CONVOLUTION POWERS OF COMPLEX-VALUED FUNCTIONS AND RELATED TOPICS IN PARTIAL DIFFERENTIAL EQUATIONS

A Dissertation
Presented to the Faculty of the Graduate School
of Cornell University
in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

by

Evan Daniel Randles

May 2016

CONVOLUTION POWERS OF COMPLEX-VALUED FUNCTIONS AND RELATED TOPICS IN PARTIAL DIFFERENTIAL EQUATIONS

Evan Daniel Randles, Ph.D.

Cornell University 2016

The study of convolution powers of a finitely supported probability distribution ϕ on the d-dimensional square lattice is central to random walk theory. For instance, the nth convolution power $\phi^{(n)}$ is the distribution of the nth step of the associated random walk. In the case that the random walk is aperiodic and irreducible, $\phi^{(n)}$ is well approximated by a single and appropriately scaled Gaussian density; this is the local (central) limit theorem. When such functions are allowed to be complex-valued, their convolution powers are seen to exhibit rich and disparate behavior, much of which never appears in the probabilistic setting. In the first half of this thesis, we study the asymptotic behavior of the convolution powers of complex-valued functions on $\mathbb{Z}^d$. This problem, originally motivated by the problem of Erastus L. De Forest in data smoothing, has found applications to the theory of stability of numerical difference schemes in partial differential equations. For a complex-valued function ϕ on $\mathbb{Z}^d$, we ask and address four basic and fundamental questions about the convolution powers $\phi^{(n)}$ which concern sup-norm estimates, generalized local limit theorems, pointwise estimates, and stability. In one dimension, we give a complete theory of sup-norm estimates and local limit theorems for the entire class of finitely supported complex-valued functions. This work extends results of I. J. Schoenberg, T. N. E. Greville, P. Diaconis and L. Saloff-Coste and, in the context of stability theory, results by V. Thomée and M. V. Fedoryuk.

In the second half of this thesis, we consider a class of "higher order" homogeneous partial differential operators on a finite-dimensional vector space and study their associated heat kernels. The heat kernels for this general class of operators are seen to arise naturally as the limiting objects of the convolution powers of complex-valued functions on the square lattice in the way that the classical heat kernel arises in the (local) central limit theorem. These so-called positive-homogeneous operators generalize the class of semi-elliptic operators in the sense that the definition is coordinate-free. We then introduce a class of variable-coefficient operators, each of which is uniformly comparable to a positive-homogeneous operator, and we study the corresponding Cauchy problem for the heat equation. Under the assumption that such an operator has Hölder continuous coefficients, we construct a fundamental solution to its heat equation by the method of E. E. Levi, adapted to parabolic systems by A. Friedman and S. D. Eidelman. Though our results in this direction are implied by the long-known results of S. D. Eidelman for $2\vec{b}$-parabolic systems, our focus is to highlight the role played by the Legendre-Fenchel transform in heat kernel estimates. Specifically, we show that the fundamental solution satisfies an off-diagonal estimate, i.e., a heat kernel estimate, written in terms of the Legendre-Fenchel transform of the operator's principal symbol – an estimate which is seen to be sharp in many cases. We then turn to the study of such variable-coefficient operators whose coefficients are, at worst, bounded and measurable and we study their associated heat kernels. Following functional-analytic techniques of E. B. Davies and G. Barbatis, we prove heat kernel estimates in terms of the Legendre-Fenchel transform subject to a dimension-order restriction. Our work in this measurable-coefficient setting extends results of E. B. Davies and partially extends results of A. F. M. ter Elst and D. Robinson. All work in this thesis was done in collaboration with Laurent Saloff-Coste.

BIOGRAPHICAL SKETCH

Evan Randles was born in Mission Hills, California. After attending welding school at College of the Canyons in Valencia, California, he worked as a gas tungsten arc welder. Later, he entered California State University, Northridge where he earned a B.S. in physics, a B.A. in mathematics in 2010 and, working under the advisement of Professor David Klein, he received an M.S. in mathematics in 2011. His Master's thesis is entitled "Spacelike foliations of Robertson-Walker spacetime by Fermi space slices." In 2011, Evan began his doctoral studies in the Center for Applied Mathematics at Cornell University, working under the advisement of Professor Laurent Saloff-Coste. In the fall of 2016, he will begin a postdoctoral position in the Department of Mathematics at the University of California, Los Angeles.

To Bernie

ACKNOWLEDGEMENTS

First and foremost, I owe my deepest gratitude to my advisor, Laurent Saloff-Coste. Not only have I benefited immensely from his continued support, deep knowledge and incredible intuition, but it is by his guidance that I have begun to develop my own intuition and aesthetic appreciation of mathematics in a way I would have not previously thought possible. I am greatly indebted to Len Gross for his help and guidance throughout my time at Cornell; he is and will always be an inspiration to me as a mathematician and an expositor. I am very grateful to Veit Elser for being part of this committee. I wish to thank Persi Diaconis who, in collaboration with Laurent, began the initial exploration which led to much of the work presented in this thesis. I also want to thank Rodrigo Bañuelos, Bob Strichartz, Tim Healey, Steve Strogatz, Maria Terrell, Pete Diamessis, David Klein, Jerry Rosen, Ana Cristina Cadavid, Nate Eldredge and the late Lars Wahlbin.

I am indebted to many of my peers at Cornell in the Center for Applied Mathematics and in the Department of Mathematics. Most importantly, I want to thank Mathav Murugan, Robert Kesler, David Eriksson and Sumedh Joshi. I own them greatly for so many fruitful mathematical discussions and I feel lucky to call them friends. I want to also thank Tianyi Zheng, Isabel Kloumann, Zach Clawson, Aditya Vaidyanathan, Stephen Cowpar, Hyung Joo Park and Matt Holden.

I wish to thank Cornell University and the National Science Foundation who supported me so generously throughout the course of this work.

I want to also thank my friends in Ithaca and beyond: Ben Moss, Te-Wen Lo, Amy Cheatle, Shrutarshi Basu, Alyce Daubenspeck, Jon Beyeler, Tyler Rasmussen, Adam Gelfand, Sam Havens, Kelly Margaritis, Ben Jose.

I am greatly indebted to my Mom, Dad and Grandmother for their love and support. I want to thank my late cat Mischief who spent countless hours by my side as I worked. And most of all, I want to thank my wife Bernie Randles for her love, support, encouragement and endless proofreading.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

The study of convolution powers of a finitely supported probability distribution ϕ on the d-dimensional square lattice is central to random walk theory. For instance, the nth convolution power $\phi^{(n)}$ is the distribution of the nth step of the associated random walk. In the case that the random walk is aperiodic and irreducible, $\phi^{(n)}$ is well approximated by a single and appropriately scaled Gaussian density; this is the local (central) limit theorem. When such functions are allowed to be complex-valued, their convolution powers are seen to exhibit rich and disparate behavior, much of which never appears in the probabilistic setting. In this thesis, we study the convolution powers of complex-valued functions on $\mathbb{Z}^d$ and some related topics on heat kernels of higher-order partial differential operators.

The limiting behavior of the convolution powers of complex-valued functions (or, simply non-positive functions) was originally investigated by Erastus L. De Forest in its connection to statistical data smoothing procedures in the late nineteenth century. De Forest posed the problem of determining the pointwise limiting behavior of the convolution powers of any suitably normalized and finitely supported function $\phi : \mathbb{Z} \to \mathbb{R}$. De Forest's problem was later investigated by I. J. Schoenberg and T. N. E. Greville, both of whom proved (generalized) local limit theorems under varying hypotheses. Beyond local limit theorems, the study of convolution powers of complex-valued functions saw explosive investigation in the 1960's, paralleled by advancements in scientific computing, in its application to numerical solutions for partial differential equations. Arising from this study, the so-called stability theory (for finite difference schemes) is concerned with finding conditions under which a finitely supported function $\phi : \mathbb{Z}^d \mapsto \mathbb{C}$ is *stable* in the sense that its convolution powers $\phi^{(n)}$, for $n \geq 1$, are uniformly bounded in $\ell^1(\mathbb{Z}^d)$. This property is seen to have profound implications for finite difference schemes in partial differential equations. Namely, for a finite difference scheme (given by ϕ) to an initial value problem, von Neumann's theorem states that ϕ is stable if and only if the corresponding finite difference scheme converges to a classical solution (in a pointwise sense). In 1965, V. Thomée characterized stability (and instability) when $d = 1$; Thomée's result was partially extended to higher dimensions by M. V. Fedoryuk in 1967.

In the second chapter of this thesis, we consider the general class of finitely supported functions $\phi : \mathbb{Z} \to \mathbb{C}$. Herein, we give a complete account of local limit theorems for this general class. As the classical heat kernel arises as the limiting object of the convolution powers of probability distributions, our local limit theorems show that the convolution powers of the general class are similarly attracted to a certain class of analytic functions which includes the Airy function and the heat kernel evaluated at purely imaginary time. Extending the results of I. J. Schoenberg, T. N. E. Greville, P. Diaconis and L. Saloff-Coste, our main result in this direction (Theorem 2.1.3) presents a complete solution to De Forest's problem. Using our local limit theorems, we then give a complete account of the asymptotic behavior for the sup-norms of convolution powers.

In the third chapter, we study the convolution powers of complex-valued functions on $\mathbb{Z}^d$. Here, we ask and address four basic and fundamental questions concerning sup-norm estimates, local limit theorems, pointwise space-time estimates, and stability for convolution powers. Our results in this chapter pertain to a large, though not exhaustive, class of complex-valued functions on $\mathbb{Z}^d$ and our hypotheses are naturally stated in terms of local properties of Fourier transforms. The results concerning sup-norm estimates and local limit theorems presented in this chapter partially extend the one-dimensional results of Chapter 2. The attractors which appear in our local limit theorems in this chapter are seen to also arise as the heat kernels corresponding to higher order partial differential operators–those which are studied in the second half of this thesis. Following and extending work of P. Diaconis and L. Saloff-Coste, we prove a number of results concerning pointwise space-time estimates for convolution powers and discrete derivatives thereof; these estimates make essential use of the Legendre-Fenchel transform and motivate our subsequent study of heat kernel estimates. In the context of stability theory, we extend the affirmative results of V. Thomée and M. V. Fedoryuk.

In the fourth chapter, we consider a class of homogeneous partial differential operators on a finite-dimensional vector space and study their associated heat kernels. The heat kernels for this general class of operators are those which were seen to arise in the local limit theorems presented in Chapter 3. These so-called positive-homogeneous operators generalize the class of semi-elliptic operators, introduced by F. Browder, in the sense that the definition is coordinate-free. More generally, we introduce a class of variable-coefficient operators, each of which is uniformly comparable to a positive-homogeneous operator, and we study the corresponding Cauchy problem for the heat equation. Under the assumption that such an operator has Hölder continuous coefficients, we construct a fundamental solution to its heat equation by the method of E. E. Levi, adapted to parabolic systems by A. Friedman and S. D. Eidelman. Though our results in this direction are implied by the long-known results of S. D. Eidelman for $2\vec{b}$-parabolic systems, our focus is to highlight the role played by the Legendre-Fenchel transform in heat kernel estimates. Specifically, we show that the fundamental solution satisfies an off-diagonal estimate, i.e., a heat kernel estimate, written in terms of the Legendre-Fenchel transform of the operator's principal symbol– an estimate which is seen to be sharp in many cases.

Taking our motivation from Chapter 4, in the final chapter we study heat-kernel estimates for partial differential operators with, at worst, measurable coefficients. Following the work of E. B. Davies pertaining to higher-order uniformly elliptic operators, we present an abstract theory for heat-kernel estimates, written in terms of the Legendre-Fenchel transform, for self-adjoint partial differential operators which are uniformly comparable to positive-homogeneous operators and are subject to a necessary dimension-order restriction (written in terms of the homogeneous order of the operator). In this development, we suitably adapt Davies' method to the positive-homogeneous setting and, from this, the full d-dimensional Legendre-Fenchel transform is seen to appear naturally. Our results extend those of E. B. Davies and partially extend results of A. F. M. ter Elst and D. Robinson.

Chapters 2, 3 and 4 are based on the articles [73], [72] and [74], respectively. The material on Chapter 5 will be included in a forthcoming article. All work in this thesis was done in collboration with Laurent Saloff-Coste.

CHAPTER 2

CONVOLUTION POWERS OF COMPLEX-VALUED FUNCTIONS ON $\mathbb{Z}$

2.1 Introduction

Let $\phi : \mathbb{Z} \to \mathbb{C}$ be a finitely supported function. We wish to study the convolution powers of ϕ, that is, the functions $\phi^{(n)} : \mathbb{Z} \to \mathbb{C}$ defined iteratively by

$$\phi^{(n)}(x) = \sum_{y \in \mathbb{Z}} \phi^{(n-1)}(x-y)\phi(y),$$

where $\phi^{(1)} = \phi$. This study has been previously motivated by problems in data smoothing and numerical difference schemes for partial differential equations [42, 80, 86, 87]. We encourage the reader to see the recent article [31] for background discussion and pointers to the literature.

In the case that the support of ϕ is empty or contains a single point, the convolution powers of ϕ are rather easy to describe. The present chapter focuses on functions ϕ with finite support consisting of more than one point; in this case we say that the support of ϕ is *admissible*. When the function ϕ is a probability distribution, i.e., it is non-negative and satisfies

$$\sum_{x \in \mathbb{Z}} \phi(x) = 1,$$

the behavior of $\phi^{(n)}$ for large values of n is well-known and is the subject of the local limit theorem. A modern treatment of this classical result can be found in Chapter 2 of [63] (see also Chapter 2 of [83]). Our aim is to extend the results of [31] and describe the limiting behavior for the general class of complex valued functions on $\mathbb{Z}$ with admissible support. In particular, we give bounds on the supremum of $|\phi^{(n)}|$ and prove "generalized" local limit theorems.

As an example, we consider the function $\phi : \mathbb{Z} \to \mathbb{C}$ defined by

$$\phi(0) = \frac{1}{8}(5-2i) \qquad \phi(\pm 1) = \frac{1}{8}(2+i) \qquad \phi(\pm 2) = -\frac{1}{16} \tag{2.1}$$

and $\phi = 0$ otherwise. The convolution powers $\phi^{(n)}$ for $n = 100, 1000, 10000$ are illustrated in Figures 2.1 and 2.2. We make two crucial observations about these graphs: First, it appears that the supremum $\|\phi^{(n)}\|_\infty$ is decaying on the order of $n^{-1/2}$; this is consistent with the classical theory for probability distributions. Second, as n increases, $|\phi^{(n)}(x)|$ appears to be constant on increasingly large intervals centered at 0. This is in stark contrast to the behavior described by the classical local limit theorem for probability distributions. In the present chapter, we prove that there are constants $C, C' > 0$ for which

$$Cn^{-1/2} \leq \|\phi^{(n)}\|_\infty \leq C'n^{-1/2}.$$

We also show that

$$\phi^{(n)}(\lfloor xn^{1/2} \rfloor) = \frac{n^{-1/2}}{\sqrt{4\pi i/8}} e^{-8|x|^2/4i} + o(n^{-1/2})$$

for x in any compact subset of $\mathbb{R}$. Here, $\lfloor \cdot \rfloor$ denotes the greatest integer function. We note that this approximation cannot hold uniformly for all $x \in \mathbb{R}$ because the modulus of $(4\pi i/8)^{-1/2} \exp(-8|x|^2/4i)$ is a non-zero constant whereas $\phi^{(n)}$ has finite support for each n. For comparison with Figure 2.2, Figure 2.3 shows the graph of $\mathrm{Re}((4\pi n i/8)^{-1/2} \exp(-8|x|^2/4ni))$ for $n = 100, 1000, 10000$. We will return to this example in Subsection 2.8.2 and justify the claims made above.

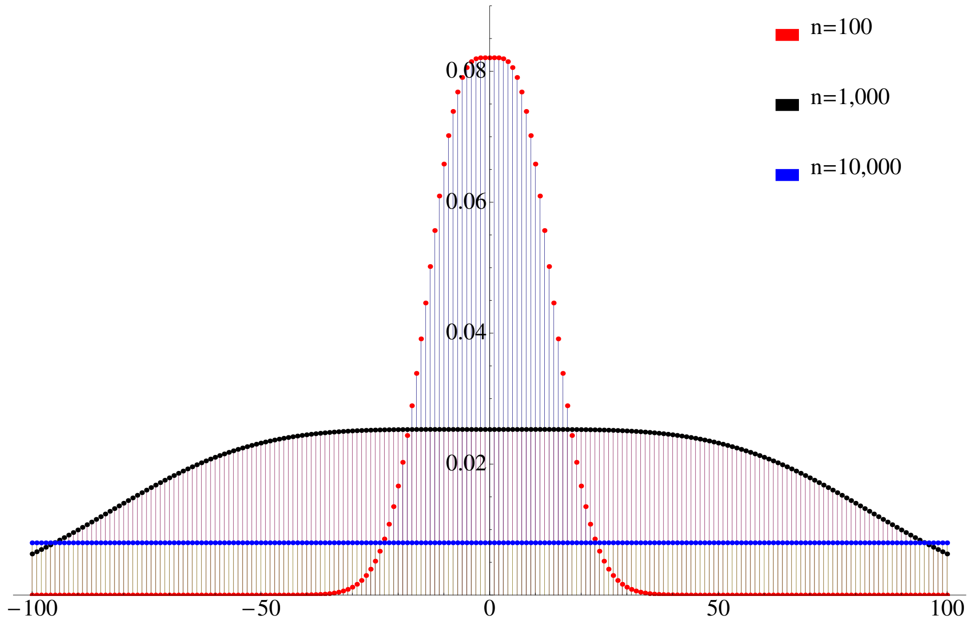

Figure 2.1: $|\phi^{(n)}|$ for $n = 100, 1000, 10000$

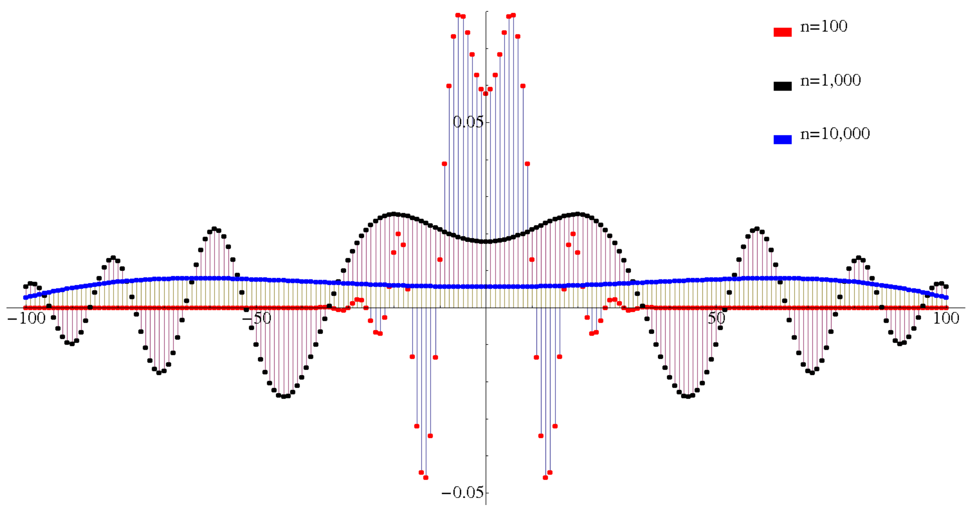

Figure 2.2: $\mathrm{Re}(\phi^{(n)})$ for $n = 100, 1000, 10000$

The Fourier transform is central to the arguments made in this thesis. We recall its definition: For $\phi : \mathbb{Z} \to \mathbb{C}$, finitely supported, the Fourier transform of ϕ is the function $\hat{\phi} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{\phi}(\xi) = \sum_{x \in \mathbb{Z}} \phi(x) e^{ix\xi} \tag{2.2}$$

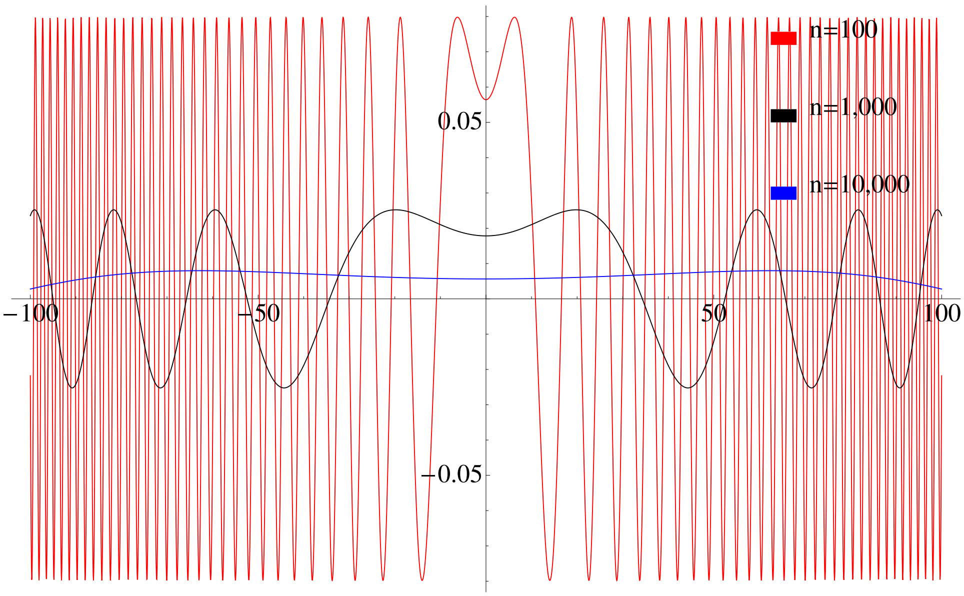

Figure 2.3: $\mathrm{Re}((4\pi i n/8)^{-1/2}\exp(-8|x|^2/4ni))$ for $n = 100, 1000, 10000$

for $\xi \in \mathbb{R}$.

Our first main result is illustrated in the following theorem.

Theorem 2.1.1. *Let $\phi : \mathbb{Z} \to \mathbb{C}$ have admissible support and let $A = \sup_\xi |\hat{\phi}(\xi)|$. Then there is a natural number $m \geq 2$, and positive constants C and C' such that*

$$Cn^{-1/m} \leq A^{-n}\|\phi^{(n)}\|_\infty \leq C'n^{-1/m} \tag{2.3}$$

for all natural numbers n.

Remark 1. *The natural number $m \geq 2$ appearing in Theorem 2.1.1 is consistent with those appearing Theorems in 2.1.2 and 2.1.3; upon dividing ϕ by A, it is defined by* (2.7).

In the classical local limit theorem, the convolution powers of a probability distribution are approximated by the heat kernel, an analytic function. In the present setting, the convolution powers $\phi^{(n)}$ are analogously approximated by certain analytic functions. We now define these so-called *attractors*: Let $m \geq 2$ be a natural number and β be a non-zero complex number for which $\mathrm{Re}(\beta) \geq 0$. We define $H_m^\beta : \mathbb{R} \to \mathbb{C}$ by

$$H_m^\beta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} e^{-\beta u^m} \, du \tag{2.4}$$

provided the integral converges as an improper Riemann integral. If additionally, for each $\epsilon > 0$ there exists $M_\epsilon > 0$ such that

$$\left| H_m^\beta(x) - \frac{1}{2\pi} \int_{M}^{M} e^{-ixu} e^{-\beta u^m} \, du \right| < \epsilon$$

for all $M \geq M_\epsilon$ and $x \in S \subseteq \mathbb{R}$, we say that the integral defining H_m^β converges uniformly in x on S. When $\mathrm{Re}(\beta) > 0$ and m is an even natural number, it is easy to see that

$$|e^{-ixu} e^{-\beta u^m}| = e^{-\mathrm{Re}(\beta) u^m} \in L^1(\mathbb{R})$$

whence the defining integral converges uniformly in x on $\mathbb{R}$. In this case, H_m^β is equivalently defined by its inverse Fourier transform, $e^{-\beta u^m}$. In the case that $\mathrm{Re}(\beta) = 0$, it is not immediately clear for which values of m or in what sense the integral in (2.4) will converge. It will be shown that when $m > 2$, the integral converges uniformly in x on $\mathbb{R}$ and, when $m = 2$, it converges uniformly in x on any compact set. This is the subject of Proposition 2.4.1. The proposition extends the results of [42] in which only odd values of m (for $\mathrm{Re}(\beta) = 0$) were considered.

In the case that $m \geq 2$ is even and $\mathrm{Re}(\beta) \geq 0$, H_m^β is the integral kernel of the bounded holomorphic semigroup $T_\beta = e^{-\beta(\Delta)^{m/2}}$ generated by the non-negative self-adjoint operator $(\Delta)^{m/2}$ on $L^2(\mathbb{R})$; here, Δ is the unique self-adjoint extension of $-(d/dx)^2$ originally defined on smooth compactly supported functions on $\mathbb{R}$. In the specific case that $m = 2$,

$$H_2^\beta(x) = \frac{1}{\sqrt{4\pi\beta}} e^{-\frac{|x|^2}{4\beta}} \tag{2.5}$$

is the heat kernel evaluated at complex time β. There is an extensive theory concerning these semigroups and generalizations thereof for $\mathrm{Re}(\beta) > 0$. In the context of $\mathbb{R}^d$, we refer the reader to the articles [9, 21] which consider general self-adjoint operators with measurable coefficients, called superelliptic operators, each comparable to $(\Delta)^{m/2}$ for some even $m \geq 2$. In the context of Lie groups, such generalizations are treated by [33, 76, 77]. An integral piece of this theory concerns off-diagonal estimates for these kernels. In our setting, this is the estimate

$$|H_m^\beta(x)| \leq C \exp(-B|x|^{\frac{m}{m-1}}) \tag{2.6}$$

for all $x \in \mathbb{R}$, where $C, B > 0$. Given (2.4), a complex change of variables via contour integration followed by a minimization argument easily yields the estimate (2.6) (see Proposition 5.3 of [77]).

Viewing things from a slightly different perspective, when $m \geq 2$ is even and $\mathrm{Re}(\beta) > 0$, the function $Z : (0, \infty) \times \mathbb{R} \to \mathbb{C}$, defined by

$$Z(t, x) = H_m^{t\beta}(x),$$

is a fundamental solution to the constant-coefficient parabolic equation

$$\frac{\partial}{\partial t} + i^m \beta \frac{\partial^m}{\partial x^m} = 0.$$

The treatise [35] by S. D. Eidelman gives an extensive treatment of such "higher order" parabolic equations with variable coefficients on $\mathbb{R}^d$. For second order parabolic systems ($m = 2$), A. Friedman's classic text [40] is an excellent reference.

Remark 2. *In the case that* $\mathrm{Re}(\beta) > 0$ *and* m *is even, the function* H_m^β *and the function* $H_{m,b}$ *used in Theorem 2.3 of [31] and defined by its Fourier transform,* $\hat{H}_{m,b}(\xi) = e^{-(1+ib)\xi^m}$*, are connected via the relation*

$$H_{m,\frac{\mathrm{Im}(\beta)}{\mathrm{Re}(\beta)}}\left(\frac{x}{(\mathrm{Re}(\beta))^{1/m}}\right) = (\mathrm{Re}(\beta))^{1/m} H_m^\beta(x)$$

which follows from the change of variables $u \mapsto (\mathrm{Re}(\beta))^{1/m} u$*.*

In the case that $m \geq 2$ is even and $\beta > 0$, the functions H_m^β are real valued and when $m > 2$ they take on both positive and negative values. As the classical Wiener measure is defined by the transition kernel H_2^1, V. Krylov [60] and later K. Hochberg [51] considered finitely additive signed measures on path space defined by H_m^1 for $m \in \{4, 6, 8, \dots\}$. Recently, D. Levin and T. Lyons [65] used rough path theory to study these measures. Both Krylov and Hochberg associated something like a process to such finitely additive measures, called signed Wiener measures in [51], to mimic the way that Brownian motion is associated to Wiener measure. This theory has been pursued recently by a number of authors [53, 61, 62, 68, 79], and such "processes" are now called pseudo-processes; the pseudo-process corresponding to H_4^1 is called the biharmonic pseudo- process. We do not pursue signed Wiener measures or pseudo-processes here.

When β is purely imaginary and $m \geq 2$, the situation is very different from those described above. The graphs of $\mathrm{Re}(H_m^i(x))$ for $m = 2, 3, 4, 5$ are illustrated in Figure 2.4. When $\beta = i/m$, $H_m^\beta = H_m^{i/m}$ satisfies the ordinary differential equation

$$\frac{d^{m-1}y}{dx^{m-1}} + (-i)^{m-1} xy = 0,$$

c.f., Remark 3 of [48]. When $m = 3$, this is Airy's equation and $H_3^{i/3}(x)$ is the famous Airy function, $\mathrm{Ai}(x)$. The study of the functions H_m^β, for β purely imag-

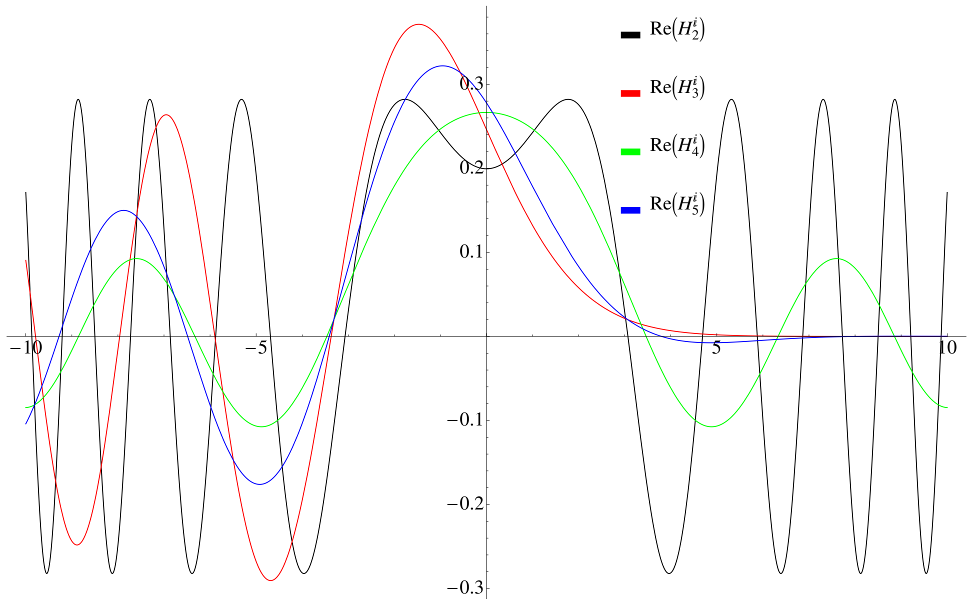

Figure 2.4: $\mathrm{Re}(H_m^i(x))$ for $m = 2, 3, 4, 5$

inary, dates back the 1920's. Such functions are closely related to those used by Hardy and Littlewood [44] in their consideration of Waring's problem. Using the method of steepest descent, Burwell [16] deduced asymptotic expansions for H_m^β for all $m > 2$ (see also [8]). Concerning global bounds for H_m^β, we cannot expect to have estimates of the form (2.6) when β is purely imaginary, for, in view of (2.5), $x \mapsto |H_2^\beta(x)|$ is constant. When $m > 2$, using oscillatory integral methods, we show that

$$|H_m^\beta(x)| \leq \frac{A}{|x|^{\frac{(m-2)}{2(m-1)}}} + \frac{B}{|x|}$$

for all real numbers x, where $A, B > 0$. This estimate can also be deduced from the asymptotic expansions of Burwell [16]. Our estimates are seen to be sharp in view of this comparison.

Returning to our discussion of convolution powers, let us momentarily view the situation with a probabilistic eye. Suppose that μ is a signed Borel measure on

$\mathbb{R}$ and $X_1, X_2, \ldots$ are independent "random" variables each with distribution μ. The distribution of the sum $S_n := X_1 + X_2 + \cdots X_n$, for $n = 1, 2, \ldots$, is the measure $\mu^{(n)}$ and can be computed by taking successive convolution powers of the measure μ. Limit theorems are seen to be affirmative answers to the following question: Does $\mu^{(n)}$, properly scaled, converge in any sense as $n \to \infty$ and if so, to what? In [51, 52], K. Hochberg proved a class of central limit theorems. They essentially state that, under certain conditions on μ, there exists an even natural number $m \geq 2$ such that the signed Borel measures $\{\nu_n\}_{n\geq 1}$, defined by $\nu_n(B) = \mu^{(n)}(n^{1/m}B)$ for any Borel set B, converge weakly to the measure with density H_m^1 with respect to Lebesgue measure. R. Hersch [48] proved a class of central limit theorems in which "random" variables are allowed to take values in an abstract algebra over $\mathbb{R}$ (see also [97]). Like Hochberg, Hersch's central limit theorems also involve weak convergence, however, the class of attractors in [48] is different. It consists of the Dirac mass and the measures with densities $H_m^{-i^m/m!}$ for all $m \geq 2$ such that $m \not\equiv 0 \bmod 4$. Local limit theorems, by contrast, focus on convergence of the density of $\mu^{(n)}$. In our case, these are statements of uniform (or local uniform) convergence of $\phi^{(n)}(x)$ as $n \to \infty$. Local limit theorems, in the case that ϕ is generally real valued, were treated by I. Schoenberg [80] and T. Greville [42] in connection to De Forest's problem in data smoothing. Their local limit theorems involve a certain subclass of our attractors, namely H_m^β for $m \geq 2$ even and $\beta > 0$, and $H_m^{i\tau}$ for $m > 1$ odd and $\tau \in \mathbb{R}$. The local limit theorems of Schoenberg and Greville involve ad hoc assumptions that are too restrictive for us; Theorem 2.1.2 extends their results. In the case that ϕ has admissible support, Theorem 2.1.2 also extends the results of [31]. We refer the reader to Section 2 of [31] for a brief review of local limit limit theorems and their connection to data smoothing and numerical difference schemes for partial differential equations.

The behavior of the convolution powers $\phi^{(n)}$ is determined by the local behavior of $\hat{\phi}$ by means of the Fourier inversion formula (2.17). The latter two main results of this chapter, Theorems 2.1.2 and 2.1.3, are both stated under the assumption that $\sup_\xi |\hat{\phi}(\xi)| = 1$; this can always be arranged by replacing ϕ by $A^{-1}\phi$ for an appropriate constant $A > 0$. Theorems 2.1.2 and 2.1.3 involve a number of constants and we now proceed to describe how they come about. First, we consider $\hat{\phi}(\xi)$ for $\xi \in (-\pi, \pi]$ and determine the set of points $\Omega(\phi) \subseteq (-\pi, \pi]$ at which $|\hat{\phi}(\xi)| = \sup |\hat{\phi}| = 1$. When ϕ is an aperiodic and irreducible random walk, this supremum is attained only at 0 (see Lemma 2.3.1 of [63] and its subsequent remark), but in general, $|\hat{\phi}(\xi)| = 1$ at multiple such points. In Section 2.2, we show that the set $\Omega(\phi)$ is finite. Second, for each $\xi_0 \in \Omega(\phi)$, we consider the Taylor expansion for $\log(\hat{\phi}(\xi + \xi_0)/\hat{\phi}(\xi_0))$ on a neighborhood of zero. In general, this series is of the form

$$i\alpha\xi - \beta\xi^m + o(\xi^m)$$

as $\xi \to 0$, where $m = m(\xi_0) \in \{2, 3, 4, \dots\}$, $\alpha = \alpha(\xi_0) \in \mathbb{R}$ and $\beta = \beta(\xi_0) \in \mathbb{C}$ with $\mathrm{Re}(\beta(\xi_0)) \geq 0$. Further, we show that $\mathrm{Re}(\beta(\xi_0)) = 0$ whenever $m(\xi_0)$ is odd. The constants $\alpha(\xi_0)$ and $\beta(\xi_0)$ play the roles of the mean and first non-vanishing moment of order $m(\xi_0) \geq 2$ for probability distributions (see Remark 4 of Section 2.2). Next, we set

$$m_\phi = \max_{\xi_0 \in \Omega(\phi)} m(\xi_0) \tag{2.7}$$

and restrict our attention to the subset of points $\{\xi_1, \xi_2, \dots, \xi_R\}$ of $\Omega(\phi)$ for which $m(\xi_q) = m_\phi$. We show that the contribution to $\phi^{(n)}$ by $\hat{\phi}$ near $\xi_0 \in \Omega(\phi)$ is on the order of $n^{-1/m(\xi_0)}$ (see Lemma 2.3.5). Because $n^{-1/m(\xi_0)} = o(n^{-1/m_\phi})$ as $n \to \infty$

whenever $m(\xi_0) < m_\phi$, the influence on $\phi^{(n)}$ from such points is not seen in local limits; it is only the points ξ_q for which $m(\xi_q) = m_\phi$ that matter. Finally, for each $q = 1, 2, \ldots, R$, we set $\beta_q = \beta(\xi_q)$ and $\alpha_q = \alpha(\xi_q)$.

We now state our second main theorem, the first to involve local limits.

Theorem 2.1.2. *Let $\phi : \mathbb{Z} \to \mathbb{C}$ have admissible support and be such that $\sup_\xi |\hat{\phi}(\xi)| = 1$. Referring to the constants above and setting $m = m_\phi$, suppose additionally that*

$$m > 2 \text{ or } \operatorname{Re}(\beta_q) > 0 \text{ for all } q = 1, 2, \ldots, R. \tag{2.8}$$

Then there exists a compact set $K \subseteq \mathbb{R}$ such that the supremum of $|\phi^{(n)}|$ is attained on

$$\left(\bigcup_{q=1}^{R} (\alpha_q n + K n^{1/m}) \right) \bigcap \mathbb{Z} \tag{2.9}$$

and

$$\phi^{(n)}(x) = \sum_{q=1}^{R} n^{-1/m} e^{-ix\xi_q} \hat{\phi}(\xi_q)^n H_m^{\beta_q} \left(\frac{x - \alpha_q n}{n^{1/m}} \right) + o(n^{-1/m}) \tag{2.10}$$

uniformly in $\mathbb{Z}$.

Remark 3. *If $\phi : \mathbb{Z} \to \mathbb{C}$ has admissible support and is such that $\sup_\xi |\hat{\phi}(\xi)| = 1$, hypothesis* (2.8) *is equivalent to the condition that, for every $\xi_0 \in (-\pi, \pi]$ for which $|\hat{\phi}(\xi_0)| = 1$,*

$$\frac{d^2}{d\xi^2} \log \hat{\phi}(\xi) \Big|_{\xi_0} \neq i\tau$$

for any non-zero real number τ.

As the conclusion (2.9) suggests, the interesting behavior of $\phi^{(n)}$ occurs on the moving sets $\alpha_q n + K n^{1/m}$ called *packets*. Each packet drifts with (and expands around) the point $\alpha_q n$ and so we call α_q a *drift constant*. There is much gained in studying $\phi^{(n)}$ by zooming in on its packets, i.e., choosing a drift constant α_q from

$\{\alpha_1, \alpha_2, \dots, \alpha_R\}$ and studying $\phi^{(n)}(\lfloor \alpha_q n + x n^{1/m} \rfloor)$ where x lives in a compact set (see Subsection 2.8.1). In doing this, we arrive at our third main result, a local limit theorem in which only the points $\xi_l \in \{\xi_1, \xi_2, \dots, \xi_R\}$ and corresponding attractors $H_m^{\beta_l}$ appear, provided $\alpha_l = \alpha_q$.

Theorem 2.1.3. *Let $\phi : \mathbb{Z} \to \mathbb{C}$ have admissible support and be such that $\sup_\xi |\hat{\phi}(\xi)| = 1$. Then, referring to the collections $\xi_1, \xi_2, \dots, \xi_R$, $\beta_1, \beta_2, \dots, \beta_R$ and $\alpha_1, \alpha_2, \dots, \alpha_R$ and setting $m = m_\phi$, the following holds: To each α_q, there exist subcollections $\xi_{j_1}, \xi_{j_2}, \dots, \xi_{j_{r(q)}}$ and $\beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_{r(q)}}$ such that*

$$\phi^{(n)}(\lfloor \alpha_q n + x n^{1/m} \rfloor) = \sum_{j=1}^{r(q)} n^{-1/m} e^{-i\lfloor \alpha_q n + x n^{1/m} \rfloor \xi_{j_l}} \hat{\phi}(\xi_{j_l})^m H_m^{\beta_{j_l}}(x) + o(n^{-1/m}) \quad (2.11)$$

uniformly for x in any compact set.

We note that Theorem 2.1.3 does not require the hypothesis (2.8) of Theorem 2.1.2. The hypothesis rules out the situation in which $\phi^{(n)}$ is approximated by H_2^β where β is purely imaginary. Correspondingly, the example where ϕ is defined by (2.1) can be treated by Theorem 2.1.3 but not Theorem 2.1.2. For the generality gained by eliminating the hypothesis (2.8) we lose the uniformity of the limit (2.10) on all of $\mathbb{Z}$. As we illustrate in Subsection 2.8.1, the conclusion (2.11) is sometimes more informative anyway. The limits (2.10) and (2.11) of Theorems 2.1.2 and 2.1.3 both involve a sum of the attractors H_m^β. We remark that these sums are not identically zero and, in fact, each sum is bounded below in absolute value by $Cn^{-1/m}$ for some constant $C > 0$; this is demonstrated in Section 2.6 and is used to establish the lower estimate in Theorem 2.1.1.

Everything in this chapter pertains to a single dimension. In Chapter 3, we will study the convolution powers $\phi^{(n)}$ where $\phi : \mathbb{Z}^d \to \mathbb{C}$ is subject to some restrictive assumptions. Let us simply note here that the situation is more complicated in the $\mathbb{Z}^d$ setting. For example, it is not clear what the analogue of Theorem 2.1.1 should be. Further, at points $\xi_0 \in (-\pi, \pi]^d$ where the Fourier transform satisfies $\sup_\xi |\hat{\phi}(\xi)| = |\hat{\phi}(\xi_0)| = 1$, $|\hat{\phi}(\xi)|$ can decay at different rates along different directions. This behavior will be seen to affect local limits in which attractors exhibit anisotropic scaling. For instance, by taking a tensor product of admissible functions (in the sense of the present article), one can easily construct $\phi : \mathbb{Z}^2 \to \mathbb{C}$ for which

$$\phi^{(n)}(x) = n^{-3/4} H_2^{\beta_1}(n^{-1/2}x_1) H_4^{\beta_2}(n^{-1/4}x_2) + o(n^{-3/4})$$

uniformly for $x = (x_1, x_2) \in \mathbb{Z}^2$, where $\mathrm{Re}(\beta_1), \mathrm{Re}(\beta_2) > 0$.

The chapter is organized as follows: In Section 2.2, we study the local behavior of the Fourier transform of ϕ. In Section 2.3, we address some technical lemmas involving oscillatory integrals and prove the estimate $A^{-n}\|\phi^{(n)}\|_\infty \leq C' n^{-1/m}$ of Theorem 2.1.1; this is Theorem 2.3.6. Section 2.4 concentrates on the attractors H_m^β where convergence, analyticity and global bounds are addressed. The local limit theorems of Theorems 2.1.3 and 2.1.2 are proven in Section 2.5. In Section 2.6, we complete the proof of Theorem 2.1.1 and in Section 2.7, the conclusion (2.9) of Theorem 2.1.2 is proven. Section 2.8 gives examples and addresses a general situation previously treated in [31].

2.2 Local behavior of $\hat{\phi}$

In this section we study the local behavior of $\hat{\phi}$ at points in $(-\pi, \pi]$ at which the supremum of $|\hat{\phi}|$ is attained. This will be seen to completely determine the limiting behavior of the convolution powers of ϕ. We proceed by making some simple observations about (2.2) under the assumptions that ϕ has admissible support and $\sup_\xi |\hat{\phi}(\xi)| = 1$. Our first observation concerns the number of points at which $|\hat{\phi}(\xi)| = 1$. Because ϕ has admissible support, $|\hat{\phi}|^2$ is a non-constant trigonometric polynomial and so $|\hat{\phi}|$ is not constant. From here we observe that $\hat{\phi}$ can only satisfy $|\hat{\phi}(\xi)| = 1$ at a finite number of points in $(-\pi, \pi]$; a simple accumulation-point argument shows the necessity of this fact. We now observe that $\hat{\phi}$ is a finite linear combination of exponentials and is therefore analytic. We use this observation to study the local behavior of $\hat{\phi}(\xi)$ about any point $\xi_0 \in (-\pi, \pi]$ for which $|\hat{\phi}(\xi_0)| = 1$. To this end we consider

$$\Gamma(\xi) = \log\left(\frac{\hat{\phi}(\xi + \xi_0)}{\hat{\phi}(\xi_0)}\right), \tag{2.12}$$

where $\log$ is taken to be the principle branch of logarithm and we allow the variable ξ to be complex, for the time being. It follows from our remarks above that Γ is analytic on an open neighborhood of 0 and we can therefore consider its convergent Taylor series

$$\Gamma(\xi) = \sum_{l=1}^{\infty} a_l \xi^l$$

on this neighborhood. The limiting behavior of the convolution powers of ϕ is characterized by the first few non-zero terms of this series.

The requirement that $|\hat{\phi}(\xi)| \leq 1$ imposes conditions on the Taylor expansion for Γ as follows: We consider the collection $\{a_l\}_{l=1}^{\infty}$ of coefficients of the series and let $k = \min\{l \geq 1 : \operatorname{Re}(a_l) \neq 0\}$, which exists, for otherwise $|\hat{\phi}|$ would be constant. We claim that k is even and $\operatorname{Re}(a_k) < 0$. To see this we observe that by only considering real values of ξ we can find a neighborhood of 0 on which

$$e^{C\xi^k} \leq |\hat{\phi}(\xi + \xi_0)| = |\hat{\phi}(\xi_0) e^{\Gamma(\xi)}|$$

and where C is a real constant having the same sign as $\mathrm{Re}(a_k)$. If it is the case that $\mathrm{Re}(a_k) > 0$ or k is odd we have that $|\hat{\phi}(\xi+\xi_0)| > 1$ for some ξ which leads to a contradiction. We will summarize the above arguments shortly. First we give the following convenient definition, originally motivated by Thomée [87].

Definition 2.2.1. *Let $\nu : \mathbb{R} \to \mathbb{C}$ be analytic on a neighborhood of a point ξ_0 for which $|\nu(\xi_0)| = 1$. Let $\Gamma : \mathcal{O} \subseteq \mathbb{R} \to \mathbb{C}$ be defined by*

$$\Gamma(\xi) = \log\left(\frac{\nu(\xi+\xi_0)}{\nu(\xi_0)}\right), \tag{2.13}$$

where $\mathcal{O}$ is an open neighborhood of 0 and is such that $\mathcal{O} \ni \xi \mapsto \nu(\xi+\xi_0)$ is non-vanishing.

1. *We say that ξ_0 is a point of type 1 and of order m for ν if the Taylor expansion for* (2.13) *yields an even integer $m \geq 2$, a real number α, and a complex number β with $\mathrm{Re}(\beta) > 0$ such that*

$$\Gamma(\xi) = i\alpha\xi - \beta\xi^m + \sum_{l=m+1}^{\infty} a_l \xi^l \tag{2.14}$$

on $\mathcal{O}$. In this case we write $\xi_0 \sim (1; m)$.

2. *We say that ξ_0 is a point of type 2 and of order m for ν if the Taylor expansion for* (2.13) *yields $m, k, \alpha, \gamma, p(\xi)$, where m and k are natural numbers with k even and $1 < m < k$; α and γ are real numbers with $\gamma > 0$; and $p(\xi)$ is a real polynomial with $p(0) \neq 0$ such that*

$$\Gamma(\xi) = i\alpha\xi - i\xi^m p(\xi) - \gamma\xi^k + \sum_{l=k+1}^{\infty} a_l \xi^l. \tag{2.15}$$

on $\mathcal{O}$. In this case we write $\xi_0 \sim (2; m)$ and set $\beta = ip(0)$.

In both cases, the order m refers to the degree of the first non-vanishing term, higher than degree one, in the Taylor expansion for Γ. The type refers to the complex phase of *coefficient of this term: it is of type* 1 *if the coefficient has a non-zero real part, otherwise it is of type* 2*. In either case we refer to the constant* α *as the drift constant for* ξ_0*.*

Let us note that the neighborhood $\mathcal{O}$ in the above definition is immaterial; it needs only to be small enough to ensure that $\log$ is defined and analytic there. Using the definition, the arguments which preceded it are summarized in the following proposition.

Proposition 2.2.2. *Let* $\phi : \mathbb{Z} \to \mathbb{C}$ *have admissible support. Suppose that the Fourier transform of* ϕ *satisfies* $\sup_\xi |\hat{\phi}(\xi)| = 1$*. Then*

$$\Omega(\phi) = \{\xi' \in (-\pi, \pi] : |\hat{\phi}(\xi')| = 1\}$$

is finite and, for $\xi_0 \in \Omega(\phi)$*, we have either* $\xi_0 \sim (1; m)$ *or* $\xi_0 \sim (2; m)$ *for some natural number* $m = m(\xi_0) \geq 2$*.*

Convention 2.2.3. *For any* $\phi : \mathbb{Z} \to \mathbb{C}$ *satisfying the hypotheses of Proposition 2.2.2, set*

$$m = \max_{\xi_0 \in \Omega(\phi)} m(\xi_0). \tag{2.16}$$

In view of the proposition, we can write

$$\Omega(\phi) = \{\xi_1, \xi_2, \dots, \xi_Q\},$$

where we shall henceforth assume that $\Omega(\phi)$ *is indexed in the following way:*

- *For each* $q = 1, 2, \dots, R$*,* $\xi_q \sim (1; m_q)$ *or* $\xi_q \sim (2; m_q)$ *with* $m_q = m$ *and associated constants* α_q *and* β_q*.*
- *For each* $q = R+1, R+2, \dots, Q$*,* $\xi_q \sim (1; m_q)$ *or* $\xi_q \sim (2; m_q)$ *with* $m_q < m$*.*

Hence, to the points $\{\xi_1, \xi_2, \dots, \xi_R\} \subseteq \Omega(\phi)$ *we have the associated collections* $\alpha_1, \alpha_2, \dots, \alpha_R$ *of real numbers and* $\beta_1, \beta_2, \dots, \beta_R$ *of non-zero complex numbers with* $\mathrm{Re}(\beta_q) \geq 0$ *for* $q = 1, 2, \dots, R$*.*

We remark that $\Omega(\phi)$, $m\,(=m_\phi)$, and the constants α_q and β_q for $q = 1, 2, \ldots, R$ of Convention 2.2.3 are consistent with those of the discussion preceding the statement of Theorem 2.1.2. Therefore, the constants appearing in Theorems 2.1.2 and 2.1.3 are those of Convention 2.2.3.

Remark 4. *There is an alternative way to find the constants m_q α_q and β_q above. For any function $f : \mathbb{Z} \to \mathbb{C}$, define*

$$\mathbb{E}f(X) = \sum_{x\in\mathbb{Z}} f(x)\phi(x),$$

where X is to be understood as a "random" variable with distribution ϕ. For each $\xi_q \in \Omega$, put

$$a(\xi_q) = \frac{\mathbb{E}Xe^{i\xi_q X}}{\hat{\phi}(\xi_q)}$$

and, for each natural number $k \geq 2$,

$$b_k(\xi_q) = \frac{i^k}{k!}\left(a(\xi_q)^k - \frac{\mathbb{E}X^k e^{i\xi_q X}}{\hat{\phi}(\xi_q)}\right).$$

It is easily shown that $\alpha_q = a(\xi_q)$, $m_q = \min\{k \geq 2 : b_k(\xi_q) \neq 0\}$ and $\beta_q = b_{m_q}(\xi_q)$. Proposition 2.4 of [31] gives a class of examples for which ϕ is real valued, $\Omega(\phi) = \{0\}$ and $m = m_1 = 2l$ for any specified $l \geq 1$. Necessarily, $b_k(0) = 0$ for all $k < 2l$.

If we further assume that $\phi \geq 0$ and $\Omega(\phi) = \{0\}$, it follows that $b_2(0) \neq 0$ and so $m = 2$. This situation is equivalent to the case in which ϕ is the distribution of a random variable X with state space $\mathbb{Z}$ such that Supp(ϕ) is not contained in any proper subgroup of $\mathbb{Z}$. Here $\alpha_1 = \mathbb{E}X$ and $2\beta_1 = \mathbb{E}X^2 - (\mathbb{E}X)^2 =$ Var(X). In this way, the standard local limit theorem is captured by Theorem 2.1.2.

To exploit the interplay between local approximations of $\hat{\phi}$ and the Fourier inversion formula, it useful to consider a domain of integration T in which $\Omega(\phi)$

sits in the interior. To this end, let $\eta \geq 0$ be such that $\Omega(\phi) \subseteq (-\pi+\eta, \pi+\eta)$ and set $T = (-\pi+\eta, \pi+\eta]$. Of course, for each natural number n and $x \in \mathbb{Z}$, we have

$$\phi^{(n)}(x) = \frac{1}{2\pi} \int_T e^{-ix\xi} \hat{\phi}(\xi)\, d\xi. \tag{2.17}$$

It is also useful to consider the following extension of $\phi^n(x)$: Define $\phi_{\mathrm{e}} : \mathbb{N} \times \mathbb{R} \to \mathbb{C}$ by

$$\phi_{\mathrm{e}}(n, x) = \frac{1}{2\pi} \int_T e^{-ix\xi} \hat{\phi}(\xi)\, d\xi \tag{2.18}$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. We note that $\phi_{\mathrm{e}}(n, x) = \phi^{(n)}(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}$.

The following lemma is seen to govern the limiting behavior of the convolution powers of ϕ.

Lemma 2.2.4. *Let $\nu : \mathbb{R} \to \mathbb{C}$ be analytic on a neighborhood of a point ξ_0 such that $|\nu(\xi_0)| = 1$.*

1. *If $\xi_0 \sim (1; m)$, then there exist $\delta > 0$ and $B, C > 0$ such that*

$$|\Gamma(\xi) - i\alpha\xi + \beta\xi^m| \leq B|\xi|^{m+1} \tag{2.19}$$

and

$$|\nu(\xi + \xi_0)| \leq e^{-C\xi^m} \tag{2.20}$$

for all $|\xi| \leq \delta$. Here Γ, α, and β are given by Definition 2.2.1.

2. *If $\xi_0 \sim (2; m)$, there exists $\delta > 0$ and $B > 0$ such that*

$$|\Gamma(\xi) - i\alpha\xi + ip(\xi)\xi^m| \leq B\xi^k \tag{2.21}$$

for all $|\xi| \leq \delta$. Moreover, there exist $C, D > 0$ such that the function

$$g(\xi) = \nu(\xi_0)^{-1}\nu(\xi + \xi_0)\exp(-i\alpha\xi + i\xi^m p(\xi))$$

satisfies

$$|g(\xi)| \le e^{-C\xi^k} \tag{2.22}$$

and

$$|g'(\xi)| \le D|\xi|^{k-1} e^{-C\xi^k} \tag{2.23}$$

for all $|\xi| \le \delta$. *Here* Γ, k, α, *and* $p(\xi)$ *are given by Definition 2.2.1.*

Proof. By our definitions, we have

$$\nu(\xi + \xi_0) = \nu(\xi_0) e^{\Gamma(\xi)},$$

where Γ is defined by (2.13). In the case that $\xi_0 \sim (1; m)$, (2.19) and (2.20) are immediate from (2.14) and the fact that the series $\sum_{l=m+1} a_l \xi^l$ converges uniformly on a neighborhood of 0.

