NUMERICAL SOLUTIONS OF A MODEL OF INFLUENZA WITH TWO STRAINS, AGE-STRUCTURE AND CROSS-IMMUNITY

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Numerical Solutions of a Model of Influenza with Two Strains, Age-Structure and Cross-Immunity

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Abstract

The epidemiological models with age structure, proportionate mixing, and cross-immunity which have been studied by Castillo Chavez et al. (1989). The removal of age-structure from the two strain models led to damped oscillations. Numerical simulations of a discrete version gave rise to sustained oscillators provided that age-structure, two co-circulating viral strains, and cross-immunity was included. The hypothesis that the interaction between cross-immunity and age-dependent survivorship may be enough to drive sustained oscillations. In this paper we simulate the continuos time version of this model using an algorithm based on the finite difference method used by Milner et al. (1993). The numerical scheme is proved to converge.

1 Introduction

1.1 Influenza

Influenza is disease we frequently refer to as the "flu ", is one of the oldest and most common diseases known in the history of disease spreading. It is a detrimental disease since it can be responsable for the largest lost of population comunities. Initially influenza was introduced by Hippocrates in 412 BC. The first well-described pandemic of influenza-like disease occurred in 1580. From the begining of the influenza existence, 31 influenza pandemics have been studied, with three occurring in this century: in 1918, 1957 and 1968.

The disease today still affects large sections of the population each year. Influenza is particularly interesting since the virus can mutate quickly, often producing new strains against which human beings have no immunity. When this occurs,the disease can have adverse effects in the population, for instance, during the "Spanish flu "pandemic of 1918-1920, when a large population of about 20 million suffered the consequences of such deadline disease.

Influenza is an important disease to study due to its ability to infect the respiratory system. The influenza virus is spread when an infected individual interacts with its surrounding contacts by coughing and/or sneezing. Outbreaks of influenza develope abruptly. Since the disease spreads through communities, the number of cases arising is of about 3 weeks and dies out after another 3 or 4 weeks. From the past years we have noted that about twenty to fifty percent of a population may be affected. Most of the infected individuals tend to be in the ages of 5 to 14 years old. Schools are a primary location that allows for the transmission of influenza virus, therefore we can observe that families with school age children have a higher rate of infection than other families.

Since the begining of research of influenza virus,three types of strains have been identified. These types can be describe by two major groups **A** or **B**. These names are given accordinly to their place where they were the strains initially become isolated. New strains of the virus appear each year. The most interesting epidemics have been associated with influenza **A** viruses. Influenza **B** viruses are also responsable for minor, local epidemics and are less detrimental. Currently, there are three different influenza strains circulating worldwide; two subtypes of influenza **A** and one of influenza **B**.

An important characteristic of the influenza virus is its surface antigenic structure which changes rapidly and makes its deletion almost impossible. There are two kinds of antigenic changes in influenza: drift, the gradual, relatively minor change in antigenics, and shift, the sudden, complete change of one or more of the antigens. Drift of surface antigens produces new variants and is the main reason why strain **A** can stay alive for several concurring years in the same host population. Shift is responsable for the new subtypes and is the major cause of influenza pandemics (Liu,W. Levin, S. 1989).

1.2 The model for two strains with age-structure

The first model with structure with age structure was introduced by Castillo-Chavez et al. (1989). Incorporates age-specific mortalities and age-specific contact. The two influenza strains are coupled by a coefficient cross-immunity (σ); the coupling is strong when σ is small (antigenically very similar strains) and weaker when σ is intermediate (different strains same subtype). In both instances, the simulations of an associated discrete model yield sustained oscillations driven mostly by the shape of survivorship function.

2 The model for two strains with partial cross-immunity

In this section, Castillo-Chavez et al. (1989), we reformulate two-strain epidemiological model for a homogeneous population. The population is divided in eight classes: x (fraction of susceptible), y (fraction of infected by strain 1), u (fraction of infected by strain 2), z (fraction of recovered from strain 1), k (fraction of recovered from strain 2), v (fraction of infected by strain 1 after recovery from the other strain), q (fraction of infected by strain 2 after recovery from the other strain), w (recovered from both strains). The interations among classes are represented in the transfer diagram show in Fig.1.

The initial boundary value problem governing the dynamics of these classes under proportionate mixing age-dependent bilinear incidence rates is:

$$\frac{\partial x}{\partial a} + \frac{\partial x}{\partial t} = -\lambda_1(t)b(a)x(a,t) - \lambda_2(t)b(a)x(a,t) - \mu(a)x(a,t) \quad (2.1)$$

$$\frac{\partial y}{\partial a} + \frac{\partial y}{\partial t} = \lambda_1(t)b(a)x(a,t) - \gamma_1(t)b(a)y(a,t) - \mu(a)y(a,t) \quad (2.2)$$

$$\frac{\partial u}{\partial a} + \frac{\partial u}{\partial t} = \lambda_2(t)b(a)x(a,t) - \gamma_2(t)b(a)u(a,t) - \mu(a)u(a,t) \quad (2.3)$$

$$\frac{\partial z}{\partial a} + \frac{\partial z}{\partial t} = \gamma_1(t)y(a,t) - \sigma_2\lambda_2(t)b(a)z(a,t) - \mu(a)z(a,t) \quad (2.4)$$

