline{K}} \\ \underline{\underline{K}}' & \underline{\underline{O}} \end{bmatrix}^{-1} = \begin{bmatrix} \underline{\underline{L}}'\underline{\underline{X}}'\underline{\underline{X}}\underline{\underline{L}} & \underline{\underline{L}}'\underline{\underline{K}} \\ \underline{\underline{K}}'\underline{\underline{L}} & \underline{\underline{O}} \end{bmatrix}. \quad (33)$$

If, without loss of generality, we assume that the first $r_{\underline{\underline{X}}}$ rows of $\underline{\underline{X}}$ are linearly independent and we represent them by $\underline{\underline{Z}}$, then $[\underline{\underline{Z}}' \quad \underline{\underline{K}}]'$ has full row rank. Augment this with some matrix $\underline{\underline{M}}$ of full row rank chosen so that $[\underline{\underline{Z}}' \quad \underline{\underline{K}} \quad \underline{\underline{M}}']'$ is nonsingular. Then, on defining

$$[\underline{\underline{Z}}' \quad \underline{\underline{K}} \quad \underline{\underline{M}}']'^{-1} = [\underline{\underline{A}} \quad \underline{\underline{B}} \quad \underline{\underline{C}}],$$

it is easily shown that

$$\underline{\underline{B}} = \underline{\underline{L}}\underline{\underline{K}}, \quad \underline{\underline{X}}\underline{\underline{L}}\underline{\underline{K}} = \underline{\underline{O}} \quad \text{and} \quad \underline{\underline{K}}\underline{\underline{L}}\underline{\underline{K}}' = \underline{\underline{I}}. \quad (34)$$

It will then be found that a solution to (31) is

$$\underline{\underline{\theta}} = \underline{\underline{O}} \quad \text{and} \quad \underline{\underline{b}}_{\underline{\underline{H}}}^{\underline{\underline{O}}} = \underline{\underline{L}}\underline{\underline{X}}'\underline{\underline{y}} + \underline{\underline{L}}\underline{\underline{K}}\underline{\underline{m}} + [\underline{\underline{I}} - \underline{\underline{L}}(\underline{\underline{X}}'\underline{\underline{X}} + \underline{\underline{K}}\underline{\underline{K}}')]\underline{\underline{z}} \quad (35)$$

for arbitrary $\underline{\underline{z}}$; and by using (34), $\underline{\underline{X}}\underline{\underline{b}}_{\underline{\underline{H}}}^{\underline{\underline{O}}} = \underline{\underline{X}}\underline{\underline{b}}^{\underline{\underline{O}}}$, so that again $\text{SSE}_{\underline{\underline{H}}} = \text{SSE}$ and $F = 0$. In this case a convenient solution for $\underline{\underline{b}}_{\underline{\underline{H}}}^{\underline{\underline{O}}}$ from (35) is

$$\underline{\underline{b}}_{\underline{\underline{H}}}^{\underline{\underline{O}}} = \underline{\underline{L}}(\underline{\underline{X}}'\underline{\underline{y}} + \underline{\underline{K}}\underline{\underline{m}}) = (\underline{\underline{X}}'\underline{\underline{X}} + \underline{\underline{K}}\underline{\underline{K}}')^{-1}(\underline{\underline{X}}'\underline{\underline{y}} + \underline{\underline{K}}\underline{\underline{m}}). \quad (36)$$

d. Comparing Solutions for $\underline{\underline{b}}_{\underline{\underline{H}}}^{\underline{\underline{O}}}$

The solution for $\underline{\underline{b}}_{\underline{\underline{H}}}^{\underline{\underline{O}}}$ when H is testable is (19), which can also be written as

$$\underline{\underline{b}}_{\underline{\underline{H}}}^{\underline{\underline{O}}} = [\underline{\underline{G}} - \underline{\underline{G}}\underline{\underline{K}}(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}\underline{\underline{K}}'\underline{\underline{G}}]\underline{\underline{X}}'\underline{\underline{y}} + \underline{\underline{G}}\underline{\underline{K}}(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}\underline{\underline{m}}.$$

