134-697-111
revised 12/80

On The Construction Of Orthogonal
F-Squares Of Order n From An Orthogonal
Array $(n,k,s,2)$ And An $OL(s,t)$ set

John P. Mandeli
Department of Mathematical Sciences
Virginia Commonwealth University
Richmond, VA. 23284

F.-C. Helen Lee and Walter T. Federer
Biometrics Unit
Cornell University
Ithaca, N.Y. 14853

Abstract

Complete sets of orthogonal F-squares of order $n = s^p$, where s is a prime or prime power and p is a positive integer have been constructed by Hedayat, Raghavarao, and Seiden (1975). Federer (1977) has constructed complete sets of orthogonal F-squares of order $n = 4t$, where t is a positive integer. We give a general procedure for constructing orthogonal F-squares of order n from an orthogonal array $(n, k, s, 2)$ and an $OL(s, t)$ set, where s is not necessarily a prime or prime power. In particular, we show how to construct sets of orthogonal F-squares of order $n = 2s^p$, where s is a prime or prime power and p is a positive integer. These sets are shown to be near complete and approach complete sets as s and/or p become large. We have also shown how to construct orthogonal arrays by these methods. In addition, the best upper bound on the number t of orthogonal $F(n, \lambda_1), F(n, \lambda_2), \dots, F(n, \lambda_t)$ squares is given.

AMS Subject Classification: 62K99, 05B15

Key words and phrases: Orthogonal F-squares, complete sets of orthogonal F-squares, orthogonal arrays, orthogonal Latin squares.

1. Introduction and Summary

In an effort to conserve space, the reader is referred to Hedayat and Seiden (1970), Hedayat, Raghavarao, and Seiden (1975), and to Federer (1977) for details and definitions concerning F-squares design. The symbol $OL(n,t)$ has been used frequently to denote a set of t mutually orthogonal latin squares of order n ; when $t = n - 1$, the set of orthogonal latin squares of order n is complete. A set of t mutually orthogonal F-squares of order n and of the form $F(n;\lambda)$, where $n = \lambda m$, is said to be complete if $t = (n-1)^2/(m-1)$. For $n = s^p$, s a prime or prime power, a complete set has been given by Hedayat, Raghavarao, and Seiden (1975); and for $n = 4t$, a complete set has been given by Federer (1977). The concept of a complete set of orthogonal F-squares given by Hedayat, Raghavarao, and Seiden (1975) needs to be extended as follows for orthogonal F-squares with varying number of symbols.

Theorem 1.1. The maximal number, t , of orthogonal F-squares of the type $F_1(n;\lambda_1), F_2(n;\lambda_2), \dots, F_t(n;\lambda_t)$, where $n = \lambda_i m_i (i = 1, 2, \dots, t)$, satisfies the inequality

$$\sum_{i=1}^t m_i - t \leq (n-1)^2 .$$

Proof: Let $F_1(n; \lambda_1), F_2(n; \lambda_2), \dots, F_t(n; \lambda_t)$ be a set of t mutually orthogonal F-squares. From $F_\alpha(n; \lambda_\alpha)$ we define an $n^2 \times m_\alpha$ matrix $N_\alpha = (n_{\alpha, ij, k})$, where $n_{\alpha, ij, k} = 1$, if the k^{th} symbol occurs in the $(i, j)^{\text{th}}$ cell ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$) of F_α and 0, otherwise. Let

$$M = (N_1 | N_2 | \dots | N_t) .$$

By the property of F-squares, the number of linearly independent rows in M is at most $(n-1)^2 + 1$ and so

