QUOTIENTS OF SPHERES BY LINEAR ACTIONS OF ABELIAN GROUPS

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We consider quotients of spheres by linear actions of real tori and finite abelian groups. To each quotient we associate a matroid or sequence of matroids. In the case of real tori, we find the integral homology groups of the resulting quotient spaces and singular sets in terms of the Tutte polynomial of the matroid(s). For finite groups, an algorithm for computing the $\mathbb{Z}_p$-homology of the quotient space is given.

BIOGRAPHICAL SKETCH

Marisa Hughes received a B.S. in mathematics from Binghamton University in 2005. While working on her undergraduate degree, Marisa attended summer REU's in Potsdamn, NY and Knoxville, TN. She also participated in the EDGE program between undergraduate and graduate school. Marisa was granted a Master's degree from Cornell University in 2009. She will be spending the 2012-2013 school year working as a Visiting Assistant Professor at Hamilton College in Clinton, NY.

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CHAPTER 1

INTRODUCTION

Let G be a compact group that acts by isometries on a Riemannian manifold Y. It is natural to ask whether the orbit space of this action is itself a topological manifold. In order to answer this question, we can consider the behavior of the action on the tangent space. Let x be a point of Y with tangent space T_xY. Denote by S_x the unit tangent vectors in T_xM which are perpendicular to the orbit Gx. The isotropy group of x, $G_x = \{g \in G : gx = x\}$, acts on S_x. In order for the overall orbit space Y/G to be a manifold, the quotient S_x/G_x must at least be a homology sphere for each point $x \in Y$. Since small metric neighborhoods of $\bar{x}$ (the image of x in Y/G) are homeomorphic to a cone on S_x/G_x, excision implies the quotient S_x/G_x must at least be a homology sphere for all $x \in Y$. Understanding the topology of quotients S^n/G where $G \subseteq O(n)$ is essential for answering questions about the general orbit space Y/G.

The above discussion demonstrates one motivation to study quotients of spheres by isometries. Let G, a subgroup of the orthogonal group $O(n)$, act on a sphere S^{n-1} with orbit space X. If G is abelian, this action becomes much easier to describe. This is because we can simultaneously diagonalize all of the elements of G over $\mathbb{C}$. This diagonalization yields a subgroup of $O(n)$ conjugate to G, whose action on S^{n-1} yields a orbit space isometric to S^{n-1}/G. We can therefore assume that the action of a finitely generated abelian G on X can be described by a list of diagonal matrices over $\mathbb{C}$. For this reason, we will focus on the case where G is abelian.

For cyclic groups G, the action can be described by a single matrix corresponding to a generator of G. We can demand that this matrix be diagonalized over $\mathbb{C}$, in which case it becomes clear that the group acts by rotations on certain invariant circles. Furthermore, the speed of these rotations determines the entire action. (Note: in order to use this geometric interpretation of the actions, we are assume that all subgroups of G, particularly those with even cardinality, preserve orientation). The cohomology rings of these generalized lens spaces were found by Stephen Willson in 1976[16]. Interestingly, his results about these spaces extended to quotients of homology spheres, rather than just actions on metric spheres.

In 1999, Ed Swartz discovered some interesting connections between quotients of spheres and matroid theory [14]. Specifically, he classified the homology of quotients of spheres by subgroups of $SO(2n)$ isomorphic to $(\mathbb{Z}_p)^r$ for p an odd prime, and quotients by subgroups of $O(n)$ by $(\mathbb{Z}_2)^r$. Note that when p is odd, every quotient $S^{2n}/(\mathbb{Z}_p)^r$ is the suspension of a quotient $S^{2n-1}/(\mathbb{Z}_p)^r$, so it suffices to study orbit spaces of odd-dimensional spheres. The diagonalized matrices corresponding to the generators of $(\mathbb{Z}_p)^r$ can be used to form a $r \times n$ matrix that described the action completely. Each row of this matrix corresponds to the action of a single generator; each column corresponds to the action of the group on a single circle in the join decomposition of S^{2n-1}.

After such success with finite tori $\mathbb{Z}_p^r$, we might suspect that quotients of spheres by topological subgroups of $SO(2n)$ isomorphic to real tori T^r may yield similarly interesting results. Here we compute the integral homology of the orbit space of any linear action of T^r on S^{2n-1}. Let $\widetilde{P}_X(t)$ denote the reduced Poincaré polynomial of the orbit space X: $\widetilde{P}_X(t) = \sum_{i=0}^{n} \dim(\widetilde{H}_i(X;\mathbb{Q}))\, t^i$. The following theorem is proven in section 4.2, where M_X is a matroid associated to the quotient space X :

Theorem 1. *Let $X = S^{2n-1}/T^r$ with associated matroid M_X. Then the reduced Poincaré polynomial $\widetilde{P}_X(t) = t^{r-1}T(M_X; 0, t^2)$. Furthermore, $\widetilde{H}_i(X;\mathbb{Z})$ has no torsion.*

The rational singular set of an orbit space of T^r is the image of the points whose isotropy groups are infinite. We will show that the singular set, being the image of certain subspheres of S^{2n-1}, is an arrangement in the sense of [18]. Furthermore, the lattice of this arrangement is the order dual of the lattice of flats of the associated matroid. Using the results on the topology of arrangements in [18], we show that the Poincaré polynomial of the rational singular set is the difference of two Tutte polynomials. The following result is proven in section 5:

Theorem 2. *Let $\mathcal{S}$ denote the rational singular set of the orbit space S^{2n-1}/T^r. Then $\widetilde{P}_{\mathcal{S}}(t) = t^{r-2}(T(M_X; 1, t^2) - T(M_X; 0, t^2))$. Furthermore, $H_i(\mathcal{S};\mathbb{Z})$ has no torsion.*

Having a formula for the Poincaré polynomial of S^{2n-1}/T^r in terms of the Tutte polynomial gives us the tools necessary to determine when these orbit spaces are manifolds. We classify all the actions of T^r whose orbit space is a manifold, and, even more specifically, we specify when the orbit space is a (homology) sphere.

Analyzing the orbit space of the linear action of an arbitrary finite abelian group proves to be more challenging. The computations are well-understood for elementary abelian p-groups[14] and cyclic groups [16]. To proceed, we once again define a matrix that describes the action in terms of the generators of the group G. This matrix is used to create a sequence of matroids with weak maps between them. We are able to define a homology theory for this sequence of matroids similar to that in [14], and we use this theory to generate an algorithm for computing the $\mathbb{Z}_p$-homology of any such quotient X.

CHAPTER 2

BACKGROUND AND NOTATION

2.1 Matroids

For a more thorough introduction to the theory of matroids, and proofs for many of the facts given below, see [11].

A *matroid* is a pair (E, I) where E is a finite set, $\mathcal{P}(E)$ the power set of E, and $\mathcal{I} \subseteq \mathcal{P}(E)$. The finite set E is known as the *ground set*, and $\mathcal{I}$ is the set of *independent* subsets of E. In order to be a matroid, the independent sets must respect the following axioms:

I1) $\emptyset \in \mathcal{I}$

I2) If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$.

I3) If $I_1, I_2, \in \mathcal{I}$ and $|I_1| < |I_2|$, then $\exists x \in I_2 \backslash I_1$ such that $I_1 \cup \{x\} \in \mathcal{I}$

An element $e \in E$ is called a *loop* if it is contained in no independent sets. We say $e \in E$ is a *coloop* if it is contained in every maximal independent set of M. A *representable* matroid is one that can be represented by a matrix; the ground set is the set of column vectors of a matrix, and the independent sets are precisely the sets of columns which are linearly independent as column vectors. Row operations, which preserve the linear independence relations of the columns, and column switches of a matrix do not change the isomorphism class of the matroid.

The *deletion* of a matroid element, denoted $M - e$, has ground set $E \setminus e$ and independent sets $\mathcal{I}_{M-e} = \{I \in \mathcal{I}_M : e \notin I\}$. Deleting e from a representable matroid can be accomplished by deleting the column corresponding to e in a representative matrix.

Another matroid construction, denoted by M/e, is the *contraction* of M by e.

If e is a loop of M, then M/e is the same as $M - e$. Otherwise, M/e has ground set $E \setminus e$ and independent sets $\mathcal{I}_{M/e} = \{I \setminus e \ : \ \{e\} \cup I \in \mathcal{I}_M\}$. In the case of a representable matroid, the contraction by e can be computed by row reducing a representative matrix A such that the column corresponding to e has only one nonzero entry. By deleting the row where this entry is located, along with the column corresponding to e, we get a new matrix that represents the contraction M/e. For a subset A of E the contraction M/A is obtained by contracting each element of A one at a time. It is not hard to show that M/A is independent of the order in which the contractions are performed.

A matroid is a *direct sum* of matroids, $M = M_1 \oplus M_2$, if the ground set of M is the disjoint union of the ground sets E_1 and E_2 and a set A is independent in M if and only if $A \cap E_1 \in \mathcal{I}_1$ and $A \cap E_2 \in \mathcal{I}_2$. Note that any matroid M with a loop or coloop e can be decomposed as $(M - e) \oplus e$.

Every matroid has a rank function $r : \mathcal{P}(E) \to \mathbb{N}_0$ that maps a set to the cardinality of its maximal independent subsets. A *flat*, or closed set, of a matroid is a subset $F \subseteq E$ such that $\forall e \in E - F,\ r(F) = r(F \cup e) - 1$. A *hyperplane* H of a matroid is a flat of M such that $r(H) = r(E) - 1$. We will frequently use $r(M)$, or just r, for $r(E)$.

2.2 The Lattice of Flats

The flats of M form a *lattice* under inclusion which we will denote L_M. A lattice is a partially ordered set in which each pair of elements has a unique least upper bound and greatest lower bound. The lattice of flats of any matroid is *coatomic*, i.e. any flat of M can be realized as an intersection of hyperplanes. If F is a flat of M, then the interval $[F, E] = \{F' \in L_M : F \subseteq F' \subseteq E\}$ is isomorphic as a poset to $L_{M/F}$.

For any finite poset P the *order complex* of P, denoted $\Delta(P)$, is the simplicial complex whose vertices are the elements of P and whose faces are chains in P. Let $\widetilde{L_M}$ be L_M with its least element, the flat of all loops, and its greatest element, E, removed. The homotopy type of $\Delta(\widetilde{L_M})$ plays a key role in Section 5; it is easily computed using Theorem 3 below.

2.3 Tutte Invariants

The *Tutte Polynomial*, written $T(M; x, y)$, is a matroid invariant that behaves well with respect to deletion and contraction. It is defined as the unique two-variable polynomial satisfying the following recursion:

1) $T(\text{a single coloop}; x, y) = x$; $T(\text{a single loop}; x, y) = y$

2) If e is a loop or a coloop, then $T(M; x, y) = T(e; x, y)T(M/e; x, y)$

3) If e is neither a loop nor a coloop, then $T(M; x, y) = T(M - e; x, y) + T(M/e; x, y)$

It is sometimes preferable to replace (2) with the following: If $M = M_1 \oplus M_2$, then $T(M; x, y) = T(M_1; x, y)T(M_2; x, y)$. This definition is equivalent.

The Tutte polynomial is well-defined and unique for any matroid. See [3] for a proof and many more applications of this polynomial.

The Möbius function of a finite poset is the function $\mu : L \times L \to \mathbb{Z}$ that satisfies:

$\forall x, y, z \in L$, $\sum_{x \leq y \leq z} \mu(x, z) = \delta(x, z)$ and $\mu(x, z) = 0$ if $x \not\leq z$. As usual, δ denotes Kronecker's Delta Function.

For proofs regarding the existence and uniqueness of μ, see [17]. The Möbius function of a matroid is defined as $\mu(M) = \mu_{L_M}(\bar{\emptyset}, E)$ where μ_{L_M} is the standard Möbius function on the lattice of flats and $\bar{\emptyset}$ is the least element in the lattice of flats (which contains all loops of the matroid). When M has no loops the Möbius function of M is related to the Tutte polynomial via the equation $|\mu(M)| = T(M; 1, 0)$ [3].

The following theorem relating the Möbius invariant to the lattice of flats of a matroid will also be useful:

Theorem 3. *[1] The order complex $\Delta(\widetilde{L_M})$ is homotopy equivalent to a wedge of $\mu(M)$ spheres all of which have dimension $r(M) - 2$.*

CHAPTER 3

THE MATROID ASSOCIATED TO THE ACTION

Denote the n-torus by $T^r = T_1^1 \times T_2^1 \times \cdots T_r^1$. We will also use the decomposition of an odd-dimensional sphere into circles: $S^{2n-1} = S_1^1 * S_2^1 * \cdots * S_n^1$, where * denotes the topological join of spaces. In the interests of notational brevity, we will leave out the repeated superscript "1" when referring to the circles in either decomposition. Given any linear action of T^r on an even-dimensional sphere there is a pair of antipodal points which are fixed by the action. Hence the quotient space is the suspension of a linear action of T^r on an odd-dimensional sphere. As all of our questions of interest are easily answered for suspensions, we will henceforth assume that the sphere is odd-dimensional.