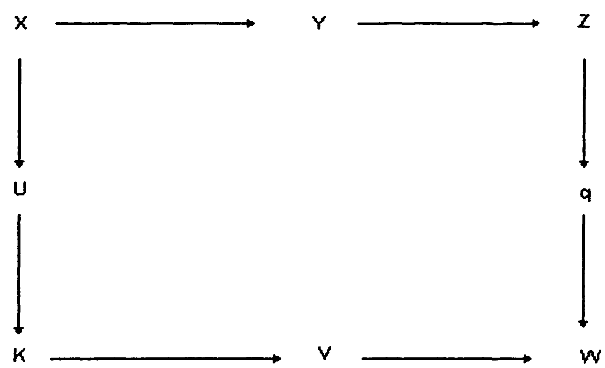

Figure 1: Transfer diagram for two co-circulating viral strain or subtypes on a single host population.

$$\frac{\partial k}{\partial a} + \frac{\partial k}{\partial t} = \gamma_2(t)u(a,t) - \sigma_1\lambda_1(t)b(a)k(a,t) - \mu(a)k(a,t) \tag{2.5}$$

$$\frac{\partial v}{\partial a} + \frac{\partial v}{\partial t} = \sigma_1\lambda_1(t)b(a)k(a,t) - \gamma_1 v(a,t) - \mu(a)v(a,t) \tag{2.6}$$

$$\frac{\partial q}{\partial a} + \frac{\partial q}{\partial t} = \sigma_2\lambda_2(t)b(a)z(a,t) - \gamma_1 q(a,t) - \mu(a)q(a,t) \tag{2.7}$$

$$\frac{\partial w}{\partial a} + \frac{\partial w}{\partial t} = \gamma_1 v(a,t) + \gamma_2 q(a,t) - \mu(a)w(a,t) \tag{2.8}$$

where

$$\lambda_1(t) = \beta_1 \int_0^\infty b(a')[y(a',t) + v(a,t)]da' \tag{2.9}$$

and

$$\lambda_2(t) = \beta_2 \int_0^\infty b(a')[u(a',t) + q(a,t)]da' \tag{2.10}$$

$$x(0,t) = \rho = \frac{1}{\int_0^\infty e^{-M(a')}da'},$$
$$y(0,t) = u(0,t) = z(0,t) = k(0,t) = v(0,t) = 0 \qquad (2.11)$$

$$\begin{array}{rclrclrcl} x(a,0) & = & x_0(a), & y(a,0) & = & y_0(a), & u(a,0) & = & u_0(a) \\ z(a,0) & = & z_0(a), & k(a,0) & = & k_0(a), & v(a,0) & = & v_0(a), \\ q(a,0) & = & q_0(a), & w(a,0) & = & w_0(a), & & & \end{array} \qquad (2.12)$$

where γ_1, γ_2 denotes constant recovery rates from strain 1 and strain 2 respectively, and $\mu(a)$ and $b(a)$ denote age-specific mortality and activity level rates. We let β_1, β_2 denote the transmission coefficients of strain 1 and 2. The susceptibility factors σ_1, σ_2 are measures of the cross-immunity. They satisfy the conditions $0 \leq \sigma_1 \leq 1$, if $\sigma_1 = 0$ (respectively $\sigma_2 = 0$) then there is no cross-infection with strain 2 (strain 1 respectively) that is strain 1 (strain 2) imparts complete immunity. If $\sigma_1 = 1$ (respectively $\sigma_2 = 1$) then there is no cross-immunity to strain 2 (strain 1). The functions λ_1, λ_2 are the instantaneous force of infection and ρ is the birth rate.

3 Discretization

We discretize the system equations (2.1)-(2.11). Let $0 \leq a \leq A$, $0 \leq t \leq T$. h to be the step-size in both directions, so that

$$Nh = A, \qquad Mh = T.$$

We approximate

$$\begin{array}{llll} x(a,t) \approx X_{ij}; & y(a,t) \approx Y_{ij}; & u(a,t) \approx U_{ij}; & z(a,t) \approx Z_{ij}; \\ k(a,t) \approx K_{ij}; & v(a,t) \approx V_{ij}; & q(a,t) \approx Q_{ij}; & w(a,t) \approx W_{ij}. \end{array}$$

We choose the same step-size in age and time since the characteristic lines of the system (2.1)-(2.11) have slope one. We approximate the derivates along the characteristic lines (see F. Milner et al. 1992). We consider the following finite difference scheme:

$$\frac{X_{ij} - X_{i-1,j-1}}{h} = -\lambda_{1,j-1} b_i X_{ij} - \lambda_{2,j-1} b_i X_{ij} - \mu_i X_{ij} \qquad (3.1)$$

$$\frac{Y_{ij} - Y_{i-1,j-1}}{h} = \lambda_{1,j-1} b_i X_{ij} - \gamma_1 Y_{ij} - \mu_i Y_{ij} \tag{3.2}$$

$$\frac{U_{ij} - U_{i-1,j-1}}{h} = \lambda_{2,j-1} b_i X_{ij} - \gamma_2 U_{ij} - \mu_i U_{ij} \tag{3.3}$$

$$\frac{Z_{ij} - Z_{i-1,j-1}}{h} = \gamma_1 Y_{ij} - \sigma_2 \lambda_{2,j-1} b_i Z_{ij} - \mu_i Z_{ij} \tag{3.4}$$

$$\frac{K_{ij} - K_{i-1,j-1}}{h} = \gamma_2 b_i U_{ij} - \sigma_1 \lambda_{1,j-1} b_i K_{ij} - \mu_i K_{ij} \tag{3.5}$$

$$\frac{V_{ij} - V_{i-1,j-1}}{h} = \sigma_1 \lambda_{1,j-1} b_i K_{ij} - \gamma_1 V_{ij} - \mu_i V_{ij} \tag{3.6}$$

$$\frac{Q_{ij} - Q_{i-1,j-1}}{h} = \sigma_2 \lambda_{2,j-1} b_i Z_{ij} - \gamma_2 Q_{ij} - \mu_i Q_{ij} \tag{3.7}$$