$$\text{Rank } (M) \leq \min [(n-1)^2 + 1, \sum_{i=1}^t m_i] .$$

Now,

$$M'M = \begin{bmatrix} n\lambda_1 I_{m_1} & \lambda_1 \lambda_2 J_{m_1 \times m_2} & \dots & \lambda_1 \lambda_t J_{m_1 \times m_t} \\ \lambda_1 \lambda_2 J_{m_2 \times m_1} & n\lambda_2 I_{m_2} & \dots & \lambda_2 \lambda_t J_{m_2 \times m_t} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 \lambda_t J_{m_t \times m_1} & \lambda_2 \lambda_t J_{m_t \times m_2} & \dots & n\lambda_t I_{m_t} \end{bmatrix}$$

where I_{m_i} is the identity matrix of order m_i ($i = 1, 2, \dots, t$) and $J_{m_i \times m_j}$ is the $m_i \times m_j$ ($i = 1, 2, \dots, t$; $j = 1, 2, \dots, t$) matrix with the element 1 everywhere. The eigenvalues of $M'M$ are

$n \sum_{i=1}^t \lambda_i$, $n\lambda_i (i=1,2,\dots,t)$ and 0 with respective multiplicities 1, $m_i - 1 (i=1,2,\dots,t)$ and $t - 1$. Therefore

$$1 + \sum_{i=1}^t m_i - t = \text{Rank}(M'M) = \text{Rank}(M) \\ \leq \min [(n-1)^2 + 1, \sum_{i=1}^t m_i] .$$

Hence $\sum_{i=1}^t m_i - t \leq (n-1)^2$.

When $\lambda_i = \lambda$ for $i = 1,2,\dots,t$ we have the result obtained by Hedayat, Raghavarao, and Seiden (1975):

Corollary 1.1: The maximal number, t , of orthogonal F-squares of the type $F(n;\lambda)$, where $n = \lambda m$, satisfies the inequality

$$t \leq (n-1)^2 / (m-1) .$$

From Theorem 1.1. we can obtain the following definition.

Definition 1.1. A set of t orthogonal F-squares of the type $F_1(n;\lambda_1), F_2(n;\lambda_2), \dots, F_t(n;\lambda_t)$ is said to be a complete set if

$$\sum_{i=1}^t m_i - t = (n-1)^2 ,$$

where $m_i = n/\lambda_i (i=1,2,\dots,t)$.

In section 2, we give a general procedure for constructing orthogonal F-squares of order n from an $OL(s,t)$ set and an orthogonal array $(n,k,s,2)$, where n is the number of assemblies (columns), k is the number of constraints (rows), s is the number of levels (elements or symbols), and 2 is the strength of the array (see e.g., Raghavarao (1971), Chapter 2, for the definition and discussion of orthogonal arrays). As a particular case, we show how to construct sets of orthogonal F-squares of order $n = 2s^p$, where s is a prime or prime power and p is a positive integer. These sets are shown to be near complete and approach complete sets as s and/or p become large.

2. Construction of Orthogonal F-squares of Order n from an Orthogonal Array $(n,k,s,2)$ and an $OL(s,t)$ Set.

Theorem 2.1. If the orthogonal array $(n,k,s,2)$ and a set of orthogonal latin squares $OL(s,t)$ exist then there exists a set of tk^2 orthogonal $F(n;n/s)$ -squares.

Proof: Suppose there exists an orthogonal array $(n,k,s,2)$ and a set of orthogonal latin squares $OL(s,t)$. Let the k constraints (rows) of the orthogonal array be denoted as $Y_1, Y_2, \dots, Y_k$ and let the t latin squares in $OL(s,t)$ be denoted as $L_1, L_2, \dots, L_t$. Let $F_{\ell,pq}$ for $\ell = 1, 2, \dots, t$ and $p, q = 1, 2, \dots, k$, be the $n \times n$ matrix in which the (i,j) -th

cell of $F_{\ell,pq}$ is equal to $(i\text{-th element of } Y_p) \otimes_{\ell}$
 $(j\text{-th element of } Y_q)$, where $\otimes_{\ell}$ is the addition defined
 by latin square L_{ℓ} . First, we claim that each $F_{\ell,pq}$
 is an $F(n;n/s)$ -square because:

1) Each row of $F_{\ell,pq}$ is $c \otimes_{\ell} Y_q$, where c is an integer
 such that $0 \leq c \leq t$, which is a permutation of Y_q . By the
 definition of the orthogonal array, each $Y_q, q=1,2,\dots,k$,
 has precisely s elements each replicated n/s times. A
 permutation of Y_q retains this property. Therefore, each
 row of $F_{\ell,pq}$ has precisely s elements each replicated n/s
 times.