We wish to study an effective linear actions $T^r \curvearrowright S^{2n-1}$ and the resulting quotient space $X = S^{2n-1}/T^r$. We associate to each such action an $r \times n$ matrix $Z = (z_{ij})$ as follows: Since T^r consists of commuting $n \times n$ orthogonal matrices we can simultaneously diagonalize all of the elements of T^r over the complexes with diagonal entries in the unit circle. Equivalently, T^r is conjugate in $SO(n)$ to a torus such that each $e^{\sqrt{-1}\,\theta} \in T_i$ acts on $e^{\sqrt{-1}\beta} \in S_j$ by $e^{\sqrt{-1}\,\theta} \cdot e^{\sqrt{-1}\,\beta} = e^{\sqrt{-1}(z_{ij}\theta+\beta)},\ z_{ij} \in \mathbb{Z}$. As conjugate tori give isometric quotient spaces, we will assume that T^r is presented in this form.

Lemma 4. *Performing any combination of the following integer matrix operations on Z does not affect the isometry type of the corresponding quotient space.*

1. *Reordering the rows of Z*
2. *Reordering the columns of Z*
3. *Multiplying any row by ± 1*
4. *Multiplying a column by ± 1.*

5. *Adding a multiple of one row to another*

Proof.

1.) Switching two rows is equivalent to changing the order of the circles in the chosen basis $T^r = T_1 \times \cdots \times T_r$.

2.) Column switching is equivalent to changing the order of the circles chosen in the join $S^{2n-1} = S_1 * \cdots * S_n$.

3.) This corresponds to the choice of a preferred orientation for the circles of the torus.

4.) This corresponds to the choice of a preferred orientation for the circles of the sphere.

5.) Let Z_i and Z_j be rows of the matrix. If we replace Z_j with $Z_j + cZ_i$, then the action $T^r \curvearrowright S^{2n-1}$ corresponding to the new matrix will be the action obtained by precomposing the original action with the group isomorphism $\phi : T^r \to T^r$ determined by the elementary matrix which is diagonal except for the ji entry which is c. $\square$

We note that if $c \in \mathbb{Z}$ divides an entire row, say row i, then the action is not effective as it has a kernel of the c-th roots of unity of T_i. However, the quotient is isometric to the orbit space of $T^r/\mathbb{Z}_c \cong T^r$, where $\mathbb{Z}_c$ acts trivially except on T_i. We therefore allow division of an entire row in the matrix by c, provided that c divides all of its entries.

As observed previously, there is a natural matroid associated to Z which we denote by M_Z. The ground set of M_Z is the columns of Z, and the independent subsets of M_Z are the linearly independent subsets of columns. An equivalent method for determining M_Z is via representation theory. The *real* irreducible representations of S^1 are isomorphic to $\mathbb{Z}/\pm 1$. So we can write the representation $\rho : T^r \to SO(n)$ given by the action as a direct sum $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ where each

$\rho_i \in (\mathbb{Z})^r / \pm 1$. This means that $\{\rho_1, \ldots, \rho_n\}$ has a matroid structure given by viewing each ρ_i as a vector in $\mathbb{Q}^r$ determined up to sign. It is not hard to see that this matroid is M_Z. Thus M_Z only depends on the action $T^r \curvearrowright S^{n-1}$, not on the chosen diagonalization. In fact, we will write M_X for this matroid. While we prefer to use matroid notation for its simplicity, it is important to keep in mind that our matroids have matrix representations derived from the action. Furthermore, when we refer to $M_X - e_j$ or M_X/e_j, we assume that there is a preferred class of representative matrices for these matroids. In particular we will use X_M to refer to a quotient space even though the matroid (without a particular representation) does not determine the quotient space up to isometry. For instance, if $Z_1 = \begin{bmatrix} 2 & 3 \end{bmatrix}$ and $Z_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}$, then the corresponding quotient spaces are non isometric two-spheres. For an example where one quotient space is $\mathbb{C}P^n$ and the other is not even a manifold, see Section 6.

CHAPTER 4

$\mathbf{H_*(X)}$ FOR QUOTIENTS BY $\mathbf{T^R}$

4.1 X as a mapping cone

Let M_X be a matroid corresponding to a quotient space X. Not surprisingly, it is possible to extract a variety of geometric and/or topological data from X through the matroid structure of M_X.

Proposition 5. *Let $X = S^{2n-1}/T^r$ and let M_X be the corresponding matroid. If M_X contains a loop e_j, then $X = S_j * X_{M-e_j}$.*

Note that $X = S_j * X_{M-e_j}$ means that X is *isometric* to the given (spherical) join.

Proof. If e_j is a loop, then the j^{th} column of any matrix representation of M_X is the zero vector. This implies that T_i fixes S_j for all i. Since S_j is fixed by the action of T^r, $X_M = S_j * X_{M-e_j}$. □

Now let us consider the situation when e_j is not a loop. Let x be a point of S_j. We will denote the stabilizer of x in T^r by T^r_x. We can decompose the quotient map on the sphere induced by the action of T^r into two parts: $f : S^{2n-1} \twoheadrightarrow S^{2n-1}/T^r_x$ and $g : S^{2n-1}/T^r_x \twoheadrightarrow S^{2n-1}/T^r$. Evidently g is just the quotient map for the action of T^r/T^r_x on $f(S^{2n-1})$. Then $g \circ f$ is the projection from S^{2n-1} to X. The entire circle S_j is fixed by f, and g identifies all of S_j to a single point $\bar{x}$. Define R_x to be the quotient of the action of T^r restricted to the $(2n-3)$-dimensional sphere $(S_1 * \cdots * \hat{S}_j * \cdots * S_n)$. As T^r respects the join decomposition of S^{2n-1} every point $\bar{y} \neq \bar{x}$ in X, but not in R_x, lies on a unique minimal geodesic from $\bar{x}$ to R_x. The minimal geodesics in X with initial value $\bar{x}$ are parameterized by $(S_1 * \cdots * \hat{S}_j * \cdots * S_n)/T^r_x$. This quotient space is usually called the space of directions of X at x and we denote it by N_x. All of the minimal geodesics from

$\bar{x}$ to R_x have length $\pi/2$. The above discussion shows that X is (homeomorphic to) the mapping cone of $g : N_x \to R_x$ with cone point $\bar{x}$. As with any mapping cone, there is an associated Mayer-Vietoris sequence.

$$\cdots \to \tilde{H}_i(R_x) \to \tilde{H}_i(X) \xrightarrow{\partial} \tilde{H}_{i-1}(N_x) \to \dots . \tag{4.1}$$

Proposition 6. *Let $X = S^{2n-1}/T^r$ and let M_X be the corresponding matroid. If M_X contains a coloop, then X is a cone.*

Proof. Let e_j be a coloop of M. Then we may row reduce the representative matrix of M using the Euclidean algorithm so that e_j is the jth column, this column contains only one nonzero entry, and that entry is in position ij. In addition, the i-th row is zero except for ij. Since the action is effective this entry must be plus or minus one. With the matrix in this form, it is clear that $T^r_x \oplus < T_i >= T^r$ for any $x \in S_j$. Hence for this x the map which determines the mapping cone structure of X is the identity. □

The above results already make it easy to compute $\pi_1(X)$. If $n = 1$, then X is homeomorphic to a circle or a point. In all other cases, X is simply connected.

Theorem 7. *If $n \geq 2$, then X is simply connected.*

Proof. If $e_1 \in M_X$ is a loop or coloop, then Propositions 5 and 6 immediately imply X is simply connected. So assume that e_1 is neither a loop nor a coloop. For the base case $n = 2$, the only remaining possibility is that $Z = [a_1 \ a_2]$ with both entries nonzero. This implies X is homeomorphic to $\mathbb{C}P^1$ and hence simply connected (see the proof of Proposition 16). For the induction step, the mapping cone presentation of X shows that X is the union of two simply connected open subsets whose intersection is connected. Apply Siefert-van Kampen. □

4.2 The Reduced Poincaré Polynomial of X

In this section we prove that $H_i(X;\mathbb{Z})$ is a free abelian group for all i and that the integral reduced Poincaré polynomial

$$\tilde{\mathbb{P}}(X,t) = \sum \operatorname{rk} \tilde{H}_i(X,\mathbb{Z})\, t^i$$

equals $t^{r-1}T(M_X;0,t^2)$. Our strategy is to use induction on n, the recursion which characterizes the Tutte polynomial, and the long exact sequence (4.1). If M_X contains a coloop, then Proposition 6 works well. However, if M_X does not contain a coloop, then an immediate obstacle to induction is that N_x may not be a quotient of a sphere by a real torus.

Let $x \in S_j$. Recall that $N_x \cong S^{2n-3}/T^r_x$, so we wish to better understand the structure of the stabilizer T^r_x. By Lemma 4 we can use the Euclidean algorithm to row reduce a representative matrix of M_X so that there is only one nonzero entry in column j, let us say it is in row i. If this ij^{th} entry is a one, then $T^r_x \cong T_1 \times \cdots \times \hat{T}_j \times \cdots \times T_r$. If the entry is some $a \neq 1$, then a is the gcd of column j. Hence, $T^r_x \cong T_1 \times \cdots \times \hat{T}_j \times \cdots \times T_r \times \mathbb{Z}_a$, where $\mathbb{Z}_a$ is the cyclic group $\mathbb{Z}/a\mathbb{Z}$. This demonstrates that $N_x \cong S^{2n-3}/T^r_x$ where $T^r_x \cong T^{r-1} \times \mathbb{Z}_a$ for some $a \in \mathbb{N}$. We can break up this action into two parts: let $\hat{N}_x \cong S^{2n-3}/T^{r-1}$ so that $N_x = \hat{N}_x/\mathbb{Z}_a$ and the matroid corresponding to $\hat{N}_x$ is M_X/e_j.

We wish to show that this extra quotient by a finite group does not affect the rational homology of $\hat{N}_x$. In order to do so, we require more information about the local structure of N_x and $\hat{N}_x$.

An *absolute neighborhood retract* (ANR) is a topological space Y with the property that for every normal space Z that embeds in Y as a closed subset, there exists an open set U in Y such that $Z \subset U \subset Y$ and Z is a retract of U. Details regarding these structures can be found in [4].

Lemma 8. *N_x and $\hat{N}_x$ are both ANRs.*

Proof. We say that an action has *finite type* if there are only a finite number of conjugacy classes of isotropy subgroups. It is shown in Conner [4] that if Γ is a compact abelian Lie group acting on a compact connected finite dimensional ANR X, and the action is of finite type, then the orbit space X/Γ is an ANR. This result also applies to all finite abelian groups Γ. It is well known that every sphere is an ANR. It remains to be shown that the linear action of T^r on S^{2n-1} has finite type. By the definition of the action, all the points x on any given invariant circle S_j have the same isotropy group T^r_x. If $x \in S^{2n-1}$ does not lie on an invariant circle, then there is some minimal subset of circles $\{S_{i_k}\}_{k=1}^m$ whose join in S^{2n-1} contains x. By choosing points $y_{i_k} \in S_{i_k}$, we see that $T^r_x = \bigcap T^r_{y_{i_k}}$. This formulation demonstrates that the toral action can only have a finite number of distinct isotropy groups and is thus of finite type. □

Lemma 9. *Suppose a finite abelian group G acts on $\hat{N}_x$. Let F be a field of characteristic 0 or of characteristic prime to the order of G. Then $H_n(\hat{N}_x/G; F) \cong [H_n(\hat{N}_x; F)]^G$, the group of invariant homology classes.*

Proof. For Čech cohomology, the lemma is a corollary of Theorem III.7.2 in Bredon's text on transformation groups [2] which states the result for more general quotient spaces. By the previous lemma, N_x and $\hat{N}_x$ are both ANRs. The lemma follows directly since singular homology and Čech cohomology are equivalent on ANRs. To see this fact, combine Theorem 1 of Milnor[10], the discussion of Čech cohomology on page 275 in Hatcher [7], and the Universal Coefficient Theorem. □

Proposition 10. $H_*(N_x;\mathbb{Z}) \cong H_*(\hat{N}_x;\mathbb{Z})$.

Proof. Let $\mathbf{k}$ be a field. If the characteristic of $\mathbf{k}$ is zero, then choose any column of Z. When the characteristic of $\mathbf{k}$ is positive, choose a column j so that the characteristic of $\mathbf{k}$ does not divide the gcd of the entries of the column. There is always such a column, otherwise the action would not be effective. Write $N_x = \hat{N}_x/\mathbb{Z}_{a_j}$ as above. The finite group $\mathbb{Z}_{a_j}$ is a subgroup of the *connected* group T_j which acts on N_x by isometries. Hence every element of $\mathbb{Z}_{a_j}$ acts on N_x by a map homotopic to the identity. Now, Lemma 9 shows that for any field $\mathbf{k}$, $H_*(N_x;\mathbf{k}) \cong H_*(\hat{N}_x;\mathbf{k})$. The universal coefficient theorem finishes the proof. □

With the main obstacle to induction out of the way we are ready to prove the main theorem of this section.