In the case that $\xi_0 \sim (2; m)$, the justification of the estimates (2.21) and (2.22) follows similarly. For the last conclusion, we observe that

$$\begin{aligned} g'(\xi) &= \frac{d}{d\xi} \exp(-i\alpha\xi + i\xi^m p(\xi)) e^{\Gamma(\xi)} \\ &= \frac{d}{d\xi} \exp\left(-\gamma\xi^k + \sum_{l=k+1}^{\infty} a_l \xi^l\right) \\ &= \left(-\gamma k \xi^{k-1} + \sum_{l=k+1}^{\infty} a_l l \xi^{l-1}\right) g(\xi) \end{aligned}$$

on a neighborhood of 0. The inequality (2.23) now follows without trouble. □

2.3 The upper bound

The goal of this section is to establish the upper bound of Theorem 2.1.1. To this end, we address a series of technical lemmas involving oscillatory integrals of the form

$$\int_a^b g(\xi)e^{if(\xi)}d\xi$$

which are used throughout the remainder of the chapter. Many of the arguments within are based on the same or slightly less general arguments made by Greville [42], Thomée [87] and, not surprisingly, van der Corput.

Lemma 2.3.1. *Let $h \in L^1([a,b])$ and $g \in \mathcal{C}^1([a,b])$ be complex valued. Then for any M such that*

$$\left|\int_a^x h(u)du\right| \leq M$$

for all $x \in [a,b]$ we have

$$\left|\int_a^b g(u)h(u)du\right| \leq M\left(\|g\|_\infty + \|g'\|_1\right).$$

Proof. For $h \in L^1([a,b])$, the function

$$f(x) = \int_b^x h(u)du$$

is absolutely continuous and $f'(x) = h(x)$ almost everywhere. Furthermore, our hypothesis guarantees that $|f(x)| \leq M$ for all $x \in [a,b]$. Integration by parts yields

$$\int_a^b g(u)h(u)du = [g(u)f(u)]_a^b - \int_a^b g'(u)f(u)du$$

and therefore

$$\begin{aligned}\left|\int_a^b g(u)h(u)du\right| &\leq |f(b)g(b)| + 0 + \int_a^b |f(u)||g'(u)|du \\ &\leq M\|g\|_\infty + M\|g'\|_1.\end{aligned}$$

□

The following two lemmas, 2.3.2 and 2.3.3, are due originally to van der Corput. The proof of Lemma 2.3.2 is a nice application of the second mean value theorem for integrals and can be found in [87]. We note that Lemma 2.3 of [87] is stated under slightly stronger hypotheses than Lemma 2.3.2, however the proof yields our statement exactly. The validity of Lemma 2.3.2 can also be seen using alternating series [42]. For the proof of Lemma 2.3.3, we refer the reader to Lemma 3.3 of [87]; its proof is relatively simple but involves checking several cases (see also Chapter 1 of [14]).

Lemma 2.3.2. *Let $f \in \mathcal{C}^1([a,b])$ be real valued and suppose that f' is a monotonic function such that $f'(x) \neq 0$ for all $x \in [a,b]$. Then,*

$$\left| \int_a^b e^{if(u)} du \right| \leq \frac{4}{\lambda}, \tag{2.24}$$

where

$$\lambda = \inf_{x \in [a,b]} |f'(x)|. \tag{2.25}$$

Lemma 2.3.3. *Let $f \in \mathcal{C}^2([a,b])$ be real valued and suppose that $f''(x) \neq 0$ for all $x \in [a,b]$. Then*

$$\left| \int_a^b e^{if(u)} du \right| \leq \frac{8}{\sqrt{\rho}},$$

where

$$\rho = \inf_{x \in [a,b]} |f''(x)|.$$

Lemma 2.3.4. *Let $g \in \mathcal{C}^1([a,b])$ be complex valued and let $f \in \mathcal{C}^2([a,b])$ be real valued and such that $f''(x) \neq 0$ for all $x \in [a,b]$. Then*

$$\left| \int_a^b g(u) e^{if(u)} du \right| \leq \min \left\{ \frac{4}{\lambda}, \frac{8}{\sqrt{\rho}} \right\} \left(\|g\|_\infty + \|g'\|_1 \right),$$

where $\lambda = \inf_{x \in [a,b]} |f'(x)|$ and $\rho = \inf_{x \in [a,b]} |f''(x)|$.

Proof. Combining the results of Lemmas 2.3.2 and 2.3.3 show

$$\left| \int_a^x e^{if(u)} du \right| \leq \min \left\{ \frac{4}{\lambda}, \frac{8}{\sqrt{\rho}} \right\}$$

for any $x \in [a, b]$. We remark that $4/\lambda$ only contributes to the upper bound provided f' is never zero, in which case the application of Lemma 2.3.2 is justified. Setting $h(u) = e^{if(u)}$ we note that the functions g and h are the subject of Lemma 2.3.1. The result now follows immediately from Lemma 2.3.1. $\square$

Lemma 2.3.5. *Let $\nu : \mathbb{R} \to \mathbb{C}$ be analytic on a neighborhood of ξ_0 where $|\nu(\xi_0)| = 1$. If ξ_0 is a point of order $m \geq 2$ for ν, then there is $\delta > 0$ such that*

$$\frac{1}{2\pi} \int_{|\xi - \xi_0| \leq \delta} \nu(\xi)^n e^{-ix\xi} d\xi = O(n^{-1/m}),$$

where the limit is uniform in $x \in \mathbb{R}$.

Proof. Let us first assume that $\xi_0 \sim (1; m)$. Our hypothesis guarantees that m is even and by Lemma 2.2.4 there are constants $C > 0$ and $\delta > 0$ such that

$$|\nu(\xi + \xi_0)| \leq e^{-C\xi^m}$$

for all $-\delta \leq \xi \leq \delta$. Therefore

$$\begin{aligned}
\left| \frac{1}{2\pi} \int_{|\xi - \xi_0| \leq \delta} \nu(\xi)^n e^{-ix\xi} d\xi \right| &\leq \int_{-\delta}^{\delta} |\nu(\xi + \xi_0)|^n d\xi \\
&\leq \int_{\mathbb{R}} e^{-nC\xi^m} d\xi \\
&\leq \frac{M}{n^{1/m}}.
\end{aligned}$$

In the second case we assume that $\xi_0 \sim (2; m)$. We set

$$g(\xi) = \left[\nu(\xi_0)^{-1} \nu(\xi + \xi_0) \exp(-i\alpha\xi + i\xi^m p(\xi))\right]$$

and

$$f_n(\xi, x) = (n\alpha - x)\xi - n\xi^m p(\xi).$$

We note that f_n is real valued. Appealing to Lemma 2.2.4, let $\delta > 0$ be chosen so that the estimates (2.22) and (2.23) hold for all $\xi \in [-\delta, \delta]$. Upon changing variables of integration and using the fact that $|\nu(\xi_0)| = 1$, we write

$$\left|\frac{1}{2\pi}\int_{|\xi-\xi_0|\le\delta} \nu(\xi)^n e^{-ix\xi} d\xi\right| \le \sum_{j=1}^{3}\left|\int_{I_j} g(\xi)^n e^{if_n(\xi,x)} d\xi\right|,$$

where $I_1 = [-\delta, -n^{1/m}]$, $I_2 = (-n^{1/m}, n^{1/m})$ and $I_3 = [n^{1/m}, \delta]$. On the interval I_2, $|g(\xi)| \le 1$ by (2.22) and therefore

$$\left|\int_{I_2} g(\xi)^n e^{if_n(\xi,x)} d\xi\right| \le \frac{2}{n^{1/m}}.$$

We now consider the integral over I_1 to which we will apply Lemma 2.3.4. First observe that the regularity requirements of Lemma 2.3.4 for f_n and g^n are met. Differentiating f_n twice with respect to ξ gives

$$\partial_\xi^2 f_n(\xi, x) = -n\frac{d^2}{d\xi^2}\xi^m p(\xi),$$

which is independent of x. Using the fact that $\xi^m p(\xi)$ is a polynomial with m being the smallest power of its terms, we may further restrict $\delta > 0$ so that

$$C^2|\xi|^{m-2} \le \left|\frac{d^2}{d\xi^2}\xi^m p(\xi)\right| \tag{2.26}$$

for some $C > 0$ and for all $\xi \in [-\delta, \delta]$. Consequently $|\partial_\xi^2 f_n(\xi, x)| > 0$ for all $\xi \in I_1$ and $x \in \mathbb{R}$. Appealing to Lemma 2.3.3 we have

$$\left|\int_{I_1} g(\xi)^n e^{if_n(\xi,x)} d\xi\right| \le \frac{8}{\sqrt{\lambda}}(\|g^n\|_\infty + \|ng'g^{n-1}\|_1), \tag{2.27}$$

where $\lambda = \inf_{\xi\in I_1} |\partial_\xi^2 f_n(\xi, x)|$. Using (2.26) and recalling that $m \ge 2$ we observe that

$$Cn^{1/m} = \sqrt{C^2 n|-n^{-1/m}|^{m-2}} \le \sqrt{\inf_{I_1} C^2 n|\xi|^{m-2}} \le \sqrt{\lambda}.$$

Now by (2.22) and (2.23) of Lemma 2.2.4 we have $\|g^n\|_\infty \leq 1$ and

$$\begin{aligned} \|ng'g^{n-1}\|_1 &= n\int_{I_1} |g'(\xi)g(\xi)^{n-1}|d\xi \\ &\leq n\int_{I_1} D|\xi|^{k-1}e^{-nC\xi^k}d\xi \\ &\leq \int_{\mathbb{R}} D|u|^{k-1}e^{-Cu^k}du = M < \infty. \end{aligned}$$

Inserting the above estimates into (2.27) gives

$$\left|\int_{I_1} g(\xi)^n e^{if_n(\xi,x)}d\xi\right| \leq \frac{8(1+M)}{Cn^{1/m}} = \frac{K_1}{n^{1/m}}.$$

A similar calculation shows that

$$\left|\int_{I_3} g_n(\xi)e^{if_n(\xi,x)}d\xi\right| \leq \frac{K_2}{n^{1/m}}$$

for some $K_2 > 0$. Putting these estimates together gives

$$\left|\frac{1}{2\pi}\int_{|\xi-\xi_0|\leq\delta} \nu(\xi)^n e^{-ix\xi}d\xi\right| \leq \frac{K_1}{n^{1/m}} + \frac{2}{n^{1/m}} + \frac{K_2}{n^{1/m}},$$

our desired inequality. $\square$

Theorem 2.3.6. *Let $\phi : \mathbb{Z} \to \mathbb{C}$ have admissible support and let $A = \sup_\xi |\hat{\phi}(\xi)|$. Then there is a natural number $m \geq 2$ and a real number $C' > 0$ such that*

$$A^{-n}\|\phi^{(n)}\|_\infty \leq C'n^{-1/m} \tag{2.28}$$

for all natural numbers n.

Proof. It suffices to prove the theorem in the case that $A = \sup_\xi |\hat{\phi}(\xi)| = 1$, for otherwise one simply multiplies the Fourier inversion formula by A^{-n}. In view of Proposition 2.2.2, we adopt Convention 2.2.3. For each $\xi_q \in \Omega(\phi)$ with associated $2 \leq m_q \leq m$, select $\delta_q > 0$ for which the conclusion of Lemma 2.3.5 holds and small enough to ensure that the intervals $I_q := [\xi_q - \delta_q, \xi_q + \delta_p] \subseteq T$

for $i = 1, 2, \dots, Q$ are disjoint. Set $J = T \setminus \cup_q I_q$ and $s = \sup_{\xi \in J} |\hat{\phi}(\xi)| < 1$. Using (2.17), we write

$$\begin{aligned} |\phi^{(n)}(x)| &= \left| \sum_{q=1}^{Q} \frac{1}{2\pi} \int_{I_q} \hat{\phi}(\xi)^n e^{-ix\xi} d\xi + \frac{1}{2\pi} \int_J \hat{\phi}(\xi)^n e^{-ix\xi} d\xi \right| \\ &\leq \sum_{q=1}^{Q} \left| \frac{1}{2\pi} \int_{I_q} \hat{\phi}(\xi)^n e^{-ix\xi} d\xi \right| + \frac{1}{2\pi} \int_J |\hat{\phi}(\xi)|^n d\xi \\ &\leq \sum_{q=1}^{Q} \left| \frac{1}{2\pi} \int_{I_q} \hat{\phi}(\xi)^n e^{-ix\xi} d\xi \right| + s^n. \end{aligned}$$

Using Lemma 2.3.5 we conclude that for every $x \in \mathbb{R}$

$$\begin{aligned} |\phi^{(n)}(x)| &\leq \sum_{q=1}^{Q} \frac{K_q}{n^{1/m_q}} + s^n \\ &\leq \frac{K}{n^{1/m}} + s^n \end{aligned}$$

from which the result follows. □

2.4 The attractors H_m^β

In this section we study the functions H_m^β defined by (2.4). Our first task is to show that the integral defining H_m^β converges in the senses indicated in the introduction.

Proposition 2.4.1. *Let $m \geq 2$ be a natural number and let β be a non-zero complex number such that $\mathrm{Re}(\beta) \geq 0$.*

1. *If m is even and $\mathrm{Re}\,\beta > 0$ then the integral defining $H_m^\beta(x)$ in* (2.4) *converges absolutely and uniformly in x on $\mathbb{R}$ as an improper Riemann integral.*

2. *If $m > 2$ and $\mathrm{Re}(\beta) = 0$ then the integral defining $H_m^\beta(x)$ converges uniformly in x on $\mathbb{R}$ as an improper Riemann integral.*

3. *If $m = 2$ and $\mathrm{Re}(\beta) = 0$ then for any compact set $K \subseteq \mathbb{R}$, the integral defining $H_m^\beta(x)$ converges uniformly in x on K as an improper Riemann integral.*

Proof. For item 1 there is nothing to prove, the result follows from the classical theory of Fourier transforms. For items *2* and *3*, let τ be the non-zero real number such that $\beta = i\tau$ and set $f(u, x) = -xu - \tau u^m$. Our job is to show that the integral

$$\int_{\mathbb{R}} e^{if(x,u)} du$$

converges in the senses indicated for $m > 2$ and $m = 2$ respectively.

We first consider the case where $m > 2$. Let $\epsilon > 0$ and choose M sufficiently large so that

$$\frac{8}{\sqrt{|\tau| m(m-1) M^{m-2}}} \leq \epsilon. \tag{2.29}$$

Observe that for any real numbers a and b such that $M < a \leq b$ or $a \leq b < -M$,

$$|\tau| m(m-1) M^{m-2} < \inf_{u \in [a,b]} |\partial_u^2 f(x,u)| =: \lambda.$$

We now apply Lemma 2.3.3 and conclude that

$$\left| \int_a^b e^{-ixu - \beta u^m} du \right| = \left| \int_a^b e^{if(x,u)} du \right| \leq \frac{8}{\sqrt{\lambda}} < \epsilon$$

for all $x \in \mathbb{R}$ and for all $a \leq b$ such that the distance from the interval $[a, b]$ to 0 is more than M. The Cauchy criterion for uniform convergence guarantees that the improper Riemann integrals

$$\int_0^\infty e^{-ixu - \beta u^m} du \text{ and } \int_{-\infty}^0 e^{-ixu - \beta u^m} du$$

converge uniformly in x on $\mathbb{R}$. This proves item *2*.

Let us now assume that $m = 2$. We remark that the above argument fails in this case because $\partial_u^2 f$ is a non-zero constant. Consequently, we need to use $\partial_u f$, which depends on both u and x, to bound our integrals. Let $\epsilon > 0$ and let $K \subseteq \mathbb{R}$ be a compact set. We choose $M > 0$ so that

$$\frac{4}{|2\tau M + x|} < \epsilon$$

for all $x \in K$. By applying Lemma 2.3.2 and making an argument analogous to that given in the previous case we conclude that

$$\left| \int_a^b e^{-ixu-\beta u^2} du \right| = \left| \int_a^b e^{if(x,u)} du \right| < \epsilon$$

for all $x \in K$ and for all $a \leq b$ such that the distance from the interval $[a, b]$ to 0 is more than M. Again, an application of the Cauchy criterion gives the desired result. □

Proposition 2.4.2. *Let β be non-zero and purely imaginary. Then for any natural number $m > 2$ there exist positive constants A, B such that*

$$|H_m^\beta(x)| \leq \frac{A}{|x|^{\frac{m-2}{2(m-1)}}} + \frac{B}{|x|} \tag{2.30}$$

for all $x \in \mathbb{R}$.

Proof. Let $\beta = i\tau$, where τ is a non-zero real number, and set $f(u, x) = -xu - \tau u^m$. For $x \neq 0$, put $M = (2m|\tau|/|x|)^{-1/(m-1)}$ and write

$$H_m^\beta(x) = \frac{1}{2\pi} \int_{-\infty}^{-M} e^{if(u,x)}\, du + \frac{1}{2\pi} \int_M^{\infty} e^{if(u,x)}\, du + \frac{1}{2\pi} \int_{-M}^{M} e^{if(u,x)}\, du. \tag{2.31}$$

Observe that

$$\begin{aligned} \inf_{u \in [-M,M]} |\partial_u f(u, x)| &= \inf_{u \in [-M,M]} |x + m\tau u^{m-1}| \\ &= m|\tau| \inf_{u \in [-M,M]} \left| \frac{x}{m\tau} + u^{m-1} \right| \geq m|\tau| M^{m-1} = \frac{|x|}{2}, \end{aligned}$$

and therefore

$$\left|\int_{-M}^{M} e^{if(u,x)}\,du\right| \le \frac{8}{|x|} \tag{2.32}$$

in view of Lemma 2.3.2. Similarly, there exists $C > 0$ such that for any $N > M$,

$$\inf_{u\in[M,N]} |\partial_u^2 f(u,x)| \ge \frac{|x|^{\frac{m-2}{m-1}}}{C^2}.$$

Thus, by appealing to Lemma 2.3.3 and Proposition 2.4.1, we have

$$\begin{aligned}\left|\int_{M}^{\infty} e^{if(u,x)}\,du\right| &= \lim_{N\to\infty}\left|\int_{M}^{N} e^{if(u,x)}\,du\right| \\ &\le \limsup_{N} \frac{8}{\sqrt{\inf_{u\in[M,N]}|\partial_u^2 f(u,x)|}} \le \frac{C}{|x|^{\frac{m-2}{2(m-1)}}}.\end{aligned}$$

By an analogous computation,

$$\left|\int_{-\infty}^{M} e^{if(u,x)}\,du\right| \le \frac{C}{|x|^{\frac{m-2}{2(m-1)}}} \tag{2.33}$$

for $C > 0$. The desired result follows by combining the estimates (2.31), (2.32) and (2.33). $\square$

The final proposition of this section, Proposition 2.4.3, asserts the analyticity and non-triviality of the functions H_m^β for all values of m and β considered above. To preface it, let's consider the case in which $m \ge 2$ is even and $\mathrm{Re}(\beta) > 0$: For any $x \in \mathbb{R}$, observe that

$$H_m^\beta(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixu}e^{-\beta u^m}\,du = \frac{1}{2\pi}\int_{\mathbb{R}}\sum_{k=0}^{\infty}\frac{(-ixu)^k}{k!}e^{-\beta u^m}\,du.$$

Setting $2b = \mathrm{Re}(\beta)$, note that

$$\int_{\mathbb{R}}\sum_{k=0}^{\infty}\left|\frac{(-ixu)^k}{k!}e^{-\beta u^m}\right|\,du = \int_{\mathbb{R}}\sum_{k=0}^{\infty}\frac{|xu|^k}{k!}e^{-2bu^m}\,du = \int_{\mathbb{R}} e^{|xu|-bu^m}e^{-bu^m}\,du.$$

By a simple maximization argument, one finds that $|xu| - bu^m \le c|x|^{m/(m-1)}$ for all $u \in \mathbb{R}$, where $c = (1-m^{-1})(mb)^{-1/(m-1)} > 0$. Therefore,

$$\int_{\mathbb{R}}\sum_{k=0}^{\infty}\left|\frac{(-ixu)^k}{k!}e^{-\beta u^m}\right|\,du \le e^{c|x|^{m/(m-1)}}\int_{\mathbb{R}} e^{-bu^m}\,du < \infty$$

and so this application of Tonelli's theorem justifies the following use of Fubini's theorem:

$$\int_{\mathbb{R}} \sum_{k=0}^{\infty} \frac{(-ixu)^k}{k!} e^{-\beta u^m}\, du = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \frac{(-iu)^k x^k}{k!} e^{-\beta u^m}\, du.$$

Therefore

$$H_m^{\beta}(x) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}} \frac{(-iu)^k}{k!} e^{-\beta u^m}\, du \right) x^k \tag{2.34}$$

for each $x \in \mathbb{R}$; note that the convergence of the series is part of the conclusion. Consequently, H_m^{β} is analytic on $\mathbb{R}$. Moreover, from the representation (2.34), it is clear that $H_m^{\beta}(x)$ is not identically zero. When $m > 1$ is odd and β is purely imaginary, the same conclusion was reached by R. Hersch [48]. His proof involves changing the contour of integration from $\mathbb{R}$ to a pair of rays on which the integrand is absolutely integrable. When $m \geq 2$ is even and β is purely imaginary, Hersh's argument pushes through with very little modification. We therefore summarize the result below and, in the case that $\mathrm{Re}(\beta) = 0$, refer the reader to Theorem 4 of [48] for the essential details.

Proposition 2.4.3. *Let $m \geq 2$ be a natural number and let β be a non-zero complex number with $\mathrm{Re}(\beta) \geq 0$. If $\mathrm{Re}(\beta) > 0$ additionally assume that m is even. Then H_m^{β} is analytic and not identically zero.*

2.5 Local limits

In this section we prove Theorem 2.1.3 and the second conclusion, (2.10), of Theorem 2.1.2. To this end, the following three lemmas, Lemmas 2.5.1, 2.5.2 and 2.5.3, focus on local approximations to Fourier-type integrals involving integer powers of an analytic function ν near a point ξ_0 at which $|\nu(\xi_0)| = 1$. The lemmas treat the cases in which $\xi_0 \sim (1;m)$, $\xi_0 \sim (2;2)$ and $\xi_0 \sim (2;m)$, respectively. The approximants are precisely the functions H_m^β studied in the previous section.

Lemma 2.5.1. *Let $\nu : \mathbb{R} \to \mathbb{C}$ be analytic on a neighborhood of a point ξ_0 for which $|\nu(\xi_0)| = 1$. Assume that $\xi_0 \sim (1;m)$ with associated constants α and β. Then for all $\epsilon > 0$ there is a $\delta > 0$ and a natural number N such that*

$$\left| \frac{n^{1/m}}{2\pi} \int_{|\xi-\xi_0|\le\delta} \nu(\xi)^n e^{-ix\xi} d\xi - e^{-ix\xi_0}\nu(\xi_0)^n H_m^\beta\left(\frac{x-\alpha n}{n^{1/m}}\right)\right| < \epsilon \tag{2.35}$$

for all $n > N$ and for all real numbers x.

Proof. Let $\epsilon > 0$ and set

$$y_n = (x - \alpha n)n^{-1/m}$$

and

$$g(u) = \left[\nu(\xi_0)^{-1} e^{-i\alpha u n^{-1/m}} \nu(\xi_0 + un^{-1/m})\right].$$

Upon changing variables of integration we have

$$\begin{aligned}
&\frac{n^{1/m}}{2\pi} \int_{|\xi-\xi_0|\le\delta} \nu(\xi)^n e^{-ix\xi} d\xi \\
&= \frac{e^{-ix\xi_0}\nu(\xi_0)^n}{2\pi} \int_{|u|\le\delta n^{1/m}} \left[\nu(\xi_0)^{-1} e^{-i\alpha u n^{-1/m}} \nu(\xi_0 + un^{-1/m})\right]^n e^{-iu\frac{x-\alpha n}{n^{1/m}}} du \\
&= \frac{e^{-ix\xi_0}\nu(\xi_0)^n}{2\pi} \int_{|u|\le\delta n^{1/m}} g(u)^n e^{-iuy_n} du.
\end{aligned}$$

Comparing the above integral with $e^{-ix\xi_0}\nu(\xi_0)^n H_m^\beta(y_n)$ gives

$$\begin{aligned}
&\left|\frac{n^{1/m}}{2\pi}\int_{|\xi-\xi_0|\leq\delta}\nu(\xi)^n e^{-ix\xi}d\xi - e^{-ix\xi_0}\nu(\xi_0)^n H_m^\beta(y_n)\right| \\
&\quad\leq \left|\frac{e^{-ix\xi_0}\nu(\xi_0)^n}{2\pi}\int_{|u|\leq M}[g(u)^n - e^{-\beta u^m}]e^{-iuy_n}du\right| \\
&\qquad + \left|\frac{e^{-ix\xi_0}\nu(\xi_0)^n}{2\pi}\int_{M<|u|\leq\delta n^{1/m}} g(u)^n e^{-iuy_n}du\right| \\
&\qquad + \left|\frac{e^{-ix\xi_0}\nu(\xi_0)^n}{2\pi}\int_{|u|>M} e^{-\beta u^m}e^{-iuy_n}du\right| \\
&\quad\leq \int_{-M}^{M}|g(u)^n - e^{-\beta u^m}|du + \int_{M<|u|\leq\delta n^{1/m}}|g(u)|^n du + \int_{|u|>M} e^{-\operatorname{Re}(\beta)u^m}du \\
&\qquad =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
\end{aligned}$$

where $M < \delta n^{1/m}$ will soon be fixed. Notice that $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ are independent of x.

In view of Lemma 2.2.4, there is $\delta > 0$ and $C > 0$ for which

$$|g(u)|^n = |v(\xi_0 + un^{-1/m})|^n \leq (e^{-C(un^{-1/m})^m})^n = e^{-Cu^m} \tag{2.36}$$

whenever $|u| \leq \delta n^{1/m}$. Therefore,

$$\mathcal{I}_2 \leq \int_{M<|u|\leq\delta n^{1/m}} e^{-Cu^m}\,du \leq \int_{|u|>M} e^{-Cu^m}\,du$$

and because $e^{-Cu^m} \in L^1(\mathbb{R})$, there exists $M > 0$ for which $\mathcal{I}_2 < \epsilon/3$. Analogously and in view of the fact that $\operatorname{Re}(\beta) > 0$, there is $M > 0$ for which $\mathcal{I}_3 < \epsilon/3$. Selecting M for which these estimates hold and restricting our attention to sufficiently large n for which $M < \delta n^{1/m}$, we move on to estimate $\mathcal{I}_1$.

Let's first observe that, for all u such that $|u| \leq M < \delta n^{1/m}$,

$$|g(u)^n - e^{-\beta u^m}| \leq |g(u)|^n + |e^{-\beta u^m}| \leq 2$$

in view of (2.36). Also, by an appeal to (2.19) of Lemma 2.2.4, for any $u \in$

$[-M, M]$,

$$
\begin{aligned}
&|n((\Gamma(un^{-1/m}) - i\alpha un^{-1/m}) + \beta u^m| \\
&= n|\Gamma(un^{-1/m}) - i\alpha un^{-1/m} + \beta(un^{-1/m})^m| \leq nB|un^{-1/m}|^{m+1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad = B|u|^m n^{-1/m}
\end{aligned}
$$

and so

$$\lim_{n\to\infty} n(\Gamma(un^{-1/m}) - i\alpha un^{-1/m}) = -\beta u^m.$$

Therefore, for each such u,

$$\lim_{n\to\infty} \left|g(u)^n - e^{-\beta u^m}\right| = \lim_{n\to\infty} \left|e^{n(\Gamma(un^{-1/m}) - i\alpha un^{-1/m})} - e^{-\beta u^m}\right| = 0.$$

Because $[-M, M]$ is a set of finite measure, an appeal to the bounded convergence theorem gives $N > (M/\delta)^m$ for which $\mathcal{I}_1 \leq \epsilon/3$ for all $n > N$. Combining the estimates for $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ gives the desired result. $\square$

Lemma 2.5.2. *Let $\nu : \mathbb{R} \to \mathbb{C}$ be analytic on a neighborhood of a point ξ_0 such that $|\nu(\xi_0)| = 1$. Assume that $\xi_0 \sim (2; 2)$ with associated constants α and β. Let $K \subseteq \mathbb{R}$ be a compact set. Then for all $\epsilon > 0$ there is a $\delta > 0$ and a natural number N such that*

$$\left|\frac{n^{1/2}}{2\pi}\int_{|\xi-\xi_0|\leq\delta} \nu(\xi)^n e^{-i(xn^{1/2}+\alpha n)\xi} d\xi - e^{-i(xn^{1/2}+\alpha n)\xi_0}\nu(\xi_0)^n H_2^{\beta}(x)\right| < \epsilon \qquad (2.37)$$

for all $n > N$ and for all $x \in K$.

Proof. Let $\epsilon > 0$, let $K \subseteq \mathbb{R}$ be a fixed compact set and choose $\delta > 0$ so that the estimates (2.21), (2.22) and (2.23) of Lemma 2.2.4 are valid. Changing variables of integration we write

$$
\begin{aligned}
&\frac{n^{1/2}}{2\pi}\int_{|\xi-\xi_0|\leq\delta} \nu(\xi)^n e^{-i(xn^{1/2}+\alpha n)\xi} d\xi = \\
&\quad e^{-i(xn^{1/2}+\alpha n)\xi_0}\nu(\xi_0)^n \frac{n^{1/2}}{2\pi}\int_{|\xi|\leq\delta} [\nu(\xi_0)^{-1}\nu(\xi+\xi_0)]^n e^{-i(xn^{1/2}+\alpha n)\xi} d\xi.
\end{aligned}
$$

Upon setting

$$\mathcal{D} = \left| \frac{n^{1/2}}{2\pi} \int_{|\xi-\xi_0|\le\delta} \nu(\xi)^n e^{-i(xn^{1/2}+\alpha n)\xi} d\xi - e^{-i(xn^{1/2}+\alpha n)\xi_0} \nu(\xi_0)^n H_2^\beta(x) \right|,$$

we have

$$\begin{aligned}
\mathcal{D} \ &\le \ \left| \frac{n^{1/2}}{2\pi} \int_{|\xi|\le Mn^{-1/2}} [\nu(\xi_0)^{-1}\nu(\xi+\xi_0)]^n e^{-i(xn^{1/2}+\alpha n)\xi} d\xi - H_2^\beta(x) \right| \\
&\quad + n^{1/2} \left| \int_{Mn^{-1/2}<|\xi|\le\delta} [\nu(\xi_0)^{-1}\nu(\xi+\xi_0)]^n e^{-i(xn^{1/2}+\alpha n)\xi} d\xi \right|,
\end{aligned}$$

where for now $0 < M < \delta n^{1/2}$ and we have used the fact that $|\nu(\xi_0)| = 1$. Continuing in this manner,

$$\begin{aligned}
\mathcal{D} \ &\le \ \left| \frac{1}{2\pi} \int_{|u|\le M} [\nu(\xi_0)^{-1}\nu(un^{-1/2}+\xi_0)]^n e^{-i(x+\alpha n^{1/2})u} du - H_2^\beta(x) \right| \\
&\quad + n^{1/2} \left| \int_{Mn^{-1/2}<|\xi|\le\delta} [\nu(\xi_0)^{-1}\nu(\xi+\xi_0)]^n e^{-i(xn^{1/2}+\alpha n)\xi} d\xi \right| \\
&\le \ \left| \int_{|u|\le M} \left([\nu(\xi_0)^{-1}\nu(un^{-1/2}+\xi_0) e^{-i\alpha un^{-1/2}}]^n - e^{-\beta u^2} \right) e^{-ixu} du \right| \\
&\quad + \left| \int_{|u|>M} e^{-ixu-\beta u^2} du \right| \\
&\quad + n^{1/2} \left| \int_{Mn^{-1/2}}^{\delta} [\nu(\xi_0)^{-1}\nu(\xi+\xi_0)]^n e^{-i(xn^{1/2}+\alpha n)\xi} d\xi \right| \\
&\quad + n^{1/2} \left| \int_{-\delta}^{-Mn^{-1/2}} [\nu(\xi_0)^{-1}\nu(\xi+\xi_0)]^n e^{-i(xn^{1/2}+\alpha n)\xi} d\xi \right| \\
&\quad =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4,
\end{aligned}$$

where we have made a change of variables and used the definition of H_2^β. We now estimate the terms $\mathcal{I}_i$ for $i = 1, 2, 3, 4$. First, using Lemma 2.4.1 we choose $M > 0$ so that $\mathcal{I}_2 \le \epsilon/4$ for all $x \in K$. Let us now focus on $\mathcal{I}_3$. We write

$$\begin{aligned}
\mathcal{I}_3 \ &= \ n^{1/2} \left| \int_{Mn^{-1/2}}^{\delta} [\nu(\xi_0)^{-1}\nu(\xi+\xi_0)]^n e^{-i(xn^{1/2}+\alpha n)\xi} d\xi \right| \\
&= \ n^{1/2} \left| \int_{Mn^{-1/2}}^{\delta} [\nu(\xi_0)^{-1}\nu(\xi+\xi_0) \exp(-i\alpha + i\xi^2 p(\xi))]^n e^{i(n\xi^2 p(\xi) - \xi x n^{1/2})} d\xi \right| \\
&= \ n^{1/2} \left| \int_{Mn^{-1/2}}^{\delta} g(\xi)^n e^{if_n(\xi)} d\xi \right|,
\end{aligned}$$

where we have put

$$g(\xi) = \nu(\xi_0)^{-1}\nu(\xi+\xi_0)\exp(-i\alpha + i\xi^2 p(\xi))$$

and

$$f_n(\xi) = (n\xi^2 p(\xi) - \xi x n^{1/2}).$$

We wish to apply Lemma 2.3.4 to the above integral. Set

$$B = 4\left(1 + \int_{\mathbb{R}} D|u|^{k-1}e^{-C|u|^k}du\right),$$

where $C, D \geq 0$ are the constants appearing in (2.22) and (2.23) of Lemma 2.2.4. Since $\xi^2 p(\xi)$ is a polynomial with 2 being the smallest power of its terms, we can further restrict $\delta > 0$ so that $f_n''(\xi) \neq 0$ and

$$c_1\xi \leq \frac{d}{d\xi}(\xi^2 p(\xi)) \leq c_2\xi$$

for all $\xi \in [Mn^{-1/2}, \delta]$, where c_1 and c_2 are non-zero real numbers of the same sign. Consequently, we can select $M > 0$ and a natural number N so that

$$\inf_{\xi\in[Mn^{1/2},\delta]} |f_n'(\xi)| > \frac{4Bn^{1/2}}{\epsilon}$$

for all x in the compact set K and for all $n > N$. Finally, an application of Lemma 2.3.4 with the above estimate and a calculation similar to that done in the proof of Lemma 2.3.5 shows

$$\mathcal{I}_3 \leq \frac{B}{\inf_{\xi\in[Mn^{1/2},\delta]} |f_n'(\xi)|} < \frac{\epsilon}{4}$$

for all $n > N$ and for all $x \in K$. An analogous argument gives the same estimate for $\mathcal{I}_4$.

Before treating $\mathcal{I}_1$, we fix M as the maximal M for which the above estimates hold simultaneously. In view of (2.22) of Lemma 2.2.4, an analogous argument to that given in the proof of Lemma 2.5.1 shows that the absolute value of integrand in $\mathcal{I}_1$ is bounded above by 2 for all n. Furthermore, for any $u \in [-M, M]$,

$$\begin{aligned}
&|n(\Gamma(un^{-1/2}) - i\alpha un^{-1/2}) + \beta u^2| \\
&\quad \leq \; n|\Gamma(un^{-1/2}) - i\alpha un^{-1/2} + ip(un^{-1/2})(un^{-1/2})^2| + |\beta u^2 - ip(un^{-1/2})u^2| \\
&\quad \leq \; Bn(un^{-1/2})^k + u^2|\beta - ip(un^{-1/2})| \\
&\quad \leq \; Bu^k n^{1-k/2} + u^2|\beta - ip(un^{-1/2})|,
\end{aligned}$$

where we have used (2.21). Because p is continuous, $ip(0) = \beta$ and $k > 2$, it follows that for all $u \in [-M, M]$,

$$\lim_{n\to\infty} |n(\Gamma(un^{-1/2}) - i\alpha un^{-1/2}) + \beta u^2| = 0$$

and hence

$$\lim_{n\to\infty} \left| \left(\nu(\xi_0)^{-1}\nu(un^{-1/2} + \xi_0)e^{-i\alpha un^{-1/2}} \right)^n - e^{-\beta u^2} \right| = 0.$$

An appeal to the bounded convergence theorem guarantees that for sufficiently large n,

$$\begin{aligned}
\mathcal{I}_1 &= \left| \int_{|u|\leq M} \left([\nu(\xi_0)^{-1}\nu(un^{-1/2} + \xi_0)e^{-i\alpha un^{-1/2}}]^n - e^{-\beta u^2} \right) e^{-ixu} du \right| \\
&\leq \int_{-M}^{M} \left| \left(\nu(\xi_0)^{-1}\nu(un^{-1/2} + \xi_0)e^{-i\alpha un^{-1/2}} \right)^n - e^{-\beta u^2} \right| du < \epsilon/4
\end{aligned}$$

for all $x \in \mathbb{R}$ and in particular for all $x \in K$. Finally, from the above arguments we choose $\delta > 0$ and a natural number N so that for each $j = 1, 2, 3, 4$, $\mathcal{I}_j < \epsilon/4$ for all $n > N$ and for all $x \in K$. Putting these estimates together shows that $\mathcal{D} < \epsilon$ as desired. □

Lemma 2.5.3. *Let $\nu : \mathbb{R} \to \mathbb{C}$ be analytic on a neighborhood of a point ξ_0 such that $|\nu(\xi_0)| = 1$. Let $m > 2$ and assume that $\xi_0 \sim (2; m)$ with associated constants α and β. Then for all $\epsilon > 0$ there is a $\delta > 0$ and a natural number N such that*

$$\left| \frac{n^{1/m}}{2\pi} \int_{|\xi-\xi_0|\leq\delta} \nu(\xi)^n e^{-ix\xi} d\xi - e^{-ix\xi_0}\nu(\xi_0)^n H_m^\beta\left(\frac{x - \alpha n}{n^{1/m}} \right) \right| < \epsilon \qquad (2.38)$$

for all $n > N$ and for all real numbers x.

The present lemma's proof is analogous to the proof of the previous lemmas in many ways. We will consequently spend less time explaining the order in which we choose our constants.

Proof. Let $\epsilon > 0$ and set

$$y_n = \frac{x - \alpha n}{n^{1/m}} \tag{2.39}$$

and

$$\mathcal{D} = \left| \frac{n^{1/m}}{2\pi} \int_{|\xi - \xi_0| \le \delta} \nu(\xi)^n e^{-ix\xi} d\xi - e^{-ix\xi_0} \nu(\xi_0)^n H_m^\beta \left(\frac{x - \alpha n}{n^{1/m}} \right) \right|.$$

Since $\xi_0 \sim (2; m)$, we choose $\delta > 0$ so that the estimates (2.21), (2.22) and (2.23) of Lemma 2.2.4 are valid, and the inequality

$$\left| \frac{d^2}{d\xi^2} \xi^m p(\xi) \right| \ge B^2 |\xi|^{m-2} \tag{2.40}$$

holds for all $-\delta \le \xi \le \delta$, where $B > 0$. Using Proposition 2.4.1, we now choose $M > 0$ such that

$$\left| \int_{|u| > M} e^{-iyu - \beta u^m} du \right| < \epsilon/4 \tag{2.41}$$

for all $y \in \mathbb{R}$ and

$$\frac{8}{BM^{m/2-1}} \left(1 + \int_{\mathbb{R}} D|u|^{k-1} e^{-Cu^k} du \right) < \epsilon/4, \tag{2.42}$$

where B was defined above and C and D are the constants appearing in (2.23).

As in the last lemma, we write

$$\begin{aligned}
\mathcal{D} &= \left|\frac{n^{1/m}}{2\pi}\int_{-\delta}^{\delta}\nu(\xi+\xi_0)^n e^{-ix(\xi+\xi_0)}d\xi - e^{-ix\xi_0}\nu(\xi_0)^n H_m^{\beta}(y)\right| \\
&\leq \left|\int_{|u|\leq M}\left([\nu(\xi_0)^{-1}\nu(un^{-1/m}+\xi_0)e^{-\alpha un^{-1/m}}]^n - e^{-\beta u^m}\right)e^{-iyu}du\right| \\
&\quad + \left|\int_{|u|>M} e^{-iyu-\beta u^m}du\right| + n^{1/m}\left|\int_{Mn^{-1/m}}^{\delta}[\nu(\xi_0)^{-1}\nu(\xi+\xi_0)]^n e^{-ix\xi}d\xi\right| \\
&\quad + n^{1/m}\left|\int_{-\delta}^{-Mn^{-1/m}}[\nu(\xi_0)^{-1}\nu(\xi+\xi_0)]^n e^{-ix\xi}d\xi\right| \\
&\quad =: \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.
\end{aligned}$$

Now we estimate the terms $\mathcal{I}_j$ for $j = 1,2,3,4$. Already from (2.41), we know that $\mathcal{I}_2 < \epsilon/4$ for all $x \in \mathbb{R}$. We have

$$\begin{aligned}
\mathcal{I}_3 &= n^{1/m}\left|\int_{Mn^{-1/m}}^{\delta}[\nu(\xi_0)^{-1}\nu(\xi+\xi_0)e^{-i\alpha\xi+i\xi^m p(\xi)}]^n e^{-i(x+\alpha n)\xi - in\xi^m p(\xi)}d\xi\right| \\
&= n^{1/m}\left|\int_{Mn^{-1/m}}^{\delta} g(\xi)^n e^{if_n(\xi)}d\xi\right|,
\end{aligned}$$

where

$$g(\xi) = [\nu(\xi_0)^{-1}\nu(\xi+\xi_0)e^{-i\alpha\xi+i\xi^m p(\xi)}]$$

and

$$f_n(\xi) = -[(x+\alpha n)\xi - in\xi^m p(\xi)].$$

With the aim of applying Lemma 2.3.4, we use (2.40) and observe that on the interval $[Mn^{-1/m},\delta]$

$$\inf|f_n''(\xi)| \geq \inf nB^2|\xi|^{m-2} \geq nB^2|Mn^{-1/m}|^{m-2} = (n^{1/m}BM^{m/2-1})^2 > 0.$$

The application of the lemma is therefore justified and we can use (2.22) and (2.23) to see that

$$\begin{aligned}
\mathcal{I}_3 &\leq \frac{8n^{1/m}}{\sqrt{(n^{1/m}BM^{m/2-1})^2}}(\|g^n\|_\infty + \|ng'g^{n-1}\|_1) \\
&\leq \frac{8}{CM^{m/2-1}}\left(1+\int_{\mathbb{R}} D|u|^{k-1}e^{-Cu^k}du\right) < \epsilon/4
\end{aligned}$$

for all $x \in \mathbb{R}$. A similar calculation gives the same estimate for $\mathcal{I}_4$.

To estimate $\mathcal{I}_1$, we essentially repeat the argument given in the proof of the previous lemma. Again, the integrand is bounded in absolute value by 2 for all n. Using (2.21), we observe that for any $u \in [-M, M]$

$$\lim_{n\to\infty}\left|\left(\nu(\xi_0)^{-1}\nu(un^{-1/m}+\xi_0)e^{-\alpha un^{-1/m}}\right)^n - e^{-\beta u^m}\right| = 0.$$

Therefore, the bounded convergence theorem gives a natural number N for which $\mathcal{I}_1 \leq \epsilon/4$ for all $n > N$ and for all $x \in \mathbb{R}$. Combining our estimates finishes the proof. $\square$

For the remainder of this section, we focus on local limit theorems. The first theorem, Theorem 2.5.4, focuses on the case in which $\phi^{(n)}$ is approximated locally on its packets by linear combinations of the attractors H_2^{β}. The second theorem, Theorem 2.5.5, isolates the second conclusion of Theorem 2.1.2. The results of both theorems are then used to prove Theorem 2.1.3.

Theorem 2.5.4. *Let $\phi : \mathbb{Z} \to \mathbb{C}$ have admissible support and suppose that $\sup_\xi |\hat{\phi}(\xi)| = 1$. Under Convention 2.2.3, suppose that $m = 2$ and for some $q = 1, 2, \ldots, R$, β_q is purely imaginary. Then, to each α_q, there exists subcollections $\xi_{j_1}, \xi_{j_2}, \ldots, \xi_{j_{r(q)}}$ and $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_{r(q)}}$, such that*

$$\phi^{(n)}(\lfloor xn^{1/2} + \alpha_q n\rfloor) = \sum_{l=1}^{r(q)} n^{-1/2} e^{-i(\lfloor xn^{1/2}+\alpha_q n\rfloor)\xi_{j_l}} \hat{\phi}(\xi_{j_l})^n H_2^{\beta_{j_l}}(x) + o(n^{-1/2}) \quad (2.43)$$

uniformly on any compact set.

Proof. Let $\epsilon > 0$ and $K \subseteq \mathbb{R}$ be a compact set. In view of Convention 2.2.3, it follows from our hypotheses that $Q = R$ and therefore $\Omega(\phi) = \{\xi_1, \xi_2, \ldots, \xi_R\}$. We note that the corresponding drift constants $\alpha_1, \alpha_2, \ldots, \alpha_R$ need not be distinct.

Let α_q be a member of the above collection and let $\{j_1, j_2, \dots, j_{r(q)}\}$ be the increasing subcollection of $\{1, 2, \dots, R\}$ for which $\alpha_{j_l} = \alpha_q$ for $l = 1, 2, \dots, r(q)$. Also, set $\Upsilon_q = \{1, 2, \dots, R\} \setminus \{j_1, j_2, \dots, j_{r(q)}\}$. It is of course possible that Υ_q is empty. For example, it might be the case that $1 = r(q) = R$ and, in this case, (2.43) consists only of the single attractor $H_2^{\beta_1}$. In fact, this is precisely the situation exemplified in the introduction in which ϕ was defined by (2.1) (see also Subsection 2.8.2).

We divide T into subintervals: For $l = 1, 2, \dots, R$, define $I_l = [\xi_l - \delta_l, \xi_l + \delta_l] \subseteq T$ where $\delta_l > 0$ are to be defined shortly; for now, let's require them to be sufficiently small to ensure that the intervals I_l, for $l = 1, 2, \dots, R$, are disjoint. In view of (2.18), put $J = T \setminus \cup I_l$ and write

$$\begin{aligned}
&n^{1/2}\phi_{\mathrm{e}}(n, xn^{1/2} + \alpha_q n) \qquad (2.44)\\
&= \frac{n^{1/2}}{2\pi}\int_T \hat{\phi}(\xi)^n e^{-i(xn^{1/2}+\alpha_q n)\xi} d\xi\\
&= \sum_{l=1}^{R} \frac{n^{1/2}}{2\pi}\int_{I_l} \hat{\phi}(\xi)^n e^{-i(xn^{1/2}+\alpha_q n)\xi} d\xi + \frac{n^{1/2}}{2\pi}\int_J \hat{\phi}(\xi)^n e^{-i(xn^{1/2}+\alpha_q n)\xi} d\xi\\
&= \sum_{l=1}^{R} \mathcal{I}_l + \mathcal{E}. \qquad (2.45)
\end{aligned}$$

We treat the integrals $\mathcal{I}_l$ in the two cases separately. First, we consider $\mathcal{I}_l$ for $l \in \Upsilon_q$. Here we show that $\mathcal{I}_l$ can be made arbitrarily small (depending on x and n) because $\alpha_q \neq \alpha_l$. If $\xi_l \sim (2; 2)$, let γ_l, k_l and $p_l(\xi)$ be associated as per

Definition 2.2.1. We have

$$\begin{aligned}
|\mathcal{I}_l| &= \left| \frac{n^{1/2}}{2\pi} \int_{I_l} \hat{\phi}(\xi)^n e^{-i(xn^{1/2}+\alpha_q n)\xi} d\xi \right| \\
&= \left| \frac{n^{1/2} e^{-i(xn^{1/2}-\alpha_q n)\xi_l} \hat{\phi}(\xi_l)^n}{2\pi} \int_{|\xi|\leq\delta} \left[\hat{\phi}(\xi_l)^{-1}\hat{\phi}(\xi_l+\xi)\right]^n e^{-i(xn^{1/2}+\alpha_q n)\xi} d\xi \right| \\
&\leq n^{1/2} \left| \int_{|\xi|\leq\delta} g_l(\xi)^n e^{if_{n,l}(\xi)} d\xi \right|,
\end{aligned}$$

where

$$g_l(\xi) = [\hat{\phi}^{-1}(\xi_l)\hat{\phi}(\xi_l+\xi)e^{-i\alpha_l\xi+i\xi^2 p_l(\xi)}]$$

and

$$f_{n,l}(\xi) = -n[(xn^{-1/2}+\alpha_q-\alpha_l)\xi+\xi^2 p_l(\xi)].$$

Now choose $\delta_l > 0$ so that, on the interval $[-\delta_l, \delta_l]$, $g_l(\xi)$ satisfies (2.22) and (2.23) for some $C_l, D_l > 0$,

$$f''_{n,l}(\xi) = -n\frac{d^2}{d\xi^2}\xi^2 p_l(\xi) \neq 0$$

and

$$B_l \leq \left| \alpha_l - \alpha_q - \frac{d}{d\xi}\xi^2 p_l(\xi) \right| \tag{2.46}$$

for some $B_l > 0$. For the first property our choice of δ_l was made using Lemma 2.2.4 and the assumption that $\xi_l \sim (2;2)$. For the second two properties we used that fact that $\xi^2 p_l(\xi)$ is a polynomial with 2 being the smallest power of its terms and $\alpha_l \neq \alpha_q$. We can therefore apply Lemma 2.3.4. This gives

$$\begin{aligned}
|\mathcal{I}_l| &\leq \frac{8n^{1/2}}{\inf_\xi |f'_{n,l}(\xi)|}(\|g_l\|_\infty + \|ng'_l g_l^{n-1}\|) \\
&\leq \frac{8}{\inf_\xi |(x-n^{1/2}(\alpha_l-\alpha_q-\frac{d}{d\xi}\xi^2 p_l(\xi))|}\left(1+\int_{\mathbb{R}} D_l|\xi|^{k_l} e^{-C_l\xi^{k_l}} d\xi\right) \\
&\leq \frac{M_l}{\inf_\xi |(x-n^{1/2}(\alpha_l-\alpha_q-\frac{d}{d\xi}\xi^2 p_l(\xi))|}
\end{aligned}$$

for some $M_l > 0$ and where the above infima are taken over the interval $[-\delta_l, \delta_l]$. Using the estimate (2.46) and recalling that x lives inside the compact set K, we can choose a natural number N_l so that

$$\inf_{\xi} |(x - n^{1/2}(\alpha_l - \alpha_q - \frac{d}{d\xi}\xi^2 p_l(\xi))| > \frac{M_l(R+1)}{\epsilon}$$

for all $n > N_l$ and for all $x \in K$. Consequently,

$$|\mathcal{I}_l| \leq \frac{M_l}{M_l(R+1)/\epsilon} = \epsilon/(R+1) \tag{2.47}$$

for all $n > N_l$ and for all $x \in K$.

If instead $\xi_l \sim (1;2)$, by an appeal to Lemma 2.5.1, we choose $\delta_l > 0$ and a natural number N_l so that

$$\begin{aligned} |\mathcal{I}_l| &\leq \epsilon/2(R+1) + |e^{-i(xn^{1/2}+\alpha_q n)\xi_l}\hat{\phi}(\xi_l)^n H_2^{\beta_l}(x + (\alpha_q - \alpha_l)n^{1/2})| \\ &\leq \epsilon/2(R+1) + |H_2^{\beta_l}(x + (\alpha_q - \alpha_l)n^{1/2})| \end{aligned}$$

for all $n > N_l$ and for all $x \in \mathbb{R}$. However, as we remarked earlier $H_2^{\beta_l}$ is the heat kernel evaluated at complex time β_l. Since $\mathrm{Re}(\beta_l) > 0$ in this case and $\alpha_q \neq \alpha_l$ we may increase our natural number N_l to ensure that

$$|H_2^{\beta_l}(x + (\alpha_q - \alpha_l)n^{1/2})| \leq \epsilon/2(R+1)$$

for any $n > N_l$ and for all x in the compact set K. These estimates together give

$$|\mathcal{I}_l| \leq \epsilon/2(R+1) + \epsilon/2(R+1) = \epsilon/(R+1) \tag{2.48}$$

for all $n > N_l$ and for all $x \in K$.

In the remaining estimates of $\mathcal{I}_l$ for $l = j_1, j_2, \dots, j_{r(q)}$, we recall that $\alpha_q = \alpha_l$. If $\xi_l \sim (2;2)$, we appeal to Lemma 2.5.2. From this we choose $\delta_l > 0$ and a natural number N_l such that

$$|\mathcal{I}_l - e^{-i(xn^{1/2}+\alpha_q n)\xi_l}\hat{\phi}(\xi_l)^n H_2^{\beta_l}(x)| \leq \epsilon/(R+1) \tag{2.49}$$

for all $n > N_l$ and for all $x \in K$. If instead $\xi_l \sim (1;2)$, we appeal to Lemma 2.5.1 and chose $\delta_l > 0$ and N_l, a natural number, such that

$$|\mathcal{I}_l - e^{-i(xn^{1/2}+\alpha_q n)\xi_l}\hat{\phi}(\xi_l)^n H_2^{\beta_l}(x)| \leq \epsilon/(R+1) \tag{2.50}$$

for all $n > N_l$ and for all $x \in \mathbb{R}$. In particular we have this estimate uniform for all $x \in K$.

After fixing our collection of δ_l's in the above arguments, the set J becomes fixed. We therefore set $s = \sup_{\xi\in J}|\hat{\phi}(\xi)| < 1$ and note that $|\mathcal{E}| \leq n^{1/2}s^n$. Thus we may choose a natural number N_0 such that $|\mathcal{E}| < \epsilon/(R+1)$ for all $n > N_0$ and for all $x \in K$.

At last, we choose N to be the maximum of N_l for $l = 0, 1, \ldots, R$. Combining the estimates (2.44), (2.47), (2.48), (2.49) and (2.50) yields

$$\begin{aligned}
&\left| n^{1/2}\phi_{\mathrm{e}}(n, xn^{1/2}+\alpha_q n) - \sum_{l\in\{j_1,j_2\ldots,j_{r(q)}\}} e^{-i(xn^{1/2}+\alpha_q n)\xi_l}\hat{\phi}(\xi_l)^n H_2^{\beta_l}(x)\right| \\
&\leq \sum_{l\in\{j_1,j_2,\ldots,j_{r(q)}\}} \left|\mathcal{I}_l - e^{-i(xn^{1/2}+\alpha_q n)\xi_l}\hat{\phi}(\xi_l)^n H_2^{\beta_l}(x)\right| + \sum_{l\in\Upsilon_q}|\mathcal{I}_l| + \mathcal{E} \\
&< \frac{(R+1)\epsilon}{R+1} = \epsilon
\end{aligned}$$

for any $n > N$ and for all $x \in K$. We have shown that

$$\phi_{\mathrm{e}}(n, xn^{1/2}+\alpha_q n) = \sum_{l=1}^{r(q)} n^{-1/2}e^{-i(xn^{1/2}+\alpha_q n)\xi_{j_l}}\hat{\phi}(\xi_{j_l})^n H_2^{\beta_{j_l}}(x) + o(n^{-1/2}) \tag{2.51}$$

uniformly for x in any compact set K.