$$\frac{W_{ij} - W_{i-1,j-1}}{h} = \gamma_1 V_{ij} + \gamma_2 Q_{ij} - \mu_i W_{ij} \tag{3.8}$$

and

$$\lambda_{1,j} = \beta_1 \sum_{k=0}^{N} w_{1,k} b_k [Y_{kj} + V_{kj}] h \tag{3.9}$$

$$\lambda_{2,j} = \beta_2 \sum_{k=0}^{N} w_{2,k} b_k [U_{kj} + Q_{kj}] h \tag{3.10}$$

where $w_{1,k}$, $w_{2,k}$, $k = 1, \ldots, N$ are weights which determine the quadrature rule. We solve the equations (3.1)-(3.10) to obtain:

$$X_{ij} = \frac{X_{i-1,j-1}}{1 + \lambda_{1,j-1} b_i h + \lambda_{2,j-1} b_i h + \mu_i h} \tag{3.11}$$

$$Y_{ij} = \frac{\lambda_{1,j-1} b_i h X_{ij} + Y_{i-1,j-1}}{1 + \gamma_1 h + \mu_i h} \tag{3.12}$$

$$U_{ij} = \frac{\lambda_{2,j-1} b_i h X_{ij} + U_{i-1,j-1}}{1 + \gamma_2 h + \mu_i h} \tag{3.13}$$

$$Z_{ij} = \frac{\gamma_1 h Y_{ij} + Z_{i-1,j-1}}{1 + \sigma_2 \lambda_{2,j-1} b_i h + \mu_i h} \tag{3.14}$$

$$K_{ij} = \frac{\gamma_2 h U_{ij} + K_{i-1,j-1}}{1 + \sigma_1 \lambda_{1,j-1} b_i h + \mu_i h} \tag{3.15}$$

$$V_{ij} = \frac{\sigma_1 \lambda_{1,j-1} b_i h K_{ij} + V_{i-1,j-1}}{1 + \gamma_1 h + \mu_i h} \tag{3.16}$$

$$Q_{ij} = \frac{\sigma_2 \lambda_{2,j-1} b_i h Z_{ij} + Q_{i-1,j-1}}{1 + \gamma_2 h + \mu_i h} \tag{3.17}$$

$$W_{ij} = \frac{\gamma_1 h V_{i,j} + \gamma_2 h Q_{i,j} + W_{i-1,j-1}}{1 + \mu_i h} \tag{3.18}$$

Clearly, the solution of (3.11)-(3.18) is no negative if the initial and boundary conditions are non-negative.

We approximate the boundary condition by

$$x(0,t) = \rho = \frac{1}{\int_0^\infty e^{-M(a')} da'}.$$

First we note that $e^{-\int_0^\tau \mu(\tau) d\tau}$ is a solution to the ODE:

$$\begin{cases} \frac{du}{da} + \mu(a) u(a) = 0, \\ u(0) = 1 \end{cases}$$

And approximate this solution numerically using the scheme

$$\begin{cases} \frac{u_i - u_{i-1}}{h} + \mu_i u_i = 0 \\ u_0 = 1 \end{cases}$$

The solution of this scheme is given inductively by

$$\begin{cases} u_i = \frac{u_{i-1}}{1+\mu_i h} \\ u_0 = 1 \end{cases}$$

or explicitly by

$$\begin{cases} u_n = \prod_{i=1}^{n} \frac{1}{1+\mu_i h} \\ u_0 = 1 \end{cases}$$

It is easy to show that if

$$u(a_i) - u_i = \vartheta_i$$

$$\mid \vartheta_i \mid \leq O(h), \quad \forall i$$

then

$$u(a_i) = u_i + O(h).$$

Hence:

$$\int_0^\infty e^{-M(a)} da = \sum_{i=0}^{N} w_i u_i h + O(h).$$

Thus

$$\int_0^\infty e^{-M(a)} da = \sum_{i=0}^{N} w_i \prod_{k=1}^{i} \frac{1}{1+\mu_j h} h + O(h)$$

therefore

$$\frac{1}{\int_0^\infty e^{-M(a)} da} = \frac{1}{\sum_{i=0}^{N} w_i \prod_{k=1}^{i} \frac{1}{1+\mu_k h} h} + O(h);$$

we see what

$$X_{0,j} = \rho = \frac{1}{w_0 h \sum_{i=0}^{N} w_i \prod_{k=1}^{i} \frac{1}{1+\mu_k h} h} + O(h)$$

$$Y_{0,j} = U_{0,j} = Z_{0,j} = K_{0,j} = V_{0,j} = Q_{0,j} = W_{0,j} = 0$$

We discretize the boundary conditions as follows

$$\begin{array}{llll} X_{i,0} = X_i^0; & Y_{i,0} = Y_i^0; & U_{i,0} = U_i^0; & Z_{i,0} = Z_i^0; \\ K_{i,0} = K_i^0; & V_{i,0} = V_i^0; & Q_{i,0} = Q_i^0; & W_{i,0} = W_i^0. \end{array}$$

To evaluate the system of differential equationsat (a_i, t_j), we observe from the Taylor's series expansion that we obtain

$$\frac{\partial f(a_i,t_j)}{\partial a}+\frac{\partial f(a_i,t_j)}{\partial t}=\frac{f(a_i,t_j)-f(a_{i-1},t_{j-1})}{h}+O(h). \quad (3.19)$$