But in this expression the coefficient of $\underline{\underline{X}}'\underline{\underline{y}}$ is not a generalized inverse of $\underline{\underline{X}}'\underline{\underline{X}}$ as it is in (27) and (36) for non-testable hypotheses. Thus $\underline{\underline{b}}_{\underline{\underline{H}}}^0$ for testable hypotheses does not have the same general form that occurs for non-testable hypotheses. This difference in solution to equations (14) and (15) is further accentuated by noting that for testable hypotheses the solution for $\underline{\underline{\theta}}$ is $(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}(\underline{\underline{K}}'\underline{\underline{b}}^0 - \underline{\underline{m}})$ of (18) which is non-null, whereas for non-testable hypotheses the solution in (24) and (35) is $\underline{\underline{\theta}} = 0$.

e. $F(H)$ when $H: \underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ is Non-testable

We showed in Section 5b that when $H: \underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ is non-testable it cannot be tested. Despite this, there are occasions when, even for $\underline{\underline{K}}'\underline{\underline{b}}$ non-estimable, $\underline{\underline{G}}$ and $\underline{\underline{K}}'$ are such that $(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}$ exists. This is so because, although $\underline{\underline{K}}'$ having full row rank is a necessary condition for $(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}$ to exist, $\underline{\underline{K}}'\underline{\underline{b}}$ being estimable is only a sufficient condition; it is not necessary. Therefore, for $\underline{\underline{K}}'\underline{\underline{b}}$ non-estimable there can exist full row rank matrices $\underline{\underline{K}}'$ with $(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}$ existing and, when this is so, Q of (9) can be computed as can $F(H) = Q/r_{\underline{\underline{K}}}^2$ of (8). We emphasize that one should not be calculating Q and $F(H)$ when $H: \underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ is non-testable. However, in cases where Q and $F(H)$ can be (and perhaps have been) calculated despite $H: \underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ being non-testable, the existence of $F(H)$ prompts the question "What hypothesis is $F(H)$ testing?". A partial answer is contained in the following theorem.

Theorem. If, when $H: \underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ is non-testable $\underline{\underline{K}}'$ and $\underline{\underline{G}}$ are nonetheless such that

$$Q = (\underline{\underline{K}}'\underline{\underline{b}}^0 - \underline{\underline{m}})'(\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}})^{-1}(\underline{\underline{K}}'\underline{\underline{b}}^0 - \underline{\underline{m}}) \quad (37)$$

can be computed, then, provided $\underline{\underline{G}}$ is symmetric and reflexive,

$$F(H) = Q/r_K \hat{\sigma}^2 \text{ is testing } H: \underline{K'Hb} = \underline{m}. \quad (38)$$

Proof. Because $\underline{H} = \underline{GX'X}$, $\underline{K'H} = \underline{K'GX'X}$ and so $\underline{K'Hb}$ is estimable.

Therefore, from (20), we can use (8) and (9) to have

$$Q = (\underline{K'Hb}^0 - \underline{m})' (\underline{K'HGH'K})^{-1} (\underline{K'Hb}^0 - \underline{m}) \quad (39)$$

with

$$\underline{K'Hb}^0 = \underline{K'GX'XGX'y} = \underline{K'GX'y} = \underline{K'b}^0. \quad (40)$$

Similarly,

$$\begin{aligned} \underline{K'HGH'K} &= \underline{K'GX'XGX'XG'K} \\ &= \underline{K'GX'XG'K} \\ &= \underline{K'GX'XGK} \text{ for } \underline{G} = \underline{G'} \\ &= \underline{K'GK} \text{ for } \underline{GX'XG} = \underline{G}. \end{aligned} \quad (41)$$

Substituting (40) and (41) into (39) gives (37) when $\underline{G}$ is symmetric and reflexive. Q.E.D.