Theorem 11. *Let $X = S^{2n-1}/T^r$ be a quotient of an odd-dimensional sphere by an effective linear action. Then $H_*(X;\mathbb{Z})$ is a finitely generated torsion-free abelian group and*

$$\tilde{\mathbb{P}}(X,t) = \sum \mathrm{rk}\, \tilde{H}_i(X,\mathbb{Z})\, t^i = t^{r-1}\, T(M_X;0,t^2). \tag{4.2}$$

Proof. It is sufficient to prove (4.2) when using arbitrary field coefficients. So let $\mathbf{k}$ be a field (of any characteristic).

We proceed by induction on n. When n is one there are only two actions to consider. The circle acting on itself and the trivial action of $T^0 = \{id\}$ on the circle. The latter is an effective action in the sense that every nonidentity element of the group acts nontrivially! In both cases (4.2) is easily verified.

For the induction step there are three cases to consider: $e_j \in M_X$ is a coloop, loop, or neither. If e_j is a coloop, then Proposition 6 tells us that X is contractible, so $\tilde{\mathbb{P}}(X,t) = 0$, while Tutte recursion insures that $T(M_X;0,t^2) = 0$. When e_j is a loop, Proposition 5 implies that $X = S^1_j * X_{M_X - e_j}$. So the induction hypothesis insures that $\tilde{\mathbb{P}}(X,t) = t^2\,\tilde{\mathbb{P}}(X_{M_X-e_j}) = t^{r-1}\,t^2\,T(M_X - e_j; 0, t^2) = t^{r-1}\,T(M_X; 0, t^2)$.

So assume that e_1 is neither a loop nor a coloop. Then we have that $r(M - e_1) = r(M)$ and $r(M_X/e_1) = r(M_X) - 1$. Now consider the long exact sequence (4.1).

$$\ldots \tilde{H}_i(N_x;\mathbf{k}) \to \tilde{H}_i(R_x;\mathbf{k}) \to \tilde{H}_i(X;\mathbf{k}) \xrightarrow{\partial} \tilde{H}_{i-1}(N_x;\mathbf{k}) \to \tilde{H}_{i-1}(R_x;\mathbf{k}) \to \ldots$$

The induction hypothesis applied to R_x and N_x (via Proposition 10) implies that, depending on the parity of i, one of two things is happening. One, $\tilde{H}_i(R_x;\mathbf{k}) = 0$ and $\tilde{H}_{i-1}(N_x;\mathbf{k}) = 0$, in which case $\tilde{H}_i(X;\mathbf{k}) = 0$. Or, $\tilde{H}_i(N_x;\mathbf{k}) = 0$ and $\tilde{H}_{i-1}(R_x;\mathbf{k}) = 0$, in which case $\tilde{H}_i(X;\mathbf{k}) \cong \tilde{H}_i(R_x;\mathbf{k}) \oplus \tilde{H}_{i-1}(N_x;\mathbf{k})$. Combining these two possibilities with the induction hypothesis gives

$$\begin{aligned}\tilde{\mathbb{P}}(X,t) = \tilde{\mathbb{P}}(R_x,t) + t\,\tilde{\mathbb{P}}(N_x,t) &= t^{r-1}T(M_X - e_1; 0, t^2) + t^{r-1}T(M_X/e_1; 0, t^2) \\ &= t^{r-1}T(M_X; 0, t^2).\end{aligned}$$

□

The above formula raises two immediate questions. Since all of the homology groups are finitely generated free abelian, $H^i(X) \cong H_i(X)$ for every i.

Problem 1. *What is the ring structure of $H^*(X)$?*

If the dimension of X is odd, or the rank of M_X is greater than n, then all products in $H^*(X)$ must be trivial for purely dimensional reasons. When M_X is rank one without loops X is a weighted projective space. (See Section 6.) In that special case the ring structure of the cohomology ring of X was determined in [8]. We do not know of any other case with nontrivial products.

As the homology of X vanishes in every other degree, it is natural to ask whether or not the following holds.

Problem 2. *Is there a CW-decomposition of X so that all boundary maps are zero?*

If so, one might hope that Tutte's theory of basis activity for graphs [15], extended to matroids by Crapo [5], might be realized with a natural bijection between the cells of the CW-structure and the bases of M_X with internal activity zero.

CHAPTER 5

THE SINGULAR SET

Given a quotient space $X = Y/T^r$, the *rational singular set* of the action is the image in the quotient space of the points of Y whose isotropy subgroups are infinite subgroups of T^r. We will denote the rational singular set of the quotient S^{2n-1}/T^r by $\mathcal{S}$ and determine its homotopy type.

Let $A = \{e_{i_1}, \ldots, e_{i_k}\} \subseteq M_X$. Define $S^A = \{(x_1, x_2, \ldots, x_{2n-1}) \in S^{2n-1} : x_{2i-1} = x_{2i} = 0 \text{ for all } e_i \notin A.\}$. Equivalently, S^A is the join $S_{i_1} * \cdots * S_{i_k}$ in S^{2n-1}. For $x \in S^{2n-1}$ set A_x to be the minimal A such that $x \in S^A$.

Any $t \in T^r_x$ must fix all of S^{A_x}. Suppose A_x is a spanning subset of M_X. Then any square submatrix of Z whose columns span and are contained in A_x can be diagonalized over the integers with nonzero diagonal entries $\{c_1, \ldots, c_r\}$ by the elementary row operations covered by Lemma 4. This implies T^r_x is contained in a subgroup of T^r isomorphic to $\mathbb{Z}_{c_1} \oplus \cdots \oplus \mathbb{Z}_{c_r}$ and hence is finite. So, for any $x \in \mathcal{S}$ we see that A_x is a nonspanning subset and hence contained in a hyperplane H of M_X.

Conversely, suppose H is a hyperplane of M_X. In the column space of Z, H is the intersection of the columns of Z with a rational hyperplane. This hyperplane is perpendicular to an integer vector. Thus there is an element γ of the row space which is an integral linear combination of the rows of Z such that the zeros of γ correspond to the columns in H. Therefore, there is an element of T^r of infinite order which fixes S^H. As a result, we now know that the preimage of the rationally singular set is the union of all S^H, H a hyperplane of M_X. Each S^H is a sphere of dimension $2|H| - 1$. Hence the image of S^H in X is of dimension $2|H| - 1 - r(H) = 2|H| - 1 - (r(M) - 1) = 2|H| - r(M)$.

We define an *arrangement* as a finite collection $\mathcal{A} = \{A_1, \ldots, A_m\}$ of closed subspaces of a topological space U such that:

i) $A, B \in \mathcal{A}$ implies that $A \cap B$ is a union of spaces in $\mathcal{A}$

ii) If $A, B \in \mathcal{A}$ and $A \subseteq B$, then the inclusion map $A \hookrightarrow B$ is a cofribration.

Given $A \subseteq M_X$ define X_A to be $g \circ f(S^A)$. Let $\mathcal{A}$ be the set generated by $\{X_H : H \text{ is a hyperplane of } M_X\}$ and all of its intersections, including the empty set if this is the intersection of all the $g \circ f(H)$. Now let P be the poset whose elements are the sets in $\mathcal{A}$, ordered by reverse inclusion. The X_H in $\mathcal{S}$ are the minimal elements of P. Furthermore, the elements of P corresponds to flats of the matroid M_X. In fact, P is isomorphic to $(L_{M_X})^*$, the *order* dual of the lattice of flats of M with the maximal element $\hat{1}$ corresponding to M_X removed. In other words, P is the poset of flats of M_X, other than M_X, ordered by reverse inclusion.

Example: Let the matrix corresponding to an orbit space S^5/T^2 be $Z = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ We see that γ_1 fixes S_3, γ_2 fixes S_1 and $\gamma_1 - \gamma_2$ fixes S_2. Each of these circles is a single point in the quotient space, and the union of these three points constitutes the singular set of the action. The matroid represented by Z is $U_{2,3}$, which itself has 3 flats.

Proposition 12. *If X_F, X_G are elements of the arrangement $\mathcal{A}$ and $X_F > X_G$, then the inclusion map $X_F \hookrightarrow X_G$ is homotopic to the constant map.*

Proof. Let $c \in S^{2n-1}$ be a point on an invariant circle S_j such that e_j is in the flat G, but not F. By Proposition 6 $X_{F\cup\{e_j\}} \subseteq X_G$ in X is a cone with base X_F in X_G. $\square$

The above proposition means that we can use the wedge lemma from [18] to compute the homotopy type of $\mathcal{S}$.

Theorem 13. *The singular set $\mathcal{S}$ is homotopy equivalent to*

$$\bigvee_{\substack{F\in L_{M_X}\\ F\neq E}} X_F * \vee_{i=1}^{\mu(M_X/F)} S^{r-r(F)-2}.$$

Proof. By the wedge lemma in [18], $\mathcal{S}$ is homotopy equivalent to

$$\bigvee_{X_F\in P} X_F * \Delta(P_{<X_F}),$$

where $P_{<X_F}$ is the subposet of P consisting of all elements of P strictly less than X_F. By definition this is the order dual of the interval $[F,E]$ in L_{M_X} with F and E removed. Since the order complex of a poset and its order dual are isomorphic, the result now follows from the fact that $[F,E]\cong L(M_X/F)$ and Theorem 3. □

With the homotopy type of singular set in hand, it is easy to compute the reduced Poincaré polynomial of $\mathcal{S}$.

Theorem 14. *The reduced Poincaré polynomial of the singular set of the action with integral coefficients is given by* $\tilde{\mathbb{P}}(\mathcal{S},t) = t^{r(M)-2}[T(M;1,t^2) - T(M;0,t^2)]$.

Proof. By the previous theorem, Theorem 11 and the results of Section 4,

$$\begin{aligned}\tilde{\mathbb{P}}(\mathcal{S},t) &= \sum_{\substack{F\in L_{M_X}\\ F\neq E}} \tilde{\mathbb{P}}(X_F * \vee_{i=1}^{\mu(M_X/F)} S^{r-r(F)-2}, t)\\ &= \sum_{\substack{F\in L_{M_X}\\ F\neq E}} t^{r-2}\,\mu(M_X/F)\,T(F;0,t^2), \qquad (5.1)\end{aligned}$$

The last equality uses the usual computation of $\tilde{\mathbb{P}}$ for the join of a space and a wedge of spheres of the same dimension via the Künneth theorem.

Recall from the end of Section 2.3 that $\mu(M_X/F) = T(M_X/F;1,0)$ whenever M_X has no loops. This is the case here, since for any flat F of any matroid M, M/F has no loops. The following convolution formula of Kook, Reiner Stanton will be of use:

Theorem [9]: The Tutte polynomial satisfies $T(M; x, y) = \sum_{F \subseteq L_M} T(M/F; x, 0)T(F; 0, y)$

The special case of this formula $T(M; 1, t^2) = \sum_{F \in L_M} T(M/F; 1, 0) \cdot T(F; 0, t^2)$. only differs from (5.1) by the inclusion of a term corresponding to $F = E$. We can therefore rewrite our formula for the Poincaré polynomial in terms of $T(M; 1, t^2)$ by subtracting this extra term. $\square$

CHAPTER 6

MANIFOLDS

One natural question to ask is, "When is X a topological manifold?"

Proposition 15. *: The Tutte polynomial specialization $T(M_X; 0, t^2)$ is always of the form*

$$t^{2(n-r)} + b_{n-r-1}t^{2(n-r-1)} + \cdots + b_1 t^2,$$

with $b_i \in \mathbb{Z}_{\geq 0}$. Furthermore, if $b_i > 0$ and $i < n$, then $b_{i+1} > 0$.

Proof. This can be shown by a deletion and contraction argument as follows: We will cite several known properties of the Tutte polynomial which can be found in [15]. For example, it is known that $\tilde{b}_{0j} = 0$ for all $j > n - r$ where $\tilde{b}_{0j}$ denotes the coefficient of y^j in $T(M; 0, t)$, that the maximal degree of y in the Tutte polynomial is the nullity, and that the coefficient of this maximal degree term is one. This completes the proof of the form. For the second statement, we use induction: let M be a matroid of size n and assume the statement holds for all smaller matroids. If M contains a coloop, $T(M; 0, t^2) = 0$. If M contains a loop e, then $T(M; 0, t^2) = t^2 T(M - e; 0, t^2)$, so the property holds. If $e \in M$ is neither a loop nor a coloop, then $T(M; 0, t^2) = T(M/e; 0, t^2) + T(M - e; , 0, t^2)$. Denote the coefficients of the Tutte polynomials of M/e by a_i and $M - e$ by c_i. If $b_i > 0$, then either $a_i > 0$ or $c_i > 0$. By the induction hypothesis, either $a_{i+1} > 0$ or $c_{i+1} > 0$. Either case forces the coefficient of $t^{2(i+1)}$ on the left-hand side to be greater than zero.

□

This fact, Poincaré duality, and our formula for $H_*(X; \mathbb{Q})$, imply that there are only two potential situations for X to be a manifold (without boundary). If $r = 1$, then M_X is the rank one uniform matroid, where a subset of M_X is independent if and only if it has cardinality one or zero. Otherwise, we have $T(M_X; 0, t^2) = t^{2(n-r)}$. If M_X does not meet one of these two criteria, then the $(2n-1-r)$-dimensional X has $H_{2n-r-3}(X;\mathbb{Q}) \neq 0$ and $H_2(X;\mathbb{Q}) = 0$, preventing the orbit space from satisfying Poincaré duality.