Set-up

$$\begin{array}{ll} x(a_i,t_j)-X_{i,j}=\xi_{i,j}; & y(a_i,t_j)-Y_{i,j}=\eta_{i,j}; \\ u(a_i,t_j)-U_{i,j}=\zeta_{i,j}; & z(a_i,t_j)-Z_{i,j}=\chi_{i,j}; \\ k(a_i,t_j)-K_{i,j}=\psi_{i,j}; & v(a_i,t_j)-V_{i,j}=\varepsilon_{i,j}; \\ q(a_i,t_j)-Q_{i,j}=\varrho_{i,j}; & w(a_i,t_j)-W_{i,j}=\theta_{i,j}; \end{array}$$

We can compute the system (2.1)-(2.8) at (a_i,t_j) using (3.19) and subtrating the corresponding equation in system (3.1)-(3.8). Hence we have that:

$$\frac{\xi_{ij}-\xi_{i-1,j-1}}{h}=-[\lambda_1(t_j)-\lambda_1(t_{j-1})]b_ix(a_i,t_j) \quad (3.20)$$

$$-\lambda_{1,j-1}b_i\xi_{ij}-[\lambda_1(t_{j-1})-\lambda_{1,j-1}]b_ix(a_i,t_j)-[\lambda_2(t_j)-\lambda_2(t_{j-1})]b_ix(a_i,t_j)$$
$$-\lambda_{2,j-1}b_i\xi_{ij}-[\lambda_2(t_{j-1})-\lambda_{2,j-1}]b_ix(a_i,t_j)-\mu_i\xi_{ij}+O(h),$$

$$\frac{\eta_{ij}-\eta_{i-1,j-1}}{h}=[\lambda_1(t_j)-\lambda_1(t_{j-1})]b_ix(a_i,t_j) \quad (3.21)$$

$$+\lambda_{1,j-1}b_i\xi_{ij}+[\lambda_1(t_{j-1})-\lambda_{1,j-1}]b_ix(a_i,t_j)-\gamma_1\eta_{ij}-\mu_i\eta_{ij}+O(h),$$

$$\frac{\zeta_{ij}-\zeta_{i-1,j-1}}{h}=[\lambda_2(t_j)-\lambda_2(t_{j-1})]b_ix(a_i,t_j) \quad (3.22)$$

$$+\lambda_{2,j-1}b_i\xi_{ij}+[\lambda_2(t_{j-1})-\lambda_{2,j-1}]b_ix(a_i,t_j)-\gamma_2\zeta_{ij}-\mu_i\zeta_{ij}+O(h),$$

$$\frac{\chi_{ij}-\chi_{i-1,j-1}}{h}=\gamma_1\eta_{ij}-\sigma_2[\lambda_2(t_j)-\lambda_2(t_{j-1})]b_iz(a_i,t_j) \quad (3.23)$$

$$-\sigma_2\lambda_{2,j-1}b_i\chi_{ij}-\sigma_2[\lambda_2(t_{j-1})-\lambda_{2,j-1}]b_iz(a_i,t_j)-\mu_i\chi_{ij}+O(h),$$

$$\frac{\psi_{ij}-\psi_{i-1,j-1}}{h}=\gamma_2\zeta_{ij}-\sigma_1[\lambda_1(t_j)-\lambda_1(t_{j-1})]b_ik(a_i,t_j) \quad (3.24)$$

$$-\sigma_1\lambda_{1,j-1}b_i\psi_{ij}-\sigma_1[\lambda_1(t_{j-1})-\lambda_{1,j-1}]b_ik(a_i,t_j)-\mu_i\psi_{ij}+O(h),$$

$$\frac{\varepsilon_{ij}-\varepsilon_{i-1,j-1}}{h}=\sigma_1[\lambda_1(t_j)-\lambda_1(t_{j-1})]b_ik(a_i,t_j) \tag{3.25}$$

$$+\sigma_1\lambda_{1,j-1}b_i\psi_{ij}+\sigma_1[\lambda_1(t_{j-1})-\lambda_{1,j-1}]b_ik(a_i,t_j)-\gamma_1\varepsilon_{ij}-\mu_i\varepsilon_{ij}+O(h),$$

$$\frac{\varrho_{ij}-\varrho_{i-1,j-1}}{h}=\sigma_2[\lambda_2(t_j)-\lambda_2(t_{j-1})]b_iz(a_i,t_j) \tag{3.26}$$

$$+\sigma_2\lambda_{2,j-1}b_i\chi_{ij}+\sigma_2[\lambda_2(t_{j-1})-\lambda_{2,j-1}]b_iz(a_i,t_j)-\gamma_2\varrho_{ij}-\mu_i\varrho_{ij}+O(h),$$

$$\frac{\theta_{ij}-\theta_{i-1,j-1}}{h}=\gamma_1\varepsilon_{ij}+\gamma_2\varrho_{ij}-\mu_i\theta_{ij}+O(h), \tag{3.27}$$

where

$$\lambda_1(t_j)-\lambda_{1,j}=\beta_1\sum_{k=0}^{N}w_{1,k}b_k[\eta_{k,j}+\varepsilon_{k,j}]h+O(h),$$

$$\lambda_2(t_j)-\lambda_{2,j}=\beta_2\sum_{k=0}^{N}w_{2,k}b_k[\zeta_{k,j}+\varrho_{k,j}]h+O(h),$$

In addition, we also have that

$$\lambda_1(t_j)-\lambda(t_{j-1})=\beta_1\sum_{k=0}^{N}w_{1,k}b_k[y(a_k,t_j)-y(a_k,t_{j-1})]h$$

$$-\beta_1\sum_{k=0}^{N}w_{1,k}b_k[v(a_k,t_j)-v(a_k,t_{j-1})]h+O(h),$$

$$=\beta_1\sum_{k=0}^{N}w_{1,k}b_k\frac{\partial y(a_k,t_j)}{\partial t}h^2-\beta_1\sum_{k=0}^{N}w_{1,k}b_k\frac{\partial v(a_k,t_j)}{\partial t}h^2+O(h).$$