This theorem corrects an error implicit in Searle (1971, p. 195) which fails to mention the requirements of $\underline{G}$ to be symmetric and reflexive.

Of course, the question will be asked "When, for a non-testable hypothesis, will $\underline{K'GK}$ nevertheless be nonsingular?", thus permitting computation of Q and interpretation thereof by means of the theorem. There seems to be no universal answer to this: nonsingularity of $\underline{K'GK}$ depends completely on the particular forms of $\underline{K'}$ and $\underline{G}$ being used. Illustrations of $\underline{K'GK}$ being nonsingular and being singular are shown in the example at the end of the paper.

f. Partially-testable Hypotheses

As illustrated in Section 5a there are some hypotheses in which some elements of $\underline{\underline{K}}'\underline{\underline{b}}$ are non-estimable but for which some linear combinations are estimable. To exclude this mixture from the definition of a non-testable hypothesis we defined it as being a hypothesis for which no linear combination of elements of $\underline{\underline{K}}'\underline{\underline{b}}$ is estimable. Since linear combinations of estimable functions are estimable, a similar definition, equivalent to that in Section 4, can be made for testable hypotheses: A hypothesis $H: \underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ is defined as testable when all linear combinations of elements of $\underline{\underline{K}}'\underline{\underline{b}}$ are estimable. The remaining forms of $\underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ lead naturally to a definition of a partially-testable hypothesis: A hypothesis $H: \underline{\underline{K}}'\underline{\underline{b}} = \underline{\underline{m}}$ is defined as partially-testable when at least one linear combination of elements of $\underline{\underline{K}}'\underline{\underline{b}}$ is estimable and at least one linear combination of elements of $\underline{\underline{K}}'\underline{\underline{b}}$ is non-estimable. Note that this requires at least one element of $\underline{\underline{K}}'\underline{\underline{b}}$ to be non-estimable. These three definitions provide a complete categorization of the possible forms of $\underline{\underline{K}}'\underline{\underline{b}}$ vis-a-vis estimable and non-estimable elements.

Often a partially-testable hypothesis can be partitioned as

$$H: \begin{cases} \underline{\underline{K}}'_1\underline{\underline{b}} = \underline{\underline{m}}_1 \\ \underline{\underline{K}}'_2\underline{\underline{b}} = \underline{\underline{m}}_2 \end{cases},$$

where $H_1: \underline{\underline{K}}'_1\underline{\underline{b}} = \underline{\underline{m}}_1$ is testable and $H_2: \underline{\underline{K}}'_2\underline{\underline{b}} = \underline{\underline{m}}_2$ is non-testable.

Suppose that $\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}}$ is nonsingular and $\underline{\underline{G}}$ is symmetric and reflexive, then $F(H)$ is testing $\underline{\underline{K}}'\underline{\underline{Hb}} = \underline{\underline{m}}$ which in this case is

$$H: \begin{cases} \underline{\underline{K}}'_1\underline{\underline{Hb}} = \underline{\underline{m}}_1 \\ \underline{\underline{K}}'_2\underline{\underline{Hb}} = \underline{\underline{m}}_2 \end{cases} \quad \text{equivalent to} \quad H: \begin{cases} \underline{\underline{K}}'_1\underline{\underline{b}} = \underline{\underline{m}}_1 \\ \underline{\underline{K}}'_2\underline{\underline{Hb}} = \underline{\underline{m}}_2 \end{cases}. \quad (42)$$

This is so because for H_1 testable,

$$\underline{K}'_1 = \underline{T}'_1 \underline{X} \quad \text{and} \quad \underline{K}'_1 \underline{H} = \underline{T}'_1 \underline{X} \underline{H} = \underline{T}'_1 \underline{X} = \underline{K}'_1 .$$

Result (42) corrects an error implicit in Searle (1971, pp. 194-195) that if Q can be calculated for a partially-testable hypothesis then $F(H)$ is testing only the testable part of it. That is not so; it is testing (42).