To complete the proof of the theorem we need to replace the argument $xn^{1/2} + \alpha_q n$ by an integer in (2.51); this is precisely where the floor function comes in. Let $K \subseteq \mathbb{R}$ be compact, set

$$y(x,n) = \frac{\lfloor \alpha_q n + x n^{1/2} \rfloor - \alpha_q n}{n^{1/2}},$$

and observe that $|x - y(n,x)| \leq n^{-1/2}$. Let $F \supseteq K$ be any compact set for which $y(x,n) \in F$ for all $x \in K$ and all natural numbers n. By Proposition 2.4.3, each function $H_2^{\beta_{j_l}}$ is uniformly continuous on F and therefore, for any $x \in K$, we have

$$\begin{aligned} &\sum_{l=1}^{r(q)} n^{-1/2} e^{-i(\lfloor x n^{1/2} + \alpha_q n \rfloor)\xi_{j_l}} \hat{\phi}(\xi_{j_l})^n H_2^{\beta_{j_l}}(y(x,n)) \\ &= \sum_{l=1}^{r(q)} n^{-1/2} e^{-i(\lfloor x n^{1/2} + \alpha_q n \rfloor)\xi_{j_l}} \hat{\phi}(\xi_{j_l})^n H_2^{\beta_{j_l}}(x) + o(n^{-1/2}). \end{aligned} \tag{2.52}$$

The result now follows from (2.51), (2.52) and the observation that

$$\phi^{(n)}(\lfloor x n^{1/2} + \alpha_q n \rfloor) = \phi_{\mathrm{e}}(n, \lfloor x n^{1/2} + \alpha_q n \rfloor).$$

□

Theorem 2.5.5. *Let $\phi : \mathbb{Z} \to \mathbb{C}$ have admissible support and suppose that* $\sup|\hat{\phi}(\xi)| = 1$. *Under Convention 2.2.3, additionally assume that $m > 2$ or $\mathrm{Re}(\beta_q) > 0$ for all $q = 1, 2, \ldots, R$ (this precisely the hypothesis* (2.8) *of Theorem 2.1.2). Then*

$$\phi^{(n)}(x) = \sum_{q=1}^{R} n^{-1/m} e^{-ix\xi_q} \hat{\phi}(\xi_q)^n H_m^{\beta_q}\left(\frac{x - \alpha_q n}{n^{1/m}}\right) + o(n^{-1/m}) \tag{2.53}$$

uniformly in $\mathbb{Z}$.

Proof. In view of Proposition 2.2.2 and under Convention 2.2.3, our hypotheses guarantee that either $m > 2$ or, in the case that $m = 2$, $\xi_q \sim (1;2)$ for each $\xi_q \in \Omega(\phi)$. Consequently to each point $\xi_q \in \Omega(\phi)$ of order m we may apply either Lemma 2.5.1 or Lemma 2.5.3.

As in the proof of the previous theorem we divide T into subintervals. For $q = 1, 2, \ldots, Q$, let $I_q = [\xi_q - \delta_q, \xi_q + \delta_q]$ for values of $\delta_q > 0$ to be chosen later (but small enough to ensure that the I_q's are disjoint) and set $J = T \setminus \cup I_q$. We again define ϕ_{e} by (2.18) and write

$$\begin{aligned} n^{1/m}\phi_{\mathrm{e}}(n,x) &= \frac{n^{1/m}}{2\pi}\int_T \hat{\phi}(\xi)^n e^{-ix\xi}d\xi \\ &= \sum_{q=1}^{Q}\frac{n^{1/m}}{2\pi}\int_{I_q}\hat{\phi}(\xi)^n e^{-ix\xi}d\xi + \frac{n^{1/m}}{2\pi}\int_J \hat{\phi}(\xi)^n e^{-ix\xi}d\xi. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left| n^{1/m}\phi_{\mathrm{e}}(n,x) - \sum_{q=1}^{R} e^{-ix\xi_q}\hat{\phi}(\xi_q)^n H_m^{\beta_q}\left(\frac{x-\alpha_q n}{n^{1/m}}\right)\right| \\ &\leq \sum_{q=1}^{R}\left|\frac{n^{1/m}}{2\pi}\int_{I_q}\hat{\phi}(\xi)^n e^{-ix\xi}d\xi - e^{-ix\xi_q}\hat{\phi}(\xi_q)^n H_m^{\beta_q}\left(\frac{x-\alpha_q n}{n^{1/m}}\right)\right| \\ &\quad + \sum_{q=R+1}^{Q} n^{1/m}\left|\frac{1}{2\pi}\int_{I_q}\hat{\phi}(\xi)^n e^{-ix\xi}d\xi\right| + \left|\frac{n^{1/m}}{2\pi}\int_J \hat{\phi}(\xi)^n e^{-ix\xi}d\xi\right|. \end{aligned} \tag{2.54}$$

As we previously noted, for $q = 1, 2, \ldots, R$, we apply either Lemma 2.5.1 or Lemma 2.5.3. We can therefore choose a natural number N_q and fix $\delta_q > 0$ so that

$$\left|\frac{n^{1/m}}{2\pi}\int_{I_q}\hat{\phi}(\xi)^n e^{-ix\xi}d\xi - e^{-ix\xi_q}\hat{\phi}(\xi_q)^n H_m^{\beta_q}\left(\frac{x-\alpha_q n}{n^{1/m}}\right)\right| < \frac{\epsilon}{(Q+1)} \tag{2.55}$$

for all $n > N_q$ and for all $x \in \mathbb{R}$.

In the case that $q = R+1, R+2, \ldots, Q$, we appeal to Lemma 2.3.5 and choose $\delta_q > 0$ and a natural number N_q such that

$$\left|\frac{1}{2\pi}\int_{I_q}\hat{\phi}(\xi)^n e^{-ix\xi}d\xi\right| \leq \frac{C_q}{n^{1/m_q}}$$

for some $C_q > 0$ and for all $n > N_q$ and $x \in \mathbb{R}$. Using the fact that $m > m_q$ we can adjust the value of N_q so that

$$n^{1/m}\left|\frac{1}{2\pi}\int_{I_q}\hat{\phi}(\xi)^n e^{-ix\xi}d\xi\right| \leq \frac{C_q}{n^{1/m_q-1/m}} < \frac{\epsilon}{(Q+1)} \tag{2.56}$$

for all $n > N_q$ and for all $x \in \mathbb{R}$.

Finally, as in the proof of the last theorem, we set $s = \inf_J |\hat{\phi}| < 1$ and observe that the last term in (2.54) is bounded by $n^{1/m}s^n$. We therefore select a natural number N_0 such that

$$\left|\frac{n^{1/2}}{2\pi}\int_{J}\hat{\phi}(\xi)^n e^{-ix\xi}d\xi\right| \leq n^{1/m}s^n < \frac{\epsilon}{(Q+1)} \tag{2.57}$$

for all $n > N_0$ and for all $x \in \mathbb{R}$.

Let us choose N to be the maximum N_q for $q = 0, 1, \ldots, Q$. Upon combining the estimates (2.55), (2.56), (2.57) and (2.54) we have

$$\left|n^{1/m}\phi_{\mathrm{e}}(n,x) - \sum_{q=1}^{R} e^{-ix\xi_q}\hat{\phi}(\xi_q)^n H_m^{\beta_q}\left(\frac{x-\alpha_q n}{n^{1/m}}\right)\right| < \epsilon \tag{2.58}$$

for all $n > N$ and for all $x \in \mathbb{R}$. In particular, (2.58) holds for all $x \in \mathbb{Z}$ and for such x, $\phi_{\mathrm{e}}(n,x) = \phi^{(n)}(x)$. This is our desired result. □

Proof of Theorem 2.1.3. Let K be a compact set. Assuming that ϕ satisfies the hypotheses of the theorem, we adopt Convention 2.2.3 by virtue of Proposition 2.2.2. There are two distinct possibilities pertaining to the constants m and $\beta_1, \beta_2, \ldots, \beta_R$: they satisfy the hypotheses of Theorem 2.5.4 or they satisfy the hypotheses of Theorem 2.5.5. A moment's thought shows that the hypotheses of Theorem 2.5.4 and the hypotheses of Theorem 2.5.5 are indeed mutually exclusive and collectively exhaustive. If the case at hand is the former there is nothing to prove for $m = 2$ and the desired result is precisely the conclusion of Theorem 2.5.4. We therefore address the latter case.

Let $\alpha_q \in \{\alpha_1, \alpha_2, \dots, \alpha_R\}$ and, exactly as was done in the proof of Theorem 2.5.4, define $\{j_1, j_2, \dots, j_r(q)\} \subseteq \{1, 2, \dots, R\}$ and Υ_q. Observe that (2.58) is uniform in $\mathbb{R}$ and we can therefore write

$$\begin{aligned}
&\phi_{\mathrm{e}}(n, \alpha_q n + x n^{1/m}) \\
&= \sum_{l=1}^{R} n^{-1/m} e^{-i(\alpha_q n + x n^{-1/m})\xi_l} \hat{\phi}(\xi_l)^n H_m^{\beta_l}\left(\frac{(\alpha_q - \alpha_l)n + x n^{1/m}}{n^{1/m}}\right) + o(n^{-1/m}) \\
&= \sum_{l \in \{j_1, j_2, \dots, j_{r(q)}\}} n^{-1/m} e^{-i(\alpha_q n + x n^{-1/m})\xi_l} \hat{\phi}(\xi_l)^n H_m^{\beta_l}(x) \\
&\quad + \sum_{l \in \Upsilon_q} n^{-1/m} e^{-i(\alpha_q n + x n^{-1/m})\xi_l} \hat{\phi}(\xi_l)^n H_m^{\beta_l}((\alpha_q - \alpha_l)n^{1-1/m} + x) + o(n^{-1/m}). \\
&= \sum_{l=1}^{r(q)} n^{-1/m} e^{-i(\alpha_q n + x n^{-1/m})\xi_{j_l}} \hat{\phi}(\xi_{j_l})^n H_m^{\beta_{j_l}}(x) + \sum_{l \in \Upsilon_q} S_l(n, x) + o(n^{-1/m}).
\end{aligned}$$

Upon requiring $x \in K$, we consider the summands $S_l(n, x)$ for $l \in \Upsilon_q$. In the case that $\mathrm{Re}(\beta_l) > 0$, we have

$$\begin{aligned}
|S_l(n, x)| &= |n^{-1/m} e^{-i(\alpha_q n + x n^{-1/m})\xi_l} \hat{\phi}(\xi_l)^n H_m^{\beta_l}((\alpha_q - \alpha_l)n^{1-1/m} + x)| \\
&= n^{-1/m} |H_m^{\beta_l}((\alpha_q - \alpha_l)n^{1-1/m} + x))| \\
&\leq n^{-1/m} C_l \exp(-B_l((\alpha_q - \alpha_l)n^{1-1/m} + x)^{m/(m-1)}) = o(n^{-1/m}).
\end{aligned}$$

If it is the case that $\mathrm{Re}(\beta_l) = 0$, we must have $m > 2$. Appealing to Proposition 2.4.2, we conclude that

$$\begin{aligned}
&|S_l(n, x)| \\
&\leq n^{-1/m} \left(\frac{A}{|(\alpha_q - \alpha_l)n^{1-1/m} + x)|^{\frac{m-2}{2(m-1)}}} + \frac{B}{|(\alpha_q - \alpha_l)n^{1-1/m} + x)|} \right) \\
&= o(n^{-1/m}).
\end{aligned}$$

Combining the above estimates shows that, for all $x \in K$,

$$\phi_{\mathrm{e}}(n, \alpha_q n + x n^{1/m}) = \sum_{l=1}^{r(q)} n^{-1/m} e^{-i(\alpha_q n + x n^{-1/m})\xi_{j_l}} \hat{\phi}(\xi_{j_l})^n H_m^{\beta_{j_l}}(x) + o(n^{-1/m}).$$

To complete the proof, it remains to replace the argument $\alpha_q n + xn^{1/m}$ by the integer $\lfloor \alpha_q n + xn^{1/m} \rfloor$ in the equation above. This can be done easily by making an argument analogous to that given in the last paragraph of the proof to Theorem 2.5.4. From this, the desired result follows without trouble. $\square$

2.6 The lower bound of $\|\phi^{(n)}\|_\infty$

In this section we complete the proof of Theorem 2.1.1.

Lemma 2.6.1. *Let $\zeta_1, \zeta_2, \cdots, \zeta_r \in (-\pi, \pi]$ be distinct, let $B > 0$ and define*

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ e^{-i\zeta_1} & e^{-i\zeta_2} & \dots & e^{-i\zeta_r} \\ e^{-i2\zeta_1} & e^{-i2\zeta_2} & \dots & e^{-i2\zeta_r} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-i(r-1)\zeta_1} & e^{-i(r-1)\zeta_2} & \dots & e^{-i(r-1)\zeta_r} \end{pmatrix}. \tag{2.59}$$

Then there is a number $C > 0$ such that for any $\rho, \sigma \in \mathbb{C}^r$ with $\|\rho\| > B$ and $\sigma = V\rho$, we have $|\sigma_j| > 3C$ for some $j = 1, 2, \dots, r$. Here $\|\cdot\|$ denotes the usual norm on $\mathbb{C}^r$.

Proof. The matrix V in (2.59) is known as Vandermonde's matrix. It is a routine exercise in linear algebra to show that

$$\det(V) = \prod_{1 \le l < k \le r} (e^{-i\zeta_k} - e^{-i\zeta_l}).$$

Noting that $e^{-i\zeta_1}, e^{-i\zeta_1}, \dots, e^{-i\zeta_r}$ are all distinct we conclude that V is invertible. The proof now follows immediately from the estimate

$$\|\rho\| \le \|V^{-1}\| \|\sigma\|.$$

$\square$

Proof of Theorem 2.1.1. Let $\phi : \mathbb{Z} \to \mathbb{C}$ have admissible support. As Theorem 2.3.6 gave the upper bound

$$A^{-n}\|\phi^{(n)}\|_\infty \leq C' n^{1/m}$$

for some $C' > 0$, our job is establish the lower bound

$$Cn^{1/m} \leq A^{-n}\|\phi^{(n)}\|_\infty$$

for some $C > 0$. This is done with the help of our local limit theorems.

As we noted in the proof of Theorem 2.3.6, it suffices to assume that $A = \sup_\xi |\hat{\phi}(\xi)| = 1$. We adopt Convention 2.2.3 by virtue of Proposition 2.2.2 and note that $m \geq 2$, defined by (2.16), is that which appears in both Theorem 2.1.3 and Theorem 2.3.6. In view of Theorem 2.1.3, set $\alpha = \alpha_1$, $r = r(1)$ and correspondingly take $\xi_{j_1}, \xi_{j_2}, \dots, \xi_{j_r} \in (-\pi, \pi]$ and $\beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_r}$ for which (2.11) holds. For notational convenience, set $b_l = \beta_{j_l}$ and $\zeta_l = \xi_{j_l}$ for $l = 1, 2, \dots, r$ and note that the points $\zeta_1, \zeta_2, \dots, \zeta_r \in (-\pi, \pi]$ are distinct. In this notation, (2.11) is the assertion that

$$\phi^{(n)}(\lfloor \alpha n + x n^{1/m} \rfloor) = \sum_{l=1}^{r} n^{-1/m} e^{-i\lfloor \alpha n + x n^{1/m} \rfloor \zeta_l} \hat{\phi}(\zeta_l)^m H_m^{b_l}(x) + o(n^{-1/m}) \tag{2.60}$$

uniformly for x in a compact set.

Appealing to Proposition 2.4.3, we know that each function $H_m^{b_1}$ is non-zero and continuous for $l = 1, 2, \dots, r$. In particular, there exists $B > 0$ and an interval $I = [a, b]$ such that $|H_m^{b_1}(x)| \geq B$ for all $x \in I$. Define V by (2.59) and let $C > 0$ as guaranteed by Lemma 2.6.1. Set

$$f(n, x) = \sum_{l=1}^{r} e^{-i(\alpha n + x n^{1/m})\zeta_l} \hat{\phi}(\zeta_l)^n H_m^{b_l}(x) \tag{2.61}$$

and

$$\sigma_k(n, x) = \sum_{l=1}^{r} e^{-ik\zeta_l} e^{-i(\alpha n + x n^{1/m})\zeta_l} \hat{\phi}(\zeta_l)^n H_m^{b_l}(x) \tag{2.62}$$

for $k = 0, 1, \ldots, r-1$. Since each function $H_m^{b_l}$ is continuous on $\mathbb{R}$ it is uniformly continuous on $[a-r, b+r] \supseteq I$. Consequently, we may choose a natural number N for which

$$|f(n, x + kn^{-1/m}) - \sigma_k(n, x)| < C \tag{2.63}$$

for all $n \geq N$, $k = 0, 1, \ldots, r-1$ and $x \in I$. By possibly further increasing N we can also guarantee that for any $n \geq N$ there is $x_0 \in I$ such that $\alpha n + x_0 n^{1/m}$ is an integer and for which $x_0 + kn^{-1/m} \in I$ for all $k = 0, 1, \ldots, r-1$. We observe that for any such k, $(\alpha n + (x_0 + kn^{-1/m})n^{1/m})$ is also an integer.

Now for any $n \geq N$, let $x_0 \in I$ be as guaranteed in the previous paragraph. Observe that

$$\begin{pmatrix} \sigma_0(n, x_0) \\ \sigma_1(n, x_0) \\ \vdots \\ \sigma_{(R-1)}(n, x_0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ e^{-i\zeta_1} & e^{-i\zeta_2} & \ldots & e^{-i\zeta_r} \\ e^{-i2\zeta_1} & e^{-i2\zeta_2} & \ldots & e^{-i2\zeta_r} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-i(r-1)\zeta_1} & e^{-i(r-1)\zeta_2} & \ldots & e^{-i(r-1)\zeta_r} \end{pmatrix} \begin{pmatrix} \rho_1(n, x_0) \\ \rho_2(n, x_0) \\ \vdots \\ \rho_r(n, x_0) \end{pmatrix},$$

where

$$\rho_l(x_0, n) = e^{-i(\alpha n + x_0 n^{1/m})\zeta_l} \hat{\phi}(\zeta_l)^n H_m^{b_l}(x_0)$$

for $l = 1, 2, \ldots, r$. Because $x_0 \in I$, $|\rho_1(x_0, n)| = |H_m^{b_1}(x_0)| > B$ and therefore

$$\|(\rho_1(n, x_0), \rho_2(n, x_0), \ldots, \rho_r(n, x_0))^\top\| > B.$$

Appealing to Lemma 2.6.1, there is some $k \in \{0, 1, 2, \ldots, r-1\}$ such that $|\sigma_k(n, x_0)| > 3C$ and so by (2.63), $|f(n, x_0 + kn^{-1/m})| > 2C$.

We have shown that there is a natural number N, a closed interval I and a constant $C > 0$ such that for any $n \geq N$

$$\sup \left| \sum_{l=1}^{r} e^{-i(\alpha n + x n^{1/m})\zeta_l} \hat{\phi}(\zeta_l)^n H_m^{b_l}(x) \right| > 2C, \tag{2.64}$$

where the above supremum is taken over the set

$$\{x : x \in I \text{ and } (\alpha n + x n^{-1/m}) \in \mathbb{Z}\}.$$

Combining (2.60) and (2.64) we conclude that

$$\sup_{x \in \mathbb{Z}} |\phi^{(n)}(x)| \geq C n^{-1/m} \tag{2.65}$$

for all $n > N$. The result now follows from the observation that $\phi^{(n)} \neq 0$ for all $n \leq N$ and so, by possibly adjusting C, (2.65) must hold for all n. □

2.7 Concentration of mass

In this section we complete the proof of Theorem 2.1.2. Recall that the theorem has two conclusions, the second of which is the subject of Theorem 2.5.5 and was already shown in the previous section. The first conclusion, (2.9), remains to be shown.

Proof of Theorem 2.1.2. We assume that ϕ satisfies the hypotheses of the theorem. By Theorem 2.5.5,

$$\phi^{(n)}(x) = \sum_{q=1}^{R} n^{-1/m} e^{-ix\xi_q} \hat{\phi}(\xi_q)^n H_m^{\beta_q}\left(\frac{x - \alpha_q n}{n^{1/m}}\right) + o(n^{-1/m}), \tag{2.66}$$

where the limit is uniform for $x \in \mathbb{Z}$ and the collections $\xi_1, \xi_2, \dots, \xi_R \in (-\pi, \pi]$, $\alpha_1, \alpha_2, \dots, \alpha_R$ and $\beta_1, \beta_2, \dots, \beta_R$ are those set by Convention 2.2.3.

Using Theorem 2.1.1 we choose $C > 0$ for which the estimate (2.3) holds. Considering all possibilities of β_q and m above, we can choose $M > 0$ such that

$$|H_m^{\beta_q}(y)| < C/(2R)$$

for all $|y| > M$ and for all $q = 1, 2, \ldots, R$. This can be done by using (2.6) or the conclusion of Proposition 2.4.2. Now let $K = [-M, M]$ and observe that, for any $q = 1, 2, \ldots, R$,

$$\left| n^{-1/m} e^{-ix\xi_q} \hat{\phi}(\xi_q)^n H_m^{\beta_q} \left(\frac{x - \alpha_q n}{n^{1/m}} \right) \right| < \frac{Cn^{-1/m}}{2R} \tag{2.67}$$

whenever $(x - \alpha_q n)/n^{1/m} > M$ or equivalently $x \notin \alpha_q n + Kn^{1/m}$. Further, by combining (2.66) and (2.67) there is some natural number N such that

$$|\phi^{(n)}(x)| < Cn^{-1/m}$$

for all $x \notin \cup_q(\alpha_q n + Kn^{1/m})$ and $n > N$. Thus by Theorem 2.1.1, the supremum $\|\phi^{(n)}\|_\infty$ must be attained on the set $(\cup_q(\alpha_q n + Kn^{1/m})) \cap \mathbb{Z}$ for all $n > N$. Lastly, observe that by enlarging the compact set K, the above dependence on N can be removed. This completes the proof.

$\square$

2.8 Examples

In this final section, we consider three examples to illustrate our results. We begin by considering a complex valued function on $\mathbb{Z}$ whose convolution powers consist of two waves drifting apart. This example cannot be treated by the results of Schoenberg, Greville or Thomée.

2.8.1 Two Airy functions with drift

Consider the function $\phi : \mathbb{Z} \to \mathbb{C}$ defined by

$$\phi(0) = \frac{3}{8} \quad \phi(\pm 2) = -\frac{1}{4} \quad \phi(\pm 3) = \frac{i}{3} \quad \phi(\pm 4) = \frac{1}{16}$$

and $\phi(x) = 0$ otherwise. The convolution powers, $\phi^{(n)}$, exhibit two distinct packets drifting apart, each with a rate of $2n$ from $x = 0$. Figure 2.5 illustrates this behavior.

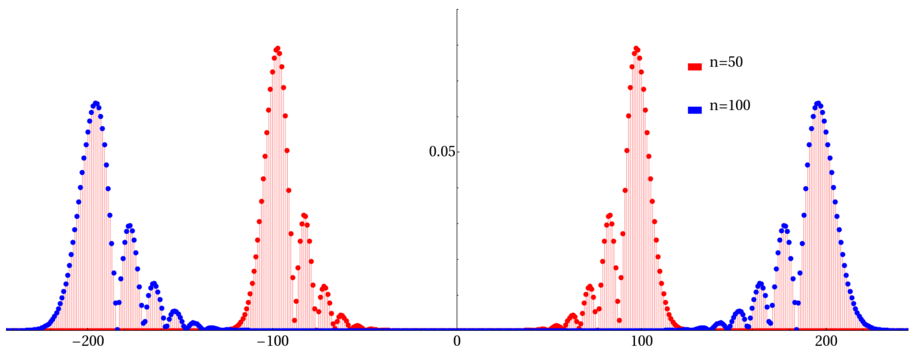

Figure 2.5: $|\phi^{(n)}|$ for $n = 50, 100$

The Fourier transform of ϕ is given by

$$\hat{\phi}(\xi) = \frac{3}{8} - \frac{1}{2}\cos(2\xi) + \frac{2i}{3}\cos(3\xi) + \frac{1}{8}\cos(4\xi).$$

Here, $\sup|\hat{\phi}| = 1$ and is attained only at $\xi_1 = \pi/2$ and $\xi_2 = -\pi/2$ in $(-\pi, \pi]$. It follows that

$$\log\left(\frac{\hat{\phi}(\xi \pm \pi/2)}{\hat{\phi}(\pm\pi/2)}\right) = \pm 2i\xi \mp \frac{5i}{3}\xi^3 - \frac{7}{3}\xi^4(1 + o(1)) \text{ as } \xi \to 0$$

and so $\alpha_1 = 2$, $\alpha_2 = -2$, $\beta_1 = 5i/3$, $\beta_2 = -5i/3$ and $m = m_1 = m_2 = 3$. In view of

Theorem 2.1.2 (or Theorem 2.5.5),

$$
\begin{aligned}
\phi^{(n)}(x) &= n^{-1/3}e^{-ix\pi/2}H_3^{\frac{5i}{3}}\left(\frac{x-2n}{n^{1/3}}\right)+n^{-1/3}e^{ix\pi/2}H_3^{\frac{-5i}{3}}\left(\frac{x+2n}{n^{1/3}}\right)+o(n^{-1/3}) \\
&= (5n)^{-1/3}(i)^x\left[(-1)^xH_3^{\frac{i}{3}}\left(\frac{x-2n}{(5n)^{1/3}}\right)+H_3^{\frac{i}{3}}\left(-\frac{x+2n}{(5n)^{1/3}}\right)\right]+o(n^{-1/3}) \\
&= (5n)^{-1/3}(i)^x\left[(-1)^x\mathrm{Ai}\left(\frac{x-2n}{(5n)^{1/3}}\right)+\mathrm{Ai}\left(-\frac{x+2n}{(5n)^{1/3}}\right)\right]+o(n^{-1/3}) \\
&= f(n,x)+o(n^{-1/3}) \qquad (2.68)
\end{aligned}
$$

uniformly for $x \in \mathbb{Z}$, where Ai denotes the standard Airy function. To appreciate Theorems 2.1.2 and 2.1.3, we consider $\phi^{(n)}(x)$ for $n = 10000$ near the right packet ($19700 \le x \le 20150$) corresponding to drift constant $\alpha_1 = \pi/2$. Figure 2.6 shows the graph of $\mathrm{Re}(\phi^{(n}(x))$ and Figure 2.7 shows the approximation, $f(n,x)$ defined by (2.68).

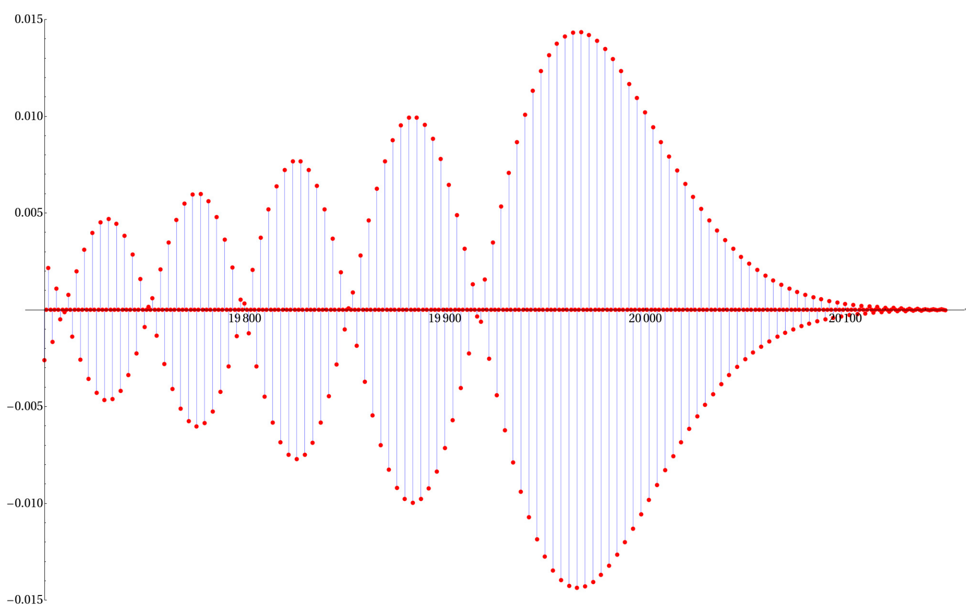

Figure 2.6: $\mathrm{Re}(\phi^{(n)})$ for $n = 10000$

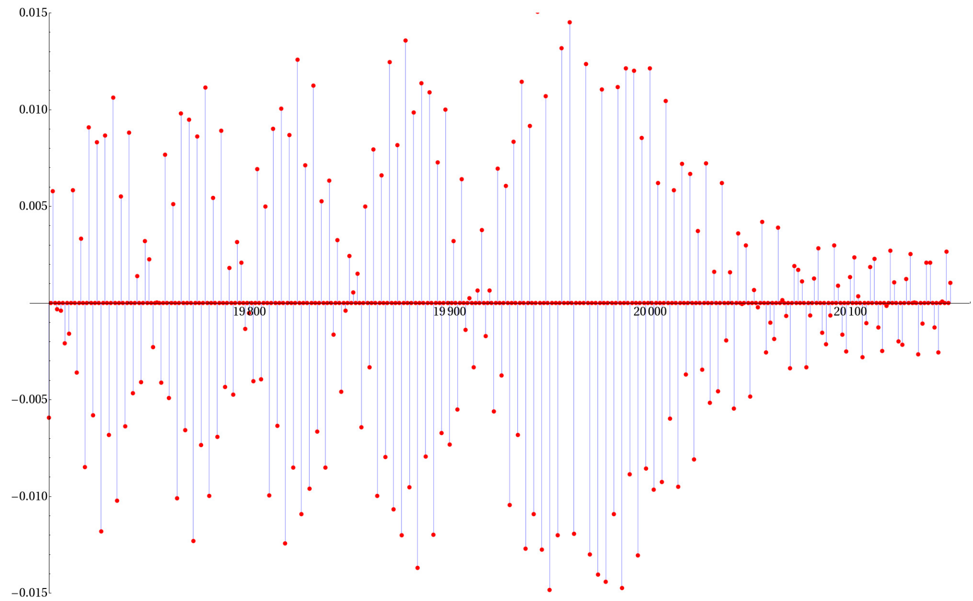

Figure 2.7: $\mathrm{Re}(f(n, x))$ for $n = 10000$

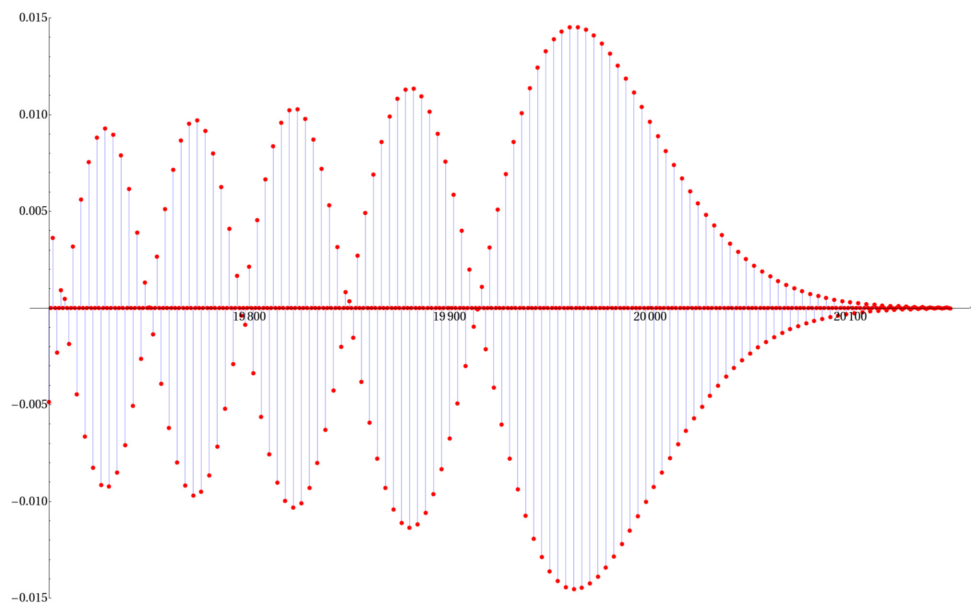

Figure 2.8: $\mathrm{Re}(g(n, x))$ for $n = 10000$

What appears to be noise in Figure 2.7 is the oscillatory tail of the term

$(5n)^{-1/3}(i)^x\mathrm{Ai}\left(-\frac{x+2n}{(5n)^{1/3}}\right)$ in (2.68). Removing this term, we consider

$$\begin{aligned} g(n,x) &= f(n,x) - (5n)^{-1/3}(i)^x\mathrm{Ai}\left(-\frac{x+2n}{(5n)^{1/3}}\right) \\ &= (5n)^{-1/3}(-i)^x\mathrm{Ai}\left(\frac{x-2n}{(5n)^{1/3}}\right). \end{aligned}$$

Upon choosing $\alpha_1 = \pi/2$, an appeal to Theorem 2.1.3 gives the approximation

$$\begin{aligned} \phi^{(n)}(\lfloor 2n + zn^{1/3}\rfloor) &= n^{-1/3}e^{-(i\lfloor 2n+zn^{1/3}\rfloor\pi/2)}H_3^{\frac{5i}{3}}(z) + o(n^{-1/3}) \\ &= (5n)^{-1/3}(-i)^{\lfloor 2n+zn^{1/3}\rfloor}\mathrm{Ai}\left(\frac{z}{5^{1/3}}\right) + o(n^{-1/3}) \end{aligned}$$

uniformly for z in any compact set; here, $\xi_{j_1} = \xi_1 = \pi/2$ and $\beta_{j_1} = \beta_1 = 5/3$. For such z, it follows that

$$\phi^{(n)}(\lfloor 2n + zn^{1/3}\rfloor) = g(n, \lfloor 2n + zn^{1/3}\rfloor) + o(n^{-1/3})$$

from which we see that g is essentially the approximation yielded by Theorem 2.1.3. As Figure 2.8 shows, $g(n,x)$ is a much better approximation to $\phi^{(n)}(x)$ at $n = 10000$ for $19700 \leq x \leq 20150$.

2.8.2 Heat kernel at purely imaginary time

We return to the example given in the introduction and justify the claims made therein. Let ϕ be given by (2.1). A quick computation shows that

$$\hat{\phi}(\xi) = 1 - \frac{i}{2}\sin^2(\xi/2) - \sin^4(\xi/2),$$

where the supremum of $|\hat{\phi}|$ on the interval $(-\pi,\pi]$ is only attained at $\xi_1 = 0$. In the notation of Proposition 2.2.2, we write

$$\Gamma(\xi) = \log\left(\frac{\hat{\phi}(\xi)}{\hat{\phi}(0)}\right) = -i\xi^2(\frac{1}{8} - \frac{1}{96}\xi^2) - \frac{7}{128}\xi^4 + \sum_{l=5}^{\infty} a_l\xi^l$$

on a neighborhood of 0 and so $m = m_1 = 2$, $\alpha_1 = 0$ and $\beta_1 = i/8$ in view of Convention 2.2.3. By Theorem 2.1.1, there are constants $C, C' > 0$ such that

$$Cn^{1/2} \leq \|\phi^{(n)}\|_\infty \leq C' n^{1/2}. \tag{2.69}$$

By Theorem 2.1.3 and using (2.5) we may also conclude that

$$\begin{aligned} \phi^{(n)}(\lfloor xn^{1/2} \rfloor) &= n^{-1/2} H_2^{i/8}(x) + o(n^{-1/2}) \\ &= \frac{n^{-1/2}}{\sqrt{4\pi i/8}} e^{-8|x|^2/4i} + o(n^{-1/2}), \end{aligned}$$

where the limit is uniform for x in any compact set.

2.8.3 A real-valued function supported on three points

In the article [31], Example 2.4 and Proposition 2.5 therein described the asymptotic behavior of the convolution powers of an arbitrary real valued function ϕ supported on three (consecutive) points. In the notation of the proposition we define ϕ by

$$\phi(0) = a_0,\ \phi(\pm 1) = a_\pm \text{ and } \phi = 0 \text{ otherwise}, \tag{2.70}$$

where $a_0, a_+, a_- \in \mathbb{R}$. As in [31], we also assume that $a_0 > 0$ and that $a_+ \neq 0$ or $a_- \neq 0$; this assumption guarantees that ϕ has admissible support. Proposition 2.5 of [31] describes the asymptotic behavior of $\phi^{(n)}$ for all values of $a_0, a_\pm$ except the special case in which $a_+a_- < 0$ and $4|a_+a_-| = a_0|a_+ + a_-|$. Theorem 2.1.2 allows us to treat this final case with ease.

Proposition 2.8.1. *Let ϕ be as above and assume additionally that $a_+a_- < 0$ and $4|a_+a_-| = a_0|a_+ + a_-|$. If $a_+ + a_- > 0$ then*

$$\phi^{(n)}(x) = n^{-1/3} A^n H_3^\beta\left(\frac{x - \alpha n}{n^{1/3}}\right) + o(A^n n^{-1/3}) \tag{2.71}$$

uniformly for $x \in \mathbb{Z}$*, where* $A = a_0 + a_+ + a_-$, $\alpha = (a_+ - a_-)/A$ *and* $\beta = i(\alpha - \alpha^3)/6$.

If $a_+ + a_- < 0$ *then*

$$\phi^{(n)}(x) = n^{-1/3} A^n e^{-ix\pi} H_3^{\beta}\left(\frac{x - \alpha n}{n^{1/3}}\right) + o(A^n n^{-1/3}) \tag{2.72}$$

uniformly for $x \in \mathbb{Z}$*, where* $A = a_0 - a_+ - a_-$, $\alpha = (a_- - a_+)/A$ *and* $\beta = i(\alpha - \alpha^3)/6$.

In either case, there is a compact set K *for which the* $\|\phi^{(n)}\|_\infty$ *is attained on the set* $(\alpha n + K n^{1/3})$.

Proof. We may write

$$\hat{\phi}(\xi) = a_0 + a_+ e^{i\xi} + a_- e^{-i\xi} = a_0 + (a_+ + a_-)\cos(\xi) + i(a_+ - a_-)\sin(\xi).$$

Under the assumption that $4|a_+ a_-| = a_0 |a_+ + a_-|$ and $a_+ + a_- > 0$, it was shown in [31] that $|\hat{\phi}|$ is maximized only at $0 = \xi_1 \in (-\pi, \pi]$ and in which case this maximum takes the value $A = a_0 + a_+ + a_-$.

Set $\psi(x) = \phi(x)/A$. It follows immediately that $A^n \psi^{(n)}(x) = \phi^{(n)}(x)$ and $\sup |\hat{\psi}| = 1$ which is taken only at $\xi_1 = 0$. In the notation of Proposition 2.2.2 we have

$$\begin{aligned}
\Gamma(\xi) &= \log\left(\frac{\hat{\psi}(\xi)}{\hat{\psi}(0)}\right) = i\left(\frac{(a_+ - a_-)}{a_0 + a_+ + a_-}\right)\xi \\
&\quad - \frac{i}{6}\left(\frac{(a_+ - a_-)(a_0^2 - a_0 a_+ - a_0 a_- - 8a_+ a_-)}{(a_0 + a_+ + a_-)^3}\right)\xi^3 - \gamma\xi^4 + \sum_{l=5}^{\infty} a_l \xi^l
\end{aligned}$$

on a neighborhood of 0, where $\gamma > 0$. Setting $\alpha = (a_+ - a_-)/A$ and using the fact that $4|a_+ a_-| = a_0 |a_+ + a_-|$, we write

$$\Gamma(\xi) = i\alpha\xi - \frac{i}{6}(\alpha - \alpha^3)\xi^3 - C\xi^4 + \sum_{l=5}^{\infty} a_l \xi^l \tag{2.73}$$

on a neighborhood of 0. By a quick inspection of (2.73) it is clear that ψ meets the hypotheses of Theorem 2.1.2 with $m = m_1 = 3$, $\alpha = \alpha_1$ and $\beta_1 = i(\alpha - \alpha^3)/6$. Therefore

$$\psi^{(n)}(x) = n^{-1/3} H_3^{\beta}\left(\frac{x - \alpha n}{n^{1/3}}\right) + o(n^{-1/3}) \tag{2.74}$$

uniformly for $x \in \mathbb{Z}$. The limit (2.71) follows immediately by multiplying (2.74) by A^n. An appeal to (2.9) of Theorem 2.1.2 shows that $\|\psi^{(n)}\|_\infty$ and hence $\|\phi^{(n)}\|_\infty$ is indeed attained on the set $(\alpha n + K n^{1/3})$ for some compact set K.

In the case that $a_+ + a_- < 0$ it was shown in [31] that $|\hat{\phi}|$ attains its only maximum at $\xi_1 = \pi \in (-\pi, \pi]$. Upon setting $A = a_0 - a_+ - a_-$, $\psi(x) = \phi(x)/A$ and considering the Taylor expansion of $\log(\hat{\psi}(\xi + \xi_1)/\hat{\psi}(\xi_1))$, the result follows by an argument similar to that given for the previous case. $\square$

CHAPTER 3

CONVOLUTION POWERS OF COMPLEX-VALUED FUNCTIONS ON $\mathbb{Z}^d$

3.1 Introduction

We denote by $\ell^1(\mathbb{Z}^d)$ the space of complex valued functions $\phi : \mathbb{Z}^d \to \mathbb{C}$ such that

$$\|\phi\|_1 = \sum_{x\in\mathbb{Z}^d} |\phi(x)| < \infty.$$

For $\psi, \phi \in \ell^1(\mathbb{Z}^d)$, the convolution product $\psi * \phi \in \ell^1(\mathbb{Z}^d)$ is defined by

$$\psi * \phi(x) = \sum_{y\in\mathbb{Z}^d} \psi(x-y)\phi(y)$$

for $x \in \mathbb{Z}^d$. Given $\phi \in \ell^1(\mathbb{Z}^d)$, we are interested in the convolution powers $\phi^{(n)} \in \ell^1(\mathbb{Z}^d)$ defined iteratively by $\phi^{(n)} = \phi^{(n-1)} * \phi^{(1)}$ for $n \in \mathbb{N}_+ =: \{1, 2, \dots\}$ where $\phi^{(1)} = \phi$. This study was originally motivated by problems in data smoothing, namely De Forest's problem, and it was later found essential to the theory of approximate difference schemes for partial differential equations [42, 80, 86, 87]; the recent article [31] gives background and pointers to the literature.

In random walk theory, the study of convolution powers is of central importance: Given an independent sequence of random vectors $X_1, X_2, \dots \in \mathbb{Z}^d$, all with distribution ϕ (here, $\phi \geq 0$), $\phi^{(n)}$ is the distribution of the random vector $S_n = X_1 + X_2 + \cdots + X_n$. Equivalently, a probability distribution ϕ on $\mathbb{Z}^d$ gives rise to a random walk whose nth-step transition kernel k_n is given by $k_n(x, y) = \phi^{(n)}(y - x)$ for $x, y \in \mathbb{Z}^d$. For an account of this theory, we encourage the reader to see the wonderful and classic book of F. Spitzer [83] and, for a more modern treatment, the recent book of G. Lawler and V. Limic [63] (see also Subsection 3.7.6). In the more general case that ϕ takes on complex values (or just simply takes on both positive and negative values), its convolution powers $\phi^{(n)}$ are seen to exhibit rich and disparate behavior, much of which never appears in the probabilistic setting. Given $\phi \in \ell^1(\mathbb{Z}^d)$, we are interested in the most basic and fundamental questions that can be asked about its convolution powers. Here are four such questions:

(i) What can be said about the decay of

$$\|\phi^{(n)}\|_\infty = \sup_{x\in\mathbb{Z}^d} |\phi^{(n)}(x)|$$

as $n \to \infty$?

(ii) Is there a simple pointwise description of $\phi^{(n)}(x)$, analogous to the local (central) limit theorem, that can be made for large n?

(iii) Are global space-time pointwise estimates obtainable for $|\phi^{(n)}|$?

(iv) Under what conditions is ϕ *stable* in the sense that

$$\sup_{n\in\mathbb{N}_+} \|\phi^{(n)}\|_1 < \infty? \tag{3.1}$$

The above questions have well-known answers in random walk theory. For simplicity we discuss the case in which ϕ is a probability distribution on $\mathbb{Z}^d$ whose associated random walk is symmetric, aperiodic, irreducible and of finite range. In this case, it is known that $n^{d/2}\phi^{(n)}(0)$ converges to a non-zero constant as $n \to \infty$ and this helps to provide an answer to Question (i) in the form of the following two-sided estimate: For positive constants C and C',

$$Cn^{-d/2} \le \sup_{x\in\mathbb{Z}^d} \phi^{(n)}(x) \le C'n^{-d/2}$$

for all $n \in \mathbb{N}_+$. Concerning the somewhat finer Question (ii), the classical local limit theorem states that

$$\phi^{(n)}(x) = n^{-d/2} G_\phi(n^{-1/2}x) + o(n^{-d/2})$$

uniformly for $x \in \mathbb{Z}^d$, where G_ϕ is the generalized Gaussian density

$$G_\phi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\big(-\xi \cdot C_\phi \xi\big) e^{-ix\cdot\xi}\, d\xi = \frac{1}{(2\pi)^{d/2}\sqrt{\det C_\phi}} \exp\left(-\frac{x \cdot {C_\phi}^{-1} x}{2}\right); \tag{3.2}$$

here, C_ϕ is the positive definite covariance matrix associated to ϕ and $\cdot$ denotes the dot product. As an application of this local limit theorem, one can easily settle the question of recurrence/transience for random walks on $\mathbb{Z}^d$ which was originally answered by G. Pólya in the context of simple random walk [71]. For general complex valued functions $\phi \in \ell^1(\mathbb{Z}^d)$, Question (ii) is a question about the validity of (generalized) local limit theorems and can be restated as follows: Under what conditions can the convolution powers $\phi^{(n)}$ be approximated pointwise by a combination (perhaps a sum) of appropriately scaled smooth functions– called *attractors*? The answer for Question (iii) for a finite range, symmetric, irreducible and aperiodic random walk is provided in terms of the so-called Gaussian estimate: For positive constants C and M,

$$\phi^{(n)}(x) \le C n^{-d/2} \exp(-M|x|^2/n)$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$; here, $|\cdot|$ is the standard euclidean norm. Such estimates, with matching lower bounds on appropriate space-time regions, are in fact valid in a much wider context, see [46]. Finally, the conservation of mass provides an obvious positive answer to Question (iv) in the case that ϕ is a probability distribution.

Beyond the probabilistic setting, the study of convolution powers for complex valued functions has centered mainly around two applications, statistical data smoothing procedures and finite difference schemes for numerical solutions to partial differential equations; the vast majority of the existing theory pertains only to one dimension. In the context of data smoothing, the earliest (known) study was motivated by a problem of Erastus L. De Forest. De Forest's problem, analogous to Question (ii), concerns the behavior of convolution powers of symmetric real valued and finitely supported functions on $\mathbb{Z}$ and was addressed by I. J. Schoenberg [80] and T. N. E. Greville [42]. In the context of numerical solutions in partial differential equations, the stability of convolution powers (Question (iv)) saw extensive investigation following World War II spurred by advancements in numerical computing. For an approximate difference scheme to an initial value problem, the property (3.1) is necessary and sufficient for convergence to a classical solution; this is the so-called Lax equivalence theorem [75, Chapter 4] (see Section 3.6). Property (3.1) is also called *power boundedness* and can be seen in the context of Banach algebras where ϕ is an element of the Banach algebra $(\ell^1(\mathbb{Z}^d), \|\cdot\|_1)$ equipped with the convolution product [57,81].

In one dimension, Questions (i-iv) were recently addressed in the articles [31] and [73]. For the general class of finitely supported complex valued functions on $\mathbb{Z}$, [73] (and the preceding chapter) completely settles Questions (i) and (ii). For instance, consider the following theorem.

Theorem 3.1.1 (Theorem 1.1 of [73]). *Let $\phi : \mathbb{Z} \to \mathbb{C}$ have finite support consisting of more than one point. Then there is a positive constant A and a natural number $m \geq 2$*

for which

$$Cn^{-1/m} \leq A^n \|\phi^{(n)}\|_\infty \leq C' n^{-1/m}$$

for all $n \in \mathbb{N}_+$, where C and C' are positive constants.

As we saw in Chapter 2, Question (ii) is completely settled in one-dimension. Specifically, Theorem 2.1.3 gives an exhaustive account of local limit theorems in which the set of possible attractors includes the Airy function and the heat kernel evaluated at purely imaginary time. In addressing Question (iii), the article [31] contains a number of results concerning global space-time estimates for $\phi^{(n)}$ for a finitely supported function ϕ – our results recapture (and extend in the case of Theorem 3.1.6) these results of [31]. The question of stability for finitely supported functions on $\mathbb{Z}$ was answered completely in 1965 by V. Thomée [87] (see Theorem 3.6.1 below). In fact, Thomée's characterization is, in some sense, the light in the dark that gives the correct framework for the study of local limit theorems in one dimension and we take it as a starting point for our study in $\mathbb{Z}^d$.

Moving beyond one dimension, the situation becomes more interesting still, the theory harder and much remains open. As we illustrate, convolution powers exhibit a significantly wider range of behaviors in $\mathbb{Z}^d$ than is seen in $\mathbb{Z}$ (see Remark 5). The focus of this chapter is to address Questions (i-iv) under some strong hypotheses on the Fourier transform – specifically, we work under the assumption that, near its extrema, the Fourier transform of ϕ is "nice" in a sense we will shortly make precise. To this end, we follow the article [31] and generalize the results therein. A complete theory for finitely supported functions on $\mathbb{Z}^d$, in which the results of the previous chapter will fit, is not presently known. Not surprisingly, our results recapture the well-known results of random walk theory on $\mathbb{Z}^d$ (see Subsection 3.7.6).

As discussed above, the theory presented in this chapter pertains to a large, though not exhaustive, class of finitely supported complex-valued functions on $\mathbb{Z}^d$. As seen in the probabilistic setting (and consistent with it), the "generic" behavior of the convolution powers of such functions is described by a single Gaussian attractor, generally evaluated at complex time, and our theory captures this typical situation with ease. The theory however describes much richer behavior which arises naturally in less "generic" examples, including varying rates of sup-norm decay, anisotropic scaling and multiple (drifting) attractors. To illustrate our results, throughout this introductory section we analyze a specific non-Gaussian example whose convolution powers exhibit a natural y-oscillation and anisotropic scaling structure (the reader is encouraged to see Section 3.7 for more examples). Consider $\phi : \mathbb{Z}^2 \to \mathbb{C}$ defined by

$$\phi(x,y) = \frac{1}{22+2\sqrt{3}} \times \begin{cases} 8 & (x,y) = (0,0) \\ 5+\sqrt{3} & (x,y) = (\pm 1, 0) \\ -2 & (x,y) = (\pm 2, 0) \\ i(\sqrt{3}-1) & (x,y) = (\pm 1, -1) \\ -i(\sqrt{3}-1) & (x,y) = (\pm 1, 1) \\ 2 \mp 2i & (x,y) = (0, \pm 1) \\ 0 & \text{otherwise.} \end{cases}$$

The graphs of $\mathrm{Re}(\phi^{(n)})$ for $(x,y) \in \mathbb{Z}^2$ for $-20 \leq x, y \leq 20$ are displayed in Figures 3.1 and 3.2 for $n = 10$ and $n = 100$ respectively. By inspection, one observes

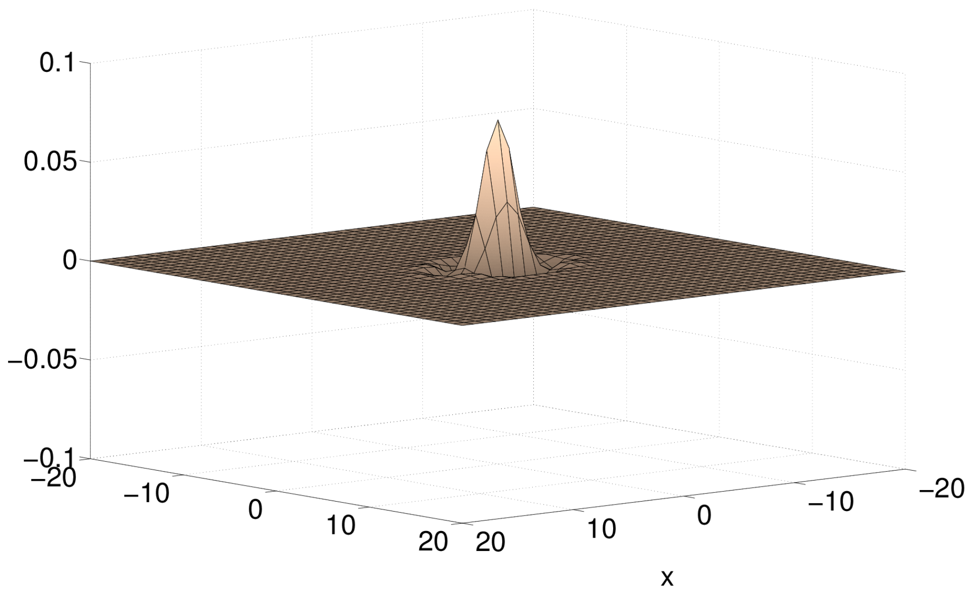

Figure 3.1: $\mathrm{Re}(\phi^{(n)})$ for $n = 10$

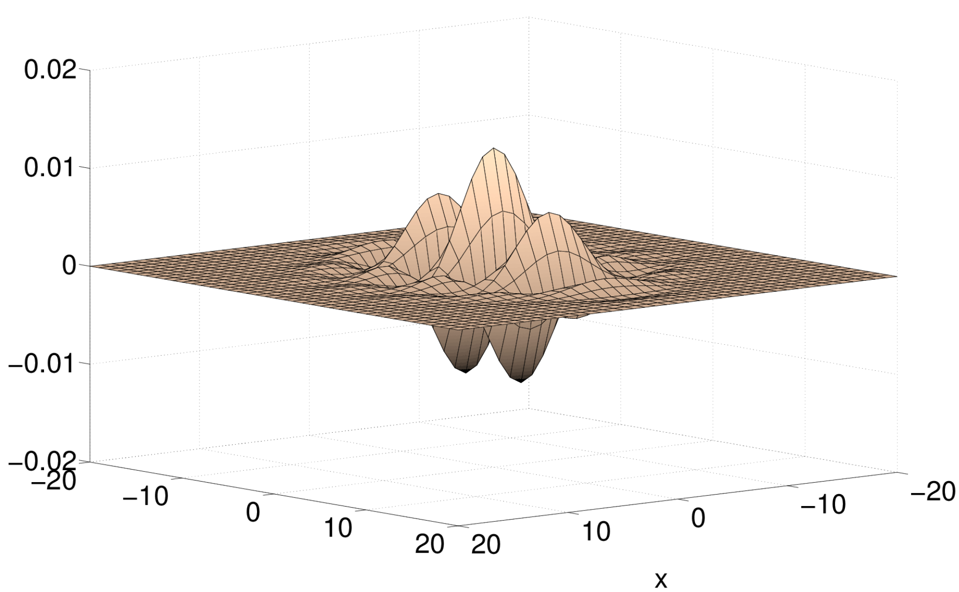

Figure 3.2: $\mathrm{Re}(\phi^{(n)})$ for $n = 100$

that $\mathrm{Re}(\phi^{(n)})$ decays in absolute value as n increases and, when $n = 100$, there is an apparent oscillation of $\mathrm{Re}(\phi^{(n)})$ in the y-direction. Our results explain these observations.

For $\phi \in \ell^1(\mathbb{Z}^d)$, its Fourier transform $\hat{\phi} : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$\hat{\phi}(\xi) = \sum_{x \in \mathbb{Z}^d} \phi(x) e^{ix\cdot\xi}$$

for $\xi \in \mathbb{R}^d$; this series is absolutely convergent. The standard Fourier inversion formula holds for all $\phi \in \ell^1(\mathbb{Z}^d)$ and moreover, for each $n \in \mathbb{N}_+$,

$$\phi^{(n)}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-ix\cdot\xi} \hat{\phi}(\xi)^n d\xi \tag{3.3}$$

for all $x \in \mathbb{Z}^d$ where $\mathbb{T}^d = (-\pi, \pi]^d$. Like the classical local limit theorem, our arguments are based on local approximations of $\hat{\phi}$ and such approximations require $\hat{\phi}$ to have a certain amount of smoothness. In our setting the order of smoothness needed in each case is not known a priori. For our purposes, it is sufficient (but not necessary) to consider only those $\phi \in \ell^1(\mathbb{Z}^d)$ with finite moments of all orders. That is, we consider the subspace of $\ell^1(\mathbb{Z}^d)$, denoted by $\mathcal{S}_d$, consisting of those ϕ for which

$$\|x^\beta \phi(x)\|_1 = \sum_{x \in \mathbb{Z}^d} |x^\beta \phi(x)| = \sum_{x \in \mathbb{Z}^d} |x_1^{\beta_1} x_2^{\beta_2} \cdots x_d^{\beta_d} \phi(x)| < \infty$$

for all multi-indices $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{N}^d$. It is straightforward to see that $\hat{\phi} \in C^\infty(\mathbb{R}^d)$ whenever $\phi \in \mathcal{S}_d$. We note that $\mathcal{S}_d$ contains all finitely supported functions mapping $\mathbb{Z}^d$ into $\mathbb{C}$; of course, when ϕ is finitely supported, $\hat{\phi}$ extends holomorphically to $\mathbb{C}^d$.