Consequently,

$$\mid\lambda_1(t_j)-\lambda(t_{j-1})\mid\leq O(h).$$

and

$$\mid\lambda_2(t_j)-\lambda(t_{j-1})\mid\leq O(h)$$

Furthermore,since the boundary condition is

$$X_{0,j} = \frac{1}{w_0 h \sum_{i=0}^{N} w_i \prod_{k=1}^{i} \frac{1}{1+\mu_k h} h} + O(h)$$

we obtain

$$\xi_{0,j} = O(h). \tag{3.28}$$

The initial age-distribution conditions are given by

$$\eta_{0,j} = \zeta_{0,j} = \chi_{0,j} = \psi_{0,j} = \varepsilon_{0,j} = \varrho_{0,j} = \theta_{0,j} = 0, \quad \forall j$$
$$\xi_{i,0} = \eta_{i,0} = \zeta_{i,0} = \chi_{i,0} = \psi_{i,0} = \varepsilon_{i,0} = \varrho_{i,0} = \theta_{i,0} = 0, \quad \forall i$$

Now,

$$x(a,t), y(a,t), z(a,t), u(a,t), v(a,t), k(a,t), q(a,t), w(a,t)$$

are continuous functions over $0 \leq a \leq A$, $0 \leq t \leq T$ so they are bounded. We assume that all of these functions are bounded by a constant M. In addition we assume that all the parameter functions are bounded except $\mu(a)$.

4 Convergence

To prove convergence of our numerical scheme rewrite equations (3.20-(3.27) by solving for ξ_{ij}, η_{ij}, ζ_{ij}, χ_{ij}, ψ_{ij}, ϱ_{ij}, ε_{ij}, θ_{ij}. Hence,

$$\xi_{ij} = \frac{p_1}{q_1} + O(h)$$

where

$$p_1 = \xi_{i-1,j-1} - [\lambda_1(t_j) - \lambda_1(t_{j-1})] h b_i x(a_i, t_j) - [\lambda_1(t_{j-1}) - \lambda_{1,j-1}] h b_i x(a_i, t_j)$$
$$- [\lambda_2(t_j) - \lambda_2(t_{j-1})] h b_i x(a_i, t_j) - [\lambda_2(t_j) - \lambda_{2,j-1})] h b_i x(a_i, t_j)$$

and

$$q_1 = 1 + \lambda_{1,j-1} h b_i + \lambda_{2,j-1} h b_i + \mu_i h;$$

$$\eta_{ij} = \frac{p_2}{q_2} + O(h)$$

where

$$p_2 = \eta_{i-1,j-1}+[\lambda_1(t_j)-\lambda_1(t_{j-1})]hb_ix(a_i,t_j)+[\lambda_1(t_{j-1})-\lambda_{1,j-1})]hb_ix(a_i,t_j)+\lambda_{1,j-1}hb_i\xi_{ij}$$

and

$$q_2 = 1+\gamma_1 h+\mu_i h;$$

$$\zeta_{ij} = \frac{p_3}{q_3}+O(h)$$

where

$$p_3 = \zeta_{i-1,j-1}+[\lambda_2(t_j)-\lambda_2(t_{j-1})]hb_ix(a_i,t_j)+[\lambda_2(t_{j-1})-\lambda_{2,j-1})]hb_ix(a_i,t_j)+\lambda_{2,j-1}hb_i\xi_{ij}$$

and

$$q_3 = 1+\gamma_2 h+\mu_i h;$$

$$\chi_{ij} = \frac{p_4}{q_4}+O(h)$$

where

$$p_4 = \chi_{i-1,j-1}+\gamma_1 h\eta_{ij}-\sigma_2[\lambda_2(t_j)-\lambda_2(t_{j-1})]hb_iz(a_i,t_j)-\sigma_2[\lambda_2(t_{j-1})-\lambda_{2,j-1})]hb_iz(a_i,t_j)$$

and

$$q_4 = 1+\sigma_2\lambda_{2,j-1}hb_i+\mu_i h;$$

$$\psi_{ij} = \frac{p_5}{q_5}+O(h)$$

where

$$p_5 = \psi_{i-1,j-1}+\gamma_2\zeta_{ij}-\sigma_1[\lambda_1(t_j)-\lambda_1(t_{j-1})]hb_ik(a_i,t_j)$$
$$-\sigma_1[\lambda_1(t_{j-1})-\lambda_{1,j-1})]hb_ik(a_i,t_j)$$

and

$$q_5 = 1+\sigma_1\lambda_{1,j-1}hb_i+\mu_i h;$$

$$\varepsilon_{ij} = \frac{p_6}{q_6}+O(h)$$

where

$$p_6 = \varepsilon_{i-1,j-1}+\sigma_1\lambda_{1,j-1}hb_i\psi_{ij}+\sigma_1[\lambda_1(t_j)-\lambda_1(t_{j-1})]hb_ik(a_i,t_j)$$

$$+\sigma_1[\lambda_1(t_{j-1})-\lambda_{1,j-1})]hb_ik(a_i,t_j)$$
$$q_6 = 1+\gamma_1 h+\mu_i h;$$

$$\varrho_{ij} = \frac{p_7}{q_7}+O(h)$$

where

$$p_7 = \varrho_{i-1,j-1}+\sigma_2\lambda_{2,j-1}hb_i\chi_{ij}+\sigma_2[\lambda_2(t_j)-\lambda_2(t_{j-1})]hb_iz(a_i,t_j);$$
$$+\sigma_2[\lambda_2(t_{j-1})-\lambda_{2,j-1})]hb_iz(a_i,t_j)$$
$$q_7 = 1+\gamma_2 h+\mu_i h$$

and finally,

$$\theta_{ij} = \frac{\theta_{i-1,j-1}+\gamma_1 h\varepsilon_{ij}+\gamma_2 h\varrho_{ij}}{1+\mu_i h}+O(h)$$