6. Example

To illustrate the preceding results we use the simple example of a 1-way classification (completely randomized design) given in Searle (1971, p. 165). The model is

$$E(y_{ij}) = \mu + \alpha_i$$

for $i = 1, 2, 3$ with $n_1 = 3$, $n_2 = 2$ and $n_3 = 1$, where y_{ij} is the j 'th observation in the i 'th class, μ is a general mean and α_i is the effect of the i 'th class. Data are as follows:

Class		
1	2	3
101	84	32
105	88	
94		
300	172	32

Then, with $\underline{y} = [101 \ 105 \ 94 \ 84 \ 88 \ 32]'$ we have

$$\underline{X}\underline{b} = \begin{bmatrix} 1 & 1 & . & . \\ 1 & 1 & . & . \\ 1 & 1 & . & . \\ 1 & . & 1 & . \\ 1 & . & 1 & . \\ 1 & . & . & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad \text{and} \quad \underline{X}'\underline{y} = \begin{bmatrix} 504 \\ 300 \\ 172 \\ 32 \end{bmatrix} ,$$

with a dot in a matrix representing zero. Thus

$$\underline{\underline{X}}'\underline{\underline{X}} = \begin{bmatrix} 6 & 3 & 2 & 1 \\ 3 & 3 & . & . \\ 2 & . & 2 & . \\ 1 & . & . & 1 \end{bmatrix}, \quad \underline{\underline{G}} = \begin{bmatrix} . & . & . & . \\ . & \frac{1}{3} & . & . \\ . & . & \frac{1}{2} & . \\ . & . & . & 1 \end{bmatrix}, \quad \underline{\underline{b}}^0 = \underline{\underline{GX}}'\underline{\underline{y}} = \begin{bmatrix} 0 \\ 100 \\ 86 \\ 32 \end{bmatrix}$$

$$\text{and } \underline{\underline{H}} = \underline{\underline{GX}}'\underline{\underline{X}} = \begin{bmatrix} . & . & . & . \\ 1 & 1 & . & . \\ 1 & . & 1 & . \\ 1 & . & . & 1 \end{bmatrix}. \quad (43)$$

And

$$\underline{\underline{Xb}}^0 = [100 \ 100 \ 100 \ 86 \ 86 \ 32]'$$

The functions $\mu + \alpha_i$ for $i = 1, 2, 3$ are estimable, as are linear combinations of them; and, for example, the numerator sum of squares, namely Q of (9), for testing the testable hypothesis

$$H: \begin{cases} \alpha_1 - \alpha_2 = 0 \\ \alpha_2 - \alpha_3 = 0 \end{cases}$$

involves

$$\underline{\underline{K}}'\underline{\underline{b}}^0 = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 86 \\ 32 \end{bmatrix} = \begin{bmatrix} 14 \\ 54 \end{bmatrix}$$

and

$$\begin{aligned}
 (\underline{\underline{K}}' \underline{\underline{G}} \underline{\underline{K}})^{-1} &= \left\{ \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{3} & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \right\}^{-1} \\
 &= \begin{bmatrix} \frac{5}{6} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}
 \end{aligned}$$

so that, from (9)

$$Q = [14 \quad 54] \frac{1}{6} \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 14 \\ 54 \end{bmatrix} = 3,480 .$$

Now consider the non-testable hypothesis

$$H: \alpha_1 = 7$$

with $\underline{\underline{K}}' = [0 \quad 1 \quad 0 \quad 0]$. Because it is non-testable we know that its $SSE_H = SSE$ and $F = 0$. This is confirmed from (28) and (29) as follows. For (28)

$$\underline{\underline{K}}' \underline{\underline{G}} = [0 \quad \frac{1}{3} \quad 0 \quad 0] \quad \text{and} \quad [\underline{\underline{K}}' (\underline{\underline{I}} - \underline{\underline{H}})]^{-} = \left\{ [0 \quad 1 \quad 0 \quad 0] \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{bmatrix} \right\}^{-} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