Before we begin to formulate our hypotheses, let us introduce some important objects by taking motivation from probability. The quadratic form $\xi \mapsto \xi \cdot C_\phi \xi$ which appears in (3.2) is a positive definite polynomial in ξ and is homogeneous in the following sense. For all $t > 0$ and $\xi \in \mathbb{R}^d$,

$$(t^{1/2}\xi) \cdot C_\phi (t^{1/2}\xi) = t\, \xi \cdot C_\phi \xi.$$

The map $(0,\infty) \ni t \mapsto t^{1/2}I \in \mathrm{Gl}_d(\mathbb{R})$ is a continuous (Lie group) homomorphism from the multiplicative group of positive real numbers into $\mathrm{Gl}_d(\mathbb{R})$; here I is the identity matrix in the set of $d\times d$ real matrices $\mathrm{M}_d(\mathbb{R})$ and $\mathrm{Gl}_d(\mathbb{R}) \subseteq \mathrm{M}_d(\mathbb{R})$ denotes the group of invertible matrices. For any such continuous homomorphism $t \mapsto T_t$, $\{T_t\}_{t>0}$ is a Lie subgroup of $\mathrm{Gl}_d(\mathbb{R})$, that is, a *continuous one-parameter group*; the Hille-Yosida construction guarantees that all such groups are of the form

$$T_t = t^E = \exp((\log t)E) = \sum_{k=0}^{\infty} \frac{(\log t)^k}{k!} E^k$$

for $t > 0$ for some $E \in \mathrm{M}_d(\mathbb{R})$. The Appendix amasses some basic properties of continuous one-parameter groups.

Definition 3.1.2. *For a continuous function $P : \mathbb{R}^d \to \mathbb{C}$ and a continuous one-parameter group $\{T_t\} \subseteq Gl_d(\mathbb{R})$, we say that P is homogeneous with respect to $T_t = t^E$ if*

$$tP(\xi) = P(T_t\xi)$$

for all $t > 0$ and $\xi \in \mathbb{R}^d$. In this case E is a member of the exponent set of P, $\mathrm{Exp}(P)$.

We say that P is positive homogeneous if the real part of P, $R = \mathrm{Re}\, P$, is positive definite (that is, $R(\xi) \geq 0$ and $R(\xi) = 0$ only when $\xi = 0$) and if $\mathrm{Exp}(P)$ contains a matrix $E \in M_d(\mathbb{R})$ whose spectrum is real.

Throughout this chapter, we concern ourselves with positive homogeneous multivariate polynomials $P : \mathbb{R}^d \to \mathbb{C}$; their appearance is seen to be natural, although not exhaustive, when considering local approximations of $\hat{\phi}$ for $\phi \in \mathcal{S}_d$. A given positive homogeneous polynomial P need not be homogeneous with respect to a unique continuous one-parameter group. For example, for each $m \in \mathbb{N}_+$, $\xi \mapsto |\xi|^{2m}$ is a positive homogeneous polynomial and it can be shown directly that

$$\operatorname{Exp}(|\cdot|^{2m}) = (2m)^{-1}I + \mathfrak{o}(d),$$

where $\mathfrak{o}(d) \subseteq \mathrm{M}_d(\mathbb{R})$ is the set of anti-symmetric matrices (these arise as the Lie algebra of the orthogonal group $\mathrm{O}_d(\mathbb{R}) \subseteq \mathrm{Gl}_d(\mathbb{R})$). It will be shown however that, for a positive homogeneous polynomial P, $\operatorname{tr} E = \operatorname{tr} E'$ whenever $E, E' \in \operatorname{Exp}(P)$; this is Corollary 3.2.4. To a given positive homogeneous polynomial P, the corollary allows us to uniquely define the number

$$\mu_P := \operatorname{tr} E \tag{3.4}$$

for any $E \in \operatorname{Exp}(P)$. This number appears in many of our results; in particular, it arises in addressing the Question (i) in which it plays the role of $1/m$ in Theorem 3.1.1.

We now begin to discuss the framework and hypotheses under which our theorems are stated. Let $\phi \in \mathcal{S}_d$ be such that $\sup_{\xi\in\mathbb{R}^d} |\hat{\phi}(\xi)| = 1$; this can always be arranged by multiplying ϕ by an appropriate constant. Set

$$\Omega(\phi) = \{\xi \in \mathbb{T}^d : |\hat{\phi}(\xi)| = 1\}$$

and, for $\xi_0 \in \Omega(\phi)$, define $\Gamma_{\xi_0} : \mathcal{U} \subseteq \mathbb{R}^d \to \mathbb{C}$ by

$$\Gamma_{\xi_0}(\xi) = \log\left(\frac{\hat{\phi}(\xi+\xi_0)}{\hat{\phi}(\xi_0)}\right)$$

where $\mathcal{U}$ is a convex open neighborhood of 0 which is small enough to ensure that $\log$, the principal branch of logarithm, is defined and continuous on $\hat{\phi}(\xi + \xi_0)/\hat{\phi}(\xi_0)$ for $\xi \in \mathcal{U}$. Because $\hat{\phi}$ is smooth, $\Gamma_{\xi_0} \in C^\infty(\mathcal{U})$ and so we can use Taylor's theorem to approximate Γ_{ξ_0} near 0. In this chapter, we focus on the case in which the Taylor expansion yields a positive homogeneous polynomial. The following definition, motivated by Thomée [87], captures this notion.

Definition 3.1.3. *Let $\phi \in \mathcal{S}_d$ be such that $\sup|\hat{\phi}(\xi)| = 1$ and let $\xi_0 \in \Omega(\phi)$. We say that ξ_0 is of positive homogeneous type for $\hat{\phi}$ if the Taylor expansion for Γ_{ξ_0} about 0 is of the form*

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - P_{\xi_0}(\xi) + \Upsilon_{\xi_0}(\xi) \tag{3.5}$$

where $\alpha_{\xi_0} \in \mathbb{R}^d$, P_{ξ_0} is a positive homogeneous polynomial and $\Upsilon_{\xi_0}(\xi) = o(R_{\xi_0}(\xi))$ as $\xi \to 0$; here $R_{\xi_0} = \operatorname{Re} P_{\xi_0}$. We say that α_{ξ_0} is the drift associated to ξ_0.

Though not obvious at first glance, α_{ξ_0} and P_{ξ_0} of the above definition are necessarily unique. When looking at any given Taylor polynomial, it will not always be apparent when the conditions of the above definition are satisfied. In Section 3.3, there is a discussion concerning this, and therein, necessary and sufficient conditions are given for $\xi_0 \in \Omega(\phi)$ to be of positive homogeneous type for $\hat{\phi}$.

Our theorems are stated under the assumption that for $\phi \in \mathcal{S}_d$, $\sup|\hat{\phi}(\xi)| = 1$ and each $\xi \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$. As we show in Section 3.3, these hypotheses ensure that the set $\Omega(\phi)$ is finite and in this case we set

$$\mu_\phi = \min_{\xi \in \Omega(\phi)} \mu_{P_\xi}. \tag{3.6}$$

This is admittedly a slight abuse of notation. We are ready to state our first main result.

Theorem 3.1.4. *Let $\phi \in \mathcal{S}_d$ be such that $\sup|\hat{\phi}(\xi)| = 1$ and suppose that each $\xi \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$. Then*

$$C' n^{-\mu_\phi} \le \|\phi^{(n)}\|_\infty \le C n^{-\mu_\phi} \tag{3.7}$$

for all $n \in \mathbb{N}_+$, where C and C' are positive constants.

The theorem above is a partial answer to Question (i) and nicely complements Theorem 3.1.1 and the results of [31]. We note however that, in view of the wider generality of Theorem 3.1.1, Theorem 3.1.4 is obviously not the final result in $\mathbb{Z}^d$ on this matter (see the discussion of tensor products in Subsection 3.7.4).

Returning to our motivating example and with the aim of applying Theorem 3.1.4, we analyze the Fourier transform of ϕ. We have

$$\begin{aligned}\hat{\phi}(\eta,\zeta) &= \frac{1}{11+\sqrt{3}}\big(4-2\cos(2\eta)+\big(5+\sqrt{3}\big)\cos(\eta)\\ &\qquad +2\big(\cos(\zeta)+\sin(\zeta)\big)+\big(2\sqrt{3}+2\big)\cos(\eta)\sin(\zeta)\big)\end{aligned}$$

for $(\eta,\zeta)\in\mathbb{R}^2$. One easily sees that $\sup|\hat{\phi}|=1$ and that $|\hat{\phi}|$ is supremized in $\mathbb{T}^2$ at only one point $(0,\pi/3)$ and here, $\hat{\phi}(0,\pi/3)=1$. As is readily computed,

$$\begin{aligned}\Gamma(\eta,\zeta) &= \log\left(\frac{\hat{\phi}(\eta,\zeta+\pi/3)}{\hat{\phi}(0,\pi/3)}\right)=\log(\hat{\phi}(\eta,\zeta+\pi/3))\\ &= -\frac{1}{11+\sqrt{3}}\eta^4+\frac{7-6\sqrt{3}}{118}\eta^2\zeta-\frac{2}{11+\sqrt{3}}\zeta^2\\ &\qquad +O(|\eta|^5)+O(|\eta^4\zeta|)+O(|\eta\zeta|^2)+O(|\zeta|^3)\end{aligned}$$

as $(\eta,\zeta)\to 0$. Let us study the polynomial

$$P(\eta,\zeta)=\frac{1}{22+2\sqrt{3}}\left(2\eta^4+\left(\sqrt{3}-1\right)\eta^2\zeta+4\zeta^2\right),$$

which leads this expansion. It is easily verified that $P=\operatorname{Re}P$ is positive definite and

$$P(t^E(\eta,\zeta))=P(t^{1/4}\eta,t^{1/2}\zeta)=tP(\eta,\zeta)\quad\text{ with }\quad E=\begin{pmatrix}1/4 & 0\\ 0 & 1/2\end{pmatrix}$$

for all $t>0$ and $(\eta,\zeta)\in\mathbb{R}^2$ and therefore P is a positive homogeneous polynomial with $E\in\operatorname{Exp}(P)$. Upon rewriting the error in the Taylor expansion, we have

$$\Gamma(\eta,\zeta) = -P(\eta,\zeta) + \Upsilon(\eta,\zeta)$$

where $\Upsilon(\eta,\zeta) = o(P(\eta,\zeta))$ as $(\eta,\zeta) \to (0,0)$ and so it follows that $(0,\pi/3)$ is of positive homogeneous type for $\hat{\phi}$ with corresponding $\alpha = (0,0) \in \mathbb{R}^2$ and positive homogeneous polynomial P. Consequently, ϕ satisfies the hypotheses of Theorem 3.1.4 with $\mu_\phi = \mu_P = \operatorname{tr} E = 3/4$ and so

$$C' n^{-3/4} \leq \|\phi^{(n)}\|_\infty \leq C n^{-3/4}$$

for all $n \in \mathbb{N}_+$, where C and C' are positive constants. With the help of a local limit theorem, we will shortly describe the pointwise behavior of ϕ.

Coming back to the general setting, we now introduce the attractors which appear in our main local limit theorem. For a positive homogeneous polynomial P, define $H_P^{(\cdot)} : (0,\infty) \times \mathbb{R}^d \to \mathbb{C}$ by

$$H_P^t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-tP(\xi)} e^{-ix\cdot\xi}\, d\xi \tag{3.8}$$

for $t > 0$ and $x \in \mathbb{R}^d$; we write $H_P(x) = H_P^1(x)$. As we show in Section 3.2, for each $t > 0$, $H_P^t(\cdot)$ belongs to the Schwartz space, $\mathcal{S}(\mathbb{R}^d)$, and moreover, for any $E \in \operatorname{Exp}(P)$,

$$H_P^t(x) = \frac{1}{t^{\operatorname{tr} E}} H_P(t^{-E^*} x) = \frac{1}{t^{\mu_P}} H_P(t^{-E^*} x) \tag{3.9}$$

for all $t > 0$ and $x \in \mathbb{R}^d$; here E^* is the adjoint of E. These function arise naturally in the study of partial differential equations. For instance, consider the partial differential operator $\partial_t + \Lambda_P$ where $\Lambda_P := P(D)$, called a *positive homogeneous operator*, is defined by replacing the d-tuple $\xi = (\xi_1, \xi_2, \ldots, \xi_d)$ in $P(\xi)$ by the d-tuple of partial derivatives $D = (i\partial_{x_1}, i\partial_{x_2}, \ldots, i\partial_{x_d})$. The associated Cauchy problem for this operator can be stated thus: Given initial data f (from a suitable class of functions), find $u(x,t)$ satisfying

$$\begin{cases} (\partial_t + \Lambda_P)u(x,t) &= 0 \qquad x \in \mathbb{R}^d,\ t > 0 \\ u(0,x) &= f(x) \quad x \in \mathbb{R}^d. \end{cases} \tag{3.10}$$

In this context, $H_P^{(\cdot)}$ is a fundamental solution to (3.10) in the sense that the representation

$$u(x,t) = (e^{-t\Lambda_P} f)(x) = \int_{\mathbb{R}^d} H_P^{(t)}(x-y) f(y)\,dy \tag{3.11}$$

satisfies $(\partial_t + \Lambda_P)u = 0$ and has $u(t,\cdot) \to f$ as $t \to 0$ in an appropriate topology. Equivalently, $H_P^{(\cdot)}$ is the integral kernel of the semigroup $e^{-t\Lambda_P}$ with infinitesimal generator Λ_P. The Cauchy problem for the setting in which Λ_P is replaced by an operator H which depends on x and is uniformly comparable to $(-\Delta)^m = \Lambda_{|\cdot|^{2m}}$ is the subject of (higher order) parabolic partial differential equations and its treatment can be found in the classic texts [35] and [40] (see also [21] and [20]). The subject of Chapter 4 treats the case in which H is uniformly comparable to a positive homogeneous operator; therein, we write $K_\Lambda = H_P$. In the present chapter, we shall only need a few basic facts concerning $H_P^{(\cdot)}$.

Remark 5. *When $d = 1$, every positive homogeneous polynomial is of the form $P(\xi) = \beta\xi^m$ where $\operatorname{Re}\beta > 0$ and m is an even natural number. In this case, H_P is equal to the function H_m^β of Chapter 2. We note that the simplicity of the dilation structure in one dimension is in complete contrast with the natural complexity of the multi-dimensional analogue seen in this chapter.*

For our next main theorem which addresses Question (ii), we restrict our attention to the set of points $\{\xi_1, \xi_2, \dots, \xi_A\} \subseteq \Omega(\phi)$ for which $\mu_{P_{\xi_q}} = \mu_\phi$ for $q = 1, 2, \dots, A$; the points $\xi \in \Omega(\phi)$ for which $\mu_{P_\xi} > \mu_\phi$ (if there are any) are not seen in local limits. Finally for each ξ_q for $q = 1, 2, \ldots, A$, we set $\alpha_q = \alpha_{\xi_q}$ and $P_q = P_{\xi_q}$. The following local limit theorem addresses Question (ii).

Theorem 3.1.5. *Let $\phi \in \mathcal{S}_d$ be such that $\sup|\hat{\phi}(\xi)| = 1$ and suppose that every point $\xi \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$. Let μ_ϕ be defined by* (3.6) *and let $\xi_1, \xi_2, \ldots, \xi_A$, $\alpha_1, \alpha_2, \ldots, \alpha_A$, and $P_1, P_2, \ldots, P_A$ be as in the previous paragraph. Then*

$$\phi^{(n)}(x) = \sum_{q=1}^{A} e^{-ix\cdot\xi_q}\hat{\phi}(\xi_q)^n H_{P_q}^n(x - n\alpha_q) + o(n^{-\mu_\phi}) \tag{3.12}$$

uniformly for $x \in \mathbb{Z}^d$.

Let us make a few remarks about this theorem. First, the attractors $H_{P_q}^n$ appearing in (3.12) are rescaled versions of $H_{P_q} = H_{P_q}^1$ in view of (3.9), and all decay in absolute value on the order $n^{-\mu_\phi}$ – this is consistent with Theorem 3.1.4. Second, the attractors $H_{P_q}(x)$ often exhibit slowly varying oscillations as $|x|$ increases (see Subsection (3.7.1)), however, the main oscillatory behavior, which is present in Figure 3.2, is a result of the prefactor $e^{-ix\cdot\xi_q}\hat{\phi}(\xi_q)$. This is, of course, a consequence of $\hat{\phi}$ being maximized away from the origin. In Subsection 3.7.6, we will see that when ϕ is a probability distribution, all of the attractors in (3.12) are identical and the prefactors collapse into a single function, Θ, which nicely describes the support of $\phi^{(n)}$ and hence periodicity of the associated random walk (see Theorems 3.7.5 and 3.7.6).

Taking another look at our motivating example, we note that the hypotheses of Theorem 3.1.4 are precisely the hypotheses of Theorem 3.1.5 and so an application of the local limit theorem is justified, where, because $\Omega(\phi)$ is a singleton, the sum in (3.12) consists only of one term. We have

$$\begin{aligned}\phi^{(n)}(x,y) &= e^{-i(x,y)\cdot(0,\pi/3)}\hat{\phi}((0,\pi/3))^n H_P^n(x,y)+o(n^{-\mu_\phi})\\ &= e^{-i\pi y/3}H_P^n(x,y)+o(n^{-3/4})\end{aligned}$$

uniformly for $(x,y) \in \mathbb{Z}^2$. To illustrate this local limit, the graphs of $\mathrm{Re}(e^{-i\pi y/3}H_P^n)$ for $(x,y)\in\mathbb{Z}^2$ for $-20\le x,y\le 20$ are displayed in Figures 3.3 and 3.4 for $n=10$ and $n=100$ respectively for comparison against Figures 3.1 and 3.2. The oscillation in the y-direction is now explained by the appearance of the multiplier $e^{-i\pi y/3}$ and is independent of n.

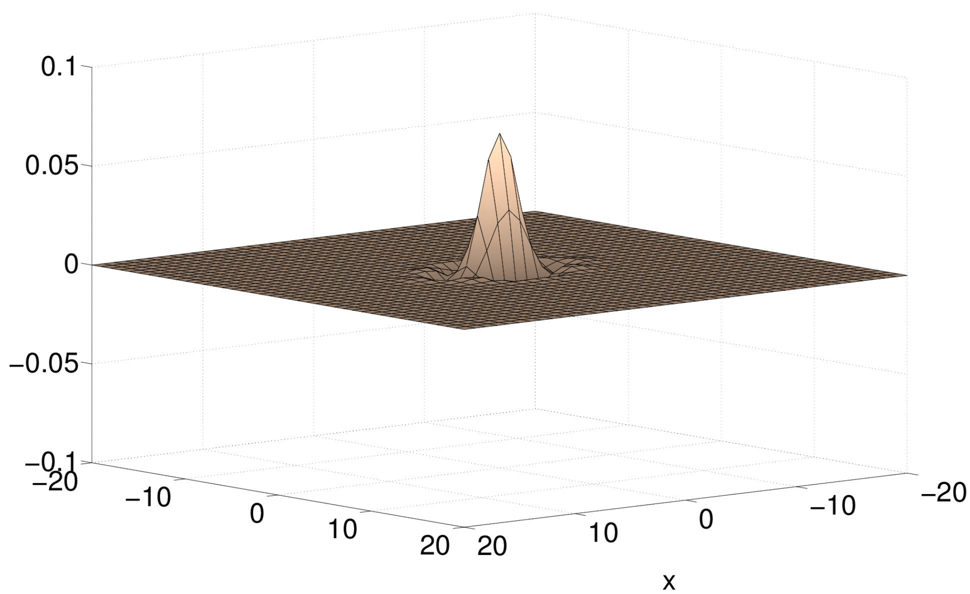

Figure 3.3: $\mathrm{Re}(e^{-i\pi y/3}H_P^n)$ for $n=10$

To address Question (iii) and obtain pointwise estimates for the $\phi^{(n)}$, we restrict our attention to those $\phi:\mathbb{Z}^d\to\mathbb{C}$ with finite support. In this Chapter, we present two theorems concerning pointwise estimates for $|\phi^{(n)}(x)|$. The most general result, in addition to requiring finite support for ϕ, assumes the hypotheses of

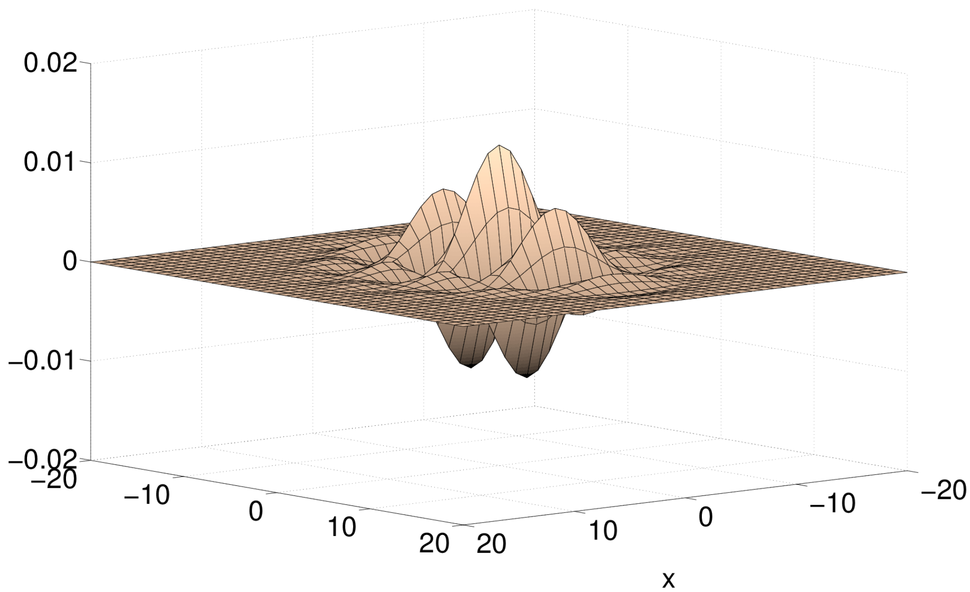

Figure 3.4: $\mathrm{Re}(e^{-i\pi y/3}H_P^n)$ for $n = 100$

Theorem 3.1.5; this is Theorem 3.5.10. The other result, Theorem 3.1.6, additionally assumes that all $\xi \in \Omega(\phi)$ have the same corresponding drift $\alpha_\xi = \alpha \in \mathbb{R}^d$ and positive homogeneous polynomial $P = P_\xi$ – a condition which is seen to be quite natural by taking a look at Subsections 3.7.3 and 3.7.6, although not necessary, see Remark 8. Theorem 3.1.6 extends the corresponding 1-dimensional result, Theorem 3.1 of [31], to d-dimensions and, even in 1-dimension, is seen to be an improvement. In addition to global pointwise estimates for $\phi^{(n)}$, in Section 3.5 we present a variety of results which give global pointwise estimates for discrete space and time derivatives of $\phi^{(n)}$. In what follows, we describe the statement of Theorem 3.1.6 as it is the simplest.

For simplicity, assume that $\phi : \mathbb{Z}^d \to \mathbb{C}$ is finitely supported, satisfies $\sup_\xi |\hat{\phi}| = 1$ and $\Omega(\phi)$ consists of only one point ξ_0 which is of positive homogeneous type for $\hat{\phi}$. In this case, we use Theorem 3.1.5 to motivate the correct form for pointwise estimated for $\phi^{(n)}$. The theorem gives the approximation

$$\phi^{(n)}(x) = e^{-ix\cdot\xi_0}\hat{\phi}(\xi_0)^n H_P^n(x - n\alpha) + o(n^{-\mu_P}) \tag{3.13}$$

uniformly for $x \in \mathbb{Z}^d$, where $P = P_{\xi_0}$ is positive homogeneous and $\alpha = \alpha_{\xi_0} \in \mathbb{R}^d$. Pointwise estimates for the attractor H_P can be deduced with the help of the Legendre-Fenchel transform, a central object in convex analysis [78, 88]. The Legendre-Fenchel transform of $R = \operatorname{Re} P$ is the function $R^{\#} : \mathbb{R}^d \to \mathbb{R}$ defined by

$$R^{\#}(x) = \sup_{\xi\in\mathbb{R}^d}\{x\cdot\xi - R(\xi)\}.$$

It is evident that $R^{\#}(x) \geq 0$ and, for $E \in \operatorname{Exp}(P)$,

$$tR^{\#}(x) = \sup_{\xi\in\mathbb{R}^d}\left\{tx\cdot\xi - R(t^E\xi)\right\} = R^{\#}\left(t^{(I-E)^*}x\right)$$

for all $t > 0$ and $x \in \mathbb{R}^d$, i.e., $(I - E)^* \in \operatorname{Exp}(R^{\#})$. It turns out that $R^{\#}$ is necessarily continuous and positive definite (Proposition A.3.2). In Section 3.2, we establish the following pointwise estimates for H_P. There exists positive constants C, M such that

$$|H_P^t(x)| \leq \frac{C}{t^{\operatorname{tr} E}}\exp(-MR^{\#}(t^{-E^*}x)) = \frac{C}{t^{\mu_P}}\exp(-tMR^{\#}(x/t)) \tag{3.14}$$

for all $x \in \mathbb{R}^d$ and $t > 0$.

Remark 6. *In the special case that $P(\xi) = |\xi|^{2m}$, $E = (2m)^{-1}I \in \operatorname{Exp}(P)$ and one can directly compute $R^{\#}(x) = C_m|x|^{2m/(2m-1)}$ where $C_m = (2m)^{-1/(2m-1)} - (2m)^{-2m/(2m-1)} > 0$. Here, the estimate* (3.14) *takes the form*

$$H_{|\cdot|^{2m}}^t(x) \leq \frac{C}{t^{d/2m}}\exp\left(-M|x|^{2m/(2m-1)}/t^{1/(2m-1)}\right)$$

for $t > 0$ and $x \in \mathbb{R}^d$ and so we recapture the well-known off-diagonal estimate for the semigroup $e^{-t(-\Delta)^m}$ [20,21,35,40]. In the context of local limit theorems, $H_{|\cdot|^{2m}}$ is seen *to be the attractor of the convolution powers of* $\kappa_m = \delta_0 - (\delta_0 - \kappa)^{(m)}$ *where* κ *is the probability distribution assigning* $1/2$ *probability to* 0 *and* $1/(4d)$ *probability to* $\pm e_j$ *for* $j = 1, 2, \ldots, d$*; here and in what follows,* $e_1, e_2, \ldots, e_d$ *denote the standard euclidean basis vectors of* $\mathbb{R}^d$*.*

In view of (3.13) and the preceding discussion, one expects an estimate of the form (3.14) to hold for $\phi^{(n)}$, although, we note that no such estimate can be established on these grounds (this is due to the error term in (3.13)). This however motivates the correct form and we are able to establish the following result which captures, as a special case, the situation described above in which $\Omega(\phi) = \{\xi_0\}$.

Theorem 3.1.6. *Let* $\phi : \mathbb{Z}^d \to \mathbb{C}$ *be finitely supported and such that* $\sup_{\xi \in \mathbb{T}^d} |\hat{\phi}(\xi)| = 1$*. Suppose that every point of* $\xi \in \Omega(\phi)$ *is of positive homogeneous type for* $\hat{\phi}$ *and every* $\xi \in \Omega(\phi)$ *has the same drift* $\alpha = \alpha_\xi \in \mathbb{R}^d$ *and positive homogeneous polynomial* $P = P_\xi$*. Also let* $\mu_\phi = \mu_P$ *be defined by* (3.4) *and let* $R^{\#}$ *be the Legendre-Fenchel transform of* $R = \operatorname{Re} P$*. Then there exists* $C, M > 0$ *for which*

$$|\phi^{(n)}(x)| \leq \frac{C}{n^{\mu_\phi}} \exp\left(-nMR^{\#}\left(\frac{x - n\alpha}{n}\right)\right) \tag{3.15}$$

for all $n \in \mathbb{N}_+$ *and* $x \in \mathbb{Z}^d$*.*

Revisiting, for a final time, our motivating example, we note that ϕ also satisfies the hypotheses of Theorem 3.1.6. An appeal to the theorem gives constants $C, M > 0$ for which

$$|\phi^{(n)}(x, y)| \leq \frac{C}{n^{3/4}} \exp\left(-nMR^{\#}((x, y)/n)\right) \tag{3.16}$$

for all $n \in \mathbb{N}_+$ and for all $(x, y) \in \mathbb{Z}^2$, where $R^{\#}$ is the Legendre-Fenchel transform of $R = \operatorname{Re} P = P$. Instead of finding a closed-form expression for $R^{\#}$, which is not particularly illuminating, we simply remark that

$$R^{\#}(x,y) \asymp |x|^{4/3} + |y|^2, \tag{3.17}$$

where $\asymp$ means that the ratio of the functions is bounded above and below by positive constants ((3.17) is straightforward to establish and can be seen as consequence of Corollary A.3.3). Upon combining (3.16) and (3.17), we obtain constants $C, M > 0$ for which

$$\begin{aligned}|\phi^{(n)}(x,y)| \\ \leq \ \frac{C}{n^{3/4}} \exp\left(-nM\left(\left|\frac{x}{n}\right|^{4/3} + \left|\frac{y}{n}\right|^2\right)\right) = \frac{C}{n^{3/4}} \exp\left(-M\left(\frac{|x|^{4/3}}{n^{1/3}} + \frac{|y|^2}{n}\right)\right)\end{aligned}$$

for all $n \in \mathbb{N}_+$ and for all $(x,y) \in \mathbb{Z}^2$. This result illustrates the anisotropic exponential decay of $n^{3/4}|\phi^{(n)}(x,y)|$ for each $n \in \mathbb{N}_+$.

Back within the general setting and continuing under the assumption that $\phi : \mathbb{Z}^d \to \mathbb{C}$ is finitely supported, we come to the final question posed at the beginning of this introduction, Question (iv). The following result extends the (affirmative) results of V. Thomée [87] and M.V. Fedoryuk [38] (see also the related result of [81, Theorem 7.5]).

Theorem 3.1.7. *Let $\phi : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported and such that $\sup_\xi |\hat{\phi}(\xi)| = 1$. Suppose additionally that each $\xi \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$. Then, there exists a positive constant C for which*

$$\|\phi^{(n)}\|_1 = \sum_{x \in \mathbb{Z}^d} |\phi^{(n)}(x)| \leq C$$

for all $n \in \mathbb{N}_+$.

This chapter is organized as follows: Section 3.2 outlines the basic theory of positive homogeneous polynomials and their corresponding attractors. Section 3.3

focuses on the local behavior of $\hat{\phi}$ wherein necessary and sufficient condition are given to ensure that a given $\xi_0 \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$. In Section 3.4, we prove the main local limit theorem, Theorem 3.1.5, and deduce from it Theorem 3.1.4. Section 3.5 focuses on global space-time bounds for $\phi^{(n)}$ in the case that ϕ is finitely supported. In addition to the proof of Theorem 3.1.6, Subsection 3.5.1 contains a number of results concerning global exponential estimates for discrete space and time differences of $\phi^{(n)}$. In Subsection 3.5.2, we prove global sub-exponential estimates for $\phi^{(n)}$ in the general case that ϕ, in addition to being finitely supported, satisfies the hypotheses of Theorem 3.1.7; this is Theorem 3.5.10. In Section 3.6, after a short discussion on stability of numerical difference schemes in partial differential equations, we present Theorem 3.1.7 as a consequence of Theorem 3.5.10. Section 3.7 contains a number of concrete examples, mostly in $\mathbb{Z}^2$, to which we apply our results; the reader is encouraged to skip ahead to this section as it can be read at any time. We end Section 3.7 by showing, from our perspective, some results on the classical theory of random walks on $\mathbb{Z}^d$. The Appendix contains a number of linear-algebraic results which highlight the interplay between one-parameter contracting groups and positive homogeneous functions.

Notation: For $y \in \mathbb{Z}^d$, $\delta_y : \mathbb{Z}^d \to \{0,1\}$ is the standard delta function defined by $\delta_y(y) = 1$ and $\delta_y(x) = 0$ for $x \neq y$. For any subset A of $\mathbb{R}$, A_+ denotes the subset of positive elements of A. Given $M \in \mathrm{M}_d(\mathbb{R})$, its corresponding linear transformation on $\mathbb{R}^d$ is denoted by L_M. For any $r > 0$, we denote the open unit ball with center $x \in \mathbb{R}^d$ by $B_r(x)$ and the closed unit ball by $\overline{B_r(x)}$. When $x = 0$, we write $B_r = B_r(0)$ and denote by $S_r = \partial B_r$ the sphere of radius r. Further, when $r = 1$, we write $B = B_1$ and $S = S_1$. We define a d-dimensional floor function by $\lfloor \cdot \rfloor : \mathbb{R}^d \to \mathbb{Z}^d$ by $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \dots, \lfloor x_d \rfloor)$ for $x \in \mathbb{R}^d$ where $\lfloor x_k \rfloor$ is the integer part of x_k for $k = 1, 2, \dots d$; this is admittedly a slight abuse of notation. Given $\mathbf{n} = (n_1, n_2, \dots, n_d) \in (\mathbb{N}_+)^d = \mathbb{N}_+^d$ and a multi-index $\beta \in \mathbb{N}^d$, put

$$|\beta : \mathbf{n}| = \sum_{i=1}^{d} \frac{\beta_i}{n_d};$$

this is consistent with Hörmander's notation for semi-elliptic operators and polynomials [55, p. 100]. For any two real functions f, g on a set X, we write $f \asymp g$ when there are positive constants C and C' for which $Cg(x) \leq f(x) \leq C'g(x)$ for all $x \in X$.

3.2 Positive homogeneous polynomials and attractors

In this section, we study positive homogeneous polynomials and their corresponding attractors; let us first give some background. In Hörmander's treatise [55], polynomials of the form

$$Q(\xi) = \sum_{|\beta:\mathbf{m}| \leq 1} a_\beta \xi^\beta$$

for $\mathbf{m} \in \mathbb{N}_+^d$ are called *semi-elliptic* provided their principal part,

$$Q_p(\xi) = \sum_{|\beta:\mathbf{m}|=1} a_\beta \xi^\beta,$$

is non-degenerate, that is, $Q_p(\xi) \neq 0$ whenever $\xi \neq 0$. For a semi-elliptic polynomial Q, its corresponding partial differential operator $\Lambda_Q = Q(D)$, called a semi-elliptic operator, is hypoelliptic in the sense that all Λ_Q-harmonic distributions are smooth, see [55, p. 200]. What appears to be the most desirable property of semi-elliptic polynomials is the way that they scale in the sense that

$$\begin{aligned} &Q_p(t^{1/m_1}\xi_1, t^{1/m_2}\xi_2, \dots, t^{1/m_d}\xi_d) \\ &= \sum_{|\beta:\mathbf{m}|=1} a_\beta \prod_{j=1}^{d} (t^{1/m_i}\xi_j)^{\beta_j} = \sum_{|\beta:\mathbf{m}|=1} t^{|\beta:\mathbf{m}|} a_\beta \xi^\beta = tQ_p(\xi) \end{aligned}$$

for all $t > 0$ and $\xi \in \mathbb{R}^d$. This property, used explicitly by Hörmander, is precisely the statement that $E = \operatorname{diag}(1/m_1, 1/m_2, \dots, 1/m_d) \in \operatorname{Exp}(Q_p)$, in view of Definition 3.1.2. Further, the associated one-parameter group $\{T_t\} = \{t^E\}$ has the useful property that it dilates and contracts space. The following definition captures this behavior in general (see [45, Section 1.1]).

Definition 3.2.1. *Let $\{T_t\}_{t>0} \subseteq Gl_d(\mathbb{R})$ be a continuous one-parameter group. We say that $\{T_t\}$ is contracting if*

$$\lim_{t \to 0} \|T_t\| = 0.$$

Here and in what follows, $\|\cdot\|$ denotes the operator norm on $Gl_d(\mathbb{R})$.

To keep in mind, the canonical example of a contracting group is $\{t^D\}$ where $D = \operatorname{diag}(\gamma_1, \gamma_2, \dots, \gamma_d) \in \mathrm{M}_d(\mathbb{R})$ with $\gamma_i > 0$ for $i = 1, 2, \dots, d$ and here, it is easily seen that $t^D = \operatorname{diag}(t^{\gamma_1}, t^{\gamma_2}, \dots, t^{\gamma_d})$ for $t > 0$. Some basic results concerning contracting groups are given in the Appendix and are used throughout this chapter. As we will see shortly, for any positive homogeneous polynomial P, t^E is a contracting group for any $E \in \operatorname{Exp}(P)$.

Of interest for us is the subclass of semi-elliptic polynomials of the form

$$P(\xi) = \sum_{|\beta:2\mathbf{m}|=1} a_\beta \xi^\beta = \sum_{|\beta:\mathbf{m}|=2} a_\beta \xi^\beta, \tag{3.18}$$

where $\mathbf{m} \in \mathbb{N}_+^d$, $\{a_\beta\} \subseteq \mathbb{C}$ and $\operatorname{Re} P$ is positive definite. For these polynomials, it is easy to see that the corresponding partial differential operator

$\partial_t + \Lambda_P$ is semi-elliptic in the sense of Hörmander and hence hypoelliptic. By a slight abuse of language, any reference to a semi-elliptic polynomial is a reference to a polynomial of the form (3.18). It is straightforward to see that all such semi-elliptic polynomials are positive homogeneous and have $D = \operatorname{diag}((2m_1)^{-1}, (2m_2)^{-1}, \dots, (2m_d)^{-1}) \in \operatorname{Exp}(P)$. However, not all positive homogeneous polynomials are semi-elliptic as the example of Subsection 3.7.3 illustrates. As our first result of this section shows, every positive homogeneous polynomial has a coordinate system in which it is semi-elliptic.

Proposition 3.2.2. *Let P be a positive homogeneous polynomial and let $E \in \operatorname{Exp}(P)$ have real spectrum. There exist $A \in Gl_d(\mathbb{R})$ and $\{m_1, m_2, \dots, m_d\} \subseteq \mathbb{N}_+$ for which*

$$A^{-1}EA = \operatorname{diag}((2m_1)^{-1}, (2m_2)^{-1}, \dots, (2m_d)^{-1}) \tag{3.19}$$

and

$$(P \circ L_A)(\xi) = \sum_{|\beta:\mathbf{m}|=2} a_\beta \xi^\beta \tag{3.20}$$

for $\xi \in \mathbb{R}^d$.

Proof. In light of the fact that the spectrum of E is real, the characteristic polynomial for E factors completely over $\mathbb{R}$ and so we may apply the Jordan-Chevally decomposition. This gives $A \in \mathrm{Gl}_d(\mathbb{R})$ for which $F := A^{-1}EA = D + N$ where D is a diagonal matrix, N is a nilpotent matrix and $ND = DN$. It is evident that $Q := (P \circ L_A)$ is a polynomial and so we can write

$$Q(\xi) = \sum_\beta a_\beta \xi^\beta \tag{3.21}$$

for all $\xi \in \mathbb{R}^d$. In fact, our hypothesis guarantees that Q is positive homogeneous and $F \in \operatorname{Exp}(Q)$. Our proof proceeds in three steps, first we show that $D \in \operatorname{Exp}(Q)$. Second, we determine the spectrum of D. In the final step we show that $N = 0$.

Step 1. We have

$$tQ(\xi) = Q(t^F\xi) = Q(t^{D+N}\xi) = Q(t^N t^D\xi) \tag{3.22}$$

for all $t > 0$ and $\xi \in \mathbb{R}^d$ where $D = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_d)$ for $\gamma_1, \gamma_2, \ldots \gamma_d \in \mathbb{R}$. Because N is nilpotent,

$$t^N = I + \frac{\log t}{1}N + \cdots + \frac{(\log t)^k}{k!}N^k$$

where $k+1$ is the index of N. Thus by (3.22), for all $t > 0$ and $\xi \in \mathbb{R}^d$,

$$\begin{aligned} tQ(t^{-D}\xi) &= Q\left(\xi + (\log t)N\xi + \cdots + \frac{(\log t)^k}{k!}N^k\xi\right) \\ &= Q(\xi) + S_N(\xi, \log t) \end{aligned} \tag{3.23}$$

where S_N is a polynomial on $\mathbb{R}^d \times \mathbb{R}$ with no constant term. Consequently, for each $\xi \in \mathbb{R}^d$ we may write

$$S_N(\xi, x) = \sum_{j=1}^{l} b_j(\xi)x^j \tag{3.24}$$

where $b_j(\xi) \in \mathbb{C}$ for each j.

Let us now fix a non-zero $\xi \in \mathbb{R}^d$. Combining (3.21), (3.23) and (3.24) yields

$$\sum_{\beta} a_\beta t^{(1-\beta\cdot\gamma)}\xi^\beta = Q(\xi) + \sum_{j=1}^{l} b_j(\xi)(\log t)^j$$

for all $t > 0$ where $\beta \cdot \gamma = \beta_1\gamma_1 + \beta_2\gamma_2 + \cdots \beta_d\gamma_d$ and necessarily $Q(\xi) \neq 0$. Since distinct real powers of t and $\log t$ are linearly independent as C^∞ functions for $t > 0$, it follows that $b_j(\xi) = 0$ for each j and more importantly,

$$Q(\xi) = \sum_{\beta\cdot\gamma=1} a_\beta \xi^\beta. \tag{3.25}$$

Since ξ was arbitrary, (3.25) must hold for all $\xi \in \mathbb{R}^d$ and from this we see that

$$\begin{aligned} Q(t^D\xi) &= \sum_{\beta\cdot\gamma=1} a_\beta (t^D\xi)^\beta \\ &= \sum_{\beta\cdot\gamma=1} a_\beta t^{\beta\cdot\gamma}(\xi)^\beta = tQ(\xi) \end{aligned} \tag{3.26}$$

for all $t > 0$ and $\xi \in \mathbb{R}^d$; hence $D \in \operatorname{Exp}(Q)$.

Step 2. Writing $R_Q = \operatorname{Re} Q$, it follows from (3.25) that

$$R_Q(\xi) = \sum_{\beta\cdot\gamma=1} c_\beta \xi^\beta \tag{3.27}$$

for all $\xi \in \mathbb{R}^d$ where $c_\beta = \operatorname{Re} a_\beta$ for each multi-index β. Now for each $i = 1, 2, \dots, d$, xe_i is an eigenvector of D with eigenvalue γ_i for all non-zero $x \in \mathbb{R}$; here e_i is that of the standard euclidean basis. Using the positive definiteness of R_Q, for all $t > 0$ and $x \neq 0$, we have

$$tR_Q(xe_i) = R_Q(t^D(xe_i)) = R_Q(t^{\gamma_i}xe_i) = t^{(|\beta|\gamma_i)}c_\beta x^{|\beta|} > 0$$

where β is the only surviving multi-index from the sum in (3.27) and necessarily β is an integer multiple of e_i. From this we see that $|\beta|$ must be even for otherwise positivity would be violated and also that $1/\gamma_i = |\beta| =: 2m_i$ as claimed.

Step 3. In view of the previous step,

$$t^D = \operatorname{diag}\left(t^{(2m_1)^{-1}}, t^{(2m_2)^{-1}}, \dots, t^{(2m_d)^{-1}}\right) \tag{3.28}$$

for all $t > 0$ and so $\{t^D\}_{t>0}$ is a one-parameter contracting group. Using the positive definiteness of R_Q, it follows from Proposition A.1.5 that

$$\lim_{|\xi|\to\infty} R_Q(\xi) \geq \lim_{t\to\infty} \inf_{\eta\in S} R_Q(t^D\eta) \geq \lim_{t\to\infty} t \inf_{\eta\in S} R_Q(\eta) = \infty. \tag{3.29}$$

Now because D commutes with F and $D \in \mathrm{Exp}(R_Q)$,

$$R_Q(\xi) = tt^{-1}R_Q(\xi) = R_Q(t^F t^{-D}\xi) = R_Q(t^N \xi)$$

for $t > 0$ and $\xi \in \mathbb{R}^d$. Our goal is to show that $N = 0$. For suppose that $N \neq 0$, then for some $\xi \in \mathbb{R}^d$, $\nu = N\xi \neq 0$ but $N\nu = 0$. Then,

$$R_Q(\xi) = R_Q(t^N\xi) = R_Q\left(\xi + (\log t)N\xi + \frac{(\log t)^2}{2!}(N)^2\xi + \cdots\right) = R_Q(\xi + (\log t)\nu)$$

for all $t > 0$. This however cannot hold for its validity would contradict (3.29) and so $N = 0$ as desired. $\square$

Proposition 3.2.3. *If P is a positive homogeneous polynomial then $\mathrm{Sym}(P) := \{O \in M_d(\mathbb{R}) : P(O\xi) = P(\xi)$ for all $\xi \in \mathbb{R}^d\}$ is a compact subgroup of $Gl_d(\mathbb{R})$ and hence a subgroup of the orthogonal group, $O_d(\mathbb{R})$.*

Proof. It is clear that $I \in \mathrm{Sym}(P)$ and that for any $O_1, O_2 \in \mathrm{Sym}(P)$, $O_1O_2 \in \mathrm{Sym}(P)$. If $O \in \mathrm{Sym}(P)$, $R(O\xi) = R(\xi)$ for all $\xi \in \mathbb{R}^d$ where $R = \mathrm{Re}\, P$. The positive definiteness of R implies that $\mathrm{Ker}\, O$ is trivial and hence $O \in \mathrm{Gl}_d(\mathbb{R})$. Consequently, $P(O^{-1}\xi) = P(OO^{-1}\xi) = P(\xi)$ for all $\xi \in \mathbb{R}^d$ and hence $O^{-1} \in \mathrm{Sym}(P)$.

It remains to show that $\mathrm{Sym}(P)$ is compact and so, in view of the Heine-Borel theorem, we show that $\mathrm{Sym}(P)$ is closed and bounded. To see that $\mathrm{Sym}(P)$ is closed, let $\{O_n\} \subseteq \mathrm{Sym}(P)$ be such that $O_n \to O \in \mathrm{M}_d(\mathbb{R})$. Then the continuity of P implies that for all $\xi \in \mathbb{R}^d$,

$$P(O\xi) = \lim_n P(O_n\xi) = P(\xi)$$

and so $O \in \mathrm{Sym}(P)$.

To show that $\mathrm{Sym}(P)$ is bounded, we first make an observation from the proof of Proposition 3.2.2. Assuming the notation therein, we conclude from (3.29) that

$$\lim_{|\xi|\to\infty} R(\xi) = \infty \tag{3.30}$$

because $R(\xi) = R_Q(A^{-1}\xi)$ for all $\xi \in \mathbb{R}^d$. Finally, to reach a contradiction, we assume that $\mathrm{Sym}(P)$ is not bounded. Then there exist sequences $\{O_n\} \subseteq \mathrm{Sym}(P)$ and $\{\xi_n\} \subseteq S$ for which $\lim_n |O_n\xi_n| = \infty$. Observe however that

$$R(O_n\xi_n) = R(\xi_n) \leq \sup_{\xi\in S} R(\xi) < \infty$$

for all n; in view of (3.30) we have obtained our desired contradiction. $\square$

Corollary 3.2.4. *Let P be a positive homogeneous polynomial. Then for any $E, E' \in \mathrm{Exp}(P)$,*

$$\mathrm{tr}(E) = \mathrm{tr}(E').$$

Proof. For $E, E' \in \mathrm{Exp}(P)$, it follows immediately that $t^E t^{-E'} \in \mathrm{Sym}(P)$ for all $t > 0$. In view of Proposition 3.2.3,

$$t^{\mathrm{tr}\,E - \mathrm{tr}\,E'} = |t^{\mathrm{tr}\,E} t^{-\,\mathrm{tr}\,E'}| = |\det(t^E)\det(t^{-E'})| = |\det(t^E t^{-E'})| = 1$$

for all $t > 0$; here we have used the fact that the trace of a real matrix is real and that the determinant maps $\mathrm{O}_d(\mathbb{R})$ into the unit circle. The corollary follows immediately. $\square$

Lemma 3.2.5. *Let P be a positive homogeneous polynomial. For any $E \in \mathrm{Exp}(P)$, the continuous one-parameter group $\{t^E\}_{t>0}$ is contracting.*

Proof. First let $E_0 \in \mathrm{Exp}(P)$ have real spectrum. In view of Proposition 3.2.2,

$$A^{-1}t^{E_0}A = \mathrm{diag}(t^{\gamma_1}, t^{\gamma_2}, \ldots, t^{\gamma_d})$$

for all $t > 0$ where $0 < \gamma_i < 1/2$ for $i = 1, 2, \dots, d$. By inspection, we can immediately conclude that $\{t^{E_0}\}_{t>0}$ is contracting. Now for any $E \in \operatorname{Exp}(P)$, $t^E t^{-E_0} \in \operatorname{Sym}(P) \subseteq \mathrm{O}_d(\mathbb{R})$ for all $t > 0$ by virtue of Proposition 3.2.3; from this it follows immediately that $\{t^E\}$ is contracting. $\square$

We now turn to the study of the attractors appearing in Theorem 3.1.5; these are of the form $H_P^{(\cdot)}$, defined by (3.8), where P is a positive homogeneous polynomial.

Proposition 3.2.6. *Let P be a positive homogeneous polynomial with $R = \operatorname{Re} P$. The following is true:*

i) For any $t > 0$, $H_P^{(t)}(\cdot) \in \mathcal{S}(\mathbb{R}^d)$.

ii) If $E \in \operatorname{Exp}(P)$ then, for all $t > 0$ and $x \in \mathbb{R}^d$,

$$H_P^{(t)}(x) = \frac{1}{t^{\operatorname{tr} E}} H_P^1(t^{-E^*}x) = \frac{1}{t^{\mu_P}} H_P(t^{-E^*}x);$$

where E^* is the adjoint of E.

iii) There exist constants $C, M > 0$ such that

$$\left| H_P^{(t)}(x) \right| \leq \frac{C}{t^{\mu_P}} \exp(-tMR^{\#}(x/t))$$

for all $t > 0$ and $x \in \mathbb{R}^d$.

Proof. To prove items i) and ii), it suffices only to show that $H_P = H_P^1 \in \mathcal{S}(\mathbb{R}^d)$. Indeed, if $H_P \in \mathcal{S}(\mathbb{R}^d)$ then, in particular, $e^{-P} \in L^1(\mathbb{R}^d)$ and so the change-of-variables formula guarantees that, for any $t>0$ and $x\in\mathbb{R}^d$,

$$\begin{aligned} H_P^t(x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-tP(\xi)}e^{-ix\cdot\xi}\,d\xi \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-P(t^E\xi)}e^{-ix\cdot\xi}\,d\xi \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-P(\xi)}e^{-ix\cdot(t^{-E}\xi)}\det(t^{-E})\,d\xi \\ &= \frac{t^{-\operatorname{tr}E}}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-P(\xi)}e^{-i(t^{-E^*}x)\cdot\xi}\,d\xi \\ &= t^{-\mu_P}H_P(t^{-E^*}x) \end{aligned}$$

whenever $E\in\operatorname{Exp}(P)$. From this the validity of item ii) is clear but moreover, the formula ensures that that $H_P^t\in\mathcal{S}(\mathbb{R}^d)$ for all $t>0$.

In view of (3.8), $H_P\in\mathcal{S}(\mathbb{R}^d)$ if and only if $e^{-P}\in\mathcal{S}(\mathbb{R}^d)$ because the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$. Also, for any $A\in\mathrm{Gl}_d(\mathbb{R})$, it is clear that $e^{-P}\in\mathcal{S}(\mathbb{R}^d)$ if and only if $e^{-P\circ L_A}$. Hence, to show that $H_P\in\mathcal{S}(\mathbb{R}^d)$ it suffices to show that $e^{-P\circ L_A}\in\mathcal{S}(\mathbb{R}^d)$ for some $A\in\mathrm{Gl}_d(\mathbb{R})$. This is precisely what we do now: Let $E\in\operatorname{Exp}(P)$ have real spectrum and correspondingly, take $A\in\mathrm{Gl}_d(\mathbb{R})$ as guaranteed by Proposition 3.2.2. As in the proof of the proposition, we write $Q=P\circ L_A$, $R_Q=\operatorname{Re}Q$ and $D=\operatorname{diag}((2m_1)^{-1},(2m_2)^{-1},\dots,(2m_d)^{-1})$. It is clear that $e^{-Q}\in C^\infty(\mathbb{R}^d)$. Let μ and β be multi-indices and observe that

$$\|e^{-Q}\|_{\mu,\beta}:=\sup_{\xi\in\mathbb{R}^d}\left|\xi^\mu D^\beta e^{-Q}\right|=\sup_{\xi\in\mathbb{R}^d}\left|Q_{\mu,\beta}(\xi)\exp(-Q(\xi))\right|$$

where $Q_{\mu,\beta}$ is a polynomial. Using Proposition A.1.5 and the continuity of $Q_{\mu,\beta}e^{-Q}$, it follows that

$$\begin{aligned} \|e^{-Q}\|_{\mu,\beta} &= \sup_{\nu\in S,t>0}\left|Q_{\mu,\beta}(t^D\nu)\exp(-Q(t^D\nu))\right| \\ &= \sup_{\nu\in S,t>0}\left|Q_{\mu,\beta}(t^D\nu)\exp(-tQ(\nu))\right|. \end{aligned}$$

Now because Q is positive homogeneous, $Q_{\mu,\beta}$ is a polynomial and t^D has the form (3.28),

$$|Q_{\mu,\beta}(t^D\nu)e^{-tQ(\nu)}| \leq M_1(1+t^m)e^{-tM_2}$$

for all $t>0$ and $\nu \in \mathcal{S}$ where m, M_1 and M_2 are positive constants. We immediately see that

$$\|e^{-Q}\|_{\mu,\beta} \leq \sup_{t>0} M_1(1+t^m)e^{-tM_2} < \infty$$

and therefore $e^{-Q} \in \mathcal{S}(\mathbb{R}^d)$.