We introduce the norm

$$||f_n|| = \sum_{j=1}^{N} f_j h,$$

the following Milner et al.(1993),

$$|\xi_{ij}| \leq \frac{1}{1+\lambda_{1,j-1}hb_i+\mu_i h}.[|\xi_{i-1,j-1}|+|\lambda_1(t_j)-\lambda_1(t_{j-1})|hb_ix(a_i,t_j)$$

$$-|\lambda_2(t_j)-\lambda_2(t_{j-1})|hb_ix(a_i,t_j)-|\lambda_2(t_j)-\lambda_{2,j-1})|hb_ix(a_i,t_j)+O(h^2)$$

Multiplying both sides by h and suming from $i = 1,\dots,N$,we have that

$$||\xi_j|| \leq \xi_{0,j-1}h+||\xi_{j-1}||+(|\lambda_1(t_j)-\lambda_1(t_{j-1})|$$

$$+|\lambda_2(t_j)-\lambda_2(t_{j-1})|+|\lambda_2(t_j)-\lambda_{2,j-1}|)\sum_{i=1}^{N} h^2 b_ix(a_i,t_j)+O(h^2)$$

$$\lambda_1(t_j)-\lambda_1(t_{j-1}) = \sum_{k=1}^{N} w_{1,k}b_k[y_{k,j}-y_{k,j-1}]h+\sum_{k=1}^{N} w_{1,k}b_k[v_{k,j}-v_{k,j-1}]h+O(h^2)$$

Expanding $y_{k,j}$ in a Taylor series we have that

$$y_{k,j} = y_{k,j-1}+hy'_{k,j-1}+h.o.t$$

then

$$y_{k,j} - y_{k,j-1} = O(h)$$

and thus

$$|\lambda_1(t_j) - \lambda_1(t_{j-1})| \leq O(h),$$

$$|\lambda_2(t_j) - \lambda_2(t_{j-1})| \leq O(h)$$

and

$$||\xi_j|| \leq \xi_{0,j-1}h + ||\xi_{j-1}|| + O(h^2)$$

Hence, from (3.28) and by induction, it follows that

$$||\xi_j|| \leq O(h).$$

Similarly,

$$|\eta_{ij}| \leq \frac{1}{1+\gamma_1 h + \mu_i h}.[|\eta_{i-1,j-1}| + |\lambda_1(t_j) - \lambda_1(t_{j-1})|hb_i x(a_i, t_j)$$

$$+\lambda_{1,j-1}hb_i|\xi_{ij}| + |\lambda_1(t_{j-1}) - \lambda_{1,j-1}|hb_i x(a_i, t_j) + O(h^2)$$

Multiplying both sides by h and summing from $i = 1, \dots, N$

$$||\eta_j|| \leq \eta_{0,j-1}h + ||\eta_{j-1}|| + (|\lambda_1(t_j) - \lambda_1(t_{j-1})|$$

$$+|\lambda_1(t_{j-1}) - \lambda_{1,j-1}|)\sum_{i=1}^{N} h^2 b_i x(a_i, t_j) + \lambda_{1,j-1}Bh||\xi_j|| + O(h^2).$$

and hence

$$||\eta_j|| \leq ||\eta_{j-1}|| + Ch||\xi_j|| + O(h^2)$$

Therefore

$$||\eta_j|| \leq ||\eta_{j-1}|| + O(h^2),$$

and

$$||\eta_j|| \leq O(h).$$

Next,

$$|\zeta_{ij}| \leq \frac{1}{1+\gamma_2 h + \mu_i h}.[|\zeta_{i-1,j-1}| + |\lambda_2(t_j) - \lambda_2(t_{j-1})|hb_i x(a_i, t_j)$$

$$+\lambda_{2,j-1}hb_i|\xi_{ij}| + |\lambda_2(t_{j-1}) - \lambda_{2,j-1}|hb_i x(a_i, t_j) + O(h^2)$$

Multiplying both sides by h and summing from $i = 1, \dots, N$, we have

$$||\zeta_j|| \leq \zeta_{0,j-1}h + ||\zeta_{j-1}|| + (|\lambda_2(t_j) - \lambda_2(t_{j-1})|$$

$$+|\lambda_2(t_{j-1}) - \lambda_{2,j-1}|)\sum_{i=1}^{N} h^2 b_i x(a_i, t_j) + \lambda_{2,j-1} Bh||\xi_j|| + O(h^2);$$

and hence

$$||\zeta_j|| \leq ||\zeta_{j-1}|| + Ch||\xi_j|| + O(h^2)$$

$$||\zeta_j|| \leq O(h).$$

Therefore,

$$|\chi_{ij}| \leq \frac{1}{1+\sigma_2\lambda_{2,j-1}b_i h + \mu_i h} \cdot [|\chi_{i-1,j-1}| + \gamma_1 h|\eta_{ij}| + \sigma_2|\lambda_2(t_j) - \lambda_2(t_{j-1})| h b_i z(a_i, t_j)$$

$$+\sigma_2|\lambda_2(t_{j-1}) - \lambda_{2,j-1}| h b_i z(a_i, t_j) + O(h^2).$$

Multiplying both sides by h and summing from $i = 1, \ldots, N$, we have

$$||\chi_j|| \leq \chi_{0,j-1} h + ||\chi_{j-1}|| + \sigma_2(|\lambda_2(t_j) - \lambda_2(t_{j-1})|$$