Hence (28) is

$$\underline{\underline{G}}_r = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{3} & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}^{-} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} -1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} [0 \quad \frac{1}{3} \quad 0 \quad 0] = \begin{bmatrix} \cdot & \frac{1}{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & -\frac{1}{3} & \frac{1}{2} & \cdot \\ \cdot & -\frac{1}{3} & \cdot & 1 \end{bmatrix} .$$

Then for (29)

$$\tilde{w} = \begin{bmatrix} 1 & . & . & . \\ -1 & . & . & . \\ -1 & . & . & . \\ -1 & . & . & . \end{bmatrix} \begin{bmatrix} -1 \\ . \\ . \\ . \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \\ 7 \\ 7 \end{bmatrix} \quad \text{and} \quad \tilde{H}_r = \begin{bmatrix} 1 & 1 & . & . \\ . & . & . & . \\ . & -1 & 1 & . \\ . & -1 & . & 1 \end{bmatrix},$$

so that (29) is

$$\tilde{b}_H^0 = \begin{bmatrix} . & \frac{1}{3} & . & . \\ . & . & . & . \\ . & -\frac{1}{3} & \frac{1}{2} & . \\ . & -\frac{1}{3} & . & 1 \end{bmatrix} \begin{bmatrix} 504 \\ 300 \\ 172 \\ 32 \end{bmatrix} + \begin{bmatrix} . & -1 & . & . \\ . & 1 & . & . \\ . & 1 & . & . \\ . & 1 & . & . \end{bmatrix} \begin{bmatrix} -7 \\ 7 \\ 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 93 \\ 7 \\ -7 \\ -61 \end{bmatrix}.$$

Hence

$$\tilde{X}\tilde{b}_H^0 = [100 \ 100 \ 100 \ 86 \ 86 \ 32]' = \tilde{X}\tilde{y}^0$$

and so $SSE_H = SSE$ and thus $F = 0$.

Similarly for (36)

$$(\tilde{X}'\tilde{X} + \tilde{K}\tilde{K}')^{-1} = \left\{ \tilde{X}'\tilde{X} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} [0 \ 1 \ 0 \ 0] \right\}^{-1} = \begin{bmatrix} 6 & 3 & 2 & 1 \\ 3 & 4 & . & . \\ 2 & . & 2 & . \\ 1 & . & . & 1 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 8 & -6 & -8 & -8 \\ -6 & 6 & 6 & 6 \\ -8 & 6 & 11 & 8 \\ -8 & 6 & 8 & 14 \end{bmatrix}$$

so that (36) is

$$\tilde{b}_H^0 = \frac{1}{6} \begin{bmatrix} 8 & -6 & -8 & -8 \\ -6 & 6 & 6 & 6 \\ -8 & 6 & 11 & 8 \\ -8 & 6 & 8 & 14 \end{bmatrix} \begin{bmatrix} 504 + 0 \\ 300 + 7 \\ 172 + 0 \\ 32 + 0 \end{bmatrix} = \begin{bmatrix} 93 \\ 7 \\ -7 \\ -61 \end{bmatrix}.$$

This is the same $\tilde{b}_H^0$ as yielded by (29), but in general this need not be so.

By the nature of this simple example, $\underline{\underline{K}}'$ cannot have more than one row; but in cases where $\underline{\underline{K}}'$ has less than its maximum number of rows, i.e., less than $p - r_{\underline{\underline{X}}}$ rows, (29) and (36) do not necessarily yield the same $\underline{\underline{b}}_{\underline{\underline{H}}}^0$; indeed $\underline{\underline{X}}'\underline{\underline{X}} + \underline{\underline{K}}\underline{\underline{K}}'$ may not be nonsingular, as it is here.