6.1 $r = 1$

In this case $Z = [a_1 a_2 \dots a_n]$ with all $a_i \neq 0$. These quotient spaces have been studied under the names twisted projective spaces or weighted projective spaces. By using Lemma 4 we can further simplify and assume all the a_i are positive and $a_1 \geq a_2 \geq \dots \geq a_n$. For instance, if $Z = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$, then X is $\mathbb{C}P^n$. As we will see below, when $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$, the quotient X is always homeomorphic to S^2. On the other hand, consider $Z = \begin{bmatrix} 3 & 1 & \dots & 1 \end{bmatrix}$. Let $x \in S_1$. Then $T_x \cong \mathbb{Z}_3$ which acts on all $S_j, j \neq 1$, by rotation. Hence N_x is a lens space and excision shows $H_*(X, X - \{\bar{x}\})$ is not isomorphic to the homology of a sphere. Thus X cannot be a manifold.

The necessity portion of Proposition 16 can easily be obtained by applying [8, Theorem 1]. However, we give a direct proof here.

Proposition 16. *If $Z = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$ with all $a_1 \geq a_2 \geq \dots \geq a_n > 0$, then X is a manifold if and only if $n = 2$, or $a_1 = a_2 = \dots = a_{n-1}$ and $a_n = 1$. Furthermore, if X is a manifold, then X is homeomorphic to $\mathbb{C}P^n$.*

Proof. Suppose $n = 2$. Consider the mapping cone structure of X. Since $a_2 \neq 0$, R_x is a point. On the other hand, $N_x = S^1/\mathbb{Z}_{a_1}$ and hence homeomorphic to the circle. So X is homeomorphic to the mapping cone of the circle mapped to a point. Thus X is homeomorphic to $\mathbb{C}P^1$.

Now we assume $n \geq 3$. Let $x \in S_1$. Then $T_x = \mathbb{Z}_{a_1}$ and T_x acts nontrivially on every circle S_j with $a_j < a_i$. The homology of N_x, and by excision, the pair $(X, X - \{\bar{x}\})$, will not be that of a sphere unless the action of T_x is trivial on all the $S_i, i \neq n$ [16]. Thus $a_1 = a_2 = \cdots = a_{n-1}$. To see that $a_n = 1$, suppose $a_n > 1$. Let $z \in S_n$, so $T_z = \mathbb{Z}_{a_n}$. If a_n divides all of the other a_i, then the action is not effective. If it does not, the T_z acts nontrivially on all of the other circles and the usual excision argument shows that X cannot be a manifold.

Lastly, we have to show that if $Z = \begin{bmatrix} a_1 & \dots & a_1 & 1 \end{bmatrix}$, then X is homeomorphic to $\mathbb{C}P^n$. Break up the action $T \curvearrowright S^{2n-1}$ into two parts. First quotient out by $\mathbb{Z}_{a_1}$. This subgroup acts trivially on all the circles except S_n. Hence it leaves a quotient space $\bar{X}$ homeomorphic to S^{2n-1}. Now act on $\bar{X}$ by T/T_{a_1}. This group is also a rank one torus and this action is equivalent to $\bar{Z} = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$. Therefore, X is homeomorphic to $\mathbb{C}P^n$. $\square$

6.2 Spheres

When is $T(M_X; 0, t^2) = t^{2(n-r)}$? Obviously this is the same as when $T(M_X; 0, t) = t^{n-r}$.

Proposition 17. *Let M be a rank r matroid with n elements. Then $T(M; 0, t) = t^{n-r}$ if and only if M is a direct sum of circuits.*

Proof. Let C_l denote the $l - circuit$. C_l contains no loops or coloops. Deleting any edge $e \in C_l$ yields a set of coloops, while contracting $e \in C_l$ yields C_{l-1}. This demonstrates that $T(C_l; 0, t) = T(C_{l-1}; 0, t) = t$. If $M = C_{i_1} \oplus C_{i_2} \oplus \cdots \oplus C_{i_m}$. Then $T(M; 0, t) = T(C_{i_1}; 0, t)T(C_{i_2}; 0, t) \dots T(C_{i_m}; 0, t) = y^m$. The highest degree of y in $T(M; 0, t)$ polynomial is always the nullity of the matroid M [15], thus $m = n - r$

Now we let M be a matroid such that $|M| = n$ and $T(M; 0, t) = t^{n-r}$. We assume by way of induction that the statement holds for all matroids with a smaller ground/edge set. Clearly, M contains no coloops or $T(M; 0, t)$ would be zero. If M contains a loop, then its removal yields a smaller matroid of rank r with the Tutte polynomial t^{n-r-1}. By our assumption, $M-e$ is a direct sum of $n-r-1$ circuits, so M is a direct sum of $n-r$ circuits. Suppose M contains no loops or coloops. We have that M can be uniquely decomposed into $n-r$ connected components, since this is the maximum j such that $b_j > 0$[15]. Since there are no loops or coloops, each component has nonzero nullity. This demonstrates that each of these $n-r$ components has a nonzero Tutte polynomial, and the product of these Tutte polynomials is t^{n-r}. We conclude that each component has the Tutte polynomial y, which is characteristic of circuits. $\square$

As noted in the introduction, one of the obvious questions when considering linear quotients of spheres is, "When is the quotient space homeomorphic to a sphere?" For real tori the answer is, at least in the language of matroids, essentially the same as for $\mathbb{Z}_2$-tori [13, Theorem 4]. In preparation for this result we consider what happens when M_X is a direct sum of smaller matroids.

Suppose $M_X = M_1 \bigoplus \cdots \bigoplus M_l$. Let n_i be the cardinality of M_i, and r_i the rank of M_i. So $\sum n_i = n$ and $\sum r_i = r$. Then it is possible, after applying Lemma 4, to write Z in block diagonal form with l blocks of size $r_i \times n_i$. Denote the blocks by Z_i. Each Z_i corresponds to a quotient space $X_i = S^{2n_i-1}/T^{r_i}$. Now it is possible to write $T^r = T^{r_1} \times \cdots \times T^{r_l}$, and $S^{2n-1} = S^{2n_1-2} * \cdots * S^{2n_l-1}$ so that each T^{r_i} acts trivially on every S^{2n_j-1} when $i \neq j$. From these decompositions the following proposition is clear.

Proposition 18. *Suppose $M_X = M_1 \bigoplus \cdots \bigoplus M_l$ with notation as above. Then $X =$*

$X_1 * \cdots * X_l$.

Theorem 19. *The following are equivalent.*

1. *M_X is a direct sum of circuits.*
2. *X is homeomorphic to a sphere.*
3. *X is an integral homology sphere.*

Proof. Obviously (2) implies (3). The implication (3) implies (1) follows immediately from Proposition 17 and our formula for the homology of X. So it remains to prove (1) implies (2). Our first simplification is to observe that since joins of spheres are spheres, Proposition 18 shows that it is sufficient to prove that if M_X is a circuit, then X is homeomorphic to a sphere. If $n = 1$, then $r = 0$ and $X = S^1$. So from here on we assume that $r + 1 = n \geq 2$ and M_X is a circuit. This implies that Z can be row reduced to something of the form

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 & b_1 \\ 0 & a_2 & 0 & \dots & 0 & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_{n-1} \end{bmatrix}$$

with all a_i and b_i nonzero.

Our strategy here is simple: prove that X is a simply connected compact manifold with the homology of a sphere. We have already seen that X is simply connected (Theorem 7) and that it has the homology of a sphere. So it remains to show that X is a manifold. We will do this by induction on n. The base case $n = 2$ was discussed in the proof of Proposition 16.

Let $\bar{x} \in X$ and x be any preimage of $\bar{x}$ in S^{2n-1}. Now let N_x be the unit tangent vectors in the tangent space of x which are orthogonal to $T^r x$, the orbit of x. Since small metric neighborhood of $\bar{x}$ are homeomorphic to a cone over N_x/T^r_x, it is sufficient to prove that this quotient space is homeomorphic to S^{n-1}.

As before, for $x \in S^{2n-1}$ let A_x be the minimal nonempty subset of M_X such that $x \in S^{A_x}$. If $A_x = M_X$, then x is in the principle isotropy group of the torus and $\bar{x}$ is a manifold point. So we can assume that $A_x \neq M_X$. For notational convenience, we can also assume that $e_n \notin A_x$ by reordering the columns if necessary. Since $T^r \cdot S^{A_x} \subseteq S^{A_x},\ T^r x \subseteq S^{A_x}$. Define three subspaces of the tangent space of x as follows: $\mathcal{T}$ are the vectors tangent to $T^r x,\ \mathcal{O}$ are vectors tangent to S^{A_x}, but orthogonal to $T^r x$, and $\mathcal{N}$ are those orthogonal to S^{A_x}, and thus also orthogonal to T^r_x. Then the tangent space at x is $\mathcal{T} \oplus \mathcal{O} \oplus \mathcal{N}$. In terms of this decomposition, N_x are the unit vectors in $\mathcal{O} \oplus \mathcal{N}$. The form of Z implies that the rank T^r_x is $|A_x|$, unless $A_x = M_X$, in which case it is only $n - 1 = r$. By construction T^r_x acts trivially on $\mathcal{O}$. Thus we have reduced the problem to showing that $\tilde{N}_x = S^{M_X - A_x}/T^r_x$ is homeomorphic to a sphere, where $\tilde{N}$ are the unit vectors in $\mathcal{N}$. The action of T^r_x on $S^{M_X - A_x}$ is the induced action of T^r. This is most easily seen by representing the unit vectors by minimal geodesics beginning at x and ending in $S^{M_X - A_x}$.

As we have seen, T^r_x is a direct sum of a finite group and a torus.

$$T^r_x = \bigoplus_{i \in A_X} \mathbb{Z}_{a_i} \oplus \bigoplus_{i \notin A_x} T_i.$$

First we quotient out by the finite left-hand summand. The action of each cyclic subgroup in this summand on S^{2n-1} is trivial except for S_n where it acts by rotation by $2\pi/a_i$. Hence, after quotienting out by the finite left-hand summand we are left with S^{2n-1} with the same join decomposition as before (except S_n is smaller) and an action of $\tilde{T}^{n-|A_x|}$ whose associated matrix is the same form as before, but of smaller size and all of the $a_i = 1$. Finally, we can apply the inductive hypothesis to see that this quotient space is homeomorphic to a sphere.

□

CHAPTER 7

QUOTIENTS BY FINITE GROUPS

We now turn our attention to quotient spaces formed by finite abelian groups Γ that act effectively and preserve orientation. For a study of the actions of $(\mathbb{Z}_2)^r$, including those that are orientation reversing, see [14]. We continue to only consider quotients of odd-dimensional spheres, since representation theory demonstrates that quotients of even dimensional spheres remain suspensions of the odd case. These quotient spaces have been studied previously where Γ is a cyclic group in [16] and where $\Gamma \cong (\mathbb{Z}_p)^r$ for some prime p in [14]. For the duration of this section, we will assume that $|\Gamma|$ is a power of a prime p, and that the highest possible order of an element of Γ is p^k.

Let $\Gamma \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^{k_2}} \times \cdots \times \mathbb{Z}_{p^{k_r}}$, $k \geq k_2 \geq \cdots \geq k_r$ be a subgroup of $SO(2n)$ whose action on the sphere S^{2n-1} is effective. Note that there is a corresponding group action of the group $G \cong (\mathbb{Z}_{p^k})^r \subseteq SO(2n)$ which acts on S^{2n-1} with the same orbits, but only the action of the first generator is required to be effective. In particular, Γ is equivalent to G/K where K represents the kernel of the action. Let $X = S^{2n-1}/G = S^{2n-1}/\Gamma$

The consideration of G rather than Γ is one of notational convenience. We use this convention to define a matrix associated to the action whose entries are modulo p^k. We begin by choosing a preferred set of complex diagonal generators of G, $\gamma_1, \gamma_2, \ldots \gamma_r$, since elements of abelian linear groups are simultaneously diagonalizable. Each generator g_i acts on the circle S_j by the rotation $2\pi a_{ij}/p^k$ for some $0 \leq a_{ij} < p^k$. Let A_1 be the matrix formed by the entries $[a_{ij}]$. The entries of this matrix can be considered to be modulo p^k. Recall that we are assuming that the action of G is effective in the first coordinate. It may be the case that an entire row of A_1 is divisible by some power of p. This indicates that the basis element corresponding to this row has order less than p^k. We must recall while working with this matrix to not divide or multiply rows by any numbers not relatively prime to p, as this may alter the group action.

7.1 A Sequence of Matroids

With the exception of multiplication/division by multiples of p, row reductions on A_1 do not alter the structure of the corresponding quotient space. One might assume that associating a matroid to A_1 as we did in the torus case would be illuminating. While such a matroid is of use and will be defined below, it does not contain enough information to determine the homology of the quotient space X. We will in fact require a *sequence* of matroids. Intuitively, the "independence" properties of the generators $\gamma_1, ..., \gamma_r$ are not sufficient: we must also study the independence of $p\gamma_1, ..., p\gamma_r$, $p^2\gamma_1, ..., p^2\gamma_r$, etc. For example, it is possible that $p \cdot \gamma_1 = 0$, even if the generators γ_1 and γ_2 initially appear "independent" as in the example below.