The key to the proof of iii) is a complex change-of-variables. For each $x \in \mathbb{R}^d$, function $z \mapsto e^{-P(z)}e^{-ix\cdot z}$ is holomorphic on $\mathbb{C}^d$ and, in view of Proposition A.2.7, satisfies

$$|e^{-P(\xi-i\nu)}e^{-ix\cdot(\xi-i\nu)}| = e^{-x\cdot\nu}|e^{-P(\xi-i\nu)}| \leq e^{-x\cdot\nu+MR(\nu)}e^{-\epsilon R(\xi)} \tag{3.31}$$

for all $z=\xi-i\nu \in \mathbb{C}^d$, where M, ϵ are positive constants. By virtue of (3.30), (3.31) ensures that the integration in the definition of H_P can be shifted to any any complex plane in $\mathbb{C}^d$ parallel to $\mathbb{R}^d$. In other words, for any $x, \nu \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} e^{-P(\xi)}e^{-ix\cdot\xi}\,d\xi = \int_{\xi\in\mathbb{R}^d} e^{-P(\xi-i\nu)}e^{-ix\cdot(\xi-i\nu)}\,d\xi$$

and therefore

$$|H_P(x)| \leq e^{-x\cdot\nu+MR(\nu)}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-\epsilon R(\xi)} = C\exp(-(x\cdot\nu - MR(\nu))),$$

where $C>0$. The natural appearance of the Legendre-Fenchel transform is now seen by infimizing over $\nu \in \mathbb{R}^d$. We have

$$\begin{aligned}|H_P(x)| \leq C\inf_{\nu\in\mathbb{R}^d}\exp(-(x\cdot\nu - MR(\nu))) &= C\exp\left(-\sup_{\nu\in\mathbb{R}^d}\{x\cdot\nu - MR(\nu)\}\right)\\ &= C\exp\left(-(MR)^{\#}(x)\right) \leq C\exp\left(-MR^{\#}(x)\right)\end{aligned}$$

for all $x \in \mathbb{R}^d$, where we have made use of Corollary A.3.4 to adjust the constant

M. Finally, an appeal to ii) and Proposition A.3.2, gives

$$\begin{aligned} |H_P^{(t)}(x)| &\leq \frac{C}{t^{\mu_P}} \exp\left(-MR^{\#}(t^{-E^*}x)\right) \\ &= \frac{C}{t^{\mu_P}} \exp\left(-MR^{\#}(t^{(I-E)^*}(x/t))\right) = \frac{C}{t^{\mu_P}} \exp\left(-tMR^{\#}(x/t)\right) \end{aligned}$$

for all $t > 0$ and $x \in \mathbb{R}^d$. $\square$

3.3 Properties of $\hat{\phi}$

Lemma 3.3.1. *Let $\phi \in \mathcal{S}_d$ be such that $\sup|\hat{\phi}| = 1$ and suppose that $\xi_0 \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$. Then the expansion* (3.5)*, with $\alpha_{\xi_0} \in \mathbb{R}^d$ and positive homogeneous polynomial P_{ξ_0}, is unique.*

Proof. The fact that $|\hat{\phi}(\xi)| \leq 1$ ensures that the linear term in the Taylor expansion for Γ_{ξ_0} is purely imaginary. This determines α_{ξ_0} uniquely. We assume that

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - P_1(\xi) + \Upsilon_1(\xi) = i\alpha_{\xi_0} \cdot \xi - P_2(\xi) + \Upsilon_2(\xi)$$

for $\xi \in \mathcal{U}$ where P_1 and P_2 are positive homogeneous polynomials with $\operatorname{Re} P_1 = R_1$, $\operatorname{Re} P_2 = R_2$ and $\Upsilon_i = o(R_i)$ as $\xi \to 0$ for $i = 1, 2$. We shall prove that $P_1 = P_2$.

Let $\epsilon > 0$ and, for a fixed non-zero $\zeta \in \mathbb{R}^d$, set $\delta_i = \epsilon/2R_i(\zeta)$ for $i = 1, 2$. Also, take $E_i \in \operatorname{Exp}(P_i)$ for $i = 1, 2$. Because $\Upsilon_i = o(R_i)$ as $\xi \to 0$ for $i = 1, 2$ there is a neighborhood $\mathcal{O}$ of 0 for which $|\Upsilon_i(\xi)| < \delta_i R_i(\xi)$ whenever $\xi \in \mathcal{O}$ for $i = 1, 2$. By virtue of Lemma 3.2.5, $t^{-E_1}\zeta, t^{-E_2}\zeta \in \mathcal{O}$ for some $t > 0$ and therefore

$$\begin{aligned} |P_1(\zeta) - P_2(\zeta)| &= t|P_1(t^{-E_1}\zeta) - P_2(t^{-E_2}\zeta)| \leq t|\Upsilon_1(t^{-E_1}\zeta)| + t|\Upsilon_2(t^{-E_2}\zeta)| \\ &< t\delta_1 R_1(t^{-E_1}\zeta) + t\delta_2 R_2(t^{-E_2}\zeta) \leq \delta_1 R_1(\zeta) + \delta_2 R_2(\zeta) \leq \epsilon \end{aligned}$$

as required. $\square$

Lemma 3.3.2. *Let $\phi \in \mathcal{S}_d$ be such that $\sup|\hat{\phi}| = 1$ and suppose that $\xi_0 \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$ with associated positive homogeneous polynomial $P = P_{\xi_0}$ and remainder $\Upsilon = \Upsilon_{\xi_0}$. Then for any $E \in \operatorname{Exp}(P)$,*

$$\lim_{t\to\infty} t\Upsilon(t^{-E}\xi) = 0.$$

for each $\xi \in \mathbb{R}^d$.

Proof. The assertion is clear when $\xi = 0$. When $\xi \in \mathbb{R}^d$ is non-zero, we note that $t^{-E}\xi \to 0$ as $t \to 0$ by virtue of Lemma 3.2.5; in particular, $t^{-E}\xi \in \mathcal{U}$ for sufficiently large t. Consequently,

$$\lim_{t\to\infty} \frac{\Upsilon(t^{-E}\xi)}{R(t^{-E}\xi)} = 0$$

because $\Upsilon(\eta) = o(R(\eta))$ as $\eta \to 0$ and so it follows that

$$\lim_{t\to\infty} t\Upsilon(t^{-E}\xi) = \lim_{t\to\infty} R(\xi)\frac{\Upsilon(t^{-E}\xi)}{t^{-1}R(\xi)} = R(\xi)\lim_{t\to\infty} \frac{\Upsilon(t^{-E}\xi)}{R(t^{-E}\xi)} = 0$$

as desired. □

Given $\xi_0 \in \Omega(\phi)$ and considering the Taylor expansion for Γ_{ξ_0}, to recognize whether or not ξ_0 is of positive homogeneous type for $\hat{\phi}$ is not always straightforward, e.g., Subsection 3.7.3). Nonetheless, it is useful to have a method based on the Taylor expansion for Γ_{ξ_0} through which we can determine if ξ_0 is of positive homogeneous type for $\hat{\phi}$ and, when it is, pick out the associated positive homogeneous polynomial P_{ξ_0}. The remainder of this section is dedicated to do just this.

Given any integer $m \geq 2$, the mth order Taylor expansion for Γ_{ξ_0} is necessarily of the form

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0}\cdot\xi - Q^m_{\xi_0}(\xi) + O(|\xi|^{m+1}) \tag{3.32}$$

for $\xi \in \mathcal{U}$ where $\alpha_{\xi_0} \in \mathbb{R}^d$ and $Q^m_{\xi_0}(\xi)$ is a polynomial given by

$$Q^m_{\xi_0}(\xi) = \sum_{1<|\alpha|\leq m} c_\alpha \xi^\alpha$$

for $\xi \in \mathbb{R}^d$, where $\{c_\alpha\} \subseteq \mathbb{C}$. No constant term appears in the expansion for Γ_{ξ_0} because $\Gamma_{\xi_0}(0) = 0$. Moreover the fact that

$$\hat{\phi}(\xi + \xi_0) = \hat{\phi}(\xi_0) e^{\Gamma_{\xi_0}(\xi)}$$

for all $\xi \in \mathcal{U}$ and the condition that $\sup |\hat{\phi}(\xi)| = 1$ ensure that

$$\mathrm{Re}(i\alpha_{\xi_0} \cdot \xi - Q^m_{\xi_0}(\xi)) = -\operatorname{Re} Q^m_{\xi_0}(\xi) \leq 0$$

for ξ sufficiently close to 0 (in fact, this is precisely why $\alpha_{\xi_0} \in \mathbb{R}^d$). Our final result of this section, Proposition 3.3.3, provides necessary and sufficient conditions for ξ_0 to be of positive homogeneous type for $\hat{\phi}$ in terms of $Q^m_{\xi_0}$. We remark that the proposition, although quite useful for examples, is not used anywhere else in this work. As the proof is lengthy and in many ways parallels the proof of Proposition 3.2.2, we have placed it in the Appendix, Subsection A.4.

Proposition 3.3.3. *Let $\phi \in \mathcal{S}_d$, suppose that $\sup |\hat{\phi}(\xi)| = 1$ and let $\xi_0 \in \Omega(\phi)$. Then the following are equivalent:*

a. The point ξ_0 is of positive homogeneous type for $\hat{\phi}$ with corresponding positive homogeneous polynomial P_{ξ_0}.

b. There exist $m \geq 2$ and a positive homogeneous polynomial P such that, for some $C, r > 0$,

$$C^{-1}R(\xi) \leq \operatorname{Re} Q^m_{\xi_0}(\xi) \leq CR(\xi)$$

and

$$|\operatorname{Im} Q^m_{\xi_0}(\xi)| \leq CR(\xi)$$

for all $\xi \in \overline{B_r}$, where $R = \operatorname{Re} P$.

c. There exist $m \geq 2$ and $E \in M_d(\mathbb{R})$ with real spectrum such that, for some $r > 0$ and sequence of positive real numbers $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$, the sequence $\{\rho_n\}$ of polynomials defined by

$$\rho_n(\xi) = t_n Q^m_{\xi_0}(t_n^{-E}\xi) \tag{3.33}$$

converges for all $\xi \in \overline{B_r}$ as $n \to \infty$ and its limit has positive real part for all $\xi \in S_r$.

When the above equivalent conditions are satisfied, for any $m' \geq m$,

$$P_{\xi_0}(\xi) = \lim_{t\to\infty} t Q^{m'}_{\xi_0}(t^{-E}\xi)$$

for all $\xi \in \mathbb{R}^d$ and this convergence is uniform on all compact subsets of $\mathbb{R}^d$.

3.4 Local limit theorems and ℓ^∞ estimates

In this section we prove Theorems 3.1.4 and 3.1.5. Our first result ensures that, under the hypotheses of Theorem 3.1.5, we can approximate the convolution powers of ϕ by a finite sum of attractors.

Proposition 3.4.1. *Let $\phi \in \mathcal{S}_d$ be such that $\sup|\hat{\phi}(\xi)| = 1$. If each $\xi \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$ then $\Omega(\phi)$ is discrete (and hence finite).*

Proof. Let $\xi_0 \in \Omega(\phi)$ be of positive homogeneous type for $\hat{\phi}$; it suffices to show that ξ_0 is an isolation point of $\Omega(\phi)$. In view of Definitions 3.1.2 and 3.1.3, let Γ_{ξ_0}, $R_{\xi_0} = \operatorname{Re} P_{\xi_0}$ and Υ_{ξ_0} be associated to ξ_0. Because R_{ξ_0} is positive definite and $\Upsilon_{\xi_0}(\eta) = o(R_{\xi_0}(\eta))$ as $\eta \to 0$, there is a neighborhood of 0 on which $\Gamma_{\xi_0}(\xi) = 0$ only when $\xi = 0$. Since $\hat{\phi}(\xi + \xi_0) = \hat{\phi}(\xi_0)\exp(\Gamma_{\xi_0}(\xi))$ for all $\xi \in \mathcal{U}$, there is a neighborhood of ξ_0 on which $|\hat{\phi}(\xi)| < 1$ for all $\xi \neq \xi_0$. Hence ξ_0 is an isolation point of $\Omega(\phi)$. □

Remark 7. *For any ϕ which satisfied the hypotheses of Proposition 3.4.1, we fix $\mathbb{T}^d_\phi = (-\pi, \pi]^d + \xi_\phi$ where $\xi_\phi \in \mathbb{R}^d$ makes $\Omega(\phi)$ live in the interior of $\mathbb{T}^d_\phi$ (as a subspace of $\mathbb{R}^d$); this can always be done in view of the proposition. We do this only to avoid non-essential technical issues arising from the difference between the topology of $\mathbb{R}^d$ and the topology of $\mathbb{T}^d$ inherited as a subspace.*

Lemma 3.4.2. *Let $\phi \in \mathcal{S}_d$ be such that $\sup |\hat{\phi}(\xi)| = 1$ and suppose that $\xi_0 \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$. Let $\alpha = \alpha_{\xi_0}$ and $P = P_{\xi_0}$ be associated to $\hat{\phi}$ in view of Definition 3.1.3 and let μ_P and $H_P^{(\cdot)}$ be defined by* (3.4) *and* (3.8) *respectively. Then there exists an open neighborhood $\mathcal{U}_{\xi_0}$ of ξ_0 such that, for any open sub-neighborhood $\mathcal{O}_{\xi_0} \subseteq \mathcal{U}_{\xi_0}$ containing ξ_0, the following limit holds. For all $\epsilon > 0$ there exists $N \in \mathbb{N}_+$ such that*

$$\left| \frac{n^{\mu_P}}{(2\pi)^d} \int_{\mathcal{O}_{\xi_0}} \hat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi - n^{\mu_P} e^{-ix\cdot\xi_0} \hat{\phi}(\xi_0)^n H_P^n(x - n\alpha) \right| < \epsilon$$

for all natural numbers $n \geq N$ and for all $x \in \mathbb{R}^d$.

Proof. Given that $\xi_0 \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$,

$$\hat{\phi}(\xi + \xi_0) = \hat{\phi}(\xi_0) e^{\Gamma(\xi)} \tag{3.34}$$

for $\xi \in \mathcal{U}$ where

$$\Gamma(\xi) = i\alpha \cdot \xi - P(\xi) + \Upsilon(\xi)$$

and where $\Upsilon(\xi) = o(R(\xi))$ and $R = \operatorname{Re} P$. If necessary, we restrict $\mathcal{U}$ further so that

$$\left| e^{\Gamma(\xi)} \right| = e^{\operatorname{Re}(i\alpha\cdot\xi - P(\xi) + \Upsilon(\xi))} \leq e^{-R(\xi)/2} \tag{3.35}$$

for all $\xi \in \mathcal{U}$ and put $\mathcal{U}_{\xi_0} = \xi_0 + \mathcal{U}$. Now, let $\mathcal{O}_{\xi_0} \subseteq \mathcal{U}_{\xi_0}$ be an open set containing ξ_0. It is clear that $\mathcal{O} := \mathcal{O}_{\xi_0} - \xi_0$ is open and is such that $0 \in \mathcal{O} \subseteq \mathcal{U}$. Of course, (3.34) and (3.35) hold for all $\xi \in \mathcal{O}$.

Observe that, for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_+$,

$$\begin{aligned}
&\frac{n^{\mu_P}}{(2\pi)^d}\int_{\mathcal{O}_{\xi_0}} \hat{\phi}(\xi)^n e^{-ix\cdot\xi}\,d\xi - e^{-ix\cdot\xi_0}\hat{\phi}(\xi_0)^n n^{\mu_P} H_P^n(x-n\alpha) \\
&= \frac{n^{\mu_P}}{(2\pi)^d}\int_{\mathcal{O}} \hat{\phi}(\xi+\xi_0)^n e^{-ix\cdot(\xi+\xi_0)}\,d\xi \\
&\quad - e^{-ix\cdot\xi_0}\hat{\phi}(\xi_0)^n \frac{n^{\mu_P}}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-nP(\xi)}e^{-i(x-n\alpha)\cdot\xi}\,d\xi \\
&= \frac{e^{-ix\cdot\xi_0}\hat{\phi}(\xi_0)^n}{(2\pi)^d}\Big(n^{\mu_P}\int_{\mathcal{O}} e^{n\Gamma(\xi)}e^{-ix\cdot\xi}\,d\xi \\
&\qquad\qquad - n^{\mu_P}\int_{\mathbb{R}^d} e^{-nP(\xi)}e^{-i(x-n\alpha)\cdot\xi}\,d\xi\Big).
\end{aligned} \tag{3.36}$$

Now for $E \in \operatorname{Exp}(P)$,

$$\begin{aligned}
&n^{\mu_P}\int_{\mathbb{R}^d} e^{-nP(\xi)}e^{-i(x-n\alpha)\cdot\xi}\,d\xi \\
&= n^{\mu_P}\int_{\mathbb{R}^d} e^{-P(n^E\xi)}e^{-i(x-n\alpha)\cdot\xi}\,d\xi \\
&= n^{\mu_P}\int_{n^E(\mathbb{R}^d)} e^{-P(\xi)}e^{-i(x-n\alpha)\cdot n^{-E}\xi}\det(n^{-E})\,d\xi \\
&= \int_{\mathbb{R}^d} e^{-P(\xi)}e^{-i(x-n\alpha)\cdot n^{-E}\xi}\,d\xi
\end{aligned}$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_+$ where, in view of Corollary 3.2.4, we have used the fact that $\det(n^{-E}) = n^{-\operatorname{tr} E} = n^{-\mu_P}$. Noting the adjoint relation $(n^{-E})^* = n^{-E^*}$, and upon putting $y(n,x) = n^{-E^*}(x-n\alpha)$, we have

$$n^{\mu_P}\int_{\mathbb{R}^d} e^{-nP(\xi)}e^{-i(x-n\alpha)\cdot\xi}\,d\xi = \int_{\mathbb{R}^d} e^{-P(\xi)}e^{-iy(n,x)\cdot\xi}\,d\xi \tag{3.37}$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_+$.

Let $\epsilon > 0$ and observe that, in view of Proposition 3.2.6, $e^{-P/2} \in L^1(\mathbb{R}^d)$ because $P(\xi)/2$ is a positive homogeneous polynomial. We can therefore choose a compact set K for which

$$\int_{\mathbb{R}^d\setminus K} \left|e^{-P}\right| d\xi \le \int_{\mathbb{R}^d\setminus K} e^{-R(\xi)}\,d\xi \le \int_{\mathbb{R}^d\setminus K} e^{-R(\xi)/2} < \epsilon/3. \tag{3.38}$$

By virtue of Proposition A.1.6 and Lemma 3.2.5, there is $N_1 \in \mathbb{N}_+$, such that $n^{-E}(K) \subseteq \mathcal{O}$ for all $n \geq N_1$. Thus

$$\begin{aligned}
&\int_{\mathcal{O}} e^{n\Gamma(\xi)} e^{-ix\cdot\xi}\, d\xi \\
&= \int_{n^{-E}(K)} e^{n\Gamma(\xi)} e^{-ix\cdot\xi}\, d\xi + \int_{\mathcal{O}\setminus n^{-E}(K)} e^{n\Gamma(\xi)} e^{-ix\cdot\xi}\, d\xi \\
&= \int_{n^{-E}(K)} e^{-P(n^E\xi)+n\Upsilon(\xi)} e^{-i(x-n\alpha)\cdot\xi}\, d\xi + \int_{\mathcal{O}\setminus n^{-E}(K)} e^{n\Gamma(\xi)} e^{-ix\cdot\xi}\, d\xi \\
&= \frac{1}{n^{\mu_P}} \int_K e^{-P(\xi)+n\Upsilon(n^{-E}\xi)} e^{-iy(n,x)\cdot\xi}\, d\xi + \int_{\mathcal{O}\setminus n^{-E}(K)} e^{n\Gamma(\xi)} e^{-ix\cdot\xi}\, d\xi
\end{aligned} \tag{3.39}$$

for all $n \geq N_1$ and $x \in \mathbb{R}^d$; here we have again used the fact that $\det(n^{-E}) = n^{-\mu_P}$. Combining (3.36),(3.37) and (3.39) yields

$$\begin{aligned}
&\left| \frac{n^{\mu_P}}{(2\pi)^d} \int_{\mathcal{O}_{\xi_0}} \hat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi - e^{-ix\cdot\xi_0} \hat{\phi}(\xi_0)^n n^{\mu_P} H_P^n(x-n\alpha) \right| \\
&\leq \left| \int_K \left(e^{-P(\xi)+n\Upsilon(n^{-E}\xi)} - e^{-P(\xi)} \right) e^{-iy(n,x)}\, d\xi \right| \\
&\quad + \int_{\mathbb{R}^d\setminus K} \left| e^{-P(\xi)} e^{-iy(n,x)\cdot\xi} \right| d\xi + n^{\mu_P} \left| \int_{\mathcal{O}\setminus n^{-E}(K)} e^{n\Gamma(\xi)} e^{-ix\cdot\xi}\, d\xi \right| \\
&\leq \int_K \left| e^{-P(\xi)+n\Upsilon(n^{-E}\xi)} - e^{-P(\xi)} \right| d\xi \\
&\quad + \int_{\mathbb{R}^d\setminus K} e^{-R(\xi)}\, d\xi + n^{\mu_P} \int_{\mathcal{O}\setminus n^{-E}(K)} \left| e^{\Gamma(\xi)} \right|^n d\xi \\
&\quad =: I_1(n) + I_2(n) + I_3(n)
\end{aligned} \tag{3.40}$$

for all $n \geq N_1$ and $x \in \mathbb{R}^d$.

It is clear that $I_2(n) < \epsilon/3$ for all $n \geq N_1$ by virtue of (3.38). Now, in view of (3.35) and (3.38),

$$I_3(n) \leq n^{\mu_P} \int_{\mathcal{O}\setminus n^{-E}(K)} e^{-nR(\xi)/2}\, d\xi \leq \int_{\mathbb{R}^d\setminus K} e^{-R(\xi)/2}\, d\xi < \epsilon/3$$

for all $n \geq N_1$; here we have used that facts that $E \in \operatorname{Exp}(P) \subseteq \operatorname{Exp}(R)$, $\det(n^{-E}) = n^{-\mu_P}$, and

$$n^E(\mathcal{O} \setminus n^{-E}(K)) = n^E(\mathcal{O}) \setminus K \subseteq \mathbb{R}^d \setminus K.$$

To estimate I_1, we recall that $n^{-E}(K) \subseteq \mathcal{O}$ for all $n \geq N_1$ and so the estimate (3.35) ensures that the integrand of $I_1(n)$ is bounded by 2 for all $n \geq N_1$. In view of Lemma 3.3.2, an appeal to the Bounded Convergence Theorem gives a natural number $N \geq N_1$ for which $I_1(n) < \epsilon/3$ for all $n \geq N$. The desired result follows by combining our estimates for I_1, I_2 and I_3 with (3.40). $\square$

The next lemma follows directly from Lemma 3.4.2 by upon recalling that $n^{\mu_P} H_P^n = H_P \circ L_{n^{-E^*}} \in \mathcal{S}(\mathbb{R}^d)$ for all $n \in \mathbb{N}_+$.

Lemma 3.4.3. *Let ϕ, ξ_0 and P be as in the statement of Lemma 3.4.2. Under the same hypotheses of the lemma, there exists an open neighborhood $\mathcal{U}_{\xi_0}$ of ξ_0 such that, for any open sub-neighborhood $\mathcal{O}_{\xi_0} \subseteq \mathcal{U}_{\xi_0}$ containing ξ_0, there exists $C > 0$ and a natural number N such that*

$$\left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\xi_0}} \hat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi \right| \leq \frac{C}{n^{\mu_P}}$$

for all $n \geq N$ and $x \in \mathbb{R}^d$.

Proof of Theorem 3.1.5. Under the hypotheses of the theorem, Proposition 3.4.1 ensures that $\Omega(\phi)$ is finite. In line with the paragraph preceding the statement of the theorem, we label

$$\Omega(\phi) = \{\xi_1, \xi_2, \dots, \xi_A, \xi_{A+1}, \dots, \xi_B\} \subseteq \mathbb{T}^d$$

where $\mu_{P_{\xi_q}} = \mu_\phi$ for $q = 1, 2, \dots A$ and $\mu_{P_{\xi_q}} > \mu_\phi$ for $q = A+1, A+2, \dots B$. Also, we assume all additional notation from the paragraph preceding the statement of the theorem and take $\mathbb{T}_\phi^d$ as in Remark 7.

Let $\{\mathcal{O}_{\xi_q}\}_{q=1,2,\dots,B}$ be a collection of disjoint open subsets of $\mathbb{T}_\phi^d$ for which the conclusions of Lemmas 3.4.2 and 3.4.3 hold for $q = 1, 2, \dots A$ and $q = A+1, A+$

$2, \dots B$ respectively. Set

$$K = \mathbb{T}_\phi^d \setminus \Big(\bigcup_{q=1}^{B} \mathcal{O}_{\xi_q} \Big)$$

and observe that

$$s := \sup_{\xi \in K} |\hat{\phi}(\xi)| < 1$$

Now, in view of the Fourier inversion formula,

$$\begin{aligned} \phi^{(n)}(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}_\phi^d} \hat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi \\ &= \sum_{q=1}^{B} \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\xi_q}} \hat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi + \frac{1}{(2\pi)^d} \int_K \hat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi \end{aligned} \tag{3.41}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$. Appealing to Lemma 3.4.2 ensures that for $q = 1, 2, \dots, A$,

$$\frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\xi_q}} \hat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi = e^{-ix\cdot\xi_q} \hat{\phi}(\xi_q)^n H_{P_q}^n(x - n\alpha_q) + o(n^{-\mu_\phi}) \tag{3.42}$$

uniformly for $x \in \mathbb{R}^d$. Now, for each $q = A+1, A+2, \dots, B$, Lemma 3.4.3 guarantees that

$$\frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\xi_q}} \hat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi = O(n^{-\mu_{P_{\xi_q}}}) = o(n^{-\mu_\phi}) \tag{3.43}$$

uniformly for $x \in \mathbb{R}^d$ because $\mu_{P_{\xi_q}} > \mu_\phi$. Finally, we note that

$$\frac{1}{(2\pi)^d} \int_K \hat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi = o(n^{-\mu_\phi}) \tag{3.44}$$

uniformly for $x \in \mathbb{R}^d$ because $s^n = o(n^{-\mu_\phi})$. The desired result is obtained by combining (3.41), (3.42),(3.43) and (3.44). $\square$

As an application to Theorem 3.1.5, we are now in a position to prove $\ell^\infty(\mathbb{Z}^d)$ estimates for $\phi^{(n)}$ and thus give a partial answer to Question (i). We first treat a basic lemma whose proof makes use of the famous theorem of R. Dedekind (generalized by E. Artin) concerning the linear independence of characters. Interestingly enough, the statement of the lemma below mirrors a result of Dedekind appearing in the Volesungen [32] where the characters $e^{-ix\cdot\xi}$ are replaced by field isomorphisms, c.f., [22, p. 6].

Lemma 3.4.4. *For any distinct* $\xi_1, \xi_2, \dots, \xi_A \in \mathbb{T}^d$, *there exists* $x_1, x_2, \dots x_A \in \mathbb{Z}^d$ *such that*

$$V = \begin{pmatrix} e^{-ix_1\cdot\xi_1} & e^{-ix_1\cdot\xi_2} & \dots & e^{-ix_1\cdot\xi_A} \\ e^{-ix_2\cdot\xi_1} & e^{-ix_2\cdot\xi_2} & \dots & e^{-ix_2\cdot\xi_A} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-ix_A\cdot\xi_1} & e^{-ix_A\cdot\xi_2} & \dots & e^{-ix_A\cdot\xi_A} \end{pmatrix}$$

is invertible.

Proof. The statement is obviously true when $A = 1$ and so we use induction on A. Let $\xi_1, \xi_2, \dots, \xi_{A+1} \in \mathbb{T}^d$ be distinct and take $x_1, x_2, \dots, x_A \in \mathbb{Z}^d$ as guaranteed by the inductive hypotheses. For any $\zeta_1, \zeta_2, \dots, \zeta_A \in \mathbb{T}^d$, we define

$$F(\zeta_1, \zeta_2, \dots, \zeta_A) = \det \begin{pmatrix} e^{-ix_1\cdot\zeta_1} & e^{-ix_1\cdot\zeta_2} & \dots & e^{-ix_1\cdot\zeta_A} \\ e^{-ix_2\cdot\zeta_1} & e^{-ix_2\cdot\zeta_2} & \dots & e^{-ix_2\cdot\zeta_A} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-ix_A\cdot\zeta_1} & e^{-ix_A\cdot\zeta_2} & \dots & e^{-ix_A\cdot\zeta_A} \end{pmatrix}.$$

In this notation, our inductive hypothesis is the condition $F(\xi_1, \xi_1, \dots, \xi_A) \neq 0$. Let $G : \mathbb{Z}^d \to \mathbb{C}$ be defined by

$$G(x) = \det \begin{pmatrix} e^{-ix_1\cdot\xi_1} & e^{-ix_1\cdot\xi_2} & \dots & e^{-ix_1\cdot\xi_A} & e^{-ix_1\cdot\xi_{A+1}} \\ e^{-ix_2\cdot\xi_1} & e^{-ix_2\cdot\xi_2} & \dots & e^{-ix_2\cdot\xi_A} & e^{-ix_1\cdot\xi_{A+1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e^{-ix_A\cdot\xi_1} & e^{-ix_A\cdot\xi_2} & \dots & e^{-ix_A\cdot\xi_A} & e^{-ix_A\cdot\xi_{A+1}} \\ e^{-ix\cdot\xi_1} & e^{-ix\cdot\xi_2} & \dots & e^{-ix\cdot\xi_A} & e^{-ix\cdot\xi_{A+1}} \end{pmatrix}$$

for $x \in \mathbb{Z}^d$. Our job is to conclude that $G(x_{A+1}) \neq 0$ for some $x_{A+1} \in \mathbb{Z}^d$. We assume to reach a contradiction that this is not the case, that is, for all $x \in \mathbb{Z}^d$,

$G(x) = 0$. Upon expanding by cofactors, we have

$$G(x) = \sum_{k=1}^{A+1} (-1)^{A+1+k} F(\xi_1, \xi_2, \dots, \widehat{\xi_k}, \dots, \xi_{A+1}) e^{-ix\cdot\xi_k} = 0$$

for all $x \in \mathbb{Z}^d$; here $\widehat{\xi_k}$ means that we have omitted ξ_k from the list $\xi_1, \xi_2, \dots, \xi_{A+1}$. Given that $\xi_1, \xi_2, \dots, \xi_{A+1}$ are all distinct, the characters $x \mapsto e^{-ix\cdot\xi_k}$ for $k = 1, 2, \dots, A+1$ are distinct and so by Dedekind's independence theorem it follows that $F(\xi_1, \xi_2, \dots, \widehat{\xi_k}, \dots, \xi_{A+1}) = 0$ for all $k = 1, 2, \dots, A+1$. This however contradicts our inductive hypotheses for $F(\xi_1, \xi_2, \dots, \xi_A, \widehat{\xi_{A+1}}) = F(\xi_1, \xi_2, \dots, \xi_A) \neq 0$. □

Proof of Theorem 3.1.4. By virtue of Theorem 3.1.5 and (3.9), we have

$$n^{\mu_\phi} \phi^{(n)}(x) = \sum_{k=1}^{A} e^{-ix\cdot\xi_k} \hat{\phi}(\xi_k)^n H_{P_k}\left(n^{-E_k^*}(x - n\alpha_k)\right) + o(1) \tag{3.45}$$

uniformly for $x \in \mathbb{Z}^d$ where $E_k \in \operatorname{Exp}(P_k)$ for $k = 1, 2, \dots A$. Upon recalling that the attractors $H_{P_k} \in \mathcal{S}(\mathbb{R}^d)$, the upper estimate of (3.7) follows directly from (3.45) and the triangle inequality. Showing the lower estimate of (3.7) is trickier, for we must ensure that the sum in (3.45) does not collapse at all $x \in \mathbb{Z}^d$ – this is precisely where Lemma 3.4.4 comes in.

For the distinct collection $\xi_1, \xi_2, \dots, \xi_A \in \mathbb{T}^d$, let $x_1, x_2, \dots, x_d \in \mathbb{Z}^d$ be as guaranteed by Lemma 3.4.4 and, by focusing on x's near $n\alpha_1$, we consider the $A \times A$ systems

$$f(n, x_j) = \sum_{k=1}^{A} \exp\left(-i(x_j + \lfloor n\alpha_1 \rfloor)\cdot\xi_k\right) \hat{\phi}(\xi_k)^n H_{P_k}\left(n^{-E_k^*}(x_j + \lfloor n\alpha_1 \rfloor - n\alpha_k)\right) \tag{3.46}$$

and

$$g_j(n) = \sum_{k=1} \exp(-ix_j \cdot \xi_k) h_k(n) \tag{3.47}$$

for $j = 1, 2, \ldots, A$, where

$$h_k(n) = \begin{cases} e^{-i\lfloor n\alpha_1 \rfloor \cdot \xi_k} \hat{\phi}(\xi_k)^n H_{P_k}(0) & \text{if } \alpha_1 = \alpha_k \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, 2, \ldots, A$. By virtue of Lemma A.1.3 and Propositions 3.2.2 and 3.2.3, it follows that

$$\lim_{n\to\infty} |n^{-E_k^*}(x_j + \lfloor n\alpha_1 \rfloor - n\alpha_k)| = \begin{cases} 0 & \text{if } \alpha_k = \alpha_1 \\ \infty & \text{otherwise.} \end{cases}$$

for all $j, k = 1, 2, \ldots, A$. Again using the fact that each $H_{P_k} \in \mathcal{S}(\mathbb{R}^d)$, the above limit ensures that, for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}_+$ for which

$$|f(n, x_j) - g_j(n)| < \epsilon \tag{3.48}$$

for all $j = 1, 2, \ldots A$ and $n \geq N_\epsilon$. The system (3.47) can be rewritten in the form

$$\begin{pmatrix} g_1(n) \\ g_2(n) \\ \vdots \\ g_A(n) \end{pmatrix} = \begin{pmatrix} e^{-ix_1\cdot\xi_1} & e^{-ix_1\cdot\xi_2} & \cdots & e^{-ix_1\cdot\xi_A} \\ e^{-ix_2\cdot\xi_1} & e^{-ix_2\cdot\xi_2} & \cdots & e^{-ix_2\cdot\xi_A} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-ix_A\cdot\xi_1} & e^{-ix_A\cdot\xi_2} & \cdots & e^{-ix_A\cdot\xi_A} \end{pmatrix} \begin{pmatrix} h_1(n) \\ h_2(n) \\ \vdots \\ h_A(n) \end{pmatrix}$$

or equivalently

$$g(n) = Vh(n) \tag{3.49}$$

for $n \in \mathbb{N}_+$ where V is that of Lemma 3.4.4. Taking $\mathbb{C}^A$ to be equipped with the maximum norm, the matrix V determines a linear operator $L_V : \mathbb{C}^A \to \mathbb{C}^A$ which is bounded below by virtue of the lemma. So, in view of (3.50), there is a constant $\delta > 0$ for which

$$\max_{j=1,2,\ldots,A} |g_j(n)| \geq \delta \max_{j=1,2,\ldots,A} |h_j(n)| \geq \delta |H_{P_1}(0)| =: 3C > 0 \tag{3.50}$$

for all $n \in \mathbb{N}_+$. Upon combining (3.45), (3.48) and (3.50), we obtain $N \in \mathbb{N}_+$ for which

$$n^{\mu_\phi}\|\phi^{(n)}\|_\infty \geq \max_{j=1,2,\dots A} |n^{\mu_\phi}\phi^{(n)}(x_j + \lfloor n\alpha_1 \rfloor)| \geq C$$

for all $n \geq N$. The theorem now follows by, if necessary, adjusting the constant C for $n < N$. $\square$

3.5 Pointwise bounds for $\phi^{(n)}$

Throughout this section, we assume that $\phi : \mathbb{Z}^d \to \mathbb{C}$ is finitely supported. In this case, $\hat{\phi}(z)$ is a trigonometric polynomial on $\mathbb{C}^d$. As usual, we assume that $\sup_{\xi \in \mathbb{T}^d} |\hat{\phi}(\xi)| = \sup_{\xi \in \mathbb{R}^d} |\hat{\phi}(\xi + 0i)| = 1$.

3.5.1 Generalized exponential bounds

In this subsection, we prove Theorem 3.1.6 and present a variety of results concerning discrete space and time differences of convolution powers. The estimate of the following lemma, Lemma 3.5.1, is crucial to our arguments to follow; its analogue when $d = 1$ can be found the proof of Theorem 3.1 of [31]. We note that in [31], the analogue of Lemma 3.5.1 is used to deduce Gevrey-type estimates from which the desired estimates follow in one dimension. Such arguments are troublesome when the decay is anisotropic for $d > 1$. By contrast, our off-diagonal estimates are found by applying Lemma 3.5.1 following a complex change-of-variables.

Lemma 3.5.1. *Let $\phi : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported and such that $\sup_{\xi \in \mathbb{T}^d} |\hat{\phi}(\xi)| = 1$. Suppose that $\xi_0 \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$ with associated $\alpha \in \mathbb{R}^d$*

and positive homogeneous polynomial P. Define $f_{\xi_0} : \mathbb{C}^d \to \mathbb{C}$ by

$$f_{\xi_0}(z) = \hat{\phi}(\xi_0)^{-1} e^{-\alpha\cdot(z+\xi_0)} \hat{\phi}(z+\xi_0) \tag{3.51}$$

for $z \in \mathbb{C}^d$. For any compact set $K \subseteq \mathbb{R}^d$ containing an open neighborhood of 0 for which $|\phi(\xi+\xi_0)| < 1$ for all non-zero $\xi \in K$, there exist $\epsilon, M > 0$ for which

$$|f_{\xi_0}(z)| \leq \exp(-\epsilon R(\xi) + MR(\nu))$$

for all $z = \xi - i\nu$ such that $\xi \in K$ and $\nu \in \mathbb{R}^d$.

Proof. Write $f = f_{\xi_0}$ and denote by π_r the canonical projection from $\mathbb{C}^d$ onto $\mathbb{R}^d$. We first estimate $f(z)$ on a neighborhood of 0 in $\mathbb{C}^d$.

Our assumption that $\xi_0 \in \Omega(\phi)$ ensures that the expansion (3.5) is valid on an open set $\mathcal{U} \in \mathbb{C}^d$ such that $0 \in \pi_r(\mathcal{U}) \subseteq K$. By virtue of Proposition A.2.7, we can further restrict $\mathcal{U}$ to ensure that, for some $\epsilon' > 0$ and $M > 0$,

$$|f(z)| \leq e^{-\epsilon' R(\xi) + MR(\nu)} \tag{3.52}$$

for $z = \xi - i\nu \in \mathcal{U}$.

We now estimate $f(z)$ on a cylinder of K in $\mathbb{C}^d$. Since $|\hat{\phi}(\xi)| < 1$ for all non-zero $\xi \in K$, the compactness $K \setminus \pi_r(\mathcal{U})$ ensures that, for some $0 < \epsilon \leq \epsilon'$, the continuous function $h : \mathbb{C}^d \to \mathbb{C}$, defined by

$$h(z) = e^{\epsilon R(\xi)} f(z) = \exp(-\epsilon(R \circ \pi_r)(z)) f(z)$$

for $z = \xi - i\nu \in \mathbb{C}^d$, is such that $|h(\xi)| < 1$ for all $\xi \in K \setminus \pi_r(\mathcal{U})$. Because h is continuous, there exists $\delta > 0$ for which $|h(z)| \leq 1$ for all $z = \xi - i\nu$ such that $\xi \in K \setminus \pi_r(\mathcal{U})$ and $|\nu| \leq \delta$. Consequently,

$$|h(z)| \leq e^{-\epsilon R(\xi)} \leq e^{-\epsilon R(\xi) + MR(\nu)} \tag{3.53}$$

for all $z = \xi - i\nu$ such that $\xi \in K \setminus \pi_r(\mathcal{U})$ and $|\nu| \leq \delta$. Upon possibly further restricting $\delta > 0$, a combination of the estimates (3.52) and (3.53) ensures that

$$|f(z)| \leq e^{-\epsilon R(\xi) + MR(\nu)} \tag{3.54}$$

for all $z = \xi - i\nu \in \mathbb{C}$ such that $\xi \in K$ and $|\nu| \leq \delta$.

Finally, we estimate $f(z) = f(\xi - i\nu)$ for unbounded ν. Because $\hat{\phi}$ is a trigonometric polynomial, $f(z)$ has exponential growth on the order of $|\nu|$ for $z = \xi - i\nu \in \mathbb{C}^d$ when ξ is restricted to K. Therefore,

$$|f(z)| \leq e^{-\epsilon R(\xi) + |\nu| + C} \tag{3.55}$$

for all $z = \xi - i\nu$ such that $\xi \in K$ and $\nu \in \mathbb{R}^d$. Because $|\nu| + C$ is dominated by $R(\nu)$ by virtue of Corollary A.2.6, the lemma follows immediately from the estimates (3.54) and (3.55). □

Lemma 3.5.2. *Let $\phi : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported and such that $\sup_{\xi \in \mathbb{T}^d} |\hat{\phi}(\xi)| = 1$. Assume additionally that $\Omega(\phi) = \{\xi_0\}$ and ξ_0 is of positive homogeneous type for $\hat{\phi}$ with corresponding $\alpha \in \mathbb{R}^d$ and positive homogeneous polynomial P and let $\mathbb{T}^d_\phi$ be as in Remark 7. Define $g_{(\cdot)} : \mathbb{N}_+ \times \mathbb{C}^d \to \mathbb{C}$ by $g_l(z) = 1 - f_{\xi_0}(z)^l$ for $l \in \mathbb{N}_+$ and $z \in \mathbb{C}^d$ where f_{ξ_0} is given by* (3.51). *There exist positive constants C and M for which*

$$|g_l(z)| \leq lC(R(\nu) + R(\xi))e^{lMR(\nu)}$$

for all $l \in \mathbb{N}_+$ and $z = \xi - i\nu$ such that $\xi \in \mathbb{T}^d_\phi$ and $\nu \in \mathbb{R}^d$.

Proof. By making similar arguments to those in the proof of the previous lemma, we obtain positive constants C and M for which $|1 - f_{\xi_0}(z)| \leq C(R(\xi) + R(\nu))e^{MR(\nu)}$ for all $z = \xi + i\nu$ such that $\xi \in \mathbb{T}^d_\phi$ and $\nu \in \mathbb{R}^d$. The desired estimate now follows from Lemma 3.5.1 (where $K = \overline{T^d_\phi}$) by writing $g_l = (1 - f_{\xi_0}) \sum_{k=0}^{l-1} f_{\xi_0}^k$ and making use of the triangle inequality. □

We are now in a position to prove Theorem 3.1.6.

Proof of Theorem 3.1.6. In view of the hypotheses, there exist $\alpha \in \mathbb{R}^d$ and a positive homogeneous polynomial P such that each $\xi \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$ with corresponding $\alpha_\xi = \alpha$ and $P_\xi = P$. We write $\Omega(\phi) = \{\xi_1, \xi_2, \ldots, \xi_Q\}$ in view of Proposition 3.4.1 and take $\mathbb{T}^d_\phi$ as in Remark 7. Because $\Omega(\phi)$ is finite and lives on the interior of $\mathbb{T}^d_\phi$, there exits a collection of mutually disjoint and relatively compact sets $\{K_q\}_{q=1}^Q$ such that $\mathbb{T}^d_\phi = \cup_{q=1}^Q K_q$ and, for each $q = 1, 2, \ldots, Q$, K_q contains an open neighborhood of ξ_q. We now establish two important uniform estimates. First, upon noting that $|\hat{\phi}(\xi+\xi_q)| < 1$ for all $\xi \in \overline{K_q - \xi_q}$ for each $q = 1, 2, \ldots, Q$, by virtue of Lemma 3.5.1 there are positive constants M and ϵ such that, for each $q = 1, 2, \ldots, Q$,

$$|f_{\xi_q}(\xi - i\nu)| \leq \exp(-\epsilon R(\xi) - MR(\nu)) \tag{3.56}$$

for all $\xi \in K_q - \xi_q$ and $\nu \in \mathbb{R}^d$. Also, by a similar argument to those given in the proof of Lemma 3.4.2, we observe that

$$\begin{aligned} n^{\mu_P} \int_{K_q - \xi_q} e^{-\epsilon n R(\xi)}\, d\xi &= n^{\mu_P} \int_{K_q - \xi_q} e^{-\epsilon R(n^{-E}\xi)}\, d\xi = \int_{n^E(K_q - \xi_q)} e^{-\epsilon R(\xi)}\, d\xi \\ &\leq \int_{\mathbb{R}^d} e^{-\epsilon R(\xi)}\, d\xi =: C < \infty \end{aligned} \tag{3.57}$$

for all $n \in \mathbb{N}_+$ and $q = 1, 2, \ldots, Q$.

Now, let $\nu \in \mathbb{R}^d$ be arbitrary but fixed. Because $\hat{\phi}$ is a trigonometric polynomial (and so periodic on $\mathbb{C}^d$), it follows that

$$\begin{aligned} \phi^{(n)}(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d_\phi} e^{-ix\cdot(\xi - i\nu)} \hat{\phi}(\xi - i\nu)^n\, d\xi \\ &= \frac{1}{(2\pi)^d} \sum_{q=1}^{Q} \int_{K_q} e^{-ix\cdot(\xi - i\nu)} \hat{\phi}(\xi - i\nu)^n\, d\xi \end{aligned} \tag{3.58}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$. Our aim is to uniformly estimate the integrals over K_q. To this end, for each $q = 1, 2, \cdots, Q$, we observe that

$$\begin{aligned}
&\int_{K_q} e^{-ix\cdot(\xi - i\nu)} \hat{\phi}(\xi - i\nu)^n \, d\xi \\
&\quad = \int_{K_q - \xi_q} e^{-ix\cdot(\xi_q + \xi - i\nu)} \hat{\phi}(\xi_q)^n e^{-in\alpha\cdot(\xi_0 + \xi - i\nu)} f_{\xi_q}(\xi - i\nu)^n \, d\xi \\
&\quad = e^{-ny_n(x)\cdot\nu} \int_{K_q - \xi_q} \left(e^{-iy_n(x)\cdot(\xi_q + \xi)} \hat{\phi}(\xi_q)\right)^n f_{\xi_q}(\xi - i\nu)^n \, d\xi
\end{aligned}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$, where $y_n(x) := (x - n\alpha)/n$. In view of the estimates (3.56) and (3.57), we have

$$\begin{aligned}
\left| \int_{K_q} e^{-ix\cdot(\xi - i\nu)} \hat{\phi}(\xi - i\nu) \, d\xi \right| &\leq e^{-ny_n(x)\cdot\nu} \int_{K_q - \xi_q} |f_{\xi_q}(\xi - i\nu)|^n \, d\xi \\
&\leq \frac{C}{n^{\mu_\phi}} \exp(-n(y_n(x)\cdot\nu - MR(\nu))) \qquad (3.59)
\end{aligned}$$

for all $x \in \mathbb{Z}^d$, $n \in \mathbb{N}_+$ and $q = 1, 2, \dots, Q$ where the constants M and C are independent of ν. Upon setting $C' = (2\pi)^d/Q$ and combining (3.58) and (3.59), we obtain the estimate

$$|\phi^{(n)}(x)| \leq \frac{C'}{n^{\mu_\phi}} \exp(-n(y_n(x)\cdot\nu - MR(\nu)))$$

which holds uniformly for $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$ and $\nu \in \mathbb{R}^d$. Consequently,

$$\begin{aligned}
|\phi^{(n)}(x)| &\leq \inf_{\nu \in \mathbb{R}^d} \frac{C'}{n^{\mu_\phi}} \exp(-n(y_n(x)\cdot\nu - MR(\nu))) \\
&\leq \frac{C'}{n^{\mu_\phi}} \exp\left(-n \sup_{\nu}(y_n(x)\cdot\nu - MR(\nu))\right) \\
&\leq \frac{C'}{n^{\mu_\phi}} \exp\left(-n(MR)^{\#}(y_n(x))\right)
\end{aligned}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$. The desired result follows upon noting that $(MR)^{\#} \asymp R^{\#}$ in view of Corollary A.3.4. $\square$

Remark 8. *The essential hypothesis of Theorem 3.1.6 (essential for a global exponential bound) is that each $\xi \in \Omega(\phi)$ has the same drift α; this can be seen by looking at* *the example of Subsection 3.7.2 wherein the convolution powers $\phi^{(n)}$ exhibit two "drift packets" which drift away from one another. The hypothesis that all of the corresponding positive homogeneous polynomials are the same can be weakened to include, at least, the condition that $R_\xi = \operatorname{Re} P_\xi \asymp R$ for all $\xi \in \Omega(\phi)$, where R is some fixed real valued positive homogeneous polynomial. In any case, the theorem's hypotheses are seen to be natural when ϕ has some form of "periodicity" as can be seen in the example of Subsection 3.7.3. Also, the hypotheses are satisfied for all finitely supported and genuinely d-dimensional probability distributions on $\mathbb{Z}^d$, see Subsection 3.7.6.*

For the remainder of this subsection, we restrict our attention further to finitely supported functions $\phi : \mathbb{Z}^d \to \mathbb{C}$ which satisfy $\sup_\xi |\hat{\phi}| = 1$ and where this supremum is attained at only one point in $\mathbb{T}^d$, i.e., $\Omega(\phi) = \{\xi_0\}$. In this setting, we obtain global estimates for discrete space and time derivatives of convolution powers. Our first result concerns only discrete spatial derivatives of $\phi^{(n)}$ and is a useful complement to Theorem 3.1.6. For related results, see Theorem 3.1 of [31] and Theorem 8.2 of [86], the latter being due to O. B. Widlund [95, 96]. For $w \in \mathbb{Z}^d$ and $\psi : \mathbb{Z}^d \to \mathbb{C}$, define $D_w\psi : \mathbb{Z}^d \to \mathbb{C}$ by

$$D_w\psi(x) = \psi(x + w) - \psi(x)$$

for $x \in \mathbb{Z}^d$.

Theorem 3.5.3. *Let $\phi : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported and such that $\sup_{\xi\in\mathbb{T}^d} |\hat{\phi}(\xi)| = 1$. Additionally assume that $\Omega(\phi) = \{\xi_0\}$ and that ξ_0 is of positive homogeneous type for $\hat{\phi}$ with corresponding $\alpha = \alpha_{\xi_0} \in \mathbb{R}^d$ and positive homogeneous polynomial $P = P_{\xi_0}$. Also let μ_ϕ be defined by* (3.6) *(or equivalently* (3.4)*), let $R^{\#}$ be the Legendre-Fenchel transform of $R = \operatorname{Re} P$ and take $E \in \operatorname{Exp}(P)$. There exists $M > 0$ such that, for any*

$B > 0$ and $m \in \mathbb{N}_+$, there exists $C_m > 0$ such that, for any $w_1, w_2, \dots, w_m \in \mathbb{Z}^d$,

$$\begin{aligned} &\left| D_{w_1} D_{w_2} \cdots D_{w_m} \left(\hat{\phi}(\xi_0)^{-n} e^{ix\cdot\xi_0} \phi^{(n)}(x) \right) \right| \\ &\qquad \leq \frac{C_m}{n^{\mu_\phi}} \left(\prod_{j=1}^{m} |n^{-E^*} w_j| \right) \exp\left(-nMR^{\#}\left(\frac{x - n\alpha}{n} \right) \right) \end{aligned} \tag{3.60}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$ such that $|n^{-E^*} w_j| \leq B$ for $j = 1, 2, \dots, m$.

We remark that all constants in the statement of the theorem are independent of $E \in \operatorname{Exp}(P)$ in view of Proposition 3.2.3. The appearance of the prefactor $\hat{\phi}(\xi_0)^{-n} e^{ix\cdot\xi_0}$ in the left hand side of the estimate is used to remove the highly oscillatory behavior which appears, for instance, in the example outlined in the introduction. That which remains of $\phi^{(n)}$ is well-behaved when this oscillatory prefactor is removed and this is loosely what the theorem asserts. Let us further note that, in contrast to Theorem 3.1.6, Theorem 3.5.3 does not apply to the example illustrated in Subsection 3.7.3 (where $\Omega(\phi)$ consists of two points) and, in fact, the latter theorem's conclusion does not hold for this ϕ. See Subsection 3.7.3 for further discussion.

Lemma 3.5.4. *Given $A > 0$, $\epsilon > 0$ and $m \in \mathbb{N}_+$, there exists $C > 0$ such that the function*

$$Q_{w_1, w_2, \dots, w_m}(z) = \prod_{i=1}^{m} (e^{iw_i\cdot z} - 1)$$

satisfies

$$|Q_{w_1, w_2, \dots, w_m}(\xi - i\nu)| \leq C \left(\prod_{i=1}^{m} |n^{-E^*} w_i| \right) e^{n(\epsilon R(\xi) + R(v))} \tag{3.61}$$

for all $z = \xi - i\nu \in \mathbb{C}^d$, $n \in \mathbb{N}_+$ and $w_1, w_2, \dots, w_m \in \mathbb{Z}^d$ for which $|n^{-E^*} w_i| \leq A$ for all $i = 1, 2, \dots, m$.

Proof. We observe that, for $M = m(B+1)$,

$$\begin{aligned}
|Q_{w_1,w_2,\ldots,w_m}(z)| &\le \prod_{j=1}^{m} |w_j \cdot z| e^{|w_j\cdot z|} \\
&\le \prod_{j=1}^{m} |n^{-E^*} w_j||n^E z| e^{B|n^E z|} \le \left(\prod_{j=1}^{m} |n^{-E^*} w_j|\right) e^{M|n^E z|} \qquad (3.62)
\end{aligned}$$

for all $z \in \mathbb{C}^d$, $n \in \mathbb{N}_+$ and $w_1, w_2, \ldots, w_m \in \mathbb{Z}^d$ for which $|n^{-E^*} w_j| \le B$ for all $j = 1, 2, \ldots, m$. Given $\epsilon > 0$, an appeal to Proposition A.2.5 ensures that, for some $M' > 0$,

$$M|n^E z| \le M' + \epsilon R(n^E \xi) + R(n^E \nu) = M' + n(\epsilon R(\xi) + R(\nu)) \qquad (3.63)$$

for all $z = \xi - i\nu \in \mathbb{C}^d$ and $n \in \mathbb{N}_+$. The desired estimate is obtained by combining (3.62) and (3.63). □

Proof of Theorem 3.5.3. By replacing $\phi(x)$ by $\hat{\phi}(\xi_0)^{-1} e^{ix\cdot\xi_0}\phi(x)$, we assume without loss of generality that $\xi_0 = 0$ and $\hat{\phi}(\xi_0) = 1$. For any $x, w_1, w_2, \ldots, w_m \in \mathbb{Z}^d$ and $\nu \in \mathbb{R}^d$, we invoke the periodicity of $\hat{\phi}$ to see that

$$\begin{aligned}
&D_{w_1} D_{w_2} \cdots D_{w_m} \phi^{(n)}(x) \\
&\quad = D_{w_1} D_{w_2} \cdots D_{w_m} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-ix\cdot(\xi - i\nu)} (\hat{\phi}(\xi - i\nu))^n \, d\xi \\
&\quad = \frac{e^{-ny_n(x)\cdot\nu}}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-iny_n(x)\cdot\xi} Q_{w_1,w_2,\ldots,w_m}(\xi - i\nu) f(\xi - i\nu)^n \, d\xi, \qquad (3.64)
\end{aligned}$$

where $y_n(x) = (x - n\alpha)/n$ and $f(z) = f_{\xi_0}(z) = e^{-i\alpha\cdot z}\hat{\phi}(z)$ is that of Lemma 3.5.1. An appeal to the lemma shows that, for some $\epsilon > 0$ and $M \ge 1$,

$$|f(\xi - i\nu)| \le e^{-2\epsilon R(\xi) + (M-1)R(\nu)} \qquad (3.65)$$

for all $\xi \in \mathbb{T}^d$ and $\nu \in \mathbb{R}^d$; note that these constants are independent of m. By combining the estimates (3.57), (3.61),(3.64) and (3.65) we obtain, for $\nu \in \mathbb{R}^d$ and

$w_1, w_2, \ldots, w_m \in \mathbb{Z}^d$,

$$
\begin{aligned}
&|D_{w_1} D_{w_2} \cdots D_{w_m} \phi^{(n)}(x)| \\
&\leq \; e^{-ny_n(x)\cdot\nu} \int_{\mathbb{T}^d} |Q_{w_1,w_2,\ldots,w_m}(\xi - i\nu)||f(\xi - i\nu)|^n \, d\xi \\
&\leq \; C'_m \left(\prod_{j=1}^m |n^{-E^*} w_j| \right) \exp(-ny_n(x)\cdot\nu + nMR(\nu)) \int_{\mathbb{T}^d} e^{-n\epsilon R(\xi)} \, d\xi \\
&\leq \; \frac{CC'_m}{n^{\mu_\phi}} \left(\prod_{j=1}^m |n^{-E^*} w_j| \right) \exp(-n(y_n(x)\cdot\nu - MR(\nu)))
\end{aligned}
$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$ for which $|n^{-E^*} w_j| \leq B$ for all $j = 1, 2, \ldots, m$. As all constants are independent of ν, the desired estimate is obtained by repeating the same line of reasoning of the proof of Theorem 3.1.6. □

For a collection $v = \{v_1, \ldots, v_d\} \in \mathbb{Z}^d$ and a multi-index β, consider the discrete spatial operator

$$
D_v^\beta = (D_{v_1})^{\beta_1} (D_{v_2})^{\beta_2} \cdots (D_{v_d})^{\beta_d}. \tag{3.66}
$$

Our next result, a corollary to Theorem 3.5.3, gives estimates for $D_v^\beta \phi^{(n)}$ in the case that n^{-E^*} acts diagonally on v_j for $j = 1, 2, \ldots, d$ and, in this case, the term involving w's in (3.60) simplifies considerably. We first give a definition.