$$+|\lambda_2(t_{j-1}) - \lambda_{2,j-1}|)\sum_{i=1}^{N} h^2 b_i z(a_i, t_j) + \gamma_1 Bh||\eta_j|| + O(h^2);$$

and hence

$$||\chi_j|| \leq ||\chi_{j-1}|| + Ch||\eta_j|| + O(h^2)$$

$$||\chi_j|| \leq O(h).$$

Therefore,

$$|\psi_{ij}| \leq \frac{1}{1+\sigma_1\lambda_{1,j-1}b_i h + \mu_i h} \cdot [|\psi_{i-1,j-1}| + \gamma_2 h|\zeta_{ij}| + \sigma_1|\lambda_1(t_j) - \lambda_1(t_{j-1})| h b_i k(a_i, t_j)$$

$$-\sigma_1|\lambda_1(t_{j-1}) - \lambda_{1,j-1}| h b_i k(a_i, t_j) + O(h^2).$$

Multiplying both sides by h and sum $i = 1, \ldots, N$, we have

$$||\psi_j|| \leq \psi_{0,j-1} h + ||\psi_{j-1}|| + \sigma_1(|\lambda_1(t_j) - \lambda_1(t_{j-1})|$$

$$+|\lambda_1(t_{j-1}) - \lambda_{1,j-1}|)\sum_{i=1}^{N} h^2 b_i k(a_i, t_j) + \gamma_2 Bh||\zeta_j|| + O(h^2);$$

and hence

$$||\psi_j|| \leq ||\psi_{j-1}|| + Ch||\zeta_j|| + O(h^2)$$

$$||\psi_j|| \leq O(h).$$

Therefore,

$$|\varepsilon_{ij}| \le \frac{1}{1+\gamma_1 h+\mu_i h}.[|\varepsilon_{i-1,j-1}|+\sigma_1\lambda_{1,j-1}b_ih|\psi_{ij}|+\sigma_1|\lambda_1(t_j)-\lambda_1(t_{j-1})|hb_ik(a_i,t_j)$$
$$+\sigma_1|\lambda_1(t_{j-1})-\lambda_{1,j-1}|hb_ik(a_i,t_j)+O(h^2).$$

Multiplying both sides by h and sum $i=1,\dots,N$

$$||\varepsilon_j|| \le \varepsilon_{0,j-1}h+||\varepsilon_{j-1}||+\sigma_1(|\lambda_1(t_j)-\lambda_1(t_{j-1})|$$
$$+|\lambda_1(t_{j-1})-\lambda_{1,j-1}|)\sum_{i=1}^{N}h^2b_ik(a_i,t_j)+\sigma_1\lambda_{1,j-1}Bh||\psi_j||+O(h^2);$$

and hence

$$||\varepsilon_j|| \le ||\varepsilon_{j-1}||+Ch||\psi_j||+O(h^2)$$
$$||\varepsilon_j|| \le O(h).$$

Therefore,

$$|\varrho_{ij}| \le \frac{1}{1+\gamma_2 h+\mu_i h}.[|\varrho_{i-1,j-1}|+\sigma_2\lambda_{2,j-1}b_ih|\chi_{ij}|+\sigma_2|\lambda_2(t_j)-\lambda_2(t_{j-1})|hb_iz(a_i,t_j)$$
$$+\sigma_2|\lambda_2(t_{j-1})-\lambda_{2,j-1}|hb_iz(a_i,t_j)+O(h^2)$$

Multiply both sides by h and sum $i=1,\dots,N$

$$||\varrho_j|| \le \varrho_{0,j-1}h+||\varrho_{j-1}||+\sigma_2(|\lambda_2(t_j)-\lambda_2(t_{j-1})|$$
$$+|\lambda_2(t_{j-1})-\lambda_{2,j-1}|)\sum_{i=1}^{N}h^2b_iz(a_i,t_j)+\sigma_2\lambda_{2,j-1}Bh||\chi_j||+O(h^2)$$

then

$$||\varrho_j|| \le ||\varrho_{j-1}||+Ch||\chi_j||+O(h^2)$$
$$||\varrho_j|| \le O(h).$$

Therefore,

$$|\theta_{ij}| \le \frac{1}{1+\mu_i h}.[|\theta_{i-1,j-1}|+\gamma_1 h|\varepsilon_{ij}|+\gamma_2 h|\eta_{ij}|+O(h^2)].$$

Multiplying both sides by h

$$||\theta_{ij}|| \le \theta_{0,j-1}h+||\theta_{j-1}||+\gamma_1 h||\varepsilon_j||+\gamma_2 h||\varrho_j||+O(h^2);$$

and hence

$$||\theta_{ij}|| \le ||\theta_{j-1}||+ch(||\varepsilon_j||+||\varrho_j||)+O(h^2)$$
$$||\theta_j|| \le O(h).$$

5 Conclusion

We have discretized a nonlinear system of eight partial differential equations by using a finite difference method.

We have proved the convergence of each aproximation to the system of the differential equations.

This work leads to open questions which could be aswered by estimating the error involved in the approximations made in the analysis.

We have simulated the cases for when $b(a)$ and $\mu(a)$ are constants for the model (2.1)-(2.8) which is reduced to the model intitially proposed by Castillo-Chavez et al.(1989) and shown damped oscillations for different cross-immunity values(σ), see appendix.