Example 1: Consider the action of $(\mathbb{Z}_{p^2})^2$ by $\begin{bmatrix} 1 & 1 \\ 0 & p \end{bmatrix}$ versus the action by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. In both cases, the two columns appear to be independent, so we could naively assign the matroid $M_1 = U_{1,2}$ to both spaces. However, the latter matrix can be row reduced to the identity and the corresponding quotient is a sphere. The former has the homology of a $\mathbb{Z}_p$ lens space (see the proof of Theorem 32 for details). This demonstrates that the single matroid we have used in the past no longer suffices.

In order to define a sequence of matroids that captures the information we require, we begin by defining a sequence of matrices A_β where $1 < \beta \leq k$ from A_1 by taking the entries of A_1 modulo $p^{k+\beta-1}$. The matrices in this sequence have entries from $\mathbb{Z}_{p^\beta}$, which is not generally a field. It is not immediately apparent how to derive a matroid structure from these matrices. We resolve this question by defining a rank function on A_β. Let B be a subset of the columns of the matrix A_β. Now, consider these columns as elements of G: e.g. the column $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ would correspond to $(\gamma_1 + \gamma_2 + 2\gamma_3)$ in G. Let $< B >$ to be the subgroup generated by the elements of B in G. Note that $< B >$ is a $\mathbb{Z}$ -module. We can then define $rank(B) := dim_{\mathbb{Z}_p}(< B > \otimes \mathbb{Z}_p)$.

Intuitively, we wish to measure the number of generators required for $< B >$. In this tensor, we are considering $< B >$ *as a group* rather than a collection of vectors. So, for example, if $B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ in a matrix representing a $\mathbb{Z}_4 \times Z_4$ action, $r(B) = dim_{\mathbb{Z}_2}(< B > \otimes \mathbb{Z}_2) = dim_{\mathbb{Z}_2}(\mathbb{Z}_2 \otimes \mathbb{Z}_2) = 1$. A similar rank function for infinite groups was introduced in [6].

Proposition 20. *The rank function on the columns of A_β described above defines a matroid M_β on the columns of A_β*

Proof. We must demonstrate that this function satisfies the axioms of a matroid rank function:

1) $r(B) \leq |B|$

2) If $B \subseteq C$, then $r(B) \leq r(C)$

3) $r(B \cup C) + r(B \cap C) \leq r(B) + r(C)$

1) Let $b_1, \ldots, b_m$ be all the elements of B. Then images of $b_1, \ldots, b_m$ in $< B > \otimes \mathbb{Z}_p$ form a spanning set of $< B > \otimes \mathbb{Z}_p$. Thus, $\dim_{\mathbb{Z}_p}(< B > \otimes \mathbb{Z}_p) \leq m$

2) If $B \subseteq C$, then it is clear that $(< B > \otimes \mathbb{Z}_p) \subseteq (< C > \otimes \mathbb{Z}_p)$, so $r(B) \leq r(C)$

3) Let $\tilde{B} = Span(< B > \otimes \mathbb{Z}_p) \subseteq \mathbb{Z}_p^r \cong G \otimes \mathbb{Z}_p$, and let $\tilde{C} = Span(< C > \otimes \mathbb{Z}_p) \subseteq \mathbb{Z}_p^r \cong G \otimes Z_p$. We have that $\dim(\tilde{B} + \tilde{C}) + \dim(\tilde{B} \cap \tilde{C}) = \dim(\tilde{B}) + \dim(\tilde{C})$ since these are subspaces of a vector space. Note that $\dim(\tilde{B}) = r(B)$, $\dim(\tilde{C}) = r(C)$, and $\dim(\tilde{B} + \tilde{C}) = r(B \cup C)$. We can see that $< B \cap C > \subsetneq < B > \cap < C >$, so $r(B \cap C) \leq \dim(\tilde{B} \cap \tilde{C})$. This inequality is strict in many cases: we recall that B and C are subsets of the columns, not subgroups, so their intersection could be empty even if $\tilde{B}$ and $\tilde{C}$ both span $G \otimes \mathbb{Z}_p$. Combining these facts with the vector space equality proves that the rank function is semimodular.

$\square$

Example: Consider the $\mathbb{Z}_4 \times \mathbb{Z}_4$ action on S^7 where $A_1 = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ with coefficients in $\mathbb{Z}_4$. Then $A_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ with coefficients in $\mathbb{Z}_2$. The corresponding matroid M_1 is $U_{2,4}$, whereas M_2 has two parallel edges. It is interesting to note that the four-point line would make an appearance when the only prime acting is 2. We could say that $U_{2,4}$ is 'representable' over $\mathbb{Z}_4$. The strong relationship between the Tutte polynomial of $U_{2,4}$ and the homology of this quotient space will be discussed later.

7.2 The Geometry of X

We wish to describe a cellular structure on the orbit space X. We begin by placing a simplicial structure on S^{2n-1}. Recall that S^{2n-1} is the join of circles: $S^{2n-1} \cong S_1 * \cdots * S_n$. Begin with a zero-skeleton formed by p^k equidistant vertices on each circle S_i of the join; name these vertices $v_{i1}, v_{i2} \dots v_{ip^k}$. Add the edges that connect two consecutive vertices on some S_i. This defines a structure on each individual S_i, the simplicial structure on S^{2n-1} is the join of these structures. We orient the cells lexicographically, negating the orientation for each appearance of an edge $(v_{i(p^k-1)}, v_{i1})$.

As G acts by rotation on the invariant circles, it is clear that G acts cellularly on the structure described above. Therefore, there is an induced cellular structure on X. Our goal will be to express the elements of homology of X in terms of this inherited geometric structure.

Given a point x in an invariant circle S_j, we let N_x be the space of directions perpendicular to S_j at x. Note that $N_x \cong S^{2n-3}/G_x$ where $S^{2n-3} = S_1 * \cdots * \hat{S}_j * \dots S_n$. Recall that we can form the quotient space by successively applying the quotient maps g and f induced by the actions of G_x and G/G_x, respectively. The resulting quotient can be broken up into two components:
$U := \{f \circ g(y) \,|d(y, S_j) < 3\pi/4\}$ and $V := \{f \circ g(y) \,|d(y, S_j) > \pi/4\}$.
U is homotopic to the circle S_j, while V is homotopic to the restriction $R_x := (S_1 * \cdots * \hat{S}_j * \cdots * S_n)/G$.

Proposition 21. *$U \cap V \simeq S^1 \times (S_1 * \cdots * \hat{S}_j * \cdots * S_n)/G_x$. We will refer to this space as $S^1 \times N_x$ where N_x denotes the space of directions perpendicular to S_j at x.*

Proof. Let S_j be a chosen eigencircle of the action $G \curvearrowright S^{2n-1}$, and G_j the isotropy group of any point x on S_j.

Define $U_0 := \{y \in S^{2n-1} : d(S_j, y) < 2\pi/3\}$.

Define $V_0 := \{y \in S^{2n-1} : d(S_j, y) > \pi/3\}$.

Then $g(U_0)$, $g(V_0)$ are the images of U_0 and V_0 in the orbit space S^{2n-1}/G_j and $f \circ g(U_0)$, $f \circ g(V_0)$ are images of U_0, V_0 in the orbit space S^{2n-1}/G

Since the action of G_j fixes S_j, geodesics between S_j and S^{2n-3} are identified in S^{2n-1}/G_j if and only if their endpoints in S^{2n-3} are identified. Thus, $g(U_0) \cap g(V_0)$ is homotopic to a trivial fiber bundle over S_j with fibers $F = S^{2n-3}/(G_{j|S^{2n-3}})$.

Lemma 22. *: $f \circ g(U_0) \cap f \circ g(V_0)$ is homotopic to $S^1 \times S^{2n-3}/(G_{j|S^{2n-3}})$.*

Proof. First, we wish to understand the action of G/G_j further. Note that G/G_j acts by orientation preserving linear isometries on S_j, hence it acts by rotations. We claim that G/G_j is cyclic for any $1 \leq j \leq n$. In particular, let $a, b \in G/G_j$ such that $a \cdot S_j$ and $b \cdot S_j$ induce the same rotation on S_j. Then $ab^{-1} \in G_j$, and is therefore the identity element in G/G_j. We conclude that every element of G/G_j rotates S_j by a different amount. Let γ_j be the element of G/G_j the represents the least nontrivial counterclockwise rotation of S_j, in particular γ_j is rotation by $2\pi/k_j$ for some $k_j \in \mathbb{N}$. Then γ_j generates G/G_j.

Note that γ_j, as defined in the above proof, induces an isometry Φ_j on $S^{2n-3}/(G_{j|S^{2n-3}})$, simply by restricting its domain to this subspace. We claim that this Φ_j is homotopic to the identity map. To see this, let $\pi : S^{2n-3} \to S^{2n-3}/G_{j|S^{2n-3}}$ be the quotient map, and let $g \in G$ be in the coset $\gamma_j G_j$. We know that γ_j can be diagonalized over $\mathbb{C}$. So for some $k_1 \dots k_n \in \mathbb{N}$,

$g = \text{diag}(e^{\frac{2\pi i}{k_1}}, \ldots, e^{\frac{2\pi i}{k_n}})$. Define $H(t,x) : I \times S^{2n-3} \to S^{2n-3}$ by $H(t,x) \mapsto [\text{diag}(e^{\frac{2\pi i}{k_1 - t(k_1-1)}}, \ldots, \widehat{e^{\frac{2\pi i}{k_j - t(k_j-1)}}}, \ldots, e^{\frac{2\pi i}{k_n - t(k_n-1)}})] \cdot x$. Let $\overline{H}(t,x) = \pi \circ H_t \circ \pi^{-1}(x)$. Note that $\overline{H}(x,t)$ is well-defined: if we choose two different elements of $\pi^{-1}(x)$, say x_0 and $g \cdot x_0$ where $g \in G_j$, then $\pi(H_t(x_0)) = \pi(g \cdot H_t(x_0)) = \pi(H_t(g \cdot x_0))$ since g and H_t are both diagonal elements of $SO(2n)$, and therefore commute. Thus $\overline{H}(t,x)$ is a well-defined homotopy between Φ_j and the identity in $S^{2n-3}/G_{j|S^{2n-3}}$.

Now we are ready to complete the proof of the lemma:
Consider the action of G/G_j on the set of points in S^{2n-1}/G_j that are $\pi/2$ away from S_j. This action is generated by γ_j, and acts coordinate-wise on the trivial bundle $g(U_0) \cap g(V_0) \simeq S^1 \times F$. We know that γ_j acts on S_j by a rotation of $2\pi i/k_j$, and that it acts on F by ϕ_j. After quotienting $S^1 \times F$ by the powers of γ_j, we get a new bundle whose transition maps are homotopic to the identity. Since we are considering a fiber bundle over a circle with transition maps homotopic to the identity, we can use Theorem 18.3 in Steenrod's text on fibre bundles[12] to conclude that $f \circ g(U_0) \cap f \circ g(V_0)$ is the trivial bundle as desired. $\square$

$\square$

The triviality of the normal bundle yields a Mayer-Vietoris sequence for the homology of the orbit space, as it did in torus case (4.1):

$$\tilde{H}_q(R_x) \oplus \tilde{H}_q(S^1) \xrightarrow{i_q} \tilde{H}_q(X) \xrightarrow{\partial_q} \tilde{H}_{q-1}(N_x) \xrightarrow{j_{q-1}} \tilde{H}_{q-1}(R_x) \oplus \tilde{H}_{q-1}(S^1)$$

7.3 Homology of a Matroid Sequence

Let $\mathfrak{M}$ be a sequence of matroids $M_1, M_2, ...M_k$ where the coefficients in M_β lie in $p^{k-\beta+1}$. These matroids are formed by starting with the original matrix corresponding to the action, A_1, and forming new matrices by taking each entry modulo p^{k-l+1}. Each of these matrices has a corresponding matroid derived from the group structure as described previously. Our goal is to define a homology theory for such sequences of matroids, which we will use to learn more about the homology of the quotient space X. The homology of matroids was developed by Swartz to study quotient spaces $S^{2n-1}/(\mathbb{Z}_p)^r$. Although some modifications are required to generalize this theory to matroid sequences, many of the arguments in this section parallel those in Swartz [14].

Let $\mathfrak{F}(\mathfrak{M}) = \{f : E \to \{0, 1, 2\}\}$. The subset of $\mathfrak{M}$ *associated* to f, denoted $f_{\mathfrak{M}}$ is $f^{-1}\{1, 2\}$. Define $f_{\hat{i}}$ be the function that is the same as f except that $f(e_i) = 0$. We say that a cell in S^{2n-1} *obeys* f if its restriction to S_i contains $f(e_i)$ vertices for every $1 \leq i \leq n$. The set of all simplices in S^{2n-1} that obey f is denoted by Θ_f. The set of CW-cells in X whose preimiages are in Θ_f is $\bar{\Theta}_f$. The cells of $\bar{\Theta}_f$ can also be characterized by the rule that the restriction of a cell to $f \circ g(S_i)$ has dimension $f(e_i) - 1$.