2) Each column of $F_{\ell,pq}$ is $Y_p \otimes_{\ell} d$, $0 \leq d \leq t$ and d an
 integer, which is a permutation of Y_p . For the same reasons
 as given in 1), each column of $F_{\ell,pq}$ has precisely s
 elements each replicated n/s times.

Since we have k orthogonal constraints and t orthogonal
 latin squares, we could construct tk^2 different $F_{\ell,pq}$.

Then we need to show that any two of the constructed squares
 are $\sim$ orthogonal. Given two different F-squares $F_{\ell,pq}$ and
 $F_{\ell',p'q'}$, the subscripts ℓ,pq and $\ell',p'q'$ fall into the
 following cases:

Case 1) $q \neq q'$

Case 2) $q = q'$ but $p \neq p'$

Case 3) $q = q'$ and $p = p'$ but $\ell \neq \ell'$.

Case 1. $q \neq q'$. Superimpose each row of $F_{\ell, pq}$ upon each row of $F_{\ell', p'q'}$ which in effect superimposes $c \otimes_{\ell} Y_q$ on $c' \otimes_{\ell'} Y_{q'}$, where c, c' are integers such that $0 \leq c, c' \leq t$. From the definition of the orthogonal array, each element of Y_q appears an equal number of times, $\lambda = n/s^2$, with each element of $Y_{q'}$. Since the addition of a constant c to Y_q and the addition of a constant c' to $Y_{q'}$ does not alter this property, each element of $c \otimes_{\ell} Y_q$ appears an equal number of times, $\lambda = n/s^2$, with each element of $c' \otimes_{\ell'} Y_{q'}$. Since there are n rows in each F-square, each element of $F_{\ell, pq}$ appears an equal number of times, $n\lambda = (n/s)^2$, with each element of $F_{\ell', p'q'}$.

Case 2. $q = q'$, $p \neq p'$. Superimpose each column of $F_{\ell, pq}$ upon each column of $F_{\ell', p'q}$, which in effect superimposes $Y_p \otimes_{\ell} d$ on to $Y_{p'} \otimes_{\ell'} d$ where d is an integer $0 \leq d \leq t$. From the definition of the orthogonal array, each element of Y_p appears an equal number of times, $\lambda = n/s^2$, with each element of $Y_{p'}$. Since the addition of a constant d to Y_p or $Y_{p'}$ does not alter this property, each element of $Y_p \otimes_{\ell} d$ appears an equal number of times, $\lambda = n/s^2$, with each element of $Y_{p'} \otimes_{\ell'} d$. Since there are n columns in each F-square,

each element of $F_{\ell, pq}$ appears an equal number of times,
 $n\lambda = (n/s)^2$, with each element of $F_{\ell', p'q}$.

Case 3. $q = q'$, $p = p'$, $\ell \neq \ell'$. Partition both $F_{\ell, pq}$ and $F_{\ell', pq}$ into $(n/s)(n/s) = (n/s)^2$ latin squares of order s . Superimpose each latin square of $F_{\ell, pq}$ upon the corresponding latin square of $F_{\ell', pq}$. Since L_{ℓ} is orthogonal to $L_{\ell'}$, each of the elements of L_{ℓ} appear an equal number of times, once, with each of the elements of $L_{\ell'}$. Since there are $(n/s)^2$ pairs of latin squares, each element of $F_{\ell, pq}$ appears $(n/s)^2$ times with each element of $F_{\ell', pq}$.