Definition 3.5.5. *Let $P : \mathbb{R}^d \to \mathbb{C}$ be a positive homogeneous polynomial and let $A \in Gl_d(\mathbb{R})$ and $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_+^d$ be as given by Proposition 3.2.2. An ordered collection $v = \{v_1, v_2, \ldots, v_d\} \subseteq \mathbb{Z}^d$ is said to be P-fitted if $A^* v_j \in span(e_j)$ for $j = 1, 2, \ldots, d$. In this case we say that $\mathbf{m}$ is the weight of v.*

Let us make a few remarks about the above definition. First, for a P-fitted collection $v = \{v_1, v_2, \ldots, v_d\}$ of weight $\mathbf{m}$, by virtue of Proposition 3.2.2, $t^{-E^*} v_j = t^{-1/(2m_j)} v_j$ for all $t > 0$ and $j = 1, 2, \ldots d$, where $E = ADA^{-1} \in \mathrm{Exp}(P)$. Our definition does not require the $v_j's$ to be non-zero and, in fact, it is possible that the only P-fitted collection to a given positive homogeneous polynomial P is the zero collection. We note however that every positive homogeneous polynomial P seen in this chapter admits a P-fitted collection v which is also a basis of $\mathbb{R}^d$ and, in fact, whenever P is semi-elliptic, every P-fitted collection is of the form $v = \{x^1 e_1, x^2 e_2, \ldots, x^d e_d\}$ where $x^1, x^2, \ldots, x^d \in \mathbb{Z}$.

Corollary 3.5.6. *Let $\phi : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported and such that $\sup_{\xi \in \mathbb{T}^d} |\hat{\phi}(\xi)| = 1$. Additionally assume that $\Omega(\phi) = \{\xi_0\}$ and that ξ_0 is of positive homogeneous type for $\hat{\phi}$ with corresponding $\alpha = \alpha_{\xi_0} \in \mathbb{R}^d$ and positive homogeneous polynomial $P = P_{\xi_0}$. Define μ_ϕ by* (3.6) *(or equivalently* (3.4)*), let $\mathbf{m}$ (and A) be as in Proposition 3.2.2 and denoted by $R^{\#}$, the Legendre-Fenchel transform of $R = \operatorname{Re} P$. There exists $M > 0$ such that, for any $B > 0$ and multi-index β, there is a positive constant C_β such that, for any P-fitted collection $v = \{v_1, v_2, \ldots, v_d\}$ of weight $\mathbf{m}$,*

$$\left| D_v^\beta \left(\hat{\phi}(\xi_0)^{-n} e^{ix\cdot\xi_0} \phi^{(n)}(x) \right) \right| \leq \frac{C_\beta \prod_{j=1}^d |v_j|^{\beta_j}}{n^{\mu_\phi + |\beta : 2\mathbf{m}|}} \exp\left(-nMR^{\#}\left(\frac{x - n\alpha}{n} \right) \right) \tag{3.67}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$ such that $|v_j| \leq B n^{1/(2m_j)}$ for $j = 1, 2, \ldots, d$.

Proof. As we previously remarked,

$$\left| n^{-E^*} v_j \right| = |a_j| \left| n^{-E^*} (A^*)^{-1} e_j \right| = |a_j| \left| (A^*)^{-1} n^{-D} e_j \right| = n^{-1/(2m_j)} |v_k|$$

for $j = 1, 2, \ldots, d$ and $n \in \mathbb{N}_+$, where $D = \operatorname{diag}\left((2m_1)^{-1}, (2m_2)^{-1}, \ldots, (2m_d)^{-1}\right)$ and $E = ADA^{-1}$. Considering the operator D_v^β, the term involving w's appearing in the right hand side of (3.60) is, in our case,

$$\begin{aligned} \left(\left|n^{-E^*} v_1\right|\right)^{\beta_1} \left(\left|n^{-E^*} v_2\right|\right)^{\beta_2} \cdots \left(\left|n^{-E^*} v_d\right|\right)^{\beta_d} &= \prod_{j=1}^d |v_j|^{\beta_j} n^{-\beta_j/(2m_j)} \\ &= n^{-|\beta : 2\mathbf{m}|} \prod_{j=1}^d |v_j|^{\beta_j} \end{aligned} \tag{3.68}$$

for all $n \in \mathbb{N}_+$. The desired estimate now follows by inserting (3.68) into (3.60).

$\square$

Our next theorem concerns discrete time estimates for convolution powers. Given $\phi : \mathbb{Z}^d \to \mathbb{C}$ which satisfies the hypotheses of Theorem 3.5.3 with corresponding $\alpha \in \mathbb{R}^d$. For any $l \in \mathbb{N}_+$, the theorem provides pointwise estimates for $\phi^{(n)} - \phi^{(l+n)}$ and analogous higher-order differences. Because, in general, the peak of the convolution powers drifts according to α, to compare $\phi^{(n)}$ and $\phi^{(l+n)}$, one needs to account for this drift by re-centering $\phi^{(l+n)}$ but, in doing this, a possible complication arises: If $l\alpha \notin \mathbb{Z}^d$, one cannot re-center $\phi^{(l+n)}$ in a way that keeps it on the lattice. For this reason, the theorem requires $l\alpha \in \mathbb{Z}^d$ and in this case $\left(\delta_{-l\alpha} * \phi^{(l)}\right) * \phi^{(n)}(x) = \phi^{(l+n)}(x + l\alpha)$ which can then be compared to $\phi^{(n)}(x)$. Assuming that ϕ satisfies the hypotheses of Theorem 3.5.3 (with $\xi_0 \in \mathbb{T}^d$ and $\alpha \in \mathbb{Z}^d$), for any $l \in \mathbb{N}_+$ such that $l\alpha \in \mathbb{Z}^d$, we define the discrete time difference operator $\partial_l = \partial_l(\phi, \xi_0, \alpha)$ by

$$\partial_l \psi = \left(\delta - \hat{\phi}(\xi_0)^{-l}\left(\delta_{-l\alpha} * \phi^{(l)}\right)\right) * \psi = \psi - \hat{\phi}(\xi_0)^{-l}\left(\delta_{-l\alpha} * \phi^{(l)}\right) * \psi \tag{3.69}$$

for $\psi \in \ell^1(\mathbb{Z}^d)$.

Theorem 3.5.7. *Let $\phi : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported and such that $\sup_{\xi \in \mathbb{T}^d} |\hat{\phi}(\xi)| = 1$. Additionally assume that $\Omega(\phi) = \{\xi_0\}$ and that ξ_0 is of positive homogeneous type for $\hat{\phi}$ with corresponding $\alpha = \alpha_{\xi_0} \in \mathbb{R}^d$ and positive homogeneous polynomial $P = P_{\xi_0}$. Define μ_ϕ by* (3.6) *(or equivalently* (3.4)*) and denote by $R^{\#}$, the Legendre-Fenchel transform of $R = \operatorname{Re} P$. There are positive constants C and M such that, for any $l_1, l_2, \dots, l_k \in \mathbb{N}_+$ such that $l_q\alpha \in \mathbb{Z}^d$ for $q = 1, 2, \dots, k$ (assume $k \geq 1$),*

$$\begin{aligned} &|\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k}\phi^{(n)}(x)| \\ &\leq \frac{C^k k! \prod_{q=1}^k l_q}{n^{\mu_\phi + k}} \exp\left(-(n + l_1 + l_2 + \cdots + l_k) M R^{\#}\left(\frac{x - n\alpha}{n + l_1 + l_2 + \cdots + l_k}\right)\right) \end{aligned} \tag{3.70}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$.

Proof. As in the proofs of Theorems 3.1.6 and 3.5.3, we fix $\nu \in \mathbb{R}^d$ and invoke the periodicity of $\hat{\phi}$ to see that

$$\begin{aligned}
&\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k}\phi^{(n)}(x)\\
&= \frac{1}{(2\pi)^d}\int_{\xi\in\mathbb{T}^d_\phi}\prod_{q=1}^{k}\left(1-\left(\hat{\phi}(\xi_0)^{-1}e^{-\alpha\cdot(\xi_0+z)}\hat{\phi}(\xi_0+z)\right)^{l_q}\right)\hat{\phi}(\xi_0+z)^n e^{-ix\cdot(\xi_0+z)}\,d\xi\\
&= \frac{1}{(2\pi)^d}\int_{\xi\in\mathbb{T}^d_\phi}\prod_{q=1}^{k}g_{l_q}(z)\hat{\phi}(\xi_0)^n f(z)^n e^{-i(x-n\alpha)\cdot(\xi_0+z)}\,d\xi
\end{aligned}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$, where $z = \xi - i\nu$; here, $f = f_{\xi_0}$ is defined by (3.51) and $g_{l_1}g_{l_2},\dots,g_{l_k}$ are those of Lemma 3.5.2. Put $s_k = l_1 + l_2 + \cdots + l_k$, take ϵ, M and C as guaranteed by Lemmas 3.5.1 and 3.5.2 and set $C_1 = (2C/\epsilon)$. Observe that

$$\begin{aligned}
&\left|\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k}\phi^{(n)}(x)\right|\\
&\leq \frac{C_1^k k!\prod_{q=1}^k l_q}{n^k}e^{-(x-n\alpha)\cdot\nu}\\
&\quad\times\int_{\xi\in\mathbb{T}^d_\phi}\frac{1}{k!}\left(\frac{n\epsilon}{2}(R(\nu)+R(\xi))\right)^k e^{s_k MR(\nu)}\exp(-n\epsilon R(\xi)+nMR(\nu))\,d\xi\\
&\leq \frac{C_1^k k!\prod_{q=1}^k l_q}{n^k}e^{-(x-n\alpha)\cdot\nu}\\
&\quad\times\int_{\xi\in\mathbb{T}^d_\phi}\exp(n\epsilon(R(\xi)+R(\nu))/2)\exp((n+s_k)MR(\nu)-n\epsilon R(\xi))\,d\xi
\end{aligned}$$

for $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$. Upon setting $y_{n,s_k}(x) = (x - n\alpha)/(n + s_k)$ and replacing M by $M + \epsilon/2$, we can write

$$\begin{aligned}
\left|\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k}\phi^{(n)}(x)\right| \leq\ & \frac{C_1^k k!\prod_{q=1}^k l_q}{n^k}\exp(-(n+s_k)\left(y_{n,s_k}(x)\cdot\nu - MR(\nu)\right))\\
&\times\int_{\xi\in\mathbb{T}^d_\phi}\exp(-n\epsilon R(\xi)/2)\,d\xi
\end{aligned}$$

Now, as we observed in the proof of Theorem 3.1.6, the integral over ξ is bounded above by $C_2 n^{-\mu_\phi} \leq C_2^k n^{-\mu_\phi}$ for some constant $C_2 \geq 1$ and so we obtain the estimate

$$\left|\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k}\phi^{(n)}(x)\right| \leq \frac{(C_1C_2)^k k!\prod_{q=1}^k l_q}{n^{\mu_\phi+k}}\exp(-(n+s_k)\left(y_{n,s_k}(x)\cdot\nu - MR(\nu)\right))$$

for all $x\in\mathbb{Z}^d$ and $n\in\mathbb{N}_+$. Once again, the desired result is obtained by infimizing over $\nu\in\mathbb{R}^d$. □

Remark 9. *If one allows the constant M to depend on $l_1,l_2,\dots,l_k$, then* (3.70) *can be written*

$$|\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k}\phi^{(n)}(x)| \leq \frac{C^k k!\prod_{q=1}^k l_q}{n^{\mu_\phi+k}}\exp\left(-nM_{l_1,l_2,\dots,l_k}R^{\#}\left(\frac{x-n\alpha}{n}\right)\right)$$

for all $x\in\mathbb{Z}^d$ and $n\in\mathbb{N}_+$. Indeed, set $s_k=l_1+l_2+\cdots+l_k$ and observe that

$$\begin{aligned}
-(n+s_k)R^{\#}\left(\frac{x-n\alpha}{n+s_k}\right) &= -n\sup_{\nu\in\mathbb{R}^d}\left\{\left(\frac{x-n\alpha}{n}\right)\cdot\nu-\frac{n+s_k}{n}R(\nu)\right\}\\
&\leq -n\sup_{\nu}\left\{\left(\frac{x-n\alpha}{n}\right)\cdot\nu-(1+k_s)R(\nu)\right\}\\
&\leq -n\Big((1+s_k)R\Big)^{\#}\left(\frac{x-n\alpha}{n}\right)\\
&\leq -nM_{s_k}R^{\#}\left(\frac{x-n\alpha}{n}\right)
\end{aligned}$$

where we have used Corollary A.3.4 to obtain $M_{s_k}=M_{l_1,l_2,\dots,l_k}$.

In view the remark above, the following corollary is a special case of Theorem 3.5.7 when $\alpha=0$, $\hat\phi(\xi_0)=1$ and we only consider one discrete time derivative; it applies to the example in the introduction and the examples of Subsections 3.7.1 and 3.7.5.

Corollary 3.5.8. *Let $\phi:\mathbb{Z}^d\to\mathbb{C}$ be finitely supported and such that $\sup|\hat\phi(\xi)|=1$. Suppose that $\Omega(\phi)=\{\xi_0\}$ is of positive homogeneous type for $\hat\phi$ with corresponding $\alpha\in\mathbb{R}^d$ and positive homogeneous polynomial P. Also let μ_ϕ be defined by* (3.6) *(or equivalently* (3.4)*) and let $R^{\#}$ be the Legendre-Fenchel transform of $R=\operatorname{Re}P$. Additionally assume that $\alpha=0$ and $\hat\phi(\xi_0)=1$. There exists a positive constant C and, to* *each $l \in \mathbb{N}_+$, a positive constant M_l such that*

$$\left|\phi^{(n)}(x) - \phi^{(l+n)}(x)\right| \leq \frac{Cl}{n^{\mu_\phi+1}} \exp(-nM_l R^{\#}(x/n))$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$.

Our final theorem of this subsection concerns both time and space differences for convolution powers.

Theorem 3.5.9. *Let $\phi : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported and such that $\sup_{\xi\in\mathbb{T}^d} |\hat{\phi}(\xi)| = 1$. Additionally assume that $\Omega(\phi) = \{\xi_0\}$ and that ξ_0 is of positive homogeneous type for $\hat{\phi}$ with corresponding $\alpha = \alpha_{\xi_0} \in \mathbb{R}^d$ and positive homogeneous polynomial $P = P_{\xi_0}$. Define μ_ϕ by* (3.6) *(or equivalently* (3.4)*), let $\mathbf{m}$ (and A) be as guaranteed by Proposition 3.2.2 and denote by $R^{\#}$, the Legendre-Fenchel transform of $R = \operatorname{Re} P$. There are positive constants M and C_0 and, to each $B > 0$ and multi-index β, a positive constant C_β such that, for any P-fitted collection $v = \{v_1, v_2, \dots, v_d\}$ of weight $\mathbf{m}$ and $l_1, l_2, \dots, l_k \in \mathbb{N}_+$ such that $l_q\alpha \in \mathbb{Z}^d$ for $q = 1, 2, \dots, k$,*

$$\begin{aligned}
&\left|\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k} D_v^\beta(\hat{\phi}(\xi_0)^{-1}e^{ix\cdot\xi_0}\phi^{(n)}(x))\right| \\
&\quad\leq \frac{C_\beta C_0^k k! \prod_{q=1}^k l_q \prod_{j=1}^d |v_j|^{\beta_j}}{n^{\mu_\phi+|\beta:2\mathbf{m}|+k}} \\
&\qquad\qquad\qquad \times \exp\left(-(n+l_1+l_2+\cdots+l_k)MR^{\#}\left(\frac{x-n\alpha}{n+l_1+l_2+\cdots+l_k}\right)\right)
\end{aligned}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$ such that $|v_k| \leq Bn^{1/(2m_k)}$ for $k = 1, 2, \dots, d$.

Proof. By replacing $\phi(x)$ by $\hat{\phi}(\xi_0)^{-1}e^{ix\cdot\xi_0}\phi(x)$ we can assume without loss of generality that $\xi_0 = 0$ and $\hat{\phi}(\xi_0) = 1$. Assuming the notation of Lemma 3.5.1 (with $f = f_{\xi_0}$) and Lemma 3.5.2, we fix $\nu \in \mathbb{R}^d$ and observe that

$$\partial_{l_1}\partial_{l_2}\cdots\partial_{l_k} D_v^\beta \phi^{(n)}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \prod_{q=1}^k g_{l_q}(z)Q(z)f(z)^n e^{-i(n+s_k)y_{s_k,n}(x)\cdot z}\, d\xi$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$, where $z = \xi - i\nu$, $s_k = l_1 + l_2 + \cdots + l_k$, $y_{s_k,n}(x) = (x - n\alpha)/(n + s_k)$ and $Q(z) = \prod_{j=1}^{d}(e^{iv_j \cdot z} - 1)^{\beta_j}$ is the subject of Lemma 3.5.4. The desired estimate is now established by virtually repeating the arguments in the proof of Theorems 3.5.3 and 3.5.7 while making use of Lemmas 3.5.1, 3.5.2 and 3.5.4 and noting, as was done in the proof of Corollary 3.5.6, that $|n^{-E^*}v_j| = n^{-1/(2m_j)}|v_j|$ for $j = 1, 2, \dots, d$. □

3.5.2 Sub-exponential bounds

In this subsection, we again consider a finitely supported function $\phi : \mathbb{Z}^d \to \mathbb{C}$ such that $\sup_{\xi \in \mathbb{T}^d} |\hat{\phi}(\xi)| = 1$ and each $\xi \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$. In contrast to the previous subsection, we do not require any relationship between the drifts α_ξ and positive homogeneous polynomials P_ξ for those $\xi \in \Omega(\phi)$; a glimpse into Subsections 3.7.2 and 3.7.4 shows this situation to be a natural one. As was noted in [31], the optimization procedure which yielded the exponential-type estimates of the previous subsection is no longer of use. Here we have the following result concerning sub-exponential estimates.

Theorem 3.5.10. *Let $\phi : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported and such that $\sup_{\xi \in \mathbb{T}^d} |\hat{\phi}(\xi)| = 1$. Suppose additionally each $\xi \in \Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$ and hence $\Omega(\phi) = \{\xi_1, \xi_2, \dots, \xi_Q\}$. Let $\alpha_q \in \mathbb{R}^d$ and positive homogeneous polynomial P_q be those associated to ξ_q for $q = 1, 2, \dots, Q$. Moreover, for each $q = 1, 2, \dots, Q$, set $\mu_q = \mu_{P_q}$ and let $E_q \in \operatorname{Exp}(P_q)$. Then, for any $N \geq 0$, there is a positive constant C_N such that*

$$|\phi^{(n)}(x)| \leq C_N \sum_{q=1}^{Q} \frac{1}{n^{\mu_q}} (1 + |n^{-E_q^*}(x - n\alpha_q)|)^{-N} \tag{3.71}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$. The constant C_N is independent of $E_q \in \operatorname{Exp}(P_q)$ for $q = 1, 2, \dots Q$.

Proof. In view of Proposition 3.4.1 and Remark 7, there exist relatively open subsets $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_Q$ of $\mathbb{T}^d_\phi$ satisfying the following properties:

1. For each $q = 1, 2, \dots Q$, $\mathcal{B}_q$ contains ξ_q.
2. $\mathcal{B}_1$ contains the boundary of $\mathbb{T}^d_\phi$ (as a subset of $\mathbb{R}^d$).
3. The closed sets $\{\overline{\mathcal{B}}_1, \overline{\mathcal{B}}_2, \dots, \overline{\mathcal{B}}_Q\}$ are mutually disjoint.

For $q = 1, 2, \dots Q$, define

$$\mathcal{O}_q = \mathbb{T}^d_\phi \setminus \left(\bigcup_{r \neq q} \overline{\mathcal{B}_r} \right)$$

and observe that each $\mathcal{O}_q$ is an open neighborhood of ξ_q (in the relative topology). Let $\{u_q\}_{q=1}^Q$ be a smooth partition of unity subordinate to $\{\mathcal{O}_q\}_{q=1}^Q$. By construction, $u_1 \equiv 1$ on the boundary of $\mathbb{T}^d_\phi$ and, for each $q = 1, 2, \dots Q$, u_q is compactly supported in $\mathcal{O}_q$. We note that, for each $q \neq 1$, $\mathrm{Supp}(u_q)$ is also a compact subset of $\mathbb{R}^d$ because the boundary of $\mathbb{T}^d_\phi$ is contained in $\mathcal{B}_1$ (the relative topology of $\mathbb{T}^d_\phi$ is only seen in $\mathrm{Supp}(u_1)$). Set

$$\delta = \frac{\min_{q=1,2,\dots,Q} \operatorname{dist}(\mathrm{Supp}(u_q), \partial \mathcal{O}_q)}{2\sqrt{d}} > 0.$$

Observe that, for any $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$,

$$\begin{aligned}\phi^{(n)}(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d_\phi} e^{-ix\cdot\xi} \hat{\phi}(\xi)^n \, d\xi = \sum_{q=1}^{Q} \frac{1}{(2\pi)^d} \int_{\mathcal{O}_q} e^{-ix\cdot\xi} \hat{\phi}(\xi)^n u_q(\xi) \, d\xi \\ &= \sum_{q=1}^{Q} \frac{e^{ix\cdot\xi_q} \hat{\phi}(\xi_q)^n}{n^{\mu_q} (2\pi)^d} \int_{\mathcal{U}_{q,n}} e^{-iy_n(x)\cdot\xi} f_{q,n}(\xi) u_{q,n}(\xi) \, d\xi = \sum_{q=1}^{Q} \frac{e^{ix\cdot\xi_q} \hat{\phi}(\xi_q)^n}{n^{\mu_q} (2\pi)^d} \mathcal{I}_{q,n}(x)\end{aligned} \tag{3.72}$$

where we have set $y_{q,n}(x) = n^{-E_q^*}(x - n\alpha_q)$, $\mathcal{U}_{q,n} = n^{E_q}(\mathcal{O}_q) - \xi_q$, defined

$$u_{q,n}(\xi) = u_q(n^{-E_q}\xi),$$

and

$$f_{q,n}(\xi) = (\hat{\phi}(\xi_q)^{-1} e^{-i\alpha\cdot n^{-E_q}\xi} \hat{\phi}(n^{-E_q}\xi + \xi_q))^n$$

for $\xi \in \mathcal{U}_{q,n}$, and put

$$\mathcal{I}_{q,n}(x) = \int_{\mathcal{U}_{q,n}} e^{-iy_n(x)\cdot\xi} f_{q,n}(\xi) u_{q,n}(\xi)\, d\xi.$$

Of course, for each n and q, $f_{n,q}$ extends to an entire function on $\mathbb{C}^d$; we make no distinction between this function and $f_{n,q}$. We will soon obtain the desired estimates by integrating $\mathcal{I}_{n,q}$ by parts. For this purpose, it is useful to estimate the derivatives of $f_{q,n}$ and this is done in the lemma below. The idea behind the lemma's proof is to look at $f_{q,n}$ on small neighborhoods in $\mathbb{C}^d$ of $\zeta \in \mathrm{Supp}(u_{q,n}) \subseteq \mathbb{R}^d$. On such complex neighborhoods, Lemma 3.5.1 gives tractable estimates for $f_{q,n}$ to which Cauchy's d-dimensional integral formula can be applied to estimate $D^\alpha f_{q,n}(\zeta)$.

Lemma 3.5.11. *For each $q = 1, 2, \dots, Q$, there exist positive constants C_q and ϵ_q such that, for each multi-index β,*

$$|D^\beta f_{q,n}(\zeta)| \le C_q \frac{\beta!}{\delta^{|\beta|}} \exp(-\epsilon_q R_q(\zeta))$$

for all $n \in \mathbb{N}_+$ and $\zeta \in \mathrm{Supp}(u_{q,n})$.

Proof of Lemma 3.5.11. Our choice of the open cover $\{\mathcal{O}_q\}$ guarantees that $|\hat{\phi}(\eta + \xi_q)| < 1$ for all non-zero η in the compact set $\overline{\mathcal{O}_q} - \xi_q$. An appeal to Lemma 3.5.1 gives $\epsilon_q', M_q' > 0$ such that

$$\begin{aligned} |f_{q,n}(z)| &\le \exp\big(-\epsilon_q' R_q\big(n^{-E_q}\eta\big) + M_q' R_q\big(n^{-E_q}\nu\big)\big)^n \\ &\le \exp(-\epsilon_q' R_q(\eta) + M_q' R_q(\nu)) \end{aligned} \tag{3.73}$$

for all $n \in \mathbb{N}_+$ and $z = \eta - i\nu \in \mathbb{C}^d$ for which $\eta \in \overline{\mathcal{U}_{q,n}}$.

We claim that there are constants $\epsilon_q, M_q > 0$ for which

$$-\epsilon_q' R_q(\eta) + M_q' R_q(\nu) \le -\epsilon_q R_q(\zeta) + M_q \tag{3.74}$$

for all $z = \eta - i\nu \in \mathbb{C}^d$ and $\zeta \in \mathbb{R}^d$ such that $|z_i - \zeta_i| = \delta$ for $i = 1, 2, \dots d$. Indeed, it is clear that $R_q(\nu)$ is bounded for all possible values of ν. An appeal to Proposition A.2.4 ensures that, there are $M_q', \epsilon_q > 0$ for which

$$-\epsilon_q' R_q(\eta) = -\epsilon_q' R_q(\zeta + (\eta - \zeta)) \leq -\epsilon_q R_q(\zeta) + M_q'$$

for all $\eta, \zeta \in \mathbb{R}^d$ provided $|\eta_i - \zeta_i| \leq |z_i - \zeta_i| = \delta$ for all $i = 1, 2, \dots d$. This proves the claim.

By combining (3.73) and (3.74), we deduce that, for all $n \in \mathbb{N}_+$, $\zeta \in \mathbb{R}^d$, and $z = \eta - i\nu \in \mathbb{C}^d$ for which $\eta \in \overline{\mathcal{U}_{q,n}}$,

$$|f_{q,n}(z)| \leq \exp(-\epsilon_q R_q(\zeta) + M_q) \tag{3.75}$$

whenever $|z_i - \zeta_i| = \delta$ for all $i = 1, 2, \dots d$. Our aim is to combine Cauchy's d-dimensional integral formula,

$$D^\beta f_{q,n}(\zeta) = \frac{\beta!}{(2\pi i)^d} \int_{C_1} \int_{C_2} \cdots \int_{C_d} \frac{f_{q,n}(z)\, dz_1 dz_2 \dots dz_d}{(z - \zeta)^{(\beta_1+1, \beta_2+1, \dots, \beta_d+1)}}, \tag{3.76}$$

with (3.75) to obtain our desired bound for $\zeta \in \mathrm{Supp}(u_{q,n})$; here, $C_i = \{z : |z_i - \zeta_i| = \delta\}$ for $i = 1, 2, \dots, d$. To do this, we must verify that $z = \eta - i\nu$ is such that $\eta \in \overline{\mathcal{U}_{q,n}}$ whenever $|z_i - \zeta_i| = \delta$ for $i = 1, 2, \dots, d$. This is easy to see, for if $\zeta \in \mathrm{Supp}(u_{q,n})$ and z is such that $|z_i - \zeta_i| = \delta$ for $i = 1, 2, \dots, d$,

$$|z - \zeta| = \sqrt{d}\delta < \text{dist}(\mathrm{Supp}(u_q), \partial\mathcal{O}_q) \leq \text{dist}(\mathrm{Supp}(u_{n,q}), \partial\mathcal{U}_{n,q})$$

for all $n \in \mathbb{N}_+$ (the distance only increases with n because $\{t^{E_q}\}$ is contracting). Consequently, a combination of (3.75) and (3.76) shows that, for any multi-index β,

$$|D^\beta f_{q,n}(\zeta)| \leq \frac{\beta!}{\delta^{|\beta|}} \exp(-\epsilon_q R_q(\zeta) + M_q)$$

for all $n \in \mathbb{N}_+$ and $\zeta \in \mathrm{Supp}(u_{q,n})$ and thus the desired result holds. //

We now finish the proof of Theorem 3.5.10. We assert that, for each $q = 1, 2, \dots, Q$ and multi-index β, there exists $C_\beta > 0$ such that

$$|y_{q,n}(x)^\beta \mathcal{I}_{q,n}(x)| \leq C_\beta \tag{3.77}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$. By inspecting (3.72), we see that the desired estimate, (3.71), follow directly from (3.77) and so we prove (3.77).

We have, for any multi-index β,

$$\begin{aligned}(iy_{q,n}(x))^\beta \mathcal{I}_{q,n}(x) &= \int_{\mathcal{U}_{q,n}} D_\xi^\beta (e^{-iy_n(x)\cdot\xi}) f_{q,n}(\xi) u_{q,n}(\xi)\, d\xi \\ &= (-1)^{|\beta|} \int_{\mathcal{U}_{q,n}} e^{-iy_n(x)\cdot\xi} D^\beta (f_{q,n}(\xi) u_{q,n}(\xi))\, d\xi\end{aligned}$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$ where we have integrated by parts and made explicit use of our partition of unity $\{u_q\}$ to ensure that all boundary terms vanished. To see the absence of boundary contributions, note that when $q \neq 1$, $u_{q,n}$ and its derivatives are identically zero on a neighborhood of $\partial\mathcal{U}_{q,n}$. When $q = 1$, $\operatorname{Supp}(u_{1,n}) \cap \partial\mathcal{U}_{1,n} = \partial(n^E \mathbb{T}^d)$ and because $u_{1,n} \equiv 1$ on a neighborhood of $\partial(n^E \mathbb{T}^d_\phi)$, the periodicity of $f_{q,n}$ and its derivatives (which are directly inherited form the periodicity of $\hat{\phi}(\xi)$) ensure that the integral over the $\partial\mathcal{U}_{1,n}$ is zero. Consequently,

$$|y_{q,n}(x))^\beta \mathcal{I}_{q,n}(x)| \leq \int_{\operatorname{Supp}(u_{q,n})} \left| D^\beta (f_{q,n}(\xi) u_{q,n}(\xi)) \right| d\xi$$

for, $q = 1, 2, \dots Q$, $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$. Once it is observed that derivatives of $u_{q,n}$ are well-behaved as n increases, the estimate (3.77) follows immediately from Lemma 3.5.11. The fact that C_N is independent of $E_q \in \operatorname{Exp}(P_q)$ for $q = 1, 2, \dots, Q$ follows by a direct application of Proposition 3.2.3. $\square$

3.6 Stability theory

We now turn to the stability of convolution operators. In this brief section, we show that Theorem 3.1.7 is a consequence of of estimates of the preceding section. Let $\phi : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported and define the operator A_ϕ on $L^p = L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$ by

$$(A_\phi f)(x) = \sum_{y \in \mathbb{Z}^d} \phi(y) f(x-y). \tag{3.78}$$

Such operators arise in the theory of finite difference schemes for partial differential equations in which they produce extremely accurate numerical approximations to solutions for initial value problems, e.g., (3.10). We encourage the reader to see [75] and [89] for readable introductions to this theory; Thomée's survey [86] is also an excellent reference. In this framework, the operator A_ϕ is known as an explicit constant-coefficient difference operator. General explicit difference operators are produced by allowing ϕ to depend on a real parameter $h > 0$ which is usually the grid size of an associated spatial discretization for the initial value problem.

The operator A_ϕ is said to be *stable* in L^p if the collection of successive powers of A_ϕ is uniformly bounded on L^p, i.e., there is a positive constant C for which

$$\|A_\phi^n f\|_{L^p} \leq C \|f\|_{L_p}$$

for all $f \in L^p$ and $n \in \mathbb{N}_+$; this property has profound consequences for difference schemes of partial differential equations as we discussed in the introduction. For example, the Lax equivalence theorem states that a consistent approximate difference scheme for (3.10) is stable in L^p if and only if the difference scheme converges to the true solution (3.11) [86, 89]. In the L^2 setting, checking stability is straightforward. Using the Fourier transform, one finds that A_ϕ is stable in L^2 if and only if $\sup_\xi |\hat{\phi}(\xi)| \leq 1$; this is a special case of the von Neumann condition [87]. When $p \neq 2$, the question of stability for A_ϕ is more subtle. It follows directly from the definition (3.78) that $A_\phi^n = A_{\phi^{(n)}}$ for all $n \in \mathbb{N}_+$ and so by Minkowski's inequality we see that

$$\|A_\phi^n f\|_{L^p} = \|A_{\phi^{(n)}}\|_{L^p} \leq \|\phi^{(n)}\|_1 \|f\|_{L^p} \tag{3.79}$$

for all $f \in L^p$ and $n \in \mathbb{N}_+$. This allows us to formulate a sufficient condition for stability in L^p for $1 \leq p \leq \infty$ in terms of the convolution powers of ϕ (which is consistent with Question (iv) of Section 3.1) as follows: A_ϕ is stable in L^p whenever there is a positive constant C for which

$$\|\phi^{(n)}\|_1 = \sum_{x \in \mathbb{Z}^d} |\phi^{(n)}(x)| \leq C \tag{3.80}$$

for all $n \in \mathbb{N}_+$. The condition (3.80) is also necessary when $p = \infty$ and so it is called the condition of max-norm stability. Originally investigated by John [56] and Strang [84], this theory for difference schemes has been further developed by many authors, see for example [38, 82, 86, 87]. In one dimension ($d = 1$), the question of stability in the max-norm was completely sorted out by Thomée [87]. Thomée showed that a sufficient condition of Strang was also necessary; this is summarized in the following theorem.

Theorem 3.6.1 (Thomée 1965)**.** *The operator A_ϕ is stable in $L^\infty(\mathbb{R})$ if and only if one of the following conditions is satisfied:*

(a) $\hat{\phi}(\xi) = ce^{ix\xi}$ for some $x \in \mathbb{Z}$ and $|c| = 1$.

(b) $|\hat{\phi}(\xi)| < 1$ except for at most a finite number of points $\xi_1, \xi_2, \dots, \xi_Q$ in $\mathbb{T}$ where $|\hat{\phi}(\xi)| = 1$. For $q = 1, 2, \dots Q$, there are constants α_q, γ_q, m_q, where $\alpha_q \in \mathbb{R}$,

$\operatorname{Re}\gamma_q > 0$ *and where* $m_q \in \mathbb{N}_+$, *such that*

$$\hat{\phi}(\xi + \xi_q) = \hat{\phi}(\xi_q) \exp(i\alpha_q \xi - \gamma_k \xi^{2m_q} + o(\xi^{2m_q})) \tag{3.81}$$

as $\xi \to 0$.

Thomée's characterization makes use of the fact that the level sets of non-constant holomorphic functions on $\mathbb{C}$ have no accumulation points – a fact that breaks down in all other dimensions, e.g., $f(z) = f(z_1, z_2) = \cos(z_1 - z_2)$. When $\phi : \mathbb{Z} \to \mathbb{C}$ is finitely supported and such that $\sup_\xi |\hat{\phi}(\xi)| = 1$, the reader should note that the condition (b) of Theorem 3.6.1 is equivalent to the hypotheses of Theorem 3.5.10 for, in one dimension, every positive homogeneous polynomial is necessarily of the form $P(\xi) = \gamma\xi^{2m}$ where $\operatorname{Re}\gamma > 0$ and $m \in \mathbb{N}_+$. In $\mathbb{Z}^d$, we have the following result.

Corollary 3.6.2. *Let* $\phi : \mathbb{Z}^d \to \mathbb{C}$ *satisfy the hypotheses of Theorem 3.5.10 and define* A_ϕ *by* (3.78). *Then* A_ϕ *is stable in* L^∞ *and hence stable in* $L^p(\mathbb{R}^d)$ *for all* $1 \leq p \leq \infty$.

Proof. An application of Theorem 3.5.10 with $N \geq d + 1$ yields the uniform estimate (3.80) after summing over $x \in \mathbb{Z}^d$. □

Proof of Theorem 3.1.7. This is simply Corollary 3.6.2 translated into the language of Section 3.1. □

In [87], Thomée also showed that when $\sup |\hat{\phi}| = 1$ but the leading non-linear term in the expansion (3.81) was purely imaginary, the corresponding difference scheme was unstable. As was discussed in [31] and [73], such expansions give rise to local limit theorems in which the corresponding attractors are bounded but not in L^2 and hence not in $S(\mathbb{R})$; for instance, the Airy function. In the spirit of [87], M. V. Fedoryuk explored stability and instability in higher dimensions [38]. Fedoryuk's affirmative result assumes that, for $\xi_0 \in \Omega(\phi)$, the leading quadratic polynomial in the expansion for Γ_{ξ_0} has positive definite real part. Because any quadratic polynomial P with positive definite real part is positive homogeneous ($2^{-1}I \in \operatorname{Exp}(P)$), Corollary 3.6.2 (equivalently, Theorem 3.1.7) extends the affirmative result of [38].

3.7 Examples

In this section we consider a number of examples, mostly in $\mathbb{Z}^2$, to which we apply our results. The first four examples, presented in Subsections 3.7.1, 3.7.2, 3.7.3 and 3.7.4, illustrate behaviors which appear only in the complex-valued setting. We present these examples, not for a love of pathology, but to demonstrate the richness of the complex-valued setting and to show that the intricacy dealt with in the theoretical development of the preceding chapters was warranted. Beyond these first four examples, in Subsection 3.7.5, we introduce a class of real-valued functions on $\mathbb{Z}^d$, each prescribed by two multi-parameters $\mathbf{m}$ and λ which precisely determine that anisotropic nature of the convolution powers. Finally, in Subsection 3.7.6, we revisit the classical theory of random walks on $\mathbb{Z}^d$.

3.7.1 A well-behaved real valued function on $\mathbb{Z}^2$

This example illustrates the case in which $\hat{\phi}$ is maximized only at 0 which is of positive homogeneous type for $\hat{\phi}$ with corresponding P. In this case, the local limit theorem for ϕ yields one attractor with no oscillatory prefactor. The positive homogeneous polynomial P is a semi-elliptic polynomial of the form (3.18) and the corresponding attractor exhibits small oscillations and decays anisotropically.

Consider $\phi : \mathbb{Z}^2 \to \mathbb{R}$ defined by $\phi = (\phi_1 + \phi_2)/512$, where

$$\phi_1(x,y) = \begin{cases} 326 & (x,y) = (0,0) \\ 20 & (x,y) = (\pm 2, 0) \\ 1 & (x,y) = (\pm 4, 0) \\ 64 & (x,y) = (0, \pm 1) \\ -16 & (x,y) = (0,\pm 2) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \phi_2(x,y) = \begin{cases} 76 & (x,y) = (1,0) \\ 52 & (x,y) = (-1,0) \\ \mp 4 & (x,y) = (\pm 3, 0) \\ \mp 6 & (x,y) = (\pm 1, 1) \\ \mp 6 & (x,y) = (\pm 1, -1) \\ \pm 2 & (x,y) = (\pm 3, 1) \\ \pm 2 & (x,y) = (\pm 3, -1) \\ 0 & \text{otherwise.} \end{cases}$$

The graphs of $\phi^{(n)}$ on the domain $[-50,50] \times [-50,50]$ for $n = 100$, $n = 1,000$ and $n = 10,000$ are shown in Figure 3.5; in particular, the figure illustrates the decay in $\|\phi^{(n)}\|_\infty$. Figure 3.6 depicts $\phi^{(n)}(x,y)$ when $n = 10,000$ from various angles and clearly illustrates its non-Gaussian anisotropic nature.

Given that ϕ is supported on 21 points, it is clear that $\phi \in \mathcal{S}_2$. An easy computation shows that $\sup|\hat{\phi}(\xi)| = 1$ and this supremum is only attained at $\xi = (\eta, \zeta) = (0,0)$, where $\phi(0,0) = 1$, and hence $\Omega(\phi) = \{(0,0)\}$. Expanding the logarithm of $\phi(\eta,\zeta)/\phi(0,0)$ about $(0,0)$ we find that, as $(\eta,\zeta) \to (0,0)$,

$$\Gamma(\eta,\zeta) = -\frac{1}{64}\left(\eta^6 + 2\zeta^4 - 2i\eta^3\zeta^2\right) + O(|\eta^7| + |\zeta^5| + |\eta^3\zeta^4| + |\eta^5\zeta^2| + |\eta^6\zeta^5|).$$

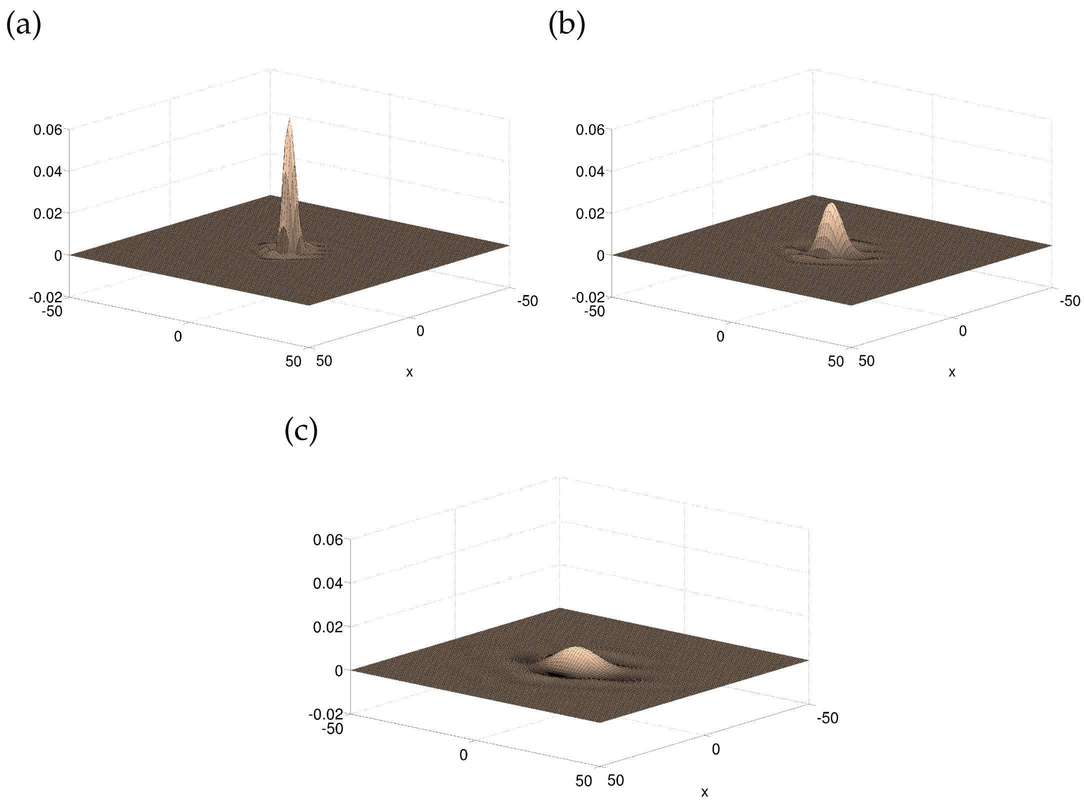

Figure 3.5: The graph of $\phi^{(n)}$ for (a) $n = 100$, (b) $n = 1,000$ and (c) $n = 10,000$

It is easy to see that the polynomial which leads the expansion,

$$P(\eta,\zeta) = \frac{1}{64}\left(\eta^6 + 2\zeta^4 - 2i\eta^3\zeta^2\right),$$

has positive definite real part,

$$R(\eta,\zeta) = \operatorname{Re} P(\eta,\zeta) = \frac{1}{64}\left(\eta^6 + 2\zeta^4\right).$$

Moreover

$$P(t^E(\eta,\zeta)) = P(t^{1/6}\eta, t^{1/4}\zeta) = tP(\eta,\zeta) \quad \text{with} \quad E = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \in \operatorname{Exp}(P)$$

for all $t > 0$ and $(\eta,\zeta) \in \mathbb{R}^2$ and therefore P is a positive homogeneous polyno-

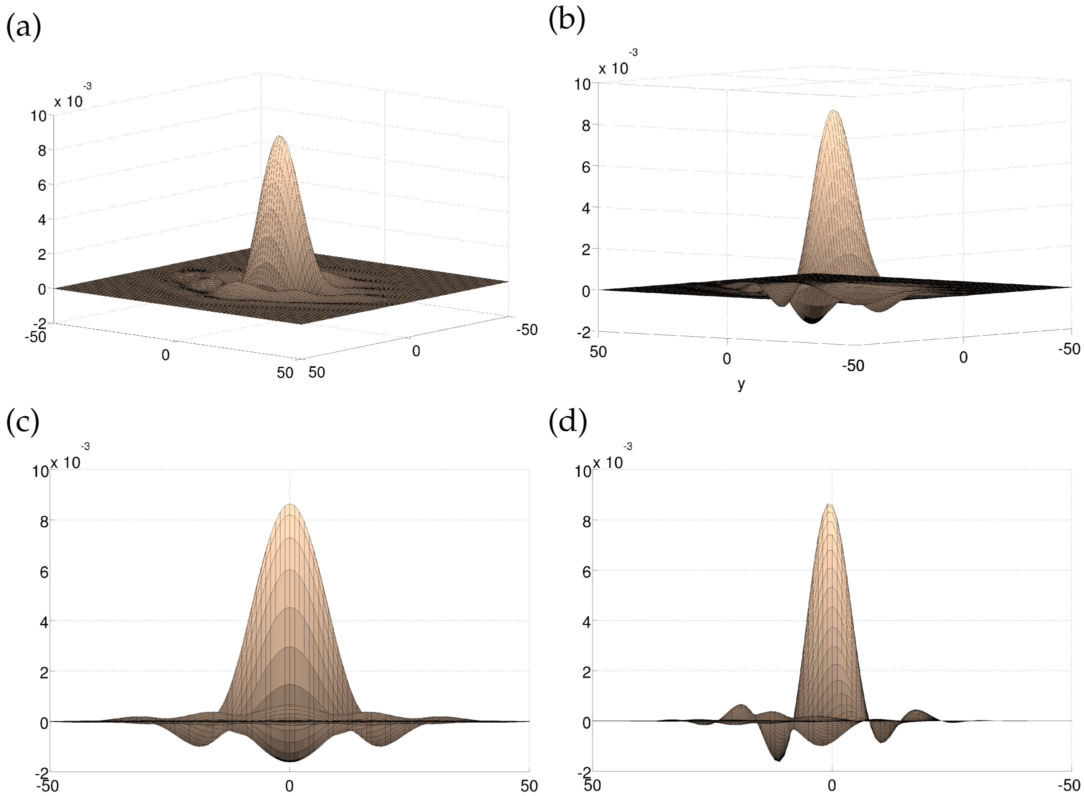

Figure 3.6: Various perspectives of $\phi^{(n)}$ for $n = 10,000$.

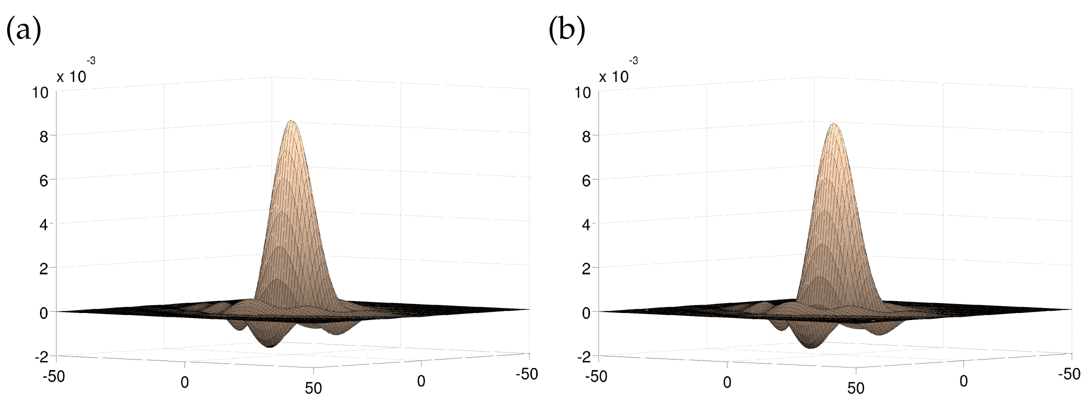

Figure 3.7: The graphs of (a) $\phi^{(n)}$ and (b) H_P^n for $n = 10,000$.

mial (it is also semi-elliptic). Further, we can rewrite the error to see that

$$\Gamma(\eta, \zeta) = -P(\eta, \zeta) + \Upsilon(\eta, \zeta)$$

where $\Upsilon(\eta,\zeta) = o(R(\eta,\zeta))$ as $(\eta,\zeta) \to (0,0)$ and so we conclude that $(0,0)$ is of positive homogeneous type for $\hat{\phi}$ with corresponding $\alpha = (0,0) \in \mathbb{R}^2$ and positive homogeneous polynomial P. Clearly, $\mu_\phi = \mu_P = \operatorname{tr} E = 5/12$ and so Theorem 3.1.4 gives positive constants C and C' for which

$$C' n^{-5/12} \leq \|\phi^{(n)}\|_\infty \leq C n^{-5/12}$$

for all $n \in \mathbb{N}_+$. An appeal to Theorem 3.1.5 shows that

$$\phi^{(n)}(x,y) = H_P^n(x,y) + o(n^{-5/12}) \tag{3.82}$$

uniformly for $(x,y) \in \mathbb{Z}^2$ where,

$$H_P^n(x,y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i(x,y)\cdot(\xi_1,\xi_2) - nP(\xi_1,\xi_2)}\, d\xi_1\, d\xi_2 = \frac{1}{n^{5/12}} H_P(n^{-1/6}x, n^{-1/4}y)$$

for $n \in \mathbb{N}_+$ and $(x,y) \in \mathbb{R}^2$. The local limit (3.82) is illustrated in Figure 3.7 when $n = 10,000$. We also make an appeal to Theorem 3.1.6 to deduce pointwise estimates for $\phi^{(n)}$ (in fact, all results of Section 3.5 are valid for this ϕ). Upon noting that

$$R^{\#}(x,y) = \frac{5}{3^{6/5}} x^{6/5} + \left(1 - \frac{1}{2^5}\right) y^{4/3}$$

for $(x,y) \in \mathbb{R}^2$, the theorem gives positive constants C and M for which

$$|\phi^{(n)}(x,y)| \leq \frac{C}{n^{5/12}} \exp\left(-nM\left(\left(\frac{x}{n}\right)^{6/5} + \left(\frac{y}{n}\right)^{4/3}\right)\right)$$

for all $n \in \mathbb{N}_+$ and $(x,y) \in \mathbb{Z}^2$.

3.7.2 Two drifting packets

In this example, we study a complex valued function on $\mathbb{Z}^2$ whose convolution powers $\phi^{(n)}$ exhibit two *packets* which drift apart as n increases. This behavior is

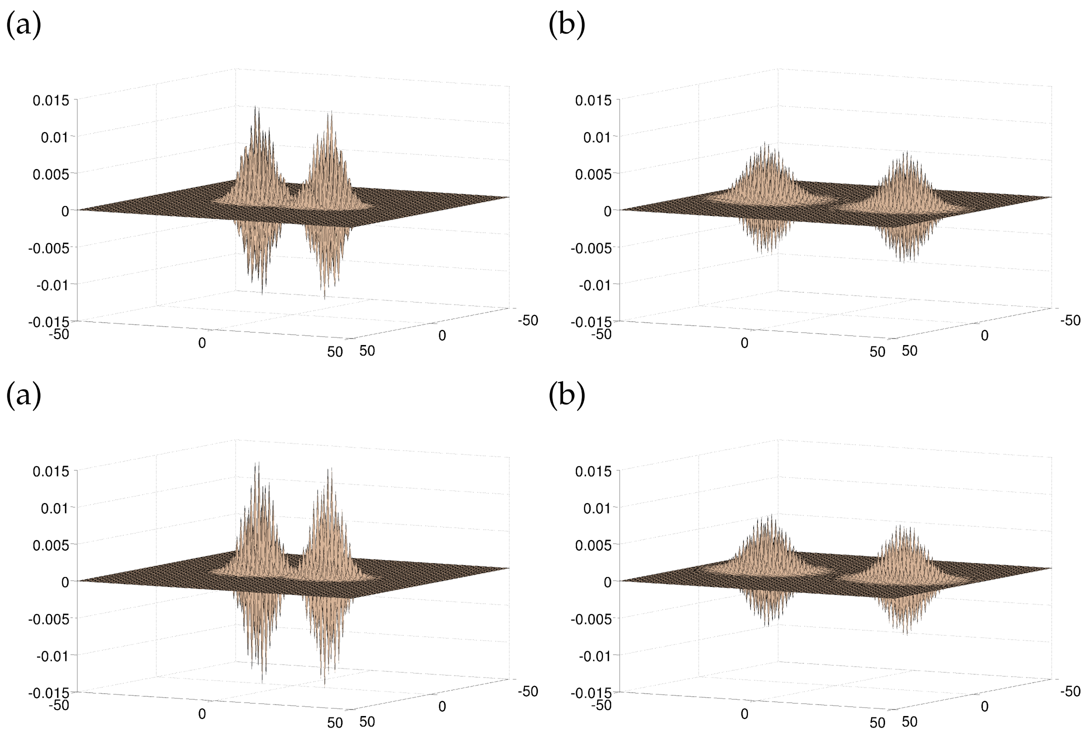

Figure 3.8: The graphs of $\mathrm{Re}(\phi^{(n)})$ and $\mathrm{Re}(f_n)$ for $n = 30, 60$.

easily described by applying Theorem 3.1.5 in which two distinct α's appear.

Consider $\phi : \mathbb{Z}^2 \to \mathbb{C}$ defined by

$$\phi(x,y) = \begin{cases} \frac{1+i}{4a} & (x,y) = (-1,\pm 1) \\ -\frac{1+i}{4a} & (x,y) = (1,\pm 1) \\ \pm\frac{1}{\sqrt{2}a} & (x,y) = (0,\pm 1) \\ 0 & \text{otherwise.} \end{cases}$$

where $a = \sqrt{2+\sqrt{2}}$. The graphs of $\mathrm{Re}(\phi^{(n)})$ for $n = 30$ and $n = 60$ are shown in Figure 3.8(a) and Figure 3.8(b), respectively; observe the appearance of the drifting packets.

In computing the Fourier transform of $\hat{\phi}$, we find that $\sup|\hat{\phi}| = 1$ and

$$\Omega(\phi) = \{\xi_1, \xi_2, \xi_3, \xi_4\} = \{(\pi/2, 3\pi/4), (\pi/2, -\pi/4), (-\pi/2, -3\pi/4), (-\pi/2, \pi/4)\},$$

where

$$\hat{\phi}(\xi_1) = \hat{\phi}(\xi_3) = (i)^{5/4} \quad \text{and} \quad \hat{\phi}(\xi_2) = \hat{\phi}(\xi_4) = -(i)^{5/4}.$$

Set $\gamma = \sqrt{2} - 1$ and

$$P(\eta, \zeta) = \frac{1 + i\gamma}{4}\eta^2 + \gamma\zeta^2.$$

As in the previous example, we expand the logarithm of $\hat{\phi}$ near ξ_j for $j = 1, 2, 3, 4$. We find that each element of $\Omega(\phi)$ is of positive homogeneous type for $\hat{\phi}$ with $\alpha_{\xi_1} = \alpha_{\xi_2} = (0, \gamma)$, $\alpha_{\xi_3} = \alpha_{\xi_4} = (0, -\gamma)$ and $P_{\xi_1} = P_{\xi_2} = P_{\xi_3} = P_{\xi_4} = P$. Note that P is obviously positive homogeneous with $E = (1/2)I \in \operatorname{Exp}(P)$ and hence

$$\mu_\phi = \mu_{P_{\xi_1}} = \mu_{P_{\xi_2}} = \mu_{P_{\xi_3}} = \mu_{P_{\xi_4}} = \mu_P = 1. \tag{3.83}$$

An appeal to Theorem 3.1.4 gives positive constants C and C' for which

$$Cn^{-1} \leq \|\phi^{(n)}\|_\infty \leq C'n^{-1}$$

for all $n \in \mathbb{N}_+$. In view of (3.83), let us note that the contribution from all points $\xi_1, \xi_2, \xi_3, \xi_4 \in \Omega(\phi)$ appear in the local limit given by Theorem 3.1.5. An application of the theorem gives

$$\begin{aligned}
\phi^{(n)}(x, y) &= (i)^{5n/4}\Big(e^{-i(x,y)\cdot\xi_1} H_P^n(x, y - n\gamma) + (-1)^n e^{i(x,y)\cdot\xi_2} H_P^n(x, y - n\gamma) \\
&\qquad + e^{-i(x,y)\cdot\xi_3} H_P^n(x, y + n\gamma) + (-1)^n e^{i(x,y)\cdot\xi_4} H_P^n(x, y + n\gamma)\Big) + o(n^{-1}) \\
&= (i)^{5n/4}\Big((-1)^y + (-1)^n\Big)\Big(e^{-i\pi x/2} e^{i\pi y/4} H_P^n(x, y - \gamma n) \\
&\qquad + e^{i\pi x/2} e^{i3\pi y/4} H_P^n(x, y + \gamma n)\Big) + o(n^{-1})
\end{aligned}$$

which holds uniformly for $(x, y) \in \mathbb{Z}^2$. In this special case that P is of second order, we can write

$$\begin{aligned} H_P^n(x,y) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-i(\eta,\zeta)\cdot(x,y)-nP(\eta,\zeta)}\,d\eta d\zeta \\ &= \frac{1}{2\pi n\sqrt{\gamma(1+i\gamma)}}\exp\left(-\frac{x^2}{n(1+i\gamma)}-\frac{y^2}{4n\gamma}\right) \end{aligned}$$

for $(x,y)\in\mathbb{R}^2$ and from this, it is easily seen that $\phi^{(n)}$ is approximated by two generalized Gaussian packets respectively centered at $\pm(0,\gamma n)$ for $n\in\mathbb{N}_+$. For comparison, Figure 3.8(c) and Figure 3.8(d) illustrate the approximation

$$\begin{aligned} f_n(x,y) &:= (i)^{5n/4}\Big((-1)^y+(-1)^n\Big) \\ &\qquad\times\Big(e^{-i\pi x/2}e^{i\pi y/4}H_P^n(x,y-\gamma n)+e^{i\pi x/2}e^{i3\pi y/4}H_P^n(x,y+\gamma n)\Big) \end{aligned}$$

to $\phi^{(n)}$ for $n=30$ and 60.

3.7.3 A supporting lattice misaligned with $\mathbb{Z}^2$

In this example, we study a real valued function ϕ whose support is not well-aligned with the principal coordinate axes. Here, the points at which $\hat{\phi}$ is maximized are of positive homogeneous type for $\hat{\phi}$ but the corresponding positive homogeneous polynomials are not semi-elliptic. In this way, we have a concrete example to illustrate the conclusion of Proposition 3.2.2. In writing out the local limit theorem for ϕ, we also see the appearance of a multiplicative prefactor which gives us information concerning the support of $\phi^{(n)}$. Finally, the validity of global space-time exponential-type estimates is discussed.

Consider $\phi : \mathbb{Z}^2 \to \mathbb{R}$ defined by

$$\phi(x,y) = \begin{cases} 3/8 & (x,y) = (0,0) \\ 1/8 & (x,y) = \pm(1,1) \\ 1/4 & (x,y) = \pm(1,-1) \\ -1/16 & (x,y) = \pm(2,-2) \\ 0 & \text{otherwise.} \end{cases}$$

Figures 3.9 and 3.10 illustrate the graph and heat map of $\phi^{(n)}$ respectively when $n = 100$.

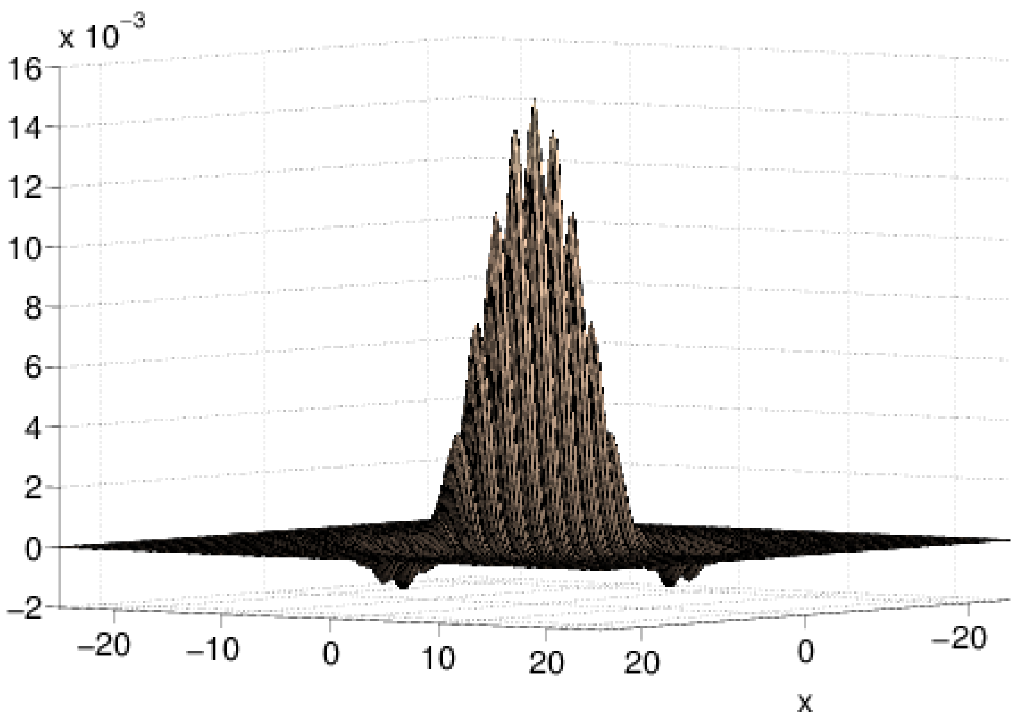

Figure 3.9: $\phi^{(n)}$ for $n = 100$

We compute the Fourier transform of ϕ and find by a routine calculation that $\sup|\hat{\phi}| = 1$ and this maximum is attained at only two points in $\mathbb{T}^2$, $(0,0)$ and

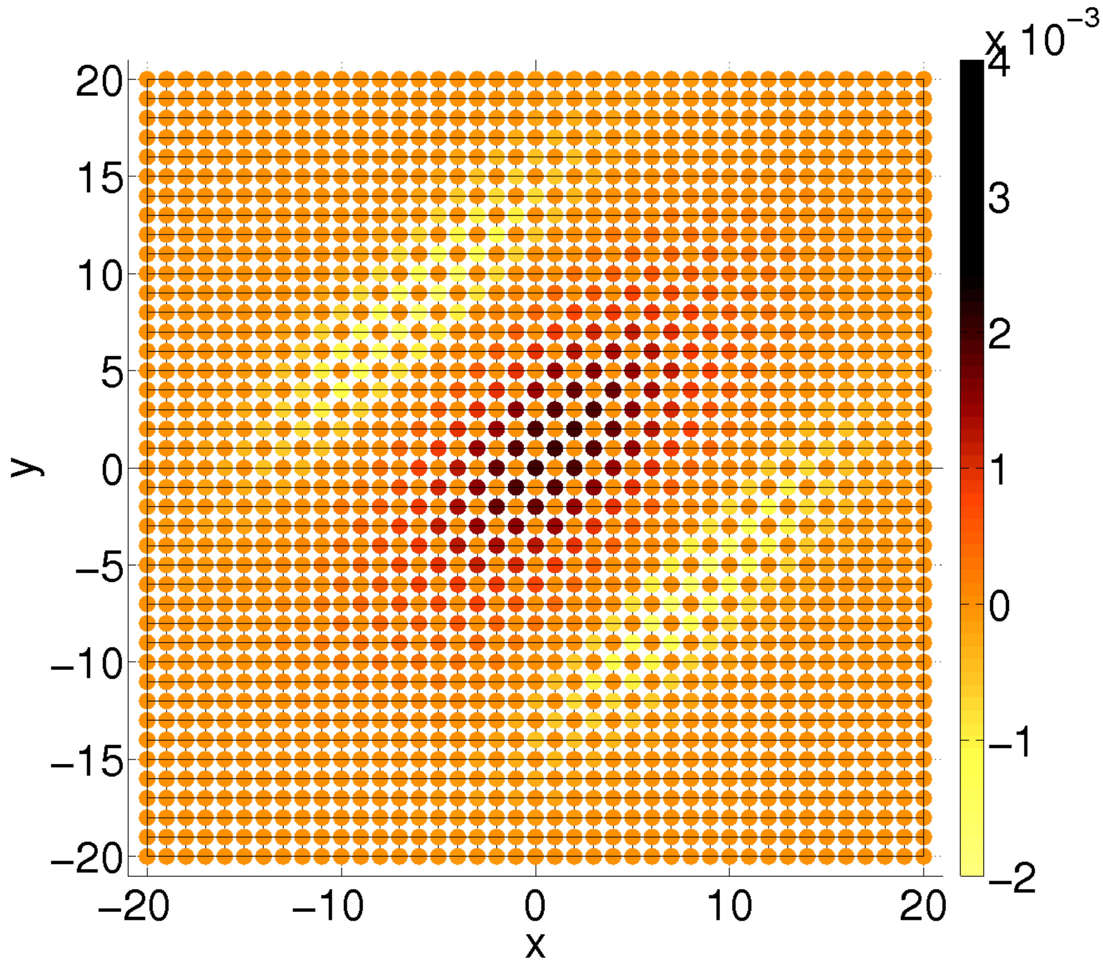

Figure 3.10: The heat map of $\phi^{(n)}$ for $n = 100$

(π, π). We write this as

$$\Omega(\phi) = \{\xi_1, \xi_2\} = \{(0,0), (\pi, \pi)\},$$

and note that $\phi(\xi_1) = \phi(\xi_2) = 1$. For $\xi_1 = (0,0)$, we have

$$\begin{aligned}
&\Gamma(\eta, \zeta) \\
&\quad = \log\left(\frac{\hat{\phi}(\xi + \xi_1)}{\phi(\xi_1)}\right) \\
&\quad = -\frac{\eta^2}{8} - \frac{23\eta^4}{384} - \frac{\eta\zeta}{4} + \frac{25\eta^3\zeta}{96} - \frac{\zeta^2}{8} - \frac{23\eta^2\zeta^2}{64} + \frac{25\eta\zeta^3}{96} - \frac{23\zeta^4}{384} + o(|(\eta, \zeta)|^4)
\end{aligned}$$

as $(\eta,\zeta)\to(0,0)$. In seeking a positive homogeneous polynomial to lead the expansion, we first note the appearance of the second order polynomial $\eta^2/8+\eta\zeta/4+\zeta^2/8$. We might be tempted to choose this as our candidate, however, it is not positive definite because it vanishes on the line $\eta=-\zeta$. Upon closer study, we write

$$\begin{aligned}\Gamma(\eta,\zeta) &= -\frac{1}{8}(\eta+\zeta)^2-\frac{23}{384}(\eta-\zeta)^4+o(|(\eta,\zeta)|^4)\\ &= -P(\eta,\zeta)+o(P(\eta,\zeta))\end{aligned}$$

as $(\eta,\zeta)\to(0,0)$, where the polynomial

$$P(\eta,\zeta)=\frac{1}{8}(\eta+\zeta)^2+\frac{23}{384}(\eta-\zeta)^4.$$

is positive definite. Fortunately, it is also a positive homogeneous polynomial as can be seen by observing that, for

$$E=\begin{pmatrix}3/8 & 1/8\\ 1/8 & 3/8\end{pmatrix},$$

$$\begin{aligned}P(t^E(\eta,\zeta)) &= P\left(t^{1/2}(\eta+\zeta)/2+t^{1/4}(\eta-\zeta)/2,t^{1/2}(\eta+\zeta)/2-t^{1/4}(\eta-\zeta)/2\right)\\ &= \frac{1}{8}\left(t^{1/2}(\eta+\zeta)\right)^2+\frac{23}{384}\left(t^{1/4}(\eta-\zeta)\right)^4\\ &= tP(\eta,\zeta)\end{aligned}$$

for all $t>0$ and $(\eta,\zeta)\in\mathbb{R}^2$. In contrast to the previous examples, P is not semi-elliptic. However, observe that

$$A^{-1}EA=\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}3/8 & 1/8\\ 1/8 & 3/8\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=\begin{pmatrix}1/2 & 0\\ 0 & 1/4\end{pmatrix}$$

and

$$(P\circ L_A)(\eta,\zeta)=P\left(\frac{\eta-\zeta}{\sqrt{2}},\frac{\eta+\zeta}{\sqrt{2}}\right)=\frac{1}{8}(\sqrt{2}\eta)^2+\frac{23}{384}(-\sqrt{2}\zeta)^4=\frac{1}{4}\eta^2+\frac{23}{96}\zeta^4$$

which is semi-elliptic; this illustrates the conclusion of Proposition 3.2.2.

We have shown that ξ_1 is of positive homogeneous type for $\hat{\phi}$ with corresponding $\alpha_{\xi_1} = (0,0)$ and positive homogeneous polynomial $P = P_{\xi_1}$. By expanding the logarithm of $\hat{\phi}$ near ξ_2, a similar argument shows that ξ_2 is also of positive homogeneous type for $\hat{\phi}$ with corresponding $\alpha_{\xi_2} = (0,0)$ and the same positive homogeneous polynomial $P = P_{\xi_2}$. It then follows immediately that ϕ meets they hypotheses of Theorems 3.1.4 and 3.1.5 where

$$\mu_\phi = \mu_P = \operatorname{tr} E = 3/4.$$

An appeal to Theorem 3.1.4 gives positive constants C and C' for which

$$C' n^{-3/4} \leq \|\phi^{(n)}\|_\infty \leq C n^{-3/4}$$

for all $n \in \mathbb{N}_+$. By an appeal to Theorem 3.1.5, we also have

$$\begin{aligned} \phi^{(n)}(x,y) &= \hat{\phi}(\xi_1)^n e^{-i\xi_1 \cdot (x,y)} H_P^n(x,y) + \hat{\phi}(\xi_2) e^{-i\xi_2 \cdot (x,y)} H_P^n(x,y) + o(n^{-3/4}) \\ &= \left(1 + e^{i\pi(x+y)}\right) H_P^n(x,y) + o(n^{-3/4}) \\ &= (1 + \cos(\pi(x+y))) H_P^n(x,y) + o(n^{-3/4}) \end{aligned} \tag{3.84}$$

uniformly for $(x,y) \in \mathbb{Z}^2$. Upon closely inspecting the prefactor $1 + \cos(\pi(x+y)$, it is reasonable to assert that

$$\operatorname{Supp}\left(\phi^{(n)}\right) \subseteq \{(x,y) \in \mathbb{Z}^2 : x \pm y \in 2\mathbb{Z}\} =: \mathcal{L}$$

for all $n \in \mathbb{N}_+$ (Figure 3.10 also gives evidence for this when $n = 100$). The assertion is indeed true, for it is easily verified that $\operatorname{Supp}(\phi) \subseteq \mathcal{L}$ and, because $\mathcal{L}$ is an additive group, induction shows that

$$\operatorname{Supp}\left(\phi^{(n+1)}\right) = \operatorname{Supp}\left(\phi^{(n)} * \phi\right) \subseteq \operatorname{Supp}\left(\phi^{(n)}\right) + \operatorname{Supp}(\phi) \subseteq \mathcal{L} + \mathcal{L} = \mathcal{L}$$

for all $n \in \mathbb{N}_+$. Thus, the prefactor $(1+\cos(\pi(x+y))$ gives us information about the support of the convolution powers. In Section 3.7.6, we will see that this situation is commonplace when ϕ is a probability distribution.

Let us finally note that, because $\alpha_{\xi_1} = \alpha_{\xi_2} = (0,0)$ and $P_{\xi_1} = P_{\xi_2} = P$, ϕ satisfies the hypotheses of Theorem 3.1.6. A straightforward computation shows that $R^{\#}(x,y) \asymp |x+y|^2 + |x-y|^{4/3}$ where $R = \operatorname{Re} P$ and so, by an appeal to Theorem 3.1.6, there are positive constants C and M for which

$$|\phi^{(n)}(x,y)| \leq \frac{C}{n^{3/4}} \exp\left(-M\left(\frac{|x+y|^2}{n} + \frac{|x-y|^{4/3}}{n^{1/3}}\right)\right)$$

for all $(x,y) \in \mathbb{Z}^2$ and $n \in \mathbb{N}_+$. We note however that because $\Omega(\phi) = \{\xi_1, \xi_2\}$, ϕ does not satisfy the hypotheses of Theorem 3.5.3 and, by closely inspecting Figure 3.9, this should come at no surprise. In fact, by a direct application of (3.84), it is easily shown that $|\phi^{(n)}(0,0)| \geq \epsilon n^{-3/4}$ for some $\epsilon > 0$ whereas $\phi^{(n)}(0,1) = 0$ for all $n \in \mathbb{N}_+$. Consequently, $|D_{(0,1)}\phi^{(n)}(0,0)| \geq \epsilon n^{-3/4}$ for all $n \in \mathbb{N}_+$ from which it is evident that the conclusion to Theorem 3.5.3, (3.60), doesn't hold.

3.7.4 Contribution from non-minimal decay exponent

In the present example, we study a real valued function ϕ on $\mathbb{Z}^2$ with $\Omega(\phi) = \{\xi_1, \xi_2\}$. Although both ξ_1 and ξ_2 are of positive homogeneous type for $\hat{\phi}$ with corresponding positive homogeneous polynomials P_{ξ_1} and P_{ξ_2}, we find that $\mu_\phi = \mu_{P_{\xi_1}} < \mu_{P_{\xi_2}}$ which is in contrast to the preceding examples. Consequently, only the contribution from ξ_1 appears in the local limit.

Consider $\phi : \mathbb{Z}^2 \to \mathbb{R}$ be defined by

$$\phi(x,y) = \begin{cases} 19/128 & (x,y) = (0,0) \\ 19/256 & (x,y) = (0,\pm 1) \\ 1/4 & (x,y) = (\pm 1, 0) \\ 1/8 & (x,y) = (\pm 1, \pm 1) \\ -5/64 & (x,y) = (\pm 2, 0) \\ -5/128 & (x,y) = (\pm 2, \pm 1) \\ 1/256 & (x,y) = (\pm 4, 0) \\ 1/512 & (x,y) = (\pm 4, \pm 1) \\ 0 & \text{otherwise.} \end{cases} \tag{3.85}$$

The graphs of $\phi^{(n)}$ for $(x,y) \in \mathbb{Z}^2$ such that $-100 \leq x, y \leq 100$ are displayed in Figures 3.11(a) and 3.11(c) for $n = 100$ and Figures 3.12(a) and 3.12(c) for $n = 1,000$. Upon considering the Fourier transform of ϕ, we find that $\sup |\hat{\phi}| = 1$ and this maximum is attained at exactly two points in $\mathbb{T}^2$. Specifically,

$$\Omega(\phi) = \{\xi_1, \xi_2\} = \{(0,0), (\pi, 0)\},$$

where $\hat{\phi}(\xi_1) = 1$ and $\hat{\phi}(\xi_2) = -1$. In expanding the logarithm of $\hat{\phi}(\xi + \xi_1)/\hat{\phi}(\xi_1)$ about $(0,0)$, we find that $\xi_1 = (0,0)$ is of positive homogeneous type for $\hat{\phi}$ with $\alpha_{\xi_1} = (0,0)$ and

$$P_{\xi_1}(\eta, \zeta) = \frac{\eta^6}{16} + \frac{\zeta^2}{4}.$$

Clearly P_{ξ_1} is positive homogeneous with $E_1 = \operatorname{diag}(1/6, 1/2) \in \operatorname{Exp}(P_{\xi_1})$ thus $\mu_{P_{\xi_1}} = \operatorname{tr} E_1 = 2/3$. Now, upon expanding the logarithm of $\hat{\phi}(\xi + \xi_2)/\hat{\phi}(\xi_2)$ we find that $\xi_2 = (\pi, 0)$ is also of positive homogeneous type for $\hat{\phi}$ with $\alpha_{\xi_2} = (0,0)$

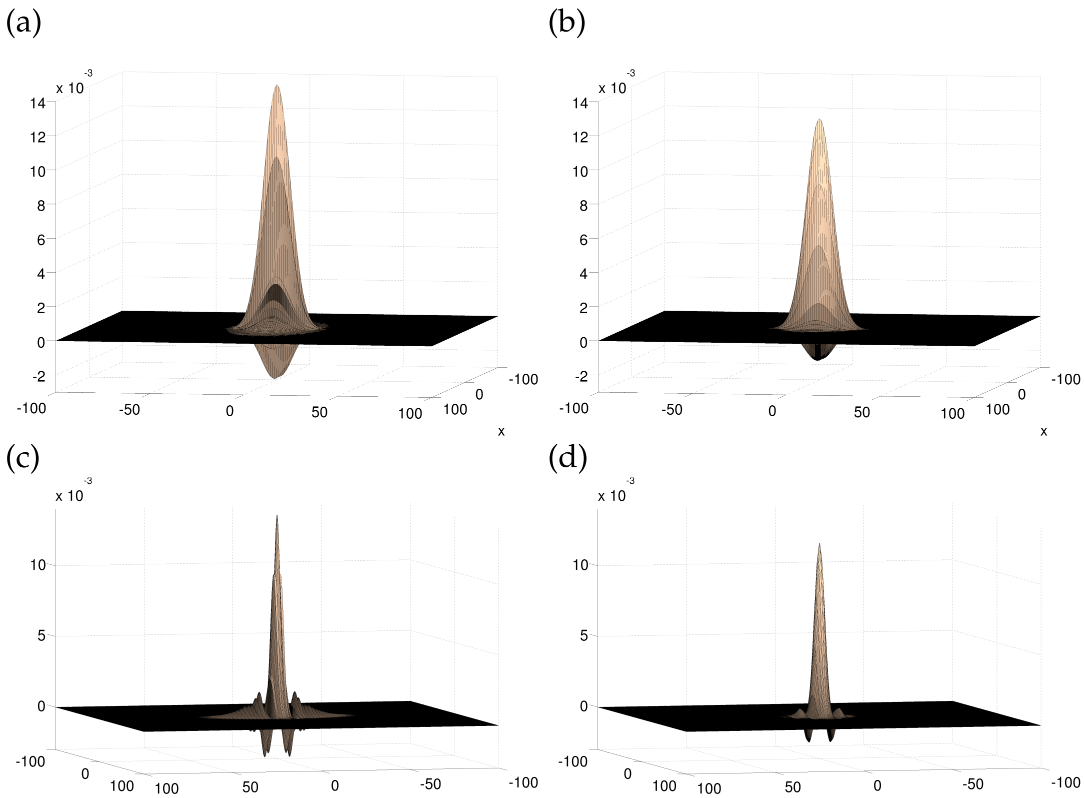

Figure 3.11: $\phi^{(n)}$, (a) and (c), $H^n_{P_{\xi_1}}$, (b) and (d), for $n = 100$

and positive homogeneous polynomial

$$P_{\xi_2}(\eta, \zeta) = \eta^2 + \frac{\zeta^2}{4};$$

Here, $E_2 = (1/2)I \in \operatorname{Exp}(P_{\xi_2})$ and thus $\mu_{P_{\xi_2}} = \operatorname{tr} E_2 = 1$. In this case

$$\mu_\phi = \min_{i=1,2} \mu_{P_{\xi_i}} = \mu_{P_{\xi_1}} = 2/3$$

and so, in light of the paragraph preceding the statement of Theorem 3.1.5, we restrict our attention to ξ_1, in which case the theorem describes the approximation of $\phi^{(n)}$ by a single attractor $H_{P_{\xi_1}}$. This is the local limit

$$\phi^{(n)}(x, y) = H^n_{P_{\xi_1}}(x, y) + o(n^{-2/3}) \tag{3.86}$$

which holds uniformly for $(x, y) \in \mathbb{Z}^2$. Figures 3.11(b), 3.11(d), 3.12(b) and 3.12(d) illustrate this result. It should be noted that the approximations shown

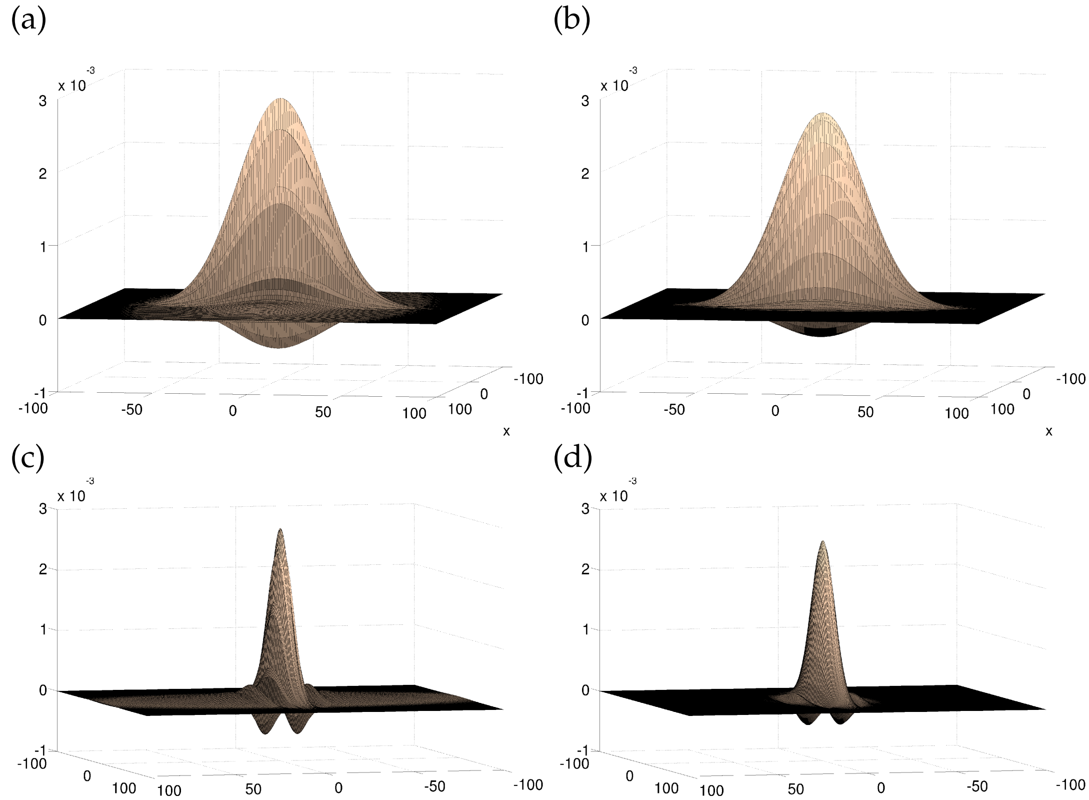

Figure 3.12: $\phi^{(n)}$, (a) and (c), $H^n_{P_{\xi_1}}$, (b) and (d), for $n = 1,000$

in Figures 3.11 and 3.12 appear more coarse than those of the previous examples. For instance, Figure 3.11(c) depicts minor oscillations in the graph of $\phi^{(n)}$ which do not appear in the approximation illustrated in Figure 3.11(d). These oscillations are due to the influence on $\phi^{(n)}$ by $\hat{\phi}$ near ξ_2 which for $n = 1,000$ is not yet sufficiently scaled out. As demonstrated in the proof of Theorem 3.1.5, this influence dies out on the relative order of $n^{1-2/3} = n^{-1/3}$ and thus the influence is not negligible when $n = 1,000$.

As a final remark, we note that ϕ is the tensor product of two functions mapping

$\mathbb{Z}$ into $\mathbb{C}$. Specifically, $\phi = \phi_1 \otimes \phi_2$ where,

$$\phi_1(x) = \begin{cases} 19/64 & x = 0 \\ 1/2 & x = \pm 1 \\ -5/32 & x = \pm 2 \\ 1/128 & x = \pm 4 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \phi_2(y) = \begin{cases} 1/2 & y = 0 \\ 1/4 & y = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

In fact, had we studied the functions ϕ_1 and ϕ_2 separately, we would have found that

$$\phi_1^{(n)}(x) = H^n_{\eta^6/16}(x) + o(n^{-1/6}) \qquad \text{and} \qquad \phi_1^{(n)}(y) = H^n_{\zeta^2/4}(y) + o(n^{-1/2})$$

uniformly for $x, y \in \mathbb{Z}$ and from this deduced the local limit (3.86) because $\phi^{(n)} = \phi_1^{(n)} \otimes \phi_2^{(n)}$ and $H_{P_{\xi_1}} = H_{\eta^6/16} \otimes H_{\zeta^2/4}$ (note also that $\mu_\phi = 1/6 + 1/2 = \mu_{\phi_1} + \mu_{\phi_2}$). In general, tensor products can be used to create a wealth of examples in any dimension to which the results of lower dimensions can be applied. For instance, by applying the much stronger theory of one dimension (in light of [73]), one can deduce stronger results than are given here for the class of finitely supported functions on $\mathbb{Z}^d$ of the form

$$\phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_d$$

where $\phi_k : \mathbb{Z} \mapsto \mathbb{C}$ is finitely supported for $k = 1, 2, \ldots, d$. How to place these examples in a d-dimensional theory is an open question.

3.7.5 A simple class of real valued functions

In this subsection we consider a class of real valued and finitely supported functions $\phi_{\mathbf{m},\lambda}$ determined by two multi-parameters $\mathbf{m} \in \mathbb{N}_+$ and $\lambda \in \mathbb{R}^d_+$, c.f, Subsection 2.4 of [31]. Here, $\Omega(\phi_{\mathbf{m},\lambda})$ contains only $0 \in \mathbb{T}^d$ which is of positive homogeneous type for $\hat{\phi}_{\mathbf{m},\lambda}$ with no drift and whose associated positive homogeneous polynomial is a semi-elliptic polynomial with no "mixed" terms. In this setting, our methods yield easily ℓ^∞-asymptotics and local limit theorems for $\phi_{\mathbf{m},\lambda}^{(n)} = (\phi_{\mathbf{m},\lambda})^{(n)}$. Moreover, all of the results of Section 3.5 concerning global space-estimates for $\phi_{\mathbf{m},\lambda}^{(n)}$ and its discrete differences are valid and we apply them.

Let $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_+^d$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ be such that $\lambda_j \in (0, 2^{1-m_j}/d]$ for $j = 1, 2, \dots, d$ with at least one $\lambda_j < 2^{1-m_j}/d$. Define

$$\phi_{\mathbf{m},\lambda} = \delta_0 - \sum_{j=1}^{d} \lambda_j (\delta_0 - \rho_j)^{(m_j)} \tag{3.87}$$

where $\rho_j = (1/2)(\delta_{e_j} + \delta_{-e_j})$ is the Bernoulli walk on the jth coordinate axis. By a straightforward computation, we have

$$\hat{\phi}_{\mathbf{m},\lambda}(\xi) = 1 - \sum_{j=1}^{d} \lambda_j (1 - \cos(\xi_j))^{m_j}$$

for $\xi = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{R}^d$ and from this it is easily seen that $\sup_\xi |\hat{\phi}_{\mathbf{m},\lambda}(\xi)| = 1$ which is attained only at $0 \in \mathbb{T}^d$, i.e., $\Omega(\phi_{\mathbf{m},\lambda}) = \{0\}$. Here, $\hat{\phi}_{\mathbf{m},\lambda}(0) = 1$ and it is easily seen that

$$\Gamma(\xi) = \log(\hat{\phi}_{\mathbf{m},\lambda}(\xi)) = -P_{\mathbf{m},\lambda}(\xi) + o(P_{\mathbf{m},\lambda}(\xi))$$

as $\xi \to 0$, where

$$P_{\mathbf{m},\lambda}(\xi) = \sum_{j=1}^{d} \frac{\lambda_j}{2^{m_j}} \xi_j^{2m_j}$$

for $\xi = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{R}^d$. Note that $P_{\mathbf{m},\lambda}(\xi)$ is a semi-elliptic polynomial of the form (3.18) with $D_{\mathbf{m}} = \operatorname{diag}((2m_1)^{-1}, (2m_2)^{-1}, \dots, (2m_d)^{-1}) \in \operatorname{Exp}(P_{\mathbf{m},\lambda})$ and hence

$$\mu_{\phi_{\mathbf{m},\lambda}} = \mu_{P_{\mathbf{m},\lambda}} = (2m_1)^{-1} + (2m_2)^{-1} + \cdots + (2m_d)^{-1} = |\mathbf{1} : 2\mathbf{m}|,$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{N}^d$.

For any $l \in \mathbb{N}$, recall from Section 3.5 the discrete time difference operator $\partial_l = \partial_l(\phi_{\mathbf{m},\lambda}, \xi_0, \alpha)$ which, in this case, is given by

$$\partial_l \psi = (\delta - \phi_{\mathbf{m},\lambda}^{(l)}) * \psi$$

for $\psi \in \ell^1(\mathbb{Z}^d)$. For any multi-index $\beta \in \mathbb{N}^d$, consider the difference operator $D^\beta = D_e^\beta$ defined for any $\psi \in \ell^1(\mathbb{Z}^d)$ by

$$D^\beta \psi = (D_{e_1})^{\beta_1} (D_{e_2})^{\beta_2} \cdots (D_{e_d})^{\beta_d} \psi$$

where $D_{e_j}\psi(x) = \psi(x + e_j) - \psi(x)$ for $x \in \mathbb{Z}^d$ and $(D_{e_j})^0$ is the identity. We note that $e = \{e_1, e_2, \ldots, e_d\}$ is $P_{\mathbf{m},\lambda}$-fitted of weight $\mathbf{m}$ in view of the discussion preceding Corollary 3.5.6. Finally, define

$$|x|_{\mathbf{m}} = \sum_{j=1}^{d} |x_j|^{2m_i/(2m_i-1)} \tag{3.88}$$

for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and observe that

$$|n^{-D_{\mathbf{m}}} x|_{\mathbf{m}} = \sum_{j=1}^{d} |x_j|^{2m_j/(2m_j-1)} / n^{1/(2m_j-1)}$$

for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_+$.

Proposition 3.7.1. *Let $\phi_{\mathbf{m},\lambda}$ be defined by* (3.87)*, assume the notation above and write $(\phi_{\mathbf{m},\lambda})^{(n)} = \phi_{\mathbf{m},\lambda}^{(n)}$ for $n \in \mathbb{N}_+$. There are positive constants C and C' for which*

$$Cn^{-|\mathbf{1}:2\mathbf{m}|} \leq \|\phi_{\mathbf{m},\lambda}^{(n)}\|_\infty \leq C' n^{-|\mathbf{1}:2\mathbf{m}|} \tag{3.89}$$

for all $n \in \mathbb{N}_+$. We have

$$\begin{aligned}
&\phi^{(n)}_{\mathbf{m},\lambda}(x)\\
&\quad= n^{-|\mathbf{1}:\mathbf{2m}|}H_{P_{\mathbf{m},\lambda}}\left(n^{-D_{\mathbf{m}}}x\right)+o(n^{-|\mathbf{1}:\mathbf{2m}|})\\
&\quad= n^{-|\mathbf{1}:\mathbf{2m}|}H_{P_{\mathbf{m},\lambda}}\left(\frac{x_1}{n^{1/(2m_1)}},\frac{x_2}{n^{1/(2m_2)}},\dots,\frac{x_d}{n^{1/(2m_d)}}\right)+o(n^{-|\mathbf{1}:\mathbf{2m}|}) \qquad (3.90)
\end{aligned}$$

uniformly for $x=(x_1,x_2,\dots,x_d)\in\mathbb{Z}^d$, where $H_{P_{\mathbf{m},\lambda}}$ is defined by (3.8)*. There are positive constants C_0, C_1, M_0 and M_1 for which*

$$|\phi^{(n)}_{\mathbf{m},\lambda}(x)| \leq \frac{C_0}{n^{|\mathbf{1}:\mathbf{2m}|}}\exp\left(-M_0\left|n^{-D_{\mathbf{m}}}x\right|_{\mathbf{m}}\right) \qquad (3.91)$$

and

$$|\phi^{(n+1)}_{\mathbf{m},\lambda}(x)-\phi^{(n)}_{\mathbf{m},\lambda}(x)| \leq \frac{C_1}{n^{|\mathbf{1}+\mathbf{2m}:\mathbf{2m}|}}\exp\left(-M_1\left|n^{-D_{\mathbf{m}}}x\right|_{\mathbf{m}}\right) \qquad (3.92)$$

for all $x\in\mathbb{Z}^d$ and $n\in\mathbb{N}_+$. Further, there are positive constants C_0 and M and, to each multi-index β, a positive constant C_β such that, for any $l_1,l_2,\dots,l_k\in\mathbb{N}^d_+$,

$$|\partial_{l_1}\partial_{l_2}\cdots\partial_{l_j}D^\beta\phi^{(n)}_{\mathbf{m},\lambda}(x)| \leq \frac{C_\beta C_0^k k!\prod_{q=1}^k l_q}{n^{|\mathbf{1}+\beta+2k\mathbf{m}:\mathbf{2m}|}}\exp\left(-M\left|(n+s_k)^{-D_{\mathbf{m}}}x\right|_{\mathbf{m}}\right) \qquad (3.93)$$

for all $x\in\mathbb{Z}^d$ and $n\in\mathbb{N}_+$, where $s_k=l_1+l_2+\cdots+l_k$.

Remark 10. *For simplicity, we have not treated the critical case in which $\lambda_j = 2^{1-m_j}/d$ for $j=1,2,\dots,d$ in the proposition above, however, our methods handle this easily. In this case, the local limit* (3.90) *instead contains the prefactor $1+\exp(i\pi(n-x_1-x_2-\cdots-x_d))$. The estimate* (3.92) *is also valid here but* (3.93) *and* (3.91) *fail to hold (for reasons similar to those of Subsection 3.7.3).*

Proof. In view of the discussion proceeding the proposition, straightforward applications of Theorems 3.1.4 and 3.1.5 yield (3.89) and (3.90) respectively. To see the global space-time estimates, we first observe that

$$\begin{aligned}
&P^{\#}_{\mathbf{m},\lambda}(x_1,x_2,\dots,x_d)\\
&\quad= \sum_{j=1}^d\left(\frac{2^{m_j}}{\lambda_j}\right)^{1/(2m_j-1)}\left(\left(\frac{1}{2m_j}\right)^{1/(2m_j-1)}-\left(\frac{1}{2m_j}\right)^{2m_j/(2m_j-1)}\right)|x_j|^{2m_j/(2m_j-1)}
\end{aligned}$$

for $x = (x_1, x_2, \dots, x_d)$. From this it is easily checked that $|\cdot|_{\mathbf{m}} \asymp P^{\#}_{\mathbf{m},\lambda}$ (this can also be seen with the help of Corollary A.3.3). Using the fact $0 \in \Omega(\phi_{\mathbf{m},\lambda})$ has corresponding $\alpha_0 = 0$ and $P_0 = P_{\mathbf{m},\lambda}$ which is semi-elliptic, $\phi_{\mathbf{m},\lambda}$ meets hypotheses of Theorem 3.1.6, Corollary 3.5.8 and Theorem 3.5.7. The estimates (3.91) follows immediately from Theorem 3.1.6. Upon noting that $\mu_\phi + 1 = |\mathbf{1} : 2\mathbf{m}| + |2\mathbf{m} : 2\mathbf{m}| = |\mathbf{1} + 2\mathbf{m} : 2\mathbf{m}|$, (3.92) follows from Corollary 3.5.8. Finally, the estimate (3.93) follows from Theorem 3.5.9 once it is observed that $\mu_\phi + |\beta : 2\mathbf{m}| + k = |\mathbf{1} + \beta + 2k\mathbf{m} : 2\mathbf{m}|$, $e = \{e_1, e_2, \dots, e_d\}$ is $P_{\mathbf{m},\lambda}$-fitted with weight $\mathbf{m}$ and $\prod_{j=1} |e_j|^{\beta_j} = 1$. □

3.7.6 Random walks on $\mathbb{Z}^d$: A look at the classical theory

In this short subsection, we revisit the classical theory of random walks on $\mathbb{Z}^d$. We denote by $\mathcal{M}_d^1$, the set functions $\phi : \mathbb{Z}^d \to [0,1]$ satisfying

$$\|\phi\|_1 = \sum_{x \in \mathbb{Z}^d} \phi(x) = 1,$$

i.e., $\mathcal{M}_d^1$ is the set of probability distributions on $\mathbb{Z}^d$. As discussed in the introduction, each $\phi \in \mathcal{M}_d^1$ drives a random walk on $\mathbb{Z}^d$ whose nth-step transition kernel k_n is given by $k_n(x,y) = \phi^{(n)}(y-x)$ for $x, y \in \mathbb{Z}^d$. Taking our terminology from [83, p. 72], we say that $\phi \in \mathcal{M}_d^1$ is *genuinely d-dimensional* if $\operatorname{Supp}(\phi)$ is not contained in any $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$; in this case, we also say that the associated random walk is genuinely d-dimensional. Our main focus throughout this subsection is on subset of $\phi \in \mathcal{M}_d^1$ which are genuinely d-dimensional with finite second moments. In contrast to the standard literature, we make no assumptions concerning periodicity/aperiodicity/irreducibility, c.f., [63, 83]. In this generality, our formulation of the (classical) local limit theorem, Theorem 3.7.5, naturally contains a prefactor Θ which nicely describes the support of $\phi^{(n)}$ and hence the random walk's periodic structure.

Our first two results, Lemma 3.7.2 and Proposition 3.7.3 are stated for the general class of $\phi \in \mathcal{M}_d^1$; one should note that both results fail to hold in the case that ϕ is generally complex valued. The lemma and proposition highlight the importance of the set $\Omega(\phi)$ and, in particular, its inherent group structure. This intrinsic structure (and much more) was also recognized by B. Schreiber in his study of (complex valued) measure algebras on locally compact abelian groups [81]. In fact, Schreiber's results can be used to prove Lemma 3.7.2 and Proposition 3.7.3; although, in our context, the proofs are straightforward and so we proceed directly.

Lemma 3.7.2. *Let $\phi \in \mathcal{M}_d^1$. Then $\Omega(\phi)$ depends only on $\operatorname{Supp}(\phi)$ in the sense that, if $\operatorname{Supp}(\phi_1) = \operatorname{Supp}(\phi_2)$ for $\phi_1, \phi_2 \in \mathcal{M}_d^1$, then $\Omega(\phi_1) = \Omega(\phi_2)$. Furthermore, for each $\xi \in \Omega(\phi)$, there exists $\omega(\xi) \in (-\pi, \pi]$ such that*

$$\hat{\phi}(\xi) = e^{i\omega(\xi)} = e^{ix\cdot\xi}$$

for all $x \in \operatorname{Supp}(\phi)$.

Proof. We shall use the following general property of complex numbers. If $\{z_1, z_2, \dots\} \subseteq \mathbb{C}$ satisfy

$$\sum_{k}^{\infty} |z_k| = 1 = \Big|\sum_{k=1}^{\infty} z_k\Big|$$

then, for some $\omega \in (-\pi, \pi]$, $z_k = r_k e^{i\omega}$ for all k. Thus, whenever $\xi \in \Omega(\phi)$, i.e.,

$$|\hat{\phi}(\xi)| = \Big|\sum_{x\in\mathbb{Z}^d} \phi(x) e^{ix\cdot\xi}\Big| = 1,$$

there exists $\omega = \omega(\xi) \in (-\pi, \pi]$ for which

$$e^{ix\cdot\xi} = e^{i\omega(\xi)} \tag{3.94}$$

for all $x \in \mathrm{Supp}(\phi)$. In particular, this shows that $\Omega(\phi)$ depends only on $\mathrm{Supp}(\phi)$. Further, observe that

$$\hat{\phi}(\xi) = \sum_{x\in\mathbb{Z}^d} \phi(x)e^{ix\cdot\xi} = e^{i\omega(\xi)} \sum_{x\in\mathbb{Z}^d} \phi(x) = e^{i\omega(\xi)} \tag{3.95}$$

and so the result follows upon combining (3.94) and (3.95). $\square$

Proposition 3.7.3. *Let $\phi \in \mathcal{M}_d^1$. Then $\Omega(\phi)$ is a subgroup of $\mathbb{T}^d$ and*

$$\hat{\phi}\Big|_{\Omega(\phi)} : \Omega(\phi) \to \mathbb{S}^1$$

is a homomorphism of groups; here, $\mathbb{T}^d$ is taken to have the canonical group structure and $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Proof. It is obvious that $0 \in \Omega(\phi)$; hence $\Omega(\phi)$ is non-empty. Let $\xi_1, \xi_2 \in \Omega(\phi)$ and, in view of Lemma 3.7.2,

$$\begin{aligned}
\hat{\phi}(\xi_2 - \xi_1) &= \sum_{x\in\mathrm{Supp}(\phi)} \phi(x)e^{ix\cdot(\xi_1-\xi_2)} = \sum_{x\in\mathrm{Supp}(\phi)} \phi(x)\hat{\phi}(\xi_2)\hat{\phi}(\xi_1)^{-1} \\
&= \hat{\phi}(\xi_2)\hat{\phi}(\xi_1)^{-1}\|\phi\|_1 = \hat{\phi}(\xi_2)\hat{\phi}(\xi_1)^{-1}
\end{aligned}$$

and thus $\xi_2 - \xi_1 \in \Omega(\phi)$ because $|\hat{\phi}(\xi_2 - \xi_1)| = |\hat{\phi}(\xi_2)\hat{\phi}(\xi_1)^{-1}| = 1$. As $\Omega(\phi)$ is non-empty and closed under subtraction, we conclude at once that $\Omega(\phi)$ is a subgroup of $\mathbb{T}^d$ and the restriction of $\hat{\phi}$ to $\Omega(\phi)$ is a homomorphism. $\square$

We now begin to develop what is needed to recapture and reformulate the classical local limit theorem in the general case that $\phi \in \mathcal{M}_d^1$ is genuinely d-dimensional and has finite second moments. In this case, the *mean* $\alpha_\phi \in \mathbb{R}^d$ and *covariance* $C_\phi \in \mathrm{M}_d(\mathbb{R})$ of ϕ are defined respectively by

$$\{\alpha_\phi\}_k = \sum_{x\in\mathbb{Z}^d} x_k\phi(x) \qquad \text{for } k = 1,2,\ldots,d$$

and

$$\{C_\phi\}_{k,l} = \sum_{x\in\mathbb{Z}^d} (x_k - \{\alpha_\phi\}_k)(x_l - \{\alpha_\phi\}_l)\phi(x) \qquad \text{for } k,l = 1,2,\ldots,d.$$

Proposition 3.7.4. *Let $\phi \in \mathcal{M}_d^1$ be genuinely d-dimensional with finite second moments and let α_ϕ and C_ϕ be the mean and covariance of ϕ as defined above. Set*

$$P_\phi(\xi) = \frac{1}{2}\xi \cdot C_\phi \xi$$

for $\xi \in \mathbb{R}^d$. Then each ξ_0 is of positive homogeneous type for $\hat{\phi}$ with $\alpha_{\xi_0} = \alpha_\phi$ and positive homogeneous polynomial $P_{\xi_0} = P_\phi$. In particular, $\mu_\phi = \mu_{P_\phi} = d/2$.

Proof. When ϕ is genuinely d-dimensional, it is well-known that the covariance form

$$\xi \mapsto \mathrm{Cov}(\phi)(\xi) = \xi \cdot C_\phi \xi$$

is positive definite (when $\alpha_\phi = 0$, $\mathrm{Supp}(\phi)$ contains a basis of $\mathbb{R}^d$ and when $\alpha_\phi \neq 0$, an appropriate shift does the trick). Upon noting that $2^{-1}I \in \mathrm{Exp}(P_\phi)$, we conclude that P_ϕ is a positive homogeneous polynomial. Observe that, for $\Gamma(\xi) = \log(\hat{\phi}(\xi+\xi_0)/\hat{\phi}(\xi_0))$,

$$\begin{aligned}
\partial_k\Gamma(0) &= \frac{\partial_k\hat{\phi}(\xi_0)}{\hat{\phi}(\xi_0)} = \frac{1}{\hat{\phi}(\xi_0)}\sum_{x\in\mathrm{Supp}(\phi)} ix_k\phi(x)e^{ix\cdot\xi_0} \\
&= \frac{1}{\hat{\phi}(\xi_0)}\sum_{x\in\mathrm{Supp}(\phi)} ix_k\phi(x)e^{i\omega(\xi_0)} = \frac{e^{i\omega(\xi_0)}}{\hat{\phi}(\xi_0)}\sum_{x\in\mathrm{Supp}(\phi)} ix_k\phi(x) \\
&= i\{\alpha_\phi\}_k
\end{aligned}$$

for all $k = 1,2\ldots d$, where we have used Lemma 3.7.2. By analogous reasoning, which again makes use of the lemma, $\partial_{k,l}\Gamma(0) = -\{C_\phi\}_{k,l}$ for $k,l = 1,2,\ldots,d$.

Consequently,

$$\Gamma(\xi) = \sum_{k=1}^{d} \partial_k \Gamma(0)\xi_k + \sum_{k,l=1}^{d} \frac{1}{2}\partial_{k,l}\Gamma(0)\xi_k\xi_l + o(|\xi|^2) = i\alpha_\phi \cdot \xi - P_\phi(\xi) + o(|\xi|^2), \tag{3.96}$$

as $\xi \to 0$, where we have used the positive definiteness P_ϕ to rewrite the error. From this it follows immediately that ξ_0 is of positive homogeneous type for $\hat{\phi}$ with $\alpha_{\xi_0} = \alpha_\phi$ and positive homogeneous polynomial $P_{\xi_0} = P_\phi$. □

We now present the classical local limit theorem in a new form. Assuming the notation of the previous proposition, the attractor $G_\phi = H_{P_\phi}$ which appears below is the generalized Gaussian density given by (3.2), see [63, p. 25]. Let us also note that, in view of the previous proposition and Proposition 3.4.1, $\Omega(\phi)$ is finite.

Theorem 3.7.5. *Let $\phi \in \mathcal{M}_d^1$ be genuinely d-dimensional with finite second moments. Then there exists positive constants C and C' for which*

$$Cn^{-d/2} \leq \sup_{x\in\mathbb{Z}^d} \phi^{(n)}(x) \leq C'n^{-d/2} \tag{3.97}$$

for all $n \in \mathbb{N}_+$. Furthermore,

$$\phi^{(n)}(x) = n^{-d/2}\Theta(n,x)G_\phi\left(\frac{x - n\alpha_\phi}{\sqrt{n}}\right) + o(n^{-d/2}) \tag{3.98}$$

uniformly for $x \in \mathbb{Z}^d$, where $\Theta : \mathbb{N}_+ \times \mathbb{Z}^d$ is dependent only on $\mathrm{Supp}(\phi)$ in the sense of Lemma 3.7.2 and is given (equivalently) by

$$\Theta(n,x) = \sum_{\xi\in\Omega(\phi)} e^{i(n\omega(\xi) - x\cdot\xi)} = \sum_{\xi\in\Omega(\phi)} \cos(n\omega(\xi) - x\cdot\xi); \tag{3.99}$$

here, $\omega(\xi) \in (-\pi,\pi]$ is that given by Lemma 3.7.2 for each $\xi \in \Omega(\phi)$.

Proof. The hypotheses of the present theorem are weaker than those of Theorems 3.1.4 and 3.1.5 as the latter theorems require ϕ to have finite moments of all orders. However, what is really used in the proof of the Theorem 3.1.5 is the condition that, for each $\xi_0 \in \Omega(\phi)$, Γ_{ξ_0} can be written in the form (3.96) where $P_{\xi_0} = P_\phi$ is positive definite (in the general case that ϕ is complex valued, it is not known a priori how many terms in the Taylor expansion for Γ_{ξ_0} are needed for this to be true). Under the present hypotheses and in view of Proposition 3.7.4, the proof of Theorem 3.1.5 pushes through with no modification and so we apply it (or simply its conclusion). As an immediate consequence, we obtain (3.97) because Theorem 3.1.4 follows directly from Theorem 3.1.5. It remains to show that the local limit yielded by Theorem 3.1.5 can be written in the form (3.98).

By virtue of Proposition 3.7.4, we have $\alpha_\xi = \alpha_\phi$, $P_\xi = P_\phi$ for all $\xi \in \Omega(\phi)$ and, moreover $\mu_\phi = \mu_P = d/2$. Noting that all $\xi \in \Omega(\phi)$ have corresponding positive homogeneous polynomials of the same order (because the polynomials are identical), all appear in the local limit. Consequently,

$$\begin{aligned}
\phi^{(n)}(x) &= \sum_{\xi\in\Omega(\phi)} e^{-ix\cdot\xi}(\hat{\phi}(\xi))^n H^n_{P_\phi}(x - n\alpha_\phi) + o(n^{-d/2}) \\
&= \left(\sum_{\xi\in\Omega(\phi)} e^{-ix\cdot\xi}(\hat{\phi}(\xi))^n\right) n^{-d/2} H_{P_\phi}\left(n^{-1/2}(x - n\alpha_\phi)\right) + o(n^{-d/2}) \\
&= n^{-d/2}\left(\sum_{\xi\in\Omega(\phi)} e^{i(n\omega(\xi)-x\cdot\xi)}\right) G_\phi\left(n^{-1/2}(x - n\alpha_\phi)\right) + o(n^{-d/2}) \\
&= n^{-d/2}\Theta(n,x) G_\phi\left(\frac{x - n\alpha_\phi}{\sqrt{n}}\right) + o(n^{-d/2})
\end{aligned}$$

uniformly for $x \in \mathbb{Z}^d$. In view of Lemma 3.7.2, it is clear that Θ depends only on $\mathrm{Supp}(\phi)$ and so to complete the proof, we need only to verify the second equality in (3.99). Using the fact that $\Omega(\phi)$ is a subgroup of $\mathbb{T}^d$ in view of Proposition 3.7.3, for each $\xi \in \Omega(\phi)$, $-\xi \in \Omega(\phi)$ and therefore

$$\Theta(n,x) = \frac{1}{2}\left(\sum_{\xi\in\Omega(\phi)} e^{i(n\omega(\xi)-x\cdot\xi)} + \sum_{\xi\in\Omega(\phi)} e^{i(n\omega(-\xi)-x\cdot(-\xi))}\right) = \sum_{\xi\in\Omega(\phi)} \cos(n\omega(\xi)-x\cdot\xi)$$

where we have noted that $\omega(-\xi) = -\omega(\xi)$ for each $\xi \in \Omega(\phi)$. $\square$

By close inspection of the theorem, one expects that in general Θ can help us describe the support of $\phi^{(n)}$ and hence the periodicity of the associated random walk. This turns out to be the case as our next theorem shows.

Theorem 3.7.6. *Let $\phi \in \mathcal{M}_d^1$ be genuinely d-dimensional with finite second moments and define $\Theta : \mathbb{N}_+ \times \mathbb{Z}^d \to \mathbb{R}$ by* (3.99). *Then*

$$\operatorname{Supp}\left(\phi^{(n)}\right) \subseteq \operatorname{Supp}(\Theta(n,\cdot)) \tag{3.100}$$

for all $n \in \mathbb{N}_+$. Further, if

$$\limsup_n |\Theta(n, x + \lfloor n\alpha_\phi \rfloor)| > 0$$

for $x \in \mathbb{Z}^d$, then

$$\limsup_n n^{\mu_\phi} \phi^{(n)}(x + \lfloor n\alpha_\phi \rfloor) > 0.$$

Proof. In view of Lemma 3.7.2, for any $x_0 \in \operatorname{Supp}(\phi)$, $\omega(\xi) = x_0 \cdot \xi$ for all $\xi \in \Omega(\phi)$. Therefore, for any $x_0 \in \operatorname{Supp}(\phi)$,

$$\Theta(n,x) = \sum_{\xi\in\Omega(\phi)} \cos((nx_0 - x)\cdot\xi)$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$ and, in particular,

$$\Theta(1,x_0) = \sum_{\xi\in\Omega(\phi)} \cos(0) = \#(\Omega(\phi)) > 0$$

whence $\mathrm{Supp}(\phi) \subseteq \mathrm{Supp}(\Theta(1,\cdot))$. The inclusion (3.100) follows straightforwardly by induction. For the second conclusion, an appeal to Theorem 3.7.5 shows that, for sufficiently large n,

$$n^{d/2}\phi^{(n)}(x+\lfloor n\alpha_\phi\rfloor) \geq |\Theta(n, x+\lfloor n\alpha_\phi\rfloor)G_\phi(n^{-1/2}(x+\lfloor n\alpha_\phi\rfloor - n\alpha_\phi))|/2$$

for all $x \in \mathbb{R}^d$. Of course, for any fixed x, $\lim_{n\to\infty}|G_\phi(n^{-1/2}(x+\lfloor n\alpha_\phi\rfloor - n\alpha_\phi))| = G_\phi(0) > 0$ and from this, the assertion follows without trouble.

□

To illustrate the utility of the function Θ, we consider a class of examples which generalizes simple random walk. For a fixed $\mathbf{m} = (m_1, m_2, \dots, m_2) \in \mathbb{N}_+^d$ define $\phi_{\mathbf{m}} \in \mathcal{M}_d^1$ by

$$\phi_{\mathbf{m}}(m_j e_j) = \phi_{\mathbf{m}}(-m_j e_j) = \frac{1}{2d}$$

for $j = 1, 2, \dots, d$ and set $\phi_{\mathbf{m}}(x) = 0$ otherwise; here, $\{e_1, e_2, \dots, e_d\}$ is the standard euclidean basis. This generates the random walk with statespace $\{(k_1 m_1, k_2 m_2, \dots, k_d m_d) : k_j \in \mathbb{Z}$ for $j = 1, 2, \dots, d\}$. We have:

Proposition 3.7.7. *Let $\Theta_{\mathbf{m}} : \mathbb{N}_+ \times \mathbb{Z}^d \to \mathbb{R}$ be that associated to $\phi_{\mathbf{m}}$ by* (3.99). *Then*

$$\Theta_{\mathbf{m}}(n,x) = \begin{cases} 2\left(\prod_{j=1}^d m_j\right) & \textit{if } m_j|x_j \textit{ for all } j = 1,2,\dots d \textit{ and } n - |x:\mathbf{m}| \textit{ is even} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For notational convenience, we write $\phi = \phi_{\mathbf{m}}$ and $\Theta = \Theta_{\mathbf{m}}$. Observe that $\hat{\phi}(\xi) = (1/d)\sum_{j=1}^d \cos(m_j\xi_j)$ for $\xi = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{T}^d$ and so by a direct computation,

$$\begin{aligned} \Omega(\phi) &= \Omega_{\mathbf{e}} \,\dot\cup\, \Omega_{\mathbf{o}} \\ &= \left\{\pi\left(\frac{k_1}{m_2}, \frac{k_2}{m_2}, \dots, \frac{k_d}{m_d}\right) : \mathbf{k} \in Z_{\mathbf{e}}\right\} \\ &\qquad \dot\bigcup \left\{\pi\left(\frac{k_1}{m_2}, \frac{k_2}{m_2}, \dots, \frac{k_d}{m_d}\right) : \mathbf{k} \in Z_{\mathbf{o}}\right\}, \end{aligned}$$

where

$$Z_{\mathbf{e}} = \{\mathbf{k} \in \mathbb{Z}^d : -m_j < k_j \leq m_j \text{ and } k_j \text{ is even for } j = 1, 2, \ldots, d\}$$

and

$$Z_{\mathbf{o}} = \{\mathbf{k} \in \mathbb{Z}^d : -m_j < k_j \leq k_j \text{ and } m_j \text{ is odd for } j = 1, 2, \ldots, d\}.$$

With this decomposition, we immediately observe that

$$\omega(\xi) = \begin{cases} 0 & \text{if } \xi \in \Omega_{\mathbf{e}} \\ \pi & \text{if } \xi \in \Omega_{\mathbf{o}}. \end{cases}$$

In the case that $m_j \big| x_j$ for $j = 1, 2, \ldots, d$,

$$\begin{aligned}
\Theta(n, x) &= \sum_{\xi \in \Omega_{\mathrm{e}}} e^{i(0n - x\cdot\xi)} + \sum_{\xi \in \Omega_{\mathrm{o}}} e^{i(\pi n - x\cdot\xi)} \\
&= \sum_{\mathbf{k} \in Z_{\mathrm{e}}} \exp\left(-i\pi \sum_{j=1}^{d} \frac{k_j x_j}{m_j}\right) + \sum_{\mathbf{k} \in Z_{\mathrm{o}}} \exp\left(i\pi\left(n - \sum_{j=1}^{d} \frac{k_j x_j}{m_j}\right)\right) \\
&= \#(Z_{\mathbf{e}}) + \exp\left(i\pi\left(n - \sum_{j=1}^{d} \frac{x_j}{m_j}\right)\right) \#(Z_{\mathbf{o}})
\end{aligned}$$

where we have used (3.99). Now $\#(Z_{\mathbf{e}}) = \#(Z_{\mathbf{o}}) = \prod_{j=1}^{d} m_j$ and so it follows that

$$\Theta(n, x) = \left(1 + e^{i\pi(n - |x:\mathbf{m}|)}\right) \prod_{j=1}^{d} m_j = \begin{cases} 2\left(\prod_{j=1}^{d} m_j\right) & \text{if } n - |x : \mathbf{m}| \text{ is even} \\ 0 & \text{if } n - |x : \mathbf{m}| \text{ is odd.} \end{cases}$$

In the case that $m_l \not| x_l$ for some $l = 1, 2, \ldots, d$, observe that

$$\begin{aligned}
\Theta(n, x) &= \sum_{\xi \in \Omega_{\mathrm{e}}} e^{-i\xi\cdot x} + \sum_{\xi \in \Omega_{\mathrm{o}}} e^{i(\pi n - \xi\cdot x)} \\
&= \prod_{j=1}^{d} \sum_{\substack{m_j < k_j \leq m_j \\ k_j \text{even}}} \exp\left(-i\pi \frac{x_j k_j}{m_j}\right) \\
&\qquad + e^{i\pi n} \prod_{j=1}^{d} \sum_{\substack{m_j < k_j \leq m_j \\ k_j \text{odd}}} \exp\left(-i\pi \frac{x_j k_j}{m_j}\right). \quad (3.101)
\end{aligned}$$

Focusing on the lth multiplicand in the first term, it is straightforward to see that

$$\begin{aligned}\left(e^{-2\pi i x_l/m_l}-1\right)&\sum_{\substack{m_l<k_j\leq m_l\\ k_l\text{even}}}\exp\left(-i\pi\frac{x_lk_l}{m_l}\right)\\ &=\sum_{\substack{m_l<k_j\leq m_l\\ k_l\text{even}}}\exp\left(-i\pi\frac{x_l(k_l+2)}{m_l}\right)-\exp\left(-i\pi\frac{x_lk_l}{m_l}\right)=0\end{aligned}$$

and since $m_l \not| x_l$, we can immediately conclude that

$$\sum_{\xi\in\Omega_{\mathrm{e}}}e^{-i\xi\cdot x}=\sum_{\substack{m_l<k_j\leq m_l\\ k_l\text{even}}}\exp\left(-i\pi\frac{x_lk_l}{m_l}\right)\prod_{j\neq l}\sum_{\substack{m_j<k_j\leq m_j\\ k_j\text{even}}}\exp\left(-i\pi\frac{x_jk_j}{m_j}\right)=0.$$

An analogous argument shows that $\sum_{\xi\in\Omega_{\mathrm{o}}}e^{i(\pi n-\xi\cdot x)}=0$ and so, in view of (3.101), it follows that $\Theta(n,x)=0$ as desired. □

Simple random walk is, of course, the random walk defined by $\phi_{\mathbf{m}}$ where $\mathbf{m}=(1,1,\dots,1)$. In this case, the proposition yields

$$\Theta_{(1,1,\dots,1)}(n,x)=\begin{cases}2 & \text{if } n-x_1-x_2-\cdots-x_d \text{ is even}\\ 0 & \text{if } n-x_1-x_2-\cdots-x_d \text{ is odd;}\end{cases}$$

this captures the walk's well-known periodicity.

We end this section by showing that Theorem 3.1.6 provides a Gaussian (upper) bound in the case that $\phi\in\mathcal{M}_d^1$ is finitely supported and genuinely d-dimensional. To obtain a matching lower bound, it is necessary to make some assumptions concerning aperiodicity.

Theorem 3.7.8. *Let $\phi\in\mathcal{M}_d^1$ be finitely supported and genuinely d-dimensional with mean $\alpha_\phi\in\mathbb{R}^d$. Then, there exist positive constants C and M for which*

$$\phi^{(n)}(x)\leq\frac{C}{n^{d/2}}\exp\left(-M|x-n\alpha_\phi|^2/n\right)$$

for all $n \in \mathbb{N}_+$ *and* $x \in \mathbb{Z}^d$.

Proof. In view of Proposition 3.7.4, our hypotheses guarantee that every $\xi \in \Omega(\phi)$ is of positive homogeneous type with corresponding $\alpha_\xi = \alpha_\phi$ and positive homogeneous polynomial $P_\xi = P_\phi$; here $\mu_\phi = \mu_{P_\phi} = d/2$ and $R_\phi = \operatorname{Re} P_\phi = P_\phi$. An appeal to Theorem 3.1.6 gives positive constants C and M for which

$$\phi^{(n)}(x) = |\phi^{(n)}(x)| \leq \frac{C}{n^{d/2}} \exp\left(-nMP_\phi^{\#}\left((x - n\alpha_\phi)/n\right)\right)$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$. Upon noting that $P_\phi^{\#}$ is necessarily quadratic and positive definite by virtue of Proposition A.3.2, we conclude that $P_\phi^{\#} \asymp |\cdot|^2$ and the theorem follows at once. $\square$

CHAPTER 4

POSITIVE-HOMOGENEOUS OPERATORS, HEAT KERNEL ESTIMATES AND THE LEGENDRE-FENCHEL TRANFORM

4.1 Introduction

In this chapter, we consider a class of homogeneous partial differential operators on a finite dimensional vector space and study their associated heat kernels. These operators, which we call nondegenerate-homogeneous operators, are seen to generalize the well-studied classes of semi-elliptic operators introduced by F. Browder [15], also known as quasielliptic operators [94], and a special "positive" subclass of semi-elliptic operators which appear as the spatial part of S. D. Eidelman's $2\vec{b}$-parabolic operators [36]. In particular, this class of operators contains all integer powers of the Laplacian. We begin this introduction by motivating the study of these homogeneous operators by first demonstrating the natural appearance of their heat kernels in the study of convolution powers of complex valued functions. To this end, consider a finitely supported function $\phi : \mathbb{Z}^d \to \mathbb{C}$ and define its convolution powers iteratively by

$$\phi^{(n)}(x) = \sum_{y \in \mathbb{Z}^d} \phi^{(n-1)}(x-y)\phi(y)$$

for $x \in \mathbb{Z}^d$ where $\phi^{(1)} = \phi$. In the special case that ϕ is a probability distribution, i.e., ϕ is non-negative and has unit mass, ϕ drives a random walk on $\mathbb{Z}^d$ whose nth-step transition kernels are given by $k_n(x,y) = \phi^{(n)}(y-x)$. Under certain mild conditions on the random walk, $\phi^{(n)}$ is well-approximated by a single Gaussian density; this is the classical local limit theorem. Specifically, for a symmetric, aperiodic and irreducible random walk, the theorem states that

$$\phi^{(n)}(x) = n^{-d/2} G_\phi(x/\sqrt{n}) + o(n^{-d/2})$$

uniformly for $x \in \mathbb{Z}^d$, where G_ϕ is the generalized Gaussian density

$$G_\phi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\big(- \xi \cdot C_\phi \xi \big) e^{-ix\cdot\xi} \, d\xi = \frac{1}{(2\pi)^{d/2}\sqrt{\det C_\phi}} \exp\left(-\frac{x \cdot {C_\phi}^{-1} x}{2} \right); \tag{4.1}$$

here, C_ϕ is the positive definite covariance matrix associated to ϕ and $\cdot$ denotes the dot product [63,72,83]. The canonical example is that in which $C_\phi = I$ (e.g. Simple Random Walk) and in this case $\phi^{(n)}$ is approximated by the so-called heat kernel

$$K^n_{(-\Delta)}(x) = n^{-d/2} G_\phi(x/\sqrt{n}) = (2\pi n)^{-d/2} \exp\left(-\frac{|x|^2}{2n} \right).$$

In addition to its natural appearance as the *attractor* in the local limit theorem, $K^t_{(-\Delta)}(x)$ is a fundamental solution to the heat equation

$$\partial_t + (-\Delta) = 0.$$

In fact, this connection to random walk underlies the heat equation's probabilistic/diffusive interpretation. Beyond the probabilistic setting, this link between convolution powers and fundamental solutions to partial differential equations persists as can be seen in the following examples.

Example 1. *Consider* $\phi : \mathbb{Z}^2 \to \mathbb{C}$ *defined by*

$$\phi(x_1, x_2) = \frac{1}{22 + 2\sqrt{3}} \times \begin{cases} 8 & (x_1, x_2) = (0,0) \\ 5 + \sqrt{3} & (x_1, x_2) = (\pm 1, 0) \\ -2 & (x_1, x_2) = (\pm 2, 0) \\ i(\sqrt{3} - 1) & (x_1, x_2) = (\pm 1, -1) \\ -i(\sqrt{3} - 1) & (x_1, x_2) = (\pm 1, 1) \\ 2 \mp 2i & (x_1, x_2) = (0, \pm 1) \\ 0 & \textit{otherwise}. \end{cases}$$

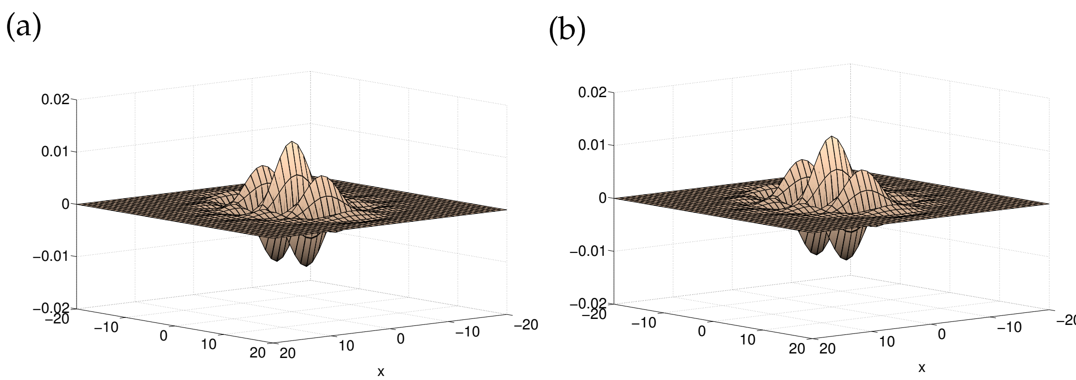

Figure 4.1: The graphs of $\mathrm{Re}(\phi^{(n)})$ (a) and $\mathrm{Re}(e^{-i\pi x_2/3} K_\Lambda^n)$ (b) for $n = 100$.

Analogous to the probabilistic setting, the large n *behavior of* $\phi^{(n)}$ *is described by a generalized local limit theorem in which the attractor is a fundamental solution to a heat-type equation. Specifically, the following local limit theorem holds (see [72] for details):*

$$\phi^{(n)}(x_1, x_2) = e^{-i\pi x_2/3} K_\Lambda^n(x_1, x_2) + o(n^{-3/4})$$

uniformly for $(x_1, x_2) \in \mathbb{Z}^2$ *where* K_Λ *is the "heat" kernel for the heat-type equation* $\partial_t + \Lambda = 0$ *where*

$$\Lambda = \frac{1}{22 + 2\sqrt{3}} \left(2\partial_{x_1}^4 - i(\sqrt{3} - 1)\partial_{x_1}^2 \partial_{x_2} - 4\partial_{x_2}^2 \right).$$

This local limit theorem is illustrated in Figure 4.1 which shows $\mathrm{Re}(\phi^{(n)})$ *and the approximation* $\mathrm{Re}(e^{-i\pi x_2/3}K_\Lambda^n)$ *when* $n = 100$.

Example 2. *Consider* $\phi : \mathbb{Z}^2 \to \mathbb{R}$ *defined by* $\phi = (\phi_1 + \phi_2)/512$*, where*

$$\phi_1(x_1, x_2) = \begin{cases} 326 & (x_1, x_2) = (0,0) \\ 20 & (x_1, x_2) = (\pm 2, 0) \\ 1 & (x_1, x_2) = (\pm 4, 0) \\ 64 & (x_1, x_2) = (0, \pm 1) \\ -16 & (x_1, x_2) = (0, \pm 2) \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\phi_2(x_1, x_2) = \begin{cases} 76 & (x_1, x_2) = (1,0) \\ 52 & (x_1, x_2) = (-1, 0) \\ \mp 4 & (x_1, x_2) = (\pm 3, 0) \\ \mp 6 & (x_1, x_2) = (\pm 1, 1) \\ \mp 6 & (x_1, x_2) = (\pm 1, -1) \\ \pm 2 & (x_1, x_2) = (\pm 3, 1) \\ \pm 2 & (x_1, x_2) = (\pm 3, -1) \\ 0 & \textit{otherwise}. \end{cases}$$

In this example, the following local limit theorem, which is illustrated by Figure 4.2, describes the limiting behavior of $\phi^{(n)}$*. We have*

$$\phi^{(n)}(x_1, x_2) = K_\Lambda^n(x_1, x_2) + o(n^{-5/12})$$

uniformly for $(x_1, x_2) \in \mathbb{Z}^2$ *where* K_Λ *is again a fundamental solution to* $\partial_t + \Lambda = 0$

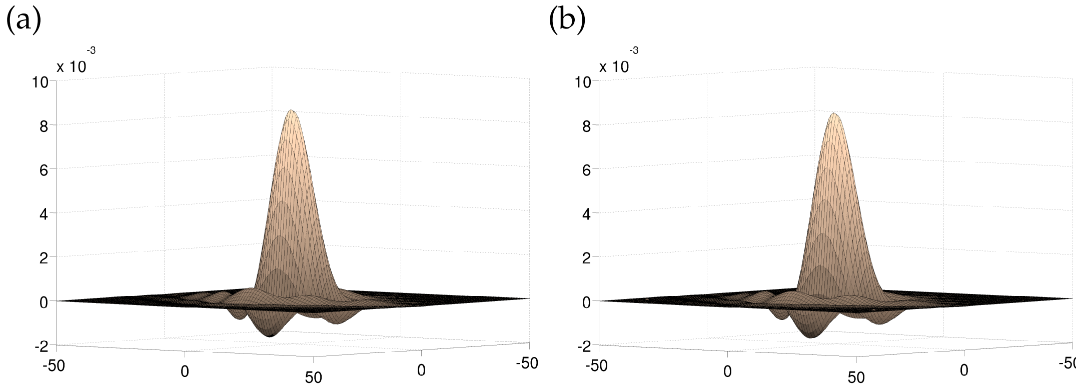

Figure 4.2: The graphs of $\mathrm{Re}(\phi^{(n)})$ (a) and K_Λ^n (b) for $n = 10,000$.

where, in this case,

$$\Lambda = \frac{1}{64}\left(-\partial_{x_1}^6 + 2\partial_{x_2}^4 + 2\partial_{x_1}^3\partial_{x_2}^2\right).$$

Example 3. *Consider $\phi : \mathbb{Z}^2 \to \mathbb{R}$ defined by*

$$\phi(x,y) = \begin{cases} 3/8 & (x_1,x_2) = (0,0) \\ 1/8 & (x_1,x_2) = \pm(1,1) \\ 1/4 & (x_1,x_2) = \pm(1,-1) \\ -1/16 & (x_1,x_2) = \pm(2,-2) \\ 0 & \textit{otherwise}. \end{cases}$$

Here, the following local limit theorem is valid:

$$\phi^{(n)}(x_1,x_2) = \left(1 + e^{i\pi(x_1+x_2)}\right) K_\Lambda^n(x_1,x_2) + o(n^{-3/4})$$

uniformly for $(x_1,x_2) \in \mathbb{Z}^2$. Here again, the attractor K_Λ is the fundamental solution to $\partial_t + \Lambda = 0$ where

$$\Lambda = -\frac{1}{8}\partial_{x_1}^2 + \frac{23}{384}\partial_{x_1}^4 - \frac{1}{4}\partial_{x_1}\partial_{x_2} - \frac{25}{96}\partial_{x_1}^3\partial_{x_2} - \frac{1}{8}\partial_{x_2}^2 + \frac{23}{64}\partial_{x_1}^2\partial_{x_2}^2 - \frac{25}{96}\partial_{x_1}\partial_{x_2}^3 + \frac{23}{384}\partial_{x_2}^4.$$

The operators appearing in the above examples share two important properties: homogeneity and positivity. While we make these notions precise in the next section, loosely speaking, homogeneity is the property that Λ "plays well" with some dilation structure on $\mathbb{R}^d$, though this structure is different in each example. Further, homogeneity for Λ is reflected by an analogous one for the corresponding heat kernel K_Λ; in fact, the specific dilation structure is, in some sense, selected by $\phi^{(n)}$ as $n \to \infty$ and leads to the corresponding local limit theorem. We encourage the reader to see the recent article [72] for a more thorough study of these examples and, in general, a more through study of local limit theorems. As we have often found–through local limit theorems and otherwise–knowledge of the attractor K_Λ informs our study of convolution powers (see Theorem 1.6 and Section 5.1 of [72]).

The prototypical examples of homogeneous operators considered in this chapter are the so-called semi-elliptic operators originally introduced by F. Browder in [15] and shortly appearing thereafter in L. Hörmander's treatise on linear partial differential operators [54, 55]. Given d-tuple of positive integers $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}_+^d$ and a multi-index $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{N}^d$, set $|\beta : \mathbf{n}| = \sum_{k=1}^d \beta_k / n_k$. Consider the constant coefficient partial differential operator

$$\Lambda = \sum_{|\beta:\mathbf{n}|\leq 1} a_\beta D^\beta$$

with principal part (relative to $\mathbf{n}$)

$$\Lambda_p = \sum_{|\beta:\mathbf{n}|=1} a_\beta D^\beta,$$

where $a_\beta \in \mathbb{C}$ and $D^\beta = (i\partial_{x_1})^{\beta_1}(i\partial_{x_2})^{\beta_2} \cdots (i\partial_{x_d})^{\beta_d}$ for each multi-index $\beta \in \mathbb{N}^d$. Such an operator is said to be *semi-elliptic* if the symbol of Λ_p, defined by $P_p(\xi) = \sum_{|\beta:\mathbf{n}|=1} a_\beta \xi^\beta$ for $\xi \in \mathbb{R}^d$, is non-vanishing away from the origin. If Λ satisfies the stronger condition that $\operatorname{Re} P_p(\xi)$ is strictly positive away from the origin, we say that it is *positive-semi-elliptic*. What seems to be the most important property of semi-elliptic operators is that their principal part Λ_p is homogeneous in the following sense: If given any smooth function f we put $\delta_t(f)(x) = f(t^{1/n_1}x_1, t^{1/n_2}x_2, \ldots, t^{1/n_d}x_d)$ for all $t > 0$ and $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, then

$$t\Lambda = \delta_{1/t} \circ \Lambda_p \circ \delta_t$$

for all $t > 0$. This homogeneous structure was used explicitly in the work of F. Browder and L. Hörmander and, in this chapter, we generalize this notion. Our generalization captures the operators appearing in Examples 1, 2 and 3.

As mentioned above, the class of semi-elliptic operators was introduced by F. Browder in [15] who studied spectral asymptotics for a related class of variable-coefficient operators (operators of constant strength). Semi-elliptic operators appeared later in L. Hörmander's text [54] as model examples of hypoelliptic operators on $\mathbb{R}^d$ beyond the class of elliptic operators. Around the same time L. R. Volevich [94] independently introduced the same class of operators but instead called them "quasi-elliptic". Since then, the theory of semi-elliptic operators, and hence quasi-elliptic operators, has reached a high level of sophistication and we refer the reader to the articles [1–5, 15, 49, 50, 54, 55, 58, 90, 92], which use the term semi-elliptic, and the articles [11–13, 17, 23–30, 41, 67, 70, 91, 93, 94], which use the term quasi-elliptic, for an account of this theory. We would also like to point to the 1971 paper of M. Troisi [91] which gives a more complete list of references (pertaining to quasi-elliptic operators).

Shortly after F. Browder's paper [15] appeared, S. D. Eidelman considered a subclass of semi-elliptic operators on $\mathbb{R}^{d+1} = \mathbb{R} \oplus \mathbb{R}^d$ (and systems thereof) of the form

$$\partial_t + \sum_{|\beta:2\mathbf{m}|\leq 1} a_\beta D^\beta = \partial_t + \sum_{|\beta:\mathbf{m}|\leq 2} a_\beta D^\beta, \tag{4.2}$$

where $\mathbf{m} \in \mathbb{N}_+^d$ and the coefficients a_β are functions of x and t. Such an operator is said to be $2\mathbf{m}$-*parabolic* if its spatial part, $\sum_{|\beta:2\mathbf{m}|\leq 1} a_\beta D^\beta$, is (uniformly) positive-semi-elliptic. We note however that Eidelman's work and the existing literature refer exclusively to $2\vec{b}$-parabolic operators, i.e., where $\mathbf{m} = \vec{b}$, and for consistency we write $2\vec{b}$-parabolic henceforth [36, 37]. The relationship between positive-semi-elliptic operators and $2\vec{b}$-parabolic operators is analogous to the relationship between the Laplacian and the heat operator and, in the context of this chapter, the relationship between nondegenerate-homogeneous and positive-homogeneous operators described by Proposition 4.2.4. The theory of $2\vec{b}$-parabolic operators, which generalizes the theory of parabolic partial differential equations (and systems), has seen significant advancement by a number of mathematicians since Eidelman's original work. We encourage the reader to see the recent text [37] which provides an account of this theory and an exhaustive list of references. It should be noted however that the literature encompassing semi-elliptic operators and quasi-elliptic operators, as far as we can tell, has very few cross-references to the literature on $2\vec{b}$-parabolic operators beyond the 1960's. We suspect that the absence of cross-references is due to the distinctness of vocabulary.

Returning to our discussion of convolution power examples, we note that the operators appearing in Examples 1 and 2 are both positive-semi-elliptic and consist only of their principal parts. This is easily verified, for $\mathbf{n} = (4, 2) = 2(2, 1)$ in Example 1 and $\mathbf{n} = (6, 4) = 2(3, 2)$ in Example 2. In contrast to Examples 1 and 2, the operator Λ which appears in Example 3 is not semi-elliptic in the given coordinate system. After careful study, the Λ appearing in Example 3 can be written equivalently as

$$\Lambda = -\frac{1}{8}\partial_{v_1}^2 + \frac{23}{384}\partial_{v_2}^4 \tag{4.3}$$

where ∂_{v_1} is the directional derivative in the $v_1 = (1,1)$ direction and ∂_{v_2} is the directional derivative in the $v_2 = (1,-1)$ direction. In this way, Λ is seen to be semi-elliptic with respect to some basis $\{v_1, v_2\}$ of $\mathbb{R}^2$. For this reason, our formulation of nondegenerate-homogeneous operators (and positive-homogeneous operators), given in the next section, is made in a basis independent way.

The subject of this chapter is an account of positive-homogeneous operators, a class of operators which generalize semi-elliptic operators, and their corresponding heat equations. In Section 4.2, we introduce positive-homogeneous operators and study their basic properties; therein, we show that each positive-homogeneous operator is semi-elliptic in some coordinate system. Section 4.3 develops the necessary background to introduce the class of variable-coefficient operators studied in this chapter; this is the class of $(2\mathbf{m}, \mathbf{v})$-positive-semi-elliptic operators introduced in Section 4.4–each of which is comparable to a constant-coefficient positive-homogeneous operator. In Section 4.5, we study the heat equations corresponding to uniformly $(2\mathbf{m}, \mathbf{v})$-positive-semi-elliptic operators with Hölder continuous coefficients. Specifically, we use the famous method of E. E. Levi, adapted to parabolic systems by A. Friedman and S. D. Eidelman, to construct a fundamental solution to the corresponding heat equation. Our results in this direction are captured by those of S. D. Eidelman [36] and the works of his collaborators, notably S. D. Ivashyshen and A. N. Kochubei [37], concerning $2\vec{b}$-parabolic systems. Our focus in this presentation is to highlight the essential role played by the Legendre-Fenchel transform in heat kernel estimates which, to our knowledge, has not been pointed out in the context of semi-elliptic operators. In a forthcoming work, we study an analogous class of operators, written in divergence form, with measurable-coefficients and their corresponding heat kernels. This class of measurable-coefficient operators does not appear to have been previously studied. The results presented here, using the Legendre-Fenchel transform, provides the background and context for our work there.

4.1.1 Preliminaries

Fourier Analysis: Our setting is a real d-dimensional vector space $\mathbb{V}$ equipped with Haar (Lebesgue) measure dx and the standard smooth structure; we do not affix $\mathbb{V}$ with a norm or basis. The dual space of $\mathbb{V}$ is denoted by $\mathbb{V}^*$ and the dual pairing is denoted by $\xi(x)$ for $x \in \mathbb{V}$ and $\xi \in \mathbb{V}^*$. Let $d\xi$ be the Haar measure on $\mathbb{V}^*$ which we take to be normalized so that our convention for the Fourier transform and inverse Fourier transform, given below, makes each unitary. For the remainder of this thesis, all functions on $\mathbb{V}$ and $\mathbb{V}^*$ are understood to be complex-valued. For a non-empty open set $\Omega \subseteq \mathbb{V}$ and $1 \leq p \leq \infty$, we denote by $L^p(\Omega) := L^p(\Omega, dx)$ the usual Lebesgue space equipped with its usual norm $\|\cdot\|_{L^p(\Omega)}$; when the context is clear, we will simply write $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$. In the case that $p = 2$, the corresponding inner product on $L^2(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$. Of course, we will also work with $L^2(\mathbb{V}^*) := L^2(\mathbb{V}^*, d\xi)$; here the L^2-norm and inner product will be denoted by $\|\cdot\|_{2^*}$ and $\langle \cdot, \cdot \rangle_*$ respectively. The Fourier transform $\mathcal{F} : L^2(\mathbb{V}) \to L^2(\mathbb{V}^*)$ and inverse Fourier transform $\mathcal{F}^{-1} : L^2(\mathbb{V}^*) \to L^2(\mathbb{V})$ are initially defined for Schwartz functions $f \in \mathcal{S}(\mathbb{V})$ and $g \in \mathcal{S}(\mathbb{V}^*)$ by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{V}} e^{i\xi(x)} f(x)\, dx \quad \text{and} \quad \mathcal{F}^{-1}(g)(x) = \check{g}(x) = \int_{\mathbb{V}^*} e^{-i\xi(x)} g(\xi)\, d\xi$$

for $\xi \in \mathbb{V}^*$ and $x \in \mathbb{V}$ respectively.

For the remainder of this chapter (mainly when duality isn't of interest), W stands for any real d-dimensional vector space (and so is interchangeable with $\mathbb{V}$ or $\mathbb{V}^*$). For a non-empty open set $\Omega \subseteq W$, we denote by $C(\Omega)$ and $C_b(\Omega)$ the set of continuous functions on Ω and bounded continuous functions on Ω, respectively. The set of smooth functions on Ω is denoted by $C^\infty(\Omega)$ and the set of compactly supported smooth functions on Ω is denoted by $C_0^\infty(\Omega)$. We denote by $\mathcal{D}'(\Omega)$ the space of distributions on Ω; this is dual to the space $C_0^\infty(\Omega)$ equipped with its usual topology given by seminorms. A partial differential operator H on W is said to be *hypoelliptic* if it satisfies the following property: Given any open set $\Omega \subseteq W$ and any distribution $u \in \mathcal{D}'(\Omega)$ which satisfies $Hu = 0$ in Ω, then necessarily $u \in C^\infty(\Omega)$.

Dilation Structure: Denote by $\mathrm{End}(W)$ and $\mathrm{Gl}(W)$ the set of endomorphisms and isomorphisms of W respectively. Given $E \in \mathrm{End}(W)$, we consider the one-parameter group $\{t^E\}_{t>0} \subseteq \mathrm{Gl}(W)$ defined by

$$t^E = \exp((\log t)E) = \sum_{k=0}^{\infty} \frac{(\log t)^k}{k!} E^k$$

for $t > 0$. These one-parameter subgroups of $\mathrm{Gl}(W)$ allow us to define continuous one-parameter groups of operators on the space of distributions as follows: Given $E \in \mathrm{End}(W)$ and $t > 0$, first define $\delta_t^E(f)$ for $f \in C_0^\infty(W)$ by $\delta_t^E(f)(x) = f(t^E x)$ for $x \in W$. Extending this to the space of distribution on W in the usual way, the collection $\{\delta_t^E\}_{t>0}$ is a continuous one-parameter group of operators on $\mathcal{D}'(W)$; it will allow us to define homogeneity for partial differential operators in the next section.

Linear Algebra and Polynomials: Given a basis $\mathbf{w} = \{w_1, w_2, \dots, w_d\}$ of W, we define the map $\phi_{\mathbf{w}} : W \to \mathbb{R}^d$ by setting $\phi_{\mathbf{w}}(w) = (x_1, x_2, \dots, x_d)$ whenever $w = \sum_{l=1}^{d} x_l w_l$. This map defines a global coordinate system on W; any such coordinate system is said to be a linear coordinate system on W. By definition, a polynomial on W is a function $P : W \to \mathbb{C}$ that is a polynomial function in every (and hence any) linear coordinate system on W. A polynomial P on W is called a nondegenerate polynomial if $P(w) \neq 0$ for all $w \neq 0$. Further, P is called a positive-definite polynomial if its real part, $R = \operatorname{Re} P$, is non-negative and has $R(w) = 0$ only when $w = 0$.

The Rest: Finally, the symbols $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ mean what they usually do, $\mathbb{N}$ denotes the set of non-negative integers and $\mathbb{I} = [0, 1] \subseteq \mathbb{R}$. The symbols $\mathbb{R}_+$, $\mathbb{N}_+$ and $\mathbb{I}_+$ denote the set of strictly positive elements of $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{I}$ respectively. Likewise, $\mathbb{R}_+^d$, $\mathbb{N}_+^d$ and $\mathbb{I}_+^d$ respectively denote the set of d-tuples of these aforementioned sets. We say that two real-valued functions f and g on a set X are comparable if, for some positive constant C, $C^{-1} f(x) \leq g(x) \leq C f(x)$ for all $x \in X$; in this case we write $f \asymp g$. Adopting the summation notation for semi-elliptic operators of L. Hörmander's treatise [55], for a fixed $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}_+^d$, we write

$$|\beta : \mathbf{n}| = \sum_{k=1}^{d} \frac{\beta_k}{m_k}.$$

for all multi-indices $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{N}^d$. Finally, throughout the estimates made in this chapter, constants denoted by C will change from line to line without explicit mention.

4.2 Homogeneous operators

In this section we introduce two important classes of homogeneous constant-coefficient on $\mathbb{V}$. These operators will serve as "model" operators in our theory in the way that integer powers of the Laplacian serves a model operators in the elliptic theory of partial differential equations. To this end, let Λ be a constant-coefficient partial differential operator on $\mathbb{V}$ and let $P : \mathbb{V}^* \to \mathbb{C}$ be its symbol. Specifically, P is the polynomial on $\mathbb{V}^*$ defined by $P(\xi) = e^{-i\xi(x)}\Lambda(e^{i\xi(x)})$ for $\xi \in \mathbb{V}^*$ (this is independent of $x \in \mathbb{V}$ precisely because Λ is a constant-coefficient operator). We first introduce the following notion of homogeneity of operators; it is mirrored by an analogous notion for symbols which we define shortly.

Definition 4.2.1. *Given $E \in End(\mathbb{V})$, we say that a constant-coefficient partial differential operator Λ is homogeneous with respect to the one-parameter group $\{\delta_t^E\}$ if*

$$\delta_{1/t}^E \circ \Lambda \circ \delta_t^E = t\Lambda$$

for all $t > 0$; in this case we say that E is a member of the exponent set of Λ and write $E \in \operatorname{Exp}(\Lambda)$.

A constant-coefficient partial differential operator Λ need not be homogeneous with respect to a unique one-parameter group $\{\delta_t^E\}$, i.e., $\operatorname{Exp}(\Lambda)$ is not necessarily a singleton. For instance, it is easily verified that, for the Laplacian $-\Delta$ on $\mathbb{R}^d$,

$$\operatorname{Exp}(-\Delta) = 2^{-1}I + \mathfrak{o}_d$$

where I is the identity and $\mathfrak{o}_d$ is the Lie algebra of the orthogonal group, i.e., is given by the set of skew-symmetric matrices. Despite this lack of uniqueness, when Λ is equipped with a nondegenerateness condition (see Definition 4.2.2), we will find that trace is the same for each member of $\mathrm{Exp}(\Lambda)$ and this allows us to uniquely define an "order" for Λ; this is Lemma 4.2.10.

Given a constant coefficient operator Λ with symbol P, one can quickly verify that $E \in \mathrm{Exp}(\Lambda)$ if and only if

$$tP(\xi) = P(t^F \xi) \tag{4.4}$$

for all $t > 0$ and $\xi \in \mathbb{V}^*$ where $F = E^*$ is the adjoint of E. More generally, if P is any continuous function on W and (4.4) is satisfied for some $F \in \mathrm{End}(\mathbb{V}^*)$, we say that P *is homogeneous with respect to* $\{t^F\}$ and write $F \in \mathrm{Exp}(P)$. This admitted slight abuse of notation should not cause confusion. In this language, we see that $E \in \mathrm{Exp}(\Lambda)$ if and only if $E^* \in \mathrm{Exp}(P)$.

We remark that the notion of homogeneity defined above is similar to that put forth for homogeneous operators on homogeneous (Lie) groups, e.g., Rockland operators [39]. The difference is mostly a matter of perspective: A homogeneous group G is equipped with a fixed dilation structure, i.e., it comes with a one-parameter group $\{\delta_t\}$, and homogeneity of operators is defined with respect to this fixed dilation structure. By contrast, we fix no dilation structure on $\mathbb{V}$ and formulate homogeneity in terms of an operator Λ and the existence of a one-parameter group $\{\delta_t^E\}$ that "plays" well with Λ in sense defined above. As seen in the study of convolution powers on the square lattice (see [72]), it useful to have this freedom.

Definition 4.2.2. *Let Λ be constant-coefficient partial differential operator on $\mathbb{V}$ with symbol P. We say that Λ is a nondegenerate-homogeneous operator if P is a nondegenerate polynomial and* $\mathrm{Exp}(\Lambda)$ *contains a diagonalizable endomorphism. We say that Λ*

is a positive-homogeneous operator if P is a positive-definite polynomial and $\mathrm{Exp}(\Lambda)$ contains a diagonalizable endomorphism.

For any polynomial P on a finite-dimensional vector space W, P is said to be *nondegenerate-homogeneous* if P is nondegenerate and $\mathrm{Exp}(P)$, defined as the set of $F \in \mathrm{End}(W)$ for which (4.4) holds, contains a diagonalizable endomorphism. We say that P is *positive-homogeneous* if it is a positive-definite polynomial and $\mathrm{Exp}(P)$ contains a diagonalizable endomorphism. In this language, we have the following proposition.

Proposition 4.2.3. *Let Λ be a positive homogeneous operator on $\mathbb{V}$ with symbol P. Then Λ is a nondegenerate-homogeneous operator if and only if P is a nondegenerate-homogeneous polynomial. Further, Λ is a positive-homogeneous operator if and only if P is a positive-homogeneous polynomial.*

Proof. Since the adjectives "nondegenerate" and "positive", in the sense of both operators and polynomials, are defined in terms of the symbol P, all that needs to be verified is that $\mathrm{Exp}(\Lambda)$ contains a diagonalizable endomorphism if and only if $\mathrm{Exp}(P)$ contains a diagonalizable endomorphism. Upon recalling that $E \in \mathrm{Exp}(\Lambda)$ if and only if $E^* \in \mathrm{Exp}(P)$, this equivalence is verified by simply noting that diagonalizability is preserved under taking adjoints. □

Remark 11. *To capture the class of nondegenerate-homogeneous operators (or positive-homogeneous operators), in addition to requiring that that the symbol P of an operator Λ be nondegenerate (or positive-definite), one can instead demand only that $\mathrm{Exp}(\Lambda)$ contains an endomorphism whose characteristic polynomial factors over $\mathbb{R}$ or, equivalently, whose spectrum is real. This a priori weaker condition is seen to be sufficient by an argument which makes use of the Jordan-Chevalley decomposition. In the positive-homogeneous case, this argument is carried out in [72] (specifically Proposition 2.2)* *wherein positive-homogeneous operators are first defined by this (a priori weaker) condition. For the nondegenerate case, the same argument pushes through with very little modification.*

We observe easily that all positive-homogeneous operators are nondegenerate-homogeneous. It is the "heat" kernels corresponding to positive-homogeneous operators that naturally appear in [72] as the attractors of convolution powers of complex-valued functions. The following proposition highlights the interplay between positive-homogeneity and nondegenerate-homogeneity for an operator Λ on $\mathbb{V}$ and its corresponding "heat" operator $\partial_t + \Lambda$ on $\mathbb{R} \oplus \mathbb{V}$.

Proposition 4.2.4. *Let Λ be a constant-coefficient partial differential operator on $\mathbb{V}$ whose exponent set $\operatorname{Exp}(\Lambda)$ contains a diagonalizable endomorphism. Let P be the symbol of Λ, set $R = \operatorname{Re} P$, and assume that there exists $\xi \in \mathbb{V}^*$ for which $R(\xi) > 0$. We have the following dichotomy: Λ is a positive-homogeneous operator on $\mathbb{V}$ if and only if $\partial_t + \Lambda$ is a nondegenerate-homogeneous operator on $\mathbb{R} \oplus \mathbb{V}$.*

Proof. Given a diagonalizable endomorphism $E \in \operatorname{Exp}(\Lambda)$, set $E_1 = I \oplus E$ where I is the identity on $\mathbb{R}$. Obviously, E_1 is diagonalizable. Further, for any $f \in C_0^\infty(\mathbb{R} \oplus \mathbb{V})$,

$$\begin{aligned}\left((\partial_t + \Lambda) \circ \delta_s^{E_1}\right)(f)(t,x) &= \left(\partial_t\left(f\left(st, s^E x\right)\right) + \Lambda\left(f\left(st, s^E x\right)\right)\right)\\ &= s(\partial_t + \Lambda)(f)(st, s^E x) = s\left(\delta_s^{E_1} \circ (\partial_t + \Lambda)\right)(f)(t,x)\end{aligned}$$

for all $s > 0$ and $(t,x) \in \mathbb{R} \oplus \mathbb{V}$. Hence

$$\delta_{1/s}^{E_1} \circ (\partial_t + \Lambda) \circ \delta_t^{E_1} = s(\partial_t + \Lambda)$$

for all $s > 0$ and therefore $E_1 \in \operatorname{Exp}(\partial_t + \Lambda)$.

It remains to show that P is positive-definite if and only if the symbol of $\partial_t + \Lambda$ is nondegenerate. To this end, we first compute the symbol of $\partial_t + \Lambda$ which we denote by Q. Since the dual space of $\mathbb{R} \oplus \mathbb{V}$ is isomorphic to $\mathbb{R} \oplus \mathbb{V}^*$, the characters of $\mathbb{R} \oplus \mathbb{V}$ are represented by the collection of maps $(\mathbb{R} \oplus \mathbb{V}) \ni (t, x) \mapsto \exp(-i(\tau t + \xi(x)))$ where $(\tau, \xi) \in \mathbb{R} \oplus \mathbb{V}^*$. Consequently,

$$Q(\tau, \xi) = e^{-i(\tau t + \xi(x))} \, (\partial_t + \Lambda) \, (e^{i(\tau t + \xi(x))}) = i\tau + P(\xi)$$

for $(\tau, \xi) \in \mathbb{R} \oplus \mathbb{V}^*$. We note that $P(0) = 0$ because $E^* \in \operatorname{Exp}(P)$; in fact, this happens whenever $\operatorname{Exp}(P)$ is non-empty. Now if P is a positive-definite polynomial, $\operatorname{Re} Q(\tau, \xi) = \operatorname{Re} P(\xi) = R(\xi) > 0$ whenever $\xi \neq 0$. Thus to verify that Q is a nondegenerate polynomial, we simply must verify that $Q(\tau, 0) \neq 0$ for all non-zero $\tau \in \mathbb{R}$. This is easy to see because, in light of the above fact, $Q(\tau, 0) = i\tau + P(0) = i\tau \neq 0$ whenever $\tau \neq 0$ and hence Q is nondegenerate. For the other direction, we demonstrate the validity of the contrapositive statement. Assuming that P is not positive-definite, an application of the intermediate value theorem, using the condition that $R(\xi) > 0$ for some $\xi \in \mathbb{V}^*$, guarantees that $R(\eta) = 0$ for some non-zero $\eta \in \mathbb{V}^*$. Here, we observe that $Q(\tau, \eta) = i(\tau + \operatorname{Im} P(\eta)) = 0$ when $(\tau, \eta) = (-\operatorname{Im} P(\eta), \eta)$ and hence Q is not nondegenerate. $\square$

We will soon return to the discussion surrounding a positive-homogeneous operator Λ and its heat operator $\partial_t + \Lambda$. It is useful to first provide representation formulas for nondegenerate-homogeneous and positive-homogeneous operators. Such representations connect our homogeneous operators to the class of semi-elliptic operators discussed in the introduction. To this end, we define the "base" operators on $\mathbb{V}$. First, for any element $u \in \mathbb{V}$, we consider the differential operator $D_u : \mathcal{D}'(\mathbb{V}) \to \mathcal{D}'(\mathbb{V})$ defined originally for $f \in C_0^\infty(\mathbb{V})$ by

$$(D_u f)(x) = i\frac{\partial f}{\partial u}(x) = i\left(\lim_{t\to 0}\frac{f(x+tu)-f(x)}{t}\right)$$

for $x \in \mathbb{V}$. Fixing a basis $\mathbf{v} = \{v_1, v_2, \dots, v_d\}$ of $\mathbb{V}$, we introduce, for each multi-index $\beta \in \mathbb{N}^d$, $D_{\mathbf{v}}^\beta = (D_{v_1})^{\beta_1}(D_{v_2})^{\beta_2}\cdots(D_{v_d})^{\beta_d}$.

Proposition 4.2.5. *Let Λ be a nondegenerate-homogeneous operator on $\mathbb{V}$. Then there exist a basis $\mathbf{v} = \{v_1, v_2, \dots, v_d\}$ of $\mathbb{V}$ and $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}_+^d$ for which*

$$\Lambda = \sum_{|\beta:\mathbf{n}|=1} a_\beta D_{\mathbf{v}}^\beta. \tag{4.5}$$

where $\{a_\beta\} \subseteq \mathbb{C}$. The isomorphism $E_{\mathbf{v}}^{\mathbf{n}} \in Gl(\mathbb{V})$, defined by $E_{\mathbf{v}}^{\mathbf{n}} v_k = (1/n_k)v_k$ for $k = 1, 2, \dots, d$, is a member of $\operatorname{Exp}(\Lambda)$. Further, if Λ is positive-homogeneous, then $\mathbf{n} = 2\mathbf{m}$ for $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_+^d$ and hence

$$\Lambda = \sum_{|\beta:\mathbf{m}|=2} a_\beta D_{\mathbf{v}}^\beta.$$

We will sometimes refer to the $\mathbf{n}$ and $\mathbf{m}$ of the proposition as *weights*. Before addressing the proposition, we first prove the following mirrored result for symbols.

Lemma 4.2.6. *Let P be a nondegenerate-homogeneous polynomial on a d-dimensional real vector space W. Then there exists a basis $\mathbf{w} = \{w_1, w_2, \dots, w_d\}$ of W and $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}_+^d$ for which*

$$P(\xi) = \sum_{|\beta:\mathbf{n}|=1} a_\beta \xi^\beta$$

for all $\xi = \xi_1 w_1 + \xi_2 w_2 + \cdots + \xi_d w_d \in W$ where $\xi^\beta := (\xi_1)^{\beta_1}(\xi_2)^{\beta_2}\cdots(\xi_d)^{\beta_d}$ and $\{a_\beta\} \subseteq \mathbb{C}$. The isomorphism $E_{\mathbf{w}}^{\mathbf{n}} \in Gl(\mathbb{V})$, defined by $E_{\mathbf{w}}^{\mathbf{n}} w_k = (1/n_k)w_k$ for $k = 1, 2, \dots, d$, is a member of $\operatorname{Exp}(P)$. Further, if P is a positive-definite polynomial, i.e., *it is positive-homogeneous, then* $\mathbf{n} = 2\mathbf{m}$ *for* $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_+^d$ *and hence*

$$P(\xi) = \sum_{|\beta:\mathbf{m}|=2} a_\beta \xi^\beta$$

for $\xi \in W$.

Proof. Let $E \in \operatorname{Exp}(P)$ be diagonalizable and select a basis $\mathbf{w} = \{w_1, w_2, \dots, w_d\}$ which diagonalizes E, i.e., $Ew_k = \delta_k w_k$ where $\delta_k \in \mathbb{R}$ for $k = 1, 2, \dots, d$. Because P is a polynomial, there exists a finite collection $\{a_\beta\} \subseteq \mathbb{C}$ for which

$$P(\xi) = \sum_\beta a_\beta \xi^\beta$$

for $\xi \in W$. By invoking the homogeneity of P with respect to E and using the fact that $t^E w_k = t^{\delta_k} w_k$ for $k = 1, 2, \dots, d$, we have

$$t \sum_\beta a_\beta \xi^\beta = \sum_\beta a_\beta (t^E \xi)^\beta = \sum_\beta a_\beta t^{\delta\cdot\beta} \xi^\beta$$

for all $\xi \in W$ and $t > 0$ where $\delta \cdot \beta = \delta_1\beta_1 + \delta_2\beta_2 + \cdots + \delta_d\beta_d$. In view of the nondegenerateness of P, the linear independence of distinct powers of t and the polynomial functions $\xi \mapsto \xi^\beta$, for distinct multi-indices β, as C^∞ functions ensures that $a_\beta = 0$ unless $\beta \cdot \delta = 1$. We can therefore write

$$P(\xi) = \sum_{\beta\cdot\delta=1} a_\beta \xi^\beta \tag{4.6}$$

for $\xi \in W$. We now determine $\delta = (\delta_1, \delta_2, \dots, \delta_d)$ by evaluating this polynomial along the coordinate axes. To this end, by fixing $k = 1, 2, \dots, d$ and setting $\xi = xw_k$ for $x \in \mathbb{R}$, it is easy to see that the summation above collapses into a single term $a_\beta x^{|\beta|}$ where $\beta = |\beta| e_k = (1/\delta_k) e_k$ (here e_k denotes the usual kth-Euclidean basis vector in $\mathbb{R}^d$). Consequently, $n_k := 1/\delta_k \in \mathbb{N}_+$ for $k = 1, 2, \dots, d$ and thus, upon setting $\mathbf{n} = (n_1, n_2, \dots, n_d)$, (4.6) yields

$$P(\xi) = \sum_{|\beta:\mathbf{n}|=1} a_\beta \xi^\beta$$

for all $\xi \in W$ as was asserted. In this notation, it is also evident that $E_{\mathbf{w}}^{\mathbf{n}} = E \in \operatorname{Exp}(P)$. Under the additional assumption that P is positive-definite, we again evaluate P at the coordinate axes to see that $\operatorname{Re} P(xw_k) = \operatorname{Re}(a_{n_k e_k})x^{n_k}$ for $x \in \mathbb{R}$. In this case, the positive-definiteness of P requires $\operatorname{Re}(a_{n_k e_k}) > 0$ and $n_k \in 2\mathbb{N}_+$ for each $k = 1, 2, \dots, d$. Consequently, $\mathbf{n} = 2\mathbf{m}$ for $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_+^d$ as desired. □

Proof of Proposition 4.2.5. Given a nondegenerate-homogeneous Λ on $\mathbb{V}$ with symbol P, P is necessarily a nondegenerate-homogeneous polynomial on $\mathbb{V}^*$ in view of Proposition 4.2.3. We can therefore apply Lemma 4.2.6 to select a basis $\mathbf{v}^* = \{v_1^*, v_2^*, \dots, v_d^*\}$ of $\mathbb{V}^*$ and $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}_+^d$ for which

$$P(\xi) = \sum_{|\beta:\mathbf{n}|=1} a_\beta \xi^\beta \tag{4.7}$$

for all $\xi = \xi_1 v_1^* + \xi_2 v_2^* + \cdots \xi_d v_d^*$ where $\{a_\beta\} \subseteq \mathbb{C}$. We will denote by $\mathbf{v}$, the dual basis to $\mathbf{v}^*$, i.e., $\mathbf{v} = \{v_1, v_2, \dots, v_d\}$ is the unique basis of $\mathbb{V}$ for which $v_k^*(v_l) = 1$ when $k = l$ and 0 otherwise. In view of the duality of the bases $\mathbf{v}$ and $\mathbf{v}^*$, it is straightforward to verify that, for each multi-index β, the symbol of $D_{\mathbf{v}}^\beta$ is ξ^β in the notation of Lemma 4.2.6. Consequently, the constant-coefficient partial differential operator defined by the right hand side of (4.5) also has symbol P and so it must be equal to Λ because operators and symbols are in one-to-one correspondence. Using (4.5), it is now straightforward to verify that $E_{\mathbf{v}}^{\mathbf{n}} \in \operatorname{Exp}(\Lambda)$. The assertion that $\mathbf{n} = 2\mathbf{m}$ when Λ is positive-homogeneous follows from the analogous conclusion of Lemma 4.2.6 by the same line of reasoning. □

In view of Proposition 4.2.5, we see that all nondegenerate-homogeneous operators are semi-elliptic in some linear coordinate system (that which is defined by v). An appeal to Theorem 11.1.11 of [55] immediately yields the following corollary.

Corollary 4.2.7. *Every nondegenerate-homogeneous operator Λ on $\mathbb{V}$ is hypoelliptic.*

Our next goal is to associate an "order" to each nondegenerate-homogeneous operator. For a positive-homogeneous operator Λ, this order will be seen to govern the on-diagonal decay of its heat kernel K_Λ and so, equivalently, the ultracontractivity of the semigroup $e^{-t\Lambda}$ (see Remark 19). With the help of Lemma 4.2.6, the few lemmas in this direction come easily.

Lemma 4.2.8. *Let P be a nondegenerate-homogeneous polynomial on a d-dimensional real vector space W. Then $\lim_{\xi\to\infty}|P(\xi)| = \infty$; here $\xi \to \infty$ means that $|\xi| \to \infty$ in any (and hence every) norm on W.*

Proof. The idea of the proof is to construct a function which bounds $|P|$ from below and obviously blows up at infinity. To this end, let $\mathbf{w}$ be a basis for W and take $\mathbf{n} \in \mathbb{N}_+^d$ as guaranteed by Lemma 4.2.6; we have $E_{\mathbf{w}}^{\mathbf{n}} \in \operatorname{Exp}(P)$ where $E_{\mathbf{w}}^{\mathbf{n}} w_k = (1/n_k)w_k$ for $k = 1, 2, \ldots, d$. Define $|\cdot|_{\mathbf{w}}^{\mathbf{n}} : W \to [0, \infty)$ by

$$|\xi|_{\mathbf{w}}^{\mathbf{n}} = \sum_{k=1}^{d} |\xi_k|^{n_k}$$

where $\xi = \xi_1 