For example, let $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ with entries modulo 4, so $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ with entries modulo 2. Let $f(e_1) = 1$ and $f(e_2) = 2$ Then $[\tilde{f}]$ is the sum of the two triangles in the orbit space including one vertex of $f \circ g(S_1)$ and one edge of $f \circ g(S_2)$.

Let $\overline{\overline{C}}_q(\mathfrak{M})$ be the free abelian group on $\{f \in \mathfrak{F}(\mathfrak{M}) : \Sigma f(e_i) = q+1\}$. We denote by $[f]$ the basis element corresponding to f.

Let $e_i \bullet_{\mathfrak{M}} E$ denote that e_i is a coloop in every matroid of $\mathfrak{M}$. We will refer to e_i as a *super-coloop* in this case. We may neglect this subscript when there is no ambiguity of the matroid structure, but it will sometimes be necessary to differentiate between structures such as $e_i \bullet_{\mathfrak{M}-e_n} (E-e_n)$ versus $e_i \bullet_{\mathfrak{M}/e_n} (E-e_n)$.

For example, the matroid sequence generated by the action- matrix $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & p \end{bmatrix}$ with coefficients in p^2 contains no super-coloops, since the two columns are parallel in the second matroid M_2.

The matroid sequence over p^2 generated by $A_1 = \begin{bmatrix} 1 & 0 & p \\ 0 & p & 1 \end{bmatrix}$ has a single super-coloop given by e_1. Note that e_1 is a coloop in M_1 since $pe_3 = e_2$ in A_1

Define $\partial_q : \overline{\overline{C}}_q(\mathfrak{M}) \to \overline{\overline{C}}_{q-1}(\mathfrak{M})$ by $\partial_q([f]) = \displaystyle\sum_{f(e_i)=1;\, e_i \bullet f_{\mathfrak{M}}} (-1)^{\sum_{i<j} f(e_j)} [f_{\hat{i}}]$

By this definition, $(\overline{\overline{C}}_q(\mathfrak{M}), \partial_q)$ is a chain complex: the removal of a super-coloop cannot create or eliminate any other super-coloops, so the standard cancellation argument for the boundary map applies.

There is a short exact sequence of chain complexes:

$$0 \to C_*(\mathfrak{M}-e) \to C_*(\mathfrak{M}) \to \frac{C_*(\mathfrak{M})}{C_*(\mathfrak{M}-e)} \to 0$$

The first map is injective since it is an inclusion, and the second map is surjective by definition.

Proposition 23. $\overline{\overline{H}}_q(\overline{\overline{C_*}}(\mathfrak{M}))/\overline{\overline{C}}_*(\mathfrak{M}-e_n) \cong \overline{\overline{H}}_{q-1}(\mathfrak{M}/e_n) \oplus \overline{\overline{H}}_{q-2}(\mathfrak{M}/e_n)$

Proof. For $[f] \in \overline{\overline{C}}_{q-1}(\mathfrak{M}/e_n)$ define $\phi_1([f])(e_i) = [f](e_i)$ for $i < n, \phi_1([f])(e_n) = 1$. For $[f] \in \overline{\overline{C}}_{q-2}(\mathfrak{M}/e_n)$ define $\phi_2([f])(e_i) = [f](e_i)$ for $i < n, \phi_2([f])(e_n) = 2$.

Note that $\phi_1 \oplus \phi_2$ is an isomorphism between $\overline{\overline{C}}_{q-1}(\mathfrak{M}/e_n) \oplus \overline{\overline{C}}_{q-2}(\mathfrak{M}/e_n)$ and $\overline{\overline{C}}_q(\mathfrak{M})/\overline{\overline{C}}_q(\mathfrak{M}-e_n)$. In particular, we can reverse the process by mapping f to $f_{\hat{n}}$ and restricting the function to $\mathfrak{M}/e_n$. This inverse is one-to-one due to the quotient by $C_q(\mathfrak{M}-e_n)$

Lemma 24. *$(\phi_1)_\star$ and $(\phi_2)_\star$ are chain maps of degree one and two, respectively.*

Proof. Let $[f] \in \overline{\overline{C}}_q(\mathfrak{M}/e_n)$, so $\phi_1([f]) \in \overline{\overline{C}}_{q+1}(\mathfrak{M})/\overline{\overline{C}}_{q+1}(\mathfrak{M}-e_n)$
Then $\partial_{q+1}(\phi_1([f])) = \displaystyle\sum_{f(e_i)=1; e_i \bullet_{\mathfrak{M}} (f_{\mathfrak{M}} \cup e_n); e_i \neq e_n} (-1)^{\sum_{j<i} f_j}[f_{\hat{i}}]$

Note that we may preclude e_n from this sum since the resulting term, even if it is a super-coloop: $(-1)^{\sum_{j<n} f_n}[f_{\hat{n}}]$ is in $\overline{\overline{C}}_q(\mathfrak{M}-e_n)$ and is thus trivial in the quotient.

By definition, we have that $\phi_1(\partial_q[f])(e_i) = \partial_q[f](e_i)$ for all $1 \leq i < n$, so $\phi_1(\partial_q[f])(e_i) = \displaystyle\sum_{f(e_i)=1; e_i \bullet_{\mathfrak{M}/e_n} f_{\mathfrak{M}}} (-1)^{\sum_{j<i} f_j}[f_{\hat{i}}]$

We rely on the following fact about (super) coloops: Let $e_i \in f_{\mathfrak{M}} \subseteq E - e_n$. Then e_i is a super-coloop of $f_{\mathfrak{M}}$ in $\mathfrak{M}/e_n$ if and only if e_i is a super-coloop of $f_{\mathfrak{M}} \cup e_n$ in $\mathfrak{M}$.

Therefore $\phi_1(\partial_q[f])(e_i) = \partial_{q+1}[f](e_i)$ for all $1 \leq i < n$, so ϕ_1 is a chain map of degree +1. The proof for ϕ_2 is the same, though without the concern of an $f_{\hat{n}}$-term when e_n is a super-coloop.

□

□

Corollary 25. *There is a long exact sequence:*

$$\xrightarrow{\partial_q \oplus \partial_{q-1}} \overline{\overline{H}}_q(\mathfrak{M} - e_n) \xrightarrow{i_q} \overline{\overline{H}}_q(\mathfrak{M}) \xrightarrow{j_q} \overline{\overline{H}}_{q-1}(\mathfrak{M}/e_n) \oplus \overline{\overline{H}}_{q-2}(\mathfrak{M}/e_n) \longrightarrow$$

The map j_q will be integral to the arguments that follow, so we will describe it more explicitly. Let $a \in \overline{\overline{H}}_q(\mathfrak{M})$ be a cycle and $a = \sum_l m_l[f_l]$, Then we can split this sum into two parts: $a = \sum_{f_l(e_n)=0} m_l[f_l] + \sum_{f_l(e_n)\neq 0} m_l[f_l]$. We observe that $j_q(a) = \sum_{f_l(e_n)\neq 0} m_l[(f_l)_{\hat{n}}]$.

Let $A \subseteq \mathfrak{M}$. Define $\rho(A) = min\{r_\beta(M_\beta) - r_\beta(A_\beta) : 1 \leq \beta \leq k\}$, where r_β is the rank function defining the matroid M_β. Define $\overline{\overline{C}}_q^s$ as the subgroup of $\overline{\overline{C}}_q$ generated by $\{[f] : \rho(f_{\mathfrak{M}}) = s\}$

Theorem 26. *For any q, $\overline{\overline{H}}_q(\mathfrak{M})$ is a free abelian group of finite rank. In addition, every element of $\overline{\overline{H}}_q(\mathfrak{M})$ has a representative in $\overline{\overline{C}}_q^0(\mathfrak{M})$. If e_n is a super-coloop of $\mathfrak{M}$, then $H_q(\mathfrak{M}) \cong H_{q-2}(\mathfrak{M}/e_n)$. If e_n is not a super-coloop of $\mathfrak{M}$, then j_q is surjective.*

Proof. By induction on n. Let f in $\mathfrak{F}(\mathfrak{M}/e_n)$ or $\mathfrak{F}(\mathfrak{M} - e_n)$, define $\overline{f}$ in $\mathfrak{F}(\mathfrak{M})$ to be the extension of f such that $f(e_n) = 1$, and $\overline{\overline{f}}$ such that $f(e_n) = 2$.

Suppose e_n is a super-coloop of $\mathfrak{M}$. Let $a = \sum_l m_l[f_l]$ be a cycle in $\overline{\overline{H}}_q(\mathfrak{M} - e_n)$. Then $\partial(\sum_l m_l[\overline{f}_l])$ can be calculated by removing the super-coloop e_n (we know the other terms cancel since all will be the summands of the previous cycle with an extra e_n term). Thus, $\partial(\sum_l m_l[\overline{f}_l]) = (-1)^q i_q(a)$, i.e. the image of the cycle a in $\overline{\overline{H}}_q(\mathfrak{M})$. This demonstrates that $i_q(a) = 0$ for all a. We can conclude that j_q is injective.

Let $a' = \sum_l m_l'[f_l']$ be a cycle in $\overline{\overline{H}}_{q-2}(\mathfrak{M}/e_n)$. Then $\overline{\overline{a'}} = \sum_l m_l'[\overline{\overline{f'}}_l]$ is a cycle in $\overline{\overline{H}}_q(\mathfrak{M})$ and $j_q(\overline{\overline{a'}}) = a'$. Since a' was chosen arbitrarily, we conclude that $\overline{\overline{H}}_{q-2}(\mathfrak{M}/e_n)$ is in the image of j_q, thus the second coordinate map of j_q gives an isomorphism from $\overline{\overline{H}}_q(\mathfrak{M})$ to $\overline{\overline{H}}_{q-2}(\mathfrak{M}/e_n)$.

Suppose there exists $a'' = \sum_l m_l''[f_l'']$ where $a'' \in \overline{\overline{C}}^0_{q-1}(\mathfrak{M}/e_n)$ in the image of j_q. Then there exists a cycle in $\overline{\overline{C}}_q(\mathfrak{M})$ of the form $b = \sum_l m_l''[\overline{f}_l''] + \sum_j m_j[g_j]$ where $g_j(e_n) = 0$ for all j. The boundary of b cannot be zero. $\partial(\sum_l m_l''[\overline{f}_l'']) = a'' \in \overline{\overline{C}}^0_{q-1}(\mathfrak{M})$, whereas $\partial(\sum_j m_j[g_j]) \in \overline{\overline{C}}^1_{q-1}(\mathfrak{M})$.

Thus j_q is a true isomorphism from $\overline{\overline{H}}_q(\mathfrak{M})$ to $\overline{\overline{H}}_{q-2}(\mathfrak{M}/e_n)$ and $\overline{\overline{H}}_q(\mathfrak{M})$ is a free abelian group by the induction hypothesis. Furthermore, since we can choose a representative of $a' \in \overline{\overline{C}}^0_{q-2}(\mathfrak{M}/e_n)$, it follows from the definition of a super-coloop that $\overline{\overline{a}}'$ has a representative in $\overline{\overline{C}}^0_q(\mathfrak{M})$. In particular, if e_n is a super-coloop and $A \subset E$ has maximal rank in every $M_j - e_n$, then $A \cup e_n$ will have maximal rank in every M_j

Now we suppose that e_n is not a super-coloop of $\mathfrak{M}$. By induction, we have that any element of $\overline{\overline{H}}_q(\mathfrak{M}-e_n)$ has a representative in $\overline{\overline{C}}^0_q(\mathfrak{M}-e_n)$, and $\partial(\overline{\overline{C}}^0_{q+1}(\mathfrak{M})) \subseteq \overline{\overline{C}}^1_q(\mathfrak{M})$.

Therefore, $\partial_q \oplus \partial_{q-1}$ is the zero map. We conclude that i_q is injective, j_q is surjective, and $\overline{\overline{H}}_q(\mathfrak{M}) \cong \overline{\overline{H}}_q(\mathfrak{M} - e_n) \oplus \overline{\overline{H}}_{q-1}(\mathfrak{M}/e_n) \oplus \overline{\overline{H}}_{q-2}(\mathfrak{M}/e_n)$ is a free abelian group of finite rank.