From Theorem 2.1 we obtain the following Corollary:

Corollary 2.1. There exists a set of $(s-1)[2(s^p-1)/(s-1)-1]^2$ orthogonal $F(2s^{p^*}, 2s^{p-1})$ -squares, for s a prime or prime power and p a positive integer.

Proof: Orthogonal arrays of the form

$\{n = 2s^{p^*}, k = [2(s^p-1)/(s-1)-1], s, 2\}$ for s a prime or prime power are known to exist and Addelman and Kempthorne (1961) give a method for constructing them. Also $OL(s, s-1)$ sets are known to exist for s a prime or prime power.

Hence by Theorem 2.1 we can construct sets of orthogonal $F(2s^{p^*}, 2s^{p-1})$ -squares from the above two combinatorial structures.

It may be worthwhile to point out that there exists one more F -square of order $2s^p$ orthogonal to the $F(2s^{p^*}, 2s^{p-1})$ -squares obtained in Corollary 2.1. This F -square is the

following $F(2s^p; s^p)$ -square:

$$\begin{array}{cccccc}
 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\
 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\
 \vdots & & & & & & & \\
 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\
 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\
 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\
 \vdots & & & & \vdots & & & \\
 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0
 \end{array}$$

It can be shown that this $F(2s^p; s^p)$ -square is orthogonal to the set of $F(2s^p; 2s^{p-1})$ -squares of Corollary 2.1. It must be emphasized that this square depends on the construction of the orthogonal array $(n, k, s, 2)$ and so will not generalize to all squares obtained in Theorem 2.1.

We define a set of t orthogonal F -squares of order n as asymptotically complete if $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^t m_i - t}{(n-1)^2} = 1$ where m_i

is defined in Theorem 1.1.

Theorem 2.2: The set of orthogonal F -squares of order $2s^p$ obtained in Corollary 2.1 and the note following it form an asymptotically complete set of orthogonal F -squares of order $2s^p$, for $p \geq 2$.

Proof: $\lim_{p \rightarrow \infty} \frac{(s-1)^2 [2(s^p-1)/(s-1)-1]^2 + 1}{(2s^p-1)^2} = 1$

$$\lim_{s \rightarrow \infty} \frac{(s-1)^2 [2(s^p-1)/(s-1)-1]^2 + 1}{(2s^p-1)^2} = \begin{cases} 1 & \text{when } p \geq 2 \\ \frac{1}{4} & \text{when } p = 1. \end{cases}$$

Hence the theorem is true for $p \geq 2$.

Closing remarks. The existence of a set of orthogonal $F(n;\lambda)$ -squares implies the existence of an orthogonal array $(n^2, r+2, n/\lambda, 2)$. Hence Theorem 2.1 can be restated in terms of orthogonal arrays:

Theorem 2.3. If the orthogonal array $(n,k,s,2)$ and a set of orthogonal latin squares $OL(s,t)$ exist then there exists an orthogonal array $(n^2, tk^2 + 2, s, 2)$.

Acknowledgement. The authors wish to thank the referee for his helpful comments.

Literature Cited

- Addelman, S. and Kempthorne, O. (1961). Some main effect plans and orthogonal arrays of strength two. *Ann. Math. Stat.* 32; 1167-1176.
- Federer, W.T. (1977). On the existence and construction of a complete set of orthogonal $F(4t; 2t, 2t)$ -squares design. *Ann. Stat.* 3, 561-564.
- Hedayat, A., Raghavarao, D., and Seiden, E. (1975). Further contributions to the theory of F-squares design. *Ann. Stat.* 3, 712-716.
- Hedayat, A. and Seiden, E. (1970). F-square and orthogonal F-squares design: A generalization of Latin square and orthogonal Latin squares design. *Ann. Math. Stat.* 41; 2035-2044.
- Raghavarao, D. (1971). Constructions and Combinatorial Problems in Design of Experiments. John Wiley and Sons, Inc., New York, London, Sydney, Toronto.