Although $H:\alpha_1 = 7$ is non-testable and hence has $F = 0$, we find that with $\underline{\underline{K}}' = [0 \ 1 \ 0 \ 0]$ and $\underline{\underline{G}}$ of (43), the Q of (9) can be computed because $\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}} = \frac{1}{3}$ is nonsingular. Hence, with $\underline{\underline{K}}'\underline{\underline{b}}^0 - \underline{\underline{m}} = 100 - 7 = 93$, $Q = 93^2(3) = 25,947$. This is confirmed from (38):

$$\underline{\underline{K}}'\underline{\underline{H}}\underline{\underline{b}} = [0 \ 1 \ 0 \ 0] \begin{bmatrix} . & . & . & . \\ 1 & 1 & . & . \\ 1 & . & 1 & . \\ 1 & . & . & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \mu + \alpha_1 \quad (44)$$

so that

$$\underline{\underline{K}}'\underline{\underline{H}}\underline{\underline{b}}^0 - \underline{\underline{m}} = 0 + 100 - 7 = 93$$

and

$$\underline{\underline{K}}'\underline{\underline{H}}\underline{\underline{G}}(\underline{\underline{K}}'\underline{\underline{H}})' = [1 \ 1 \ 0 \ 0]\underline{\underline{G}}[1 \ 1 \ 0 \ 0]' = \frac{1}{3}$$

and hence

$$Q = 93^2(3) = 25,947$$

as before.

In contrast to $\underline{\underline{K}}'\underline{\underline{G}}\underline{\underline{K}}$ being nonsingular for the choice of $\underline{\underline{G}}$ above, if we choose

$$\underline{\underline{G}} = \frac{1}{6} \begin{bmatrix} 2 & . & -2 & -2 \\ . & . & . & . \\ -2 & . & 5 & 2 \\ -2 & . & 2 & 8 \end{bmatrix} \quad (45)$$

then $\underline{\underline{K'GK}}$ is zero. This illustrates that for a given $\underline{\underline{K'}}$, the existence of Q depends on the choice of $\underline{\underline{G}}$.

Finally, consider the partially-testable hypothesis

$$H: \begin{cases} \alpha_1 = 7 \\ \alpha_1 - \alpha_2 = 3 \end{cases}$$

for which

$$\underline{\underline{K'}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \quad \underline{\underline{K'b^0}} - \underline{\underline{m}} = \begin{bmatrix} 100 \\ 14 \end{bmatrix} - \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 93 \\ 11 \end{bmatrix}$$

and

$$(\underline{\underline{K'GK}})^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{6} \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}.$$

Then

$$Q = [93 \quad 11] \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 93 \\ 11 \end{bmatrix} = 39,395.$$

From (42), with $H_1: \underline{\underline{K'_1b}} = \underline{\underline{m}}$ being $H_1: \alpha_1 - \alpha_2 = 3$ the testable part, and $H_2: \underline{\underline{K'_2b}} = \underline{\underline{m}}$ being $H_2: \alpha_1 = 7$ the non-testable part, (42) gives, using (44)

$$H: \begin{cases} \alpha_1 - \alpha_2 = 3 \\ \mu + \alpha_1 = 7 \end{cases}$$

with

$$\underline{\underline{K}}' \underline{\underline{b}} - \underline{\underline{m}} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 86 \\ 32 \end{bmatrix} - \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 11 \\ 93 \end{bmatrix}$$

and

$$(\underline{\underline{K}}' \underline{\underline{G}} \underline{\underline{K}})^{-1} = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

so that

$$Q = \begin{bmatrix} 11 & 93 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 11 \\ 93 \end{bmatrix} = 39,395$$

as before. Again, using the $\underline{\underline{G}}$ in (45) gives a singular $\underline{\underline{K}}' \underline{\underline{G}} \underline{\underline{K}}$ and leads to the nonexistence of Q .

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