All that remains is to show that the elements of $\overline{\overline{H}}_q(\mathfrak{M})$ have representatives in $\overline{\overline{C}}^0_q(\mathfrak{M})$. It suffices to find a basis of such elements. By the induction hypothesis, anything in the image of i_q has a representative. Let $\{a_1, \ldots, a_l\}$ be representatives in $\overline{\overline{C}}^0_{q-2}(\mathfrak{M}/e_n)$ of a basis of $\overline{\overline{H}}_{q-2}(\mathfrak{M}/e_n)$. Then $\{\overline{\overline{a_1}}, \ldots, \overline{\overline{a_l}}\}$ are cycles in $\overline{\overline{C}}_q(\mathfrak{M})$. Furthermore, if we define A_i to be the support of columns for a_i, we see that $(A_i)_\beta$ spans $(\mathfrak{M}/e_n)_\beta$, therefore $(A_i \cup e_n)_\beta$ spans $(\mathfrak{M})_\beta$, so $\overline{\overline{a}}_i$ lies in $\overline{\overline{C}}^0_q(\mathfrak{M})$ by definition. Lastly, let $\{b_1, \ldots, b_k\}$ be representatives in $\overline{\overline{C}}^0_{q-1}$ of a basis of $\overline{\overline{\overline{H}}}_{q-1}(\mathfrak{M}/e_n)$. Note that $b_i = \sum_l m_l[A_l]$. If $b' = \sum_l m_l[A_l \cup e_n]$ lies in $\overline{\overline{C}}^0_q$ and $j_q(b') = b_i$. However, it is not evident that b' is a cycle. We have that j_q is surjective, so there exists some cycle $B = b' + b'' + b''' \in \overline{\overline{\overline{H}}}_q(\mathfrak{M})$ such that $j_q(B) = b, b'' \in C^0_q(\mathfrak{M} - e_n)$, and $b''' \in \sum_{s=1}^{r(M)} C^s_q(\mathfrak{M})$. However, the boundary of b''' must be zero: the images of its terms under the boundary map are all of a different (lower) rank and cannot cancel with each other or any part of $\partial(b' + b'')$. Therefore, $b' + b''$ is a cycle and serves as the required representative.

□

7.4 An Algorithm for the Homology of X

Throughout this section, we will use $\mathbb{Z}_p$-coefficients

Recall that the cells of $\bar{\Theta}_f$ are characterized by the rule that the restriction of a cell to $f \circ g(S_i)$ has dimension $f(e_i) - 1$. The sum of these cells is denoted $[\tilde{f}] = \sum_{\theta \in \bar{\Theta}_f} [\theta]$. Denote by $\mathfrak{M}_X$ the sequence of matroids corresponding to the quotient space $X = S^{2n-1}/G$.

Lemma 27. *Let $f \in \mathfrak{F}(\mathfrak{M}_X)$. Then $\widetilde{\partial([f])} = \partial([\widetilde{f}])$*

Proof. Let $f \in \mathfrak{F}(\mathfrak{M}_X)$ such that $f(e_i) = 0$. It is clear that no part of circle S_i appears in $\partial([\tilde{f}])$.

This agrees with our definition: $\partial[f] = \sum_{f(e_i)=1, e_i \bullet f_M} (-1)^{\sum_{j<i} f(j)} [f_{\hat{i}}]$

Suppose $f(e_i) = 2$. Then the cells of $[\tilde{f}]$ include arcs of S_i joined with other lists of vertices $Y_l \in \bar{\Theta}_{f_{\hat{i}}}$. In particular,

$[\tilde{f}] = \sum_l \{v_{i1}, v_{i2}, Y_l\} + \cdots + \{v_{i(p^k-1)} + v_{ip^k}, Y_l\} + \{v_{ip^k}, v_{i1}, Y_l\} + \{v_{i1}, v_{i2}, Y_l\}$

Then the contribution of the circle S_i to $\partial[\tilde{f}]$ is as follows:
$\sum_l (v_{i1}, Y_l) - (v_{i2}, Y_l) + (v_{i2}, Y_l) + \cdots + (v_{i(m-1)}, Y_l) - (v_{im}, Y_l) + (v_{im}, Y_l) - (v_{i1}, Y_l)$.
Thus, in this case the circle S_i contributes nothing to the boundary.

Finally, we consider the case where $f(e_i) = 1$. We will denote by f_M the matroid M restricted to the columns on which f is nonzero. There are $\frac{p^{k|f_{M_1}|}}{p^{\sum_{j=1}^{k} r(f_{Mj})}}$ simplices in $\bar{\Theta}_f \subseteq X$ and $\frac{p^{k|f_{M_1}-e_i|}}{p^{\sum_{j=1}^{k} r(f_{M_j}-e_i)}}$ elements in $\bar{\Theta}_{f_{\hat{i}}} \in X$. The part of the boundary map sending the cells in $\bar{\Theta}_f$ to $\bar{\Theta}_{f_{\hat{i}}}$ via the removal of the vertex in S_i is thus multiplication by $p^{k-\sum_{j=1}^{k}[r(f_{M_j})-r(f_{M_j}-e_i)]}$. We know that $0 \leq r(f_{M_j}) - r(f_{M_j} - e_i) \leq 1$ for all j. In fact, this difference equals one for all j precisely when e_i is a super-coloop of $f_{\mathfrak{M}}$. Since we are working with simplicial homology in $\mathbb{Z}_p$-coefficients, the boundary map described will be the zero map unless e_i is a super-coloop of $\mathfrak{M}$, in which case the boundary map will correspond to the signed removal of the vertices in S_i . $\square$

Let $\widetilde{\Delta_*}(X)$ be the subgroup of $\Delta_*(X)$ generated by $\{[\tilde{f}] : f \in \mathfrak{F}_{\mathfrak{M}_X}\}$. The lemma above shows that $\widetilde{\Delta_*}(X)$ is a subcomplex of $\Delta(X)$ and is chain isomorphic to $\overline{\overline{C}}_*(\mathfrak{M}_X)$.

Proposition 28. *Let $x \in S_n$. Then $\mathfrak{M}_X - e_n \cong \mathfrak{M}_{R_x}$ and $\mathfrak{M}_X/e_n \cong \mathfrak{M}_{N_x}$*

Proof. We obtain $\mathfrak{M}_X - e_n \cong \mathfrak{M}_{R_x}$ from the orginal matrix associated to the action by first deleting column n and then generating the sequence of matroids as before. Since column n corresponds to the action of the group on S_n, the quotient space corresponding to this new sequence of matroids will be R_x.
The sequence of matroids $\mathfrak{M}_{N_x}$ will correspond to the quotient S^{2n-3}/G_x, where G_x is the isotropy subgroup of x. Since we are working with effective actions,

we may assume that the nth column of A has gcd relatively prime to p. We can therefore row reduce A until this column contains only one entry, a_{in}, which is relatively prime to p. Let $\tilde{A}$ be the matrix that results from deleting row i and column n from this row reduced version of A. Then $\tilde{A}$ corresponds to S^{2n-3}/G_x since column n represents the nth circle in the join and row i corresponds to the only generator of G that does not fix S_n. We claim that the matroids of the chain generated by $\tilde{A}$ will be isomorphic to M_j/e_n for each $1 \leq j \leq k$.

Let γ_i be the generator of G corresponding to row i. Let B be any set of columns of A, and let $\tilde{B}$ be the corresponding columns of $\tilde{A}$. We have that $rank(B) = dim_{\mathbb{Z}_p}(< B \cup e_n > \otimes \mathbb{Z}_p)$ where $< B \cup e_n >$ is a subgroup of G. Column operations on the elements of $B \cup e_n$ do not affect the subgroup $< B \cup e_n >$, thus we can make a_{in} the only nonzero entry in row i among the columns of $B \cup e_n$. This demonstrates that $< B \cup e_n > \cong < \tilde{B} > \oplus < e_n >$. By the definition of our rank function, $rank(\tilde{B}) = rank(B \cup e_n) - 1$. This is the expected rank of $\tilde{B}$ in the contraction. Since the rank fucntion determines the matroid, the matroids generated by $\tilde{A}$ are precisely $M_1/e_n, \ldots M_k/e_n$.

□

Proposition 29. *Suppose e_j is a loop or super-coloop of $\mathfrak{M}_X$. Then $X \cong S^1 * R_x$*

Proof. If e_j is a loop then the corresponding column in our matroid representation has only zeroes, signifying that the action fixes the circle S_i and the proposition is clear. If e_j is a super-coloop, then the gcd of the entries in the corresponding column must be relatively prime to p. Otherwise, the column would be a zero column in some matrix A_β. Using the Euclidean algorithm, we can row reduce the j^{th} column until it has only one nonzero entry a_{ij}. Suppose there is another nonzero entry in row i, specifically the integer b in column j_0. If any other entry in column j_0 is divisible by fewer powers of p than b is, it can be used to eliminate b via row reduction. If this is not the case, then all the entries of column j_0 other than b are divisible by p^α and b is divisible by p^β with $\alpha > \beta$. Then in $A_{k-\alpha}$, all the entries in column j_0 would be 0 except for $b \mod p^\alpha$. This demonstrates that column j_0 is parallel to the supercoloop e_j in $A_{k-\alpha}$, a contradiction. We conclude that is possible to row reduce A_1 such that the column corresponding to e_j has only one nonzero entry $a_{ij}, gcd(a_{ij}, p) = 1$, and the row i contains no other nonzero entries. If we call the group generator corresponding to this row γ, we can see that that $S^{2n-1}/G = S^{2n-1}/(< \gamma > \oplus (G/ < \gamma >))$ which acts coordinate-wise on S_j and $S_1 * \cdots * \hat{S}_j * \cdots S_n$. We can therefore rewrite the action as $S^{2n-1}/G \cong [S_j/ < \gamma >] * [S^{2n-3}/(G/ < \gamma >)] \cong S^1 * R_j$ □

Let ω_0 and ω_1 be generators of $H_0(S_n)$ and $H_1(S_n)$ respectively. Let x be a vertex of S_n. By the Künneth formula and triviality of the normal bundle, we have that every element in $H_q(N_x)$ is of the form $\omega_0 \times a_0 + \omega_1 \times a_1$, for some $a_0 \in H_q(N_x)$ and $a_1 \in H_{q-1}(N_x)$

Proposition 30. *Suppose e_n is neither a loop nor a super-coloop of m. Let $a \in \overline{\overline{H}}_q(\mathfrak{M}_X)$. If $j_q(a) \in \overline{\overline{H}}_{q-1}(\mathfrak{M}_X/e_n)$, then $\partial_q(\widetilde{a}) = \omega_0 \times \widetilde{j_q(a)}$*

Proof. Since $j_q(a) \in \overline{\overline{H}}_{q-1}(\mathfrak{M}_X/e_n)$, $a = \sum_{f_l(e_n)=1} m_l[f_l] + \sum_{f_k(e_n)=0} m_k[f_k]$. By cutting $\widetilde{a}$ halfway between S_n and R_x and using barycentric subdivision, as described on page 150 of Hatcher [7], we can see that $\partial_q(\widetilde{a})$ lies entirely in $N_n \cong S^1 \times N_x$ and that it corresponds to removing the vertex on circle S_n from each representative simplex in the $\widetilde{f}_l$. This is the same as $\widetilde{j_q(a)} \in H_{q-1}(N_n)$. Therefore, $\partial(\widetilde{a})$ lies in a single fiber N_x of N_n and equals $\widetilde{j_q(a)} \times w_0$ □

Proposition 31. *Suppose e_n is neither a loop nor a super-coloop of $\mathfrak{M}$. Let $a \in \overline{\overline{H}}_q(\mathfrak{M}_X)$. If $j_q(a) \in \overline{\overline{H}}_{q-2}(\mathfrak{M}_X/e_n)$, then $\partial_q(\widetilde{a}) = \omega_1 \times \widetilde{j_q(a)} + \omega_0 \times b$ where $b \in H_{q-1}(N_x)$*

Proof. Let $a = \sum_l [f_l]$ We can assume that $f_l(e_n) = 2$ for each l.

Let γ be a generator of G_x where $x \in S_n$. Let by $h_\gamma : N_x \to N_x$ be the map induced by γ on N_x, and let H be a homotopy for h_γ to the identity map.

For $Y \subseteq N_x$, let $\Psi(Y)$ be the image of $[0, 2\pi/p^\beta] \times Y$ in N_n

By cutting $\tilde{a}$ halfway between S_n and R_x we can see that $\partial(\tilde{a}) = \Psi(\widetilde{j_q(a)})$

The triviality of the normal bundle $S^1 \times N_x$ where $x \in S_n$ gives us a homotopy $F : [0,1] \times [0, 2\pi/p^k] \times N_x \to c_n \times N_x$ such that:

$$F(0, t, \nu) = (c_n(t), \nu)$$

$$F(1, t, \nu) = \Psi(c_n(t), \nu$$

$$F(s, 0, \nu) = \nu$$

$$F(s, 2\pi/p^k, \nu) = H$$

As in [14], we will use cubical singular homology. We rewrite $\widetilde{j}_q(a)$ as $\sum_l \lambda_l$ where $\lambda_l : [0,1]^{q-2} \to N_x$. Let $G_l : [0,1] \times [0,1] \times [0,1]^{q-2} \to c_n \times N_x$ be defined by $G_l(s, t, z) = F(s, t, \lambda_l(z))$ and let $G = \sum_l G_l$. This G is a $q - cubical$ singular chain in $c_n \times N_x$. By the properties of F listed above, we have that $G(1, t, z) = \Psi(\widetilde{j_q(a)})$ and $G(0, t, z) = \omega_1 \times \widetilde{j_q(a)}$. We see that $G(s, 0, z) = z$ is independent of s and is thus a degenerate cubical chain, and equals zero in the cubical singular chain complex. We also see that $G(s, 2\pi/p^k, z)$ is a singular chain complex. We demonstrated that $\widetilde{j_q(a)}$ was a cycle in Proposition 26.

$0 = \partial_q(\tilde{a}) - \widetilde{j_q(a)} \times \omega_1 - b \times \omega_0$ where $b \in H_{q-1}(N_x)$

□

Let $\iota_\star$ be the map in homology induced by $\overline{\overline{C}}_\star(\mathfrak{M}_X) \cong \tilde{\Delta}_\star(X) \hookrightarrow \Delta_\star(X)$.

Theorem 32. *The map ι_q is always surjective. If $n \geq 2$ and e_n is neither a loop nor a super-coloop of $\mathfrak{M}_X$, then for all $q > 2$, ∂_q in the Mayer-Vietoris sequence is surjective. When $q = 2$, the image of ∂_2 is $\omega_0 \times \tilde{H}_1(N_x)$*

Proof. This proof will proceed by induction on n, the number of columns in the action-matrix. We begin with the case where $n = 1$. The matrix contains only one column e_1 and acts on a circle. Let f_1 be the function $f(e_1) = 1$. Then $\tilde{\Delta}_0(X)$ is $[\tilde{f}_0]$, the sum of all vertices on S_1. Since all the vertices are included, the map $\iota_0 : \tilde{\Delta}_0(X) \to \Delta_0(X)$ induces a surjection in homology. Similarly, let f_2 be $f(e_1) = 2$. Then $\tilde{\Delta}_1(X)$ is $[\tilde{f}_1]$, the sum of all edges of S_1. This sum of edges is mapped to the circle in $\Delta_1(X)$ that generates $\tilde{H}_1(X)$, so ι_1 is surjective as well.

We must also consider the case when $n = 2$.

The matrix representing such an action can always be row reduced into the form $A_1 = \begin{bmatrix} 1 & ap^\alpha \\ 0 & bp^\beta \end{bmatrix}$ where a and b a relatively prime to p and, provided that e_2 is neither a loop nor a super-coloop, $\alpha < \beta$. If e_2 is a loop or a super-coloop then the quotient space X is homeomorphic to S^3.

It is clear from the Mayer-Vietoris sequence that $H_3(X) \cong \mathbb{Z}_p \cong H_0(X)$ and ∂_3 is surjective. To find $H_1(X)$ and $H_2(X)$ we examine the pertinent section of the Mayer-Vietoris sequence:

$$0 \to \tilde{H}_2(X) \to \tilde{H}_1([bp^\alpha] \times S_1) \to \tilde{H}_1(\begin{bmatrix} ap^\alpha \\ bp^\beta \end{bmatrix}) \oplus \tilde{H}_1(S_1) \to \tilde{H}_1(X) \to 0$$

The map $j_1 : \tilde{H}_1(N_x \times S_1) \to \tilde{H}_1(R_x) \oplus \tilde{H}_1(S_1)$ sends any element of the form $\omega_1 \times H_0(N_x)$ identically to $H_1(S_1)$. It is multiplication by $p^{\beta-\alpha}$ on all elements of the form $\omega_0 \times \tilde{H}_1(N_x)$. Since we are considering homology over $\mathbb{Z}_p$ and $\beta > \alpha$, all elements $\omega_0 \times \tilde{H}_1(N_x)$ are in the kernel of j_1 and thus are in the image of ∂_2 as desired. We can conclude that $\tilde{H}_2(X) \cong \omega_0 \times \tilde{H}_1(N_x) \cong \mathbb{Z}_p$ and $\tilde{H}_1(X) \cong \tilde{H}_1(R_x) \cong \mathbb{Z}_p$.

Smilar arguments to $n = 1$ demonstrate that ι_0 and ι_3 are surjective in homology. In particular, $\tilde{\Delta}_0(X)$ is the sum of all vertices and $\tilde{\Delta}_3(X)$ is the sum of all tetrahedra, which are geometrically both generators for $H_0(X)$ and $H_3(X)$ respectively.

Let $f_{ab} : E \to \{0, 1, 2\}$ be the function $f(e_1) = a$ and $f(e_2) = b$.
Then $\tilde{\Delta}_1(X)$ is generated by $[\tilde{f}_{20}], [\tilde{f}_{02}]$, and $[\tilde{f}_{11}]$. $[\tilde{f}_{02}]$ is the sum of all edges on the circle S_2, which is the generator for the homology of $H_1(X)$ coming from $H_1(R_x)$

$\tilde{\Delta}_2(X)$ is generated by $[\tilde{f}_{21}]$ and $[\tilde{f}_{12}]$. We see that $[\tilde{f}_{12}]$ is the sum of all triangles that are the join of one vertex in S_1 and one edge in S_2. This sum generates all elements of the homology of the form $\omega_0 \times H_1(N_x)$, which correspond to the generators of $H_2(X)$. We conclude that the Theorem holds in this base case.

Now we proceed with the induction, assuming that $n \geq 3$. If e_n is a loop or super-coloop we need only demonstrate that ι_q is surjective. We have shown that in this case, $X = S_n * R_x$. let $\tilde{a} \in \tilde{H}_q(X)$. Then there exists $\tilde{b} \in \tilde{H}_{q-2}$ such that $\tilde{a} = S_n * \tilde{b}$. By the induction hypothesis, there exists $b \in \overline{\overline{H}}_{q-2}((M/e_n)$ such that $\tilde{b} = i(b)$. Then $i(\overline{\overline{b}}) = \tilde{a}$.

Now suppose that e_n is neither a loop nor a super-coloop of $\mathfrak{M}_X$. Assume for the moment that $q > 2$. Let $\tilde{a} \in \tilde{H}_q(N_x)$ be of the form $\omega_0 \times \tilde{b}, \tilde{b} \in H_{q-1}(N_x)$. By the induction hypothesis, we have that $\tilde{b} = i(b), b \in \overline{\overline{H}}_{q-1}(\mathfrak{M}_X/e_n)$.

By Proposition 30 there is some $c \in \overline{\overline{H}}_{q-1}(\mathfrak{M}_X)$ such that $j_q(c) = b$. Furthermore, $\partial(\tilde{c}) = \omega_0 \times \tilde{b} = \tilde{a}$. So $\omega_0 \times \tilde{H}_{q-1}(N_x)$ is in the image of ∂_q. Now suppose that $\tilde{a} = \omega_1 \times \tilde{b}, \tilde{b} \in H_{q-2}(N_x)$. Again using proposition 31, we can find $c \in \overline{\overline{H}}_q(M)$ such that $\partial(\tilde{c}) = \tilde{a} + (\omega_0 \times \tilde{a}'), a' \in H_{q-1}(N_x)$. Since $\omega_0 \times \tilde{H}_{q-1}(N_x)$ is already in the image of $\partial_q, \omega_1 \times H_{q-2}(N_x)$ is in the image of ∂_q.

When $q = 2$, the same argument demonstrates that $\omega_0 \times H_1(N_x) \subseteq image(\partial_2)$. Let y be a generator of $H_0(N_x)$. To see that $\omega_1 \times y$ is not in the image of ∂_2, we note that $\phi_1(\omega_1 \times y) = \omega_1$ in $H_1(S^1)$. So $\omega_1 \times y$ is not in the kernel of ϕ_1 and thus not in the image of ∂_2.

Using the long exact sequence of homology and the surjectivity of ∂_q onto $\omega_0 \times \tilde{H}_{q-1} \bigcup \omega_1 \times \tilde{H}_{q-2}(N_x)$ in $\tilde{H}_{q-1}(N_n)$ we see that, $\tilde{H}_q(X) \cong incl_q(R_x) \oplus (\omega_0 \times \tilde{H}_{q-1}(N_x)) \oplus (\omega_1 \times \tilde{H}_{q-2}(N_x))$. Then ι_q is surjective onto the first summand by the induction hypothesis, and the second two summands because of the preimages of a basis as constructed above.

$\square$

The previous theorem describes an algorithm that can be used to find the $\mathbb{Z}_p$-coefficient homology of the quotient of any sphere by an effective orientable abelian action of the finite abelian group Γ_1, provided that $|\Gamma_1|$ is a power of p. Note, however, that every finite abelian group Γ can be decomposed as a direct sum into a component whose order is a power of p, Γ_1, and a component whose order is relatively prime to p, Γ_2. Since Γ_2 acts on S^{2n-1}/Γ_1 by rotations of the circles S_i, and $gcd(|\Gamma_2|, p) = 1$, the homology with $\mathbb{Z}_p$-coefficients of S^{2n-1}/Γ is not affected by this action. In particular, this action is a subgroup of the toral actions addressed in earlier sections. Hence, each group element of Γ_2 acts in a manner that is homotopic to the identity on the generators of S^{2n-1}/Γ. We can repeat the proof using Lemma 9 to compute the $\mathbb{Z}_p$-homology of S^{2n-1}/Γ where Γ is *any* finite abelian group with a linear, effective, orientable action on the sphere.

7.5 Examples and Problems

Based on the results of [14], we may suspect that the above algorithm yields a Poincaré polynomial of the quotient space that depends only on the Tutte polynomials of the associated matroids. Examples, counterexamples, and conjectures relating to this suspicion appear in the following section, along with other directions for further research.

For a matroid M, the doubled matroid $2M$ is formed by adding an additional element e_i' for each $1 \leq i \leq |M|$ where $e_i' \in 2M$ is parallel to $e_i \in M$. We can double a sequence of matroids $\mathfrak{M}$ by doubling each matroid $M_1, \ldots, M_k$

We first define the polynomial $\mathfrak{T}(\mathfrak{M}; 0, t)$ as follows: Let $\mathfrak{M}$ be a sequence of matroids $M_1, \ldots, M_k$. Let $T(M_i, 0, t) = \sum_{j=0} a_{ij} y^j$. Then define $\mathfrak{T}(\mathfrak{M}; 0, t) = \sum_{i=0} (max\{a_{ij}\}_i) y^j$

Problem: In the case where $G \cong (\mathbb{Z}_{p^2})^r$, is $\tilde{P}_X(t) = \mathfrak{T}(2\mathfrak{M}; 0, t)$?

This question is motivated by a large (though not exhaustive) number of examples, including the one below. It may be the case that groups composed of p^2 work out this nicely. After all, it has been shown in [14] that the Poincaré polynomial equals the Tutte polynomial for groups $(\mathbb{Z}_p)^r$.

Example: The quotient of S^7 by $\mathbb{Z}_4 \times \mathbb{Z}_4$ represented by: $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

$t^{r-1} T(2M_1; 0, t) = y^7 + 2y^6 + 3y^5 + 4y^4 + 5y^3 + 2y^2$

$t^{r-1} T(2M_2; 0, t) = y^7 + 2y^6 + 3y^5 + 3y^4 + 3y^3 + y^2$

The homology of the orbit space X, computed using Macaulay2, is given as follows:

$H_1(X;\mathbb{Z}_4) = 0$

$H_2(X;\mathbb{Z}_4) = \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$H_3(X;\mathbb{Z}_4) = (\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_4)^3$

$H_4(X;\mathbb{Z}_4) = \mathbb{Z}_2 \oplus (\mathbb{Z}_4)^3$

$H_5(X;\mathbb{Z}_4) = (\mathbb{Z}_4)^3$

$H_6(X;\mathbb{Z}_4) = (\mathbb{Z}_4)^2$

$H_7(X;\mathbb{Z}_4) = \mathbb{Z}_4$

The Tutte polynomials even seem to predict the appearance of $\mathbb{Z}_4$'s: a term is represented by a $\mathbb{Z}_4$ homology if it appears in both Tutte polynomials. Whether this formulation of the homology is true of $(\mathbb{Z}_{p^2})^r$ is unknown.

A conjecture that the maximum coefficient of the term t^k in the polynomials $t^{r-1}T(M_\beta;0,t)$ would predict the homology in dimension $t^{r-1}t^k$ is false for actions of $(\mathbb{Z}_{p^3})^r$ The following provides a counter-example.

Example: Consider the action of $G \cong \mathbb{Z}_8 \times \mathbb{Z}_8$ on S^5 wheres $A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix}$.

The Tutte polynomials associated to X are as follows:

$t^{r-1}T(2M_1;0,t) = y^5 + 2y^4 + 3y^3 + y^2$

$t^{r-1}T(2M_2;0,t) = y^5 + y^4 + y^3$

$t^{r-1}T(2M_3;0,t) = y^5 + y^4 + y^3 + y^2 + y$

The $\mathbb{Z}_2$ reduced Poincaré polynomial of X is $y^5 + 2y^4 + 3y^3 + 2y^2 + y$. In the case of y^2, the components from M_1 and M_3 are distinct. An additional problem would be to study this example further and determine for which classes of matroids or actions the Tutte polynomials are predictive.

7.6 Additional Problems

We define a new Tutte polynomials for finite abelian groups, using the matroid structure we have descibed. These polynomials, if well-defined, may lead to results for a new class of matroids. We could also define a Tutte polynomial for arbitrary sequences of matroids with weak maps between them using the theory of super-coloops.

The homology is only described here with $\mathbb{Z}_p$-coefficients. The argument for proposition 27 does not hold without this restriction. The proof of this proposition still computes the maps of the Mayer-Vietoris sequence as multiplication by powers of p, but the homology groups are difficult to compute when these maps are nonzero. Further study may yield a generalization to homology with integer coefficients.

We have not yet delved into the questions of what the singular space of $X = S^{2n-1}/G$ looks like. Its homology and structure may be an interesting avenue for further research, as the structure for the rational singular set of the torus was quite rich.

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