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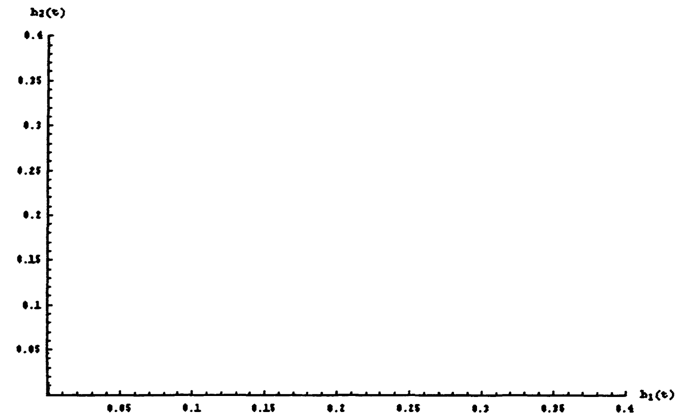
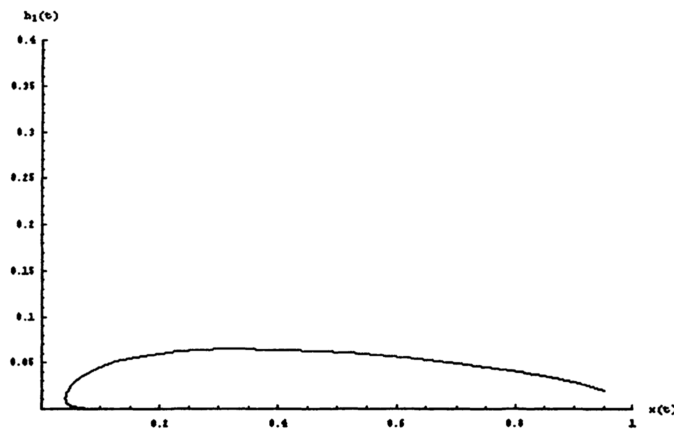
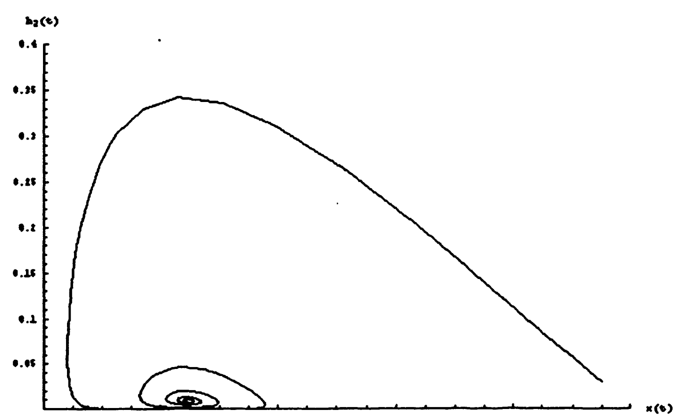
Appendix

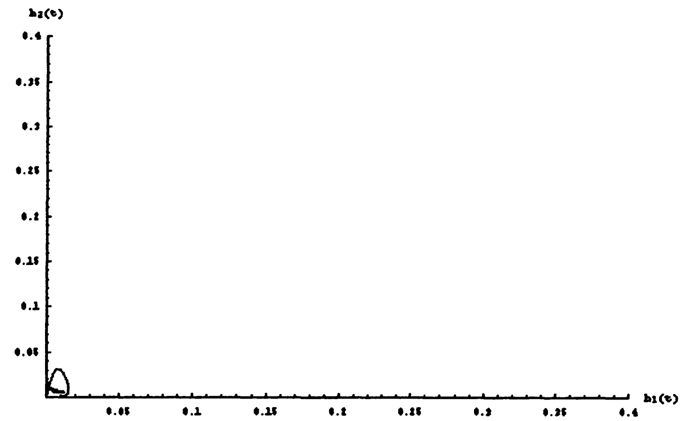
Graficas con valor de $\sigma = 0.1$

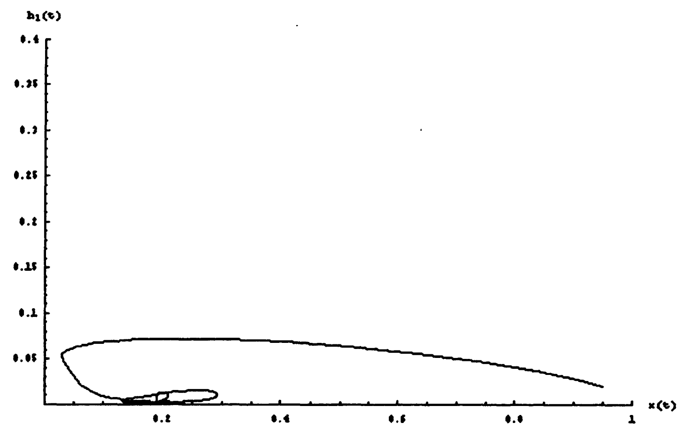


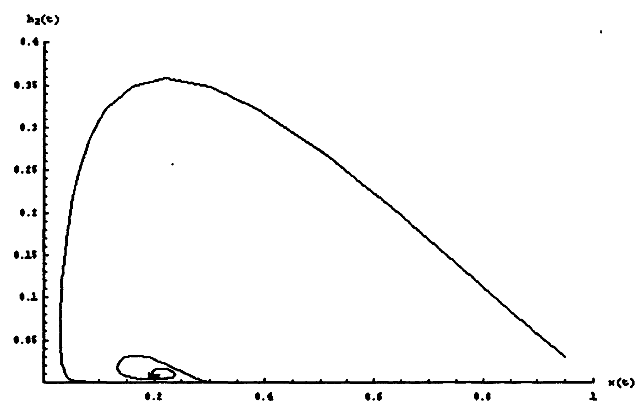




Graficas con valor de $\sigma = 0.5$







1 Graficas con valor de $\sigma = 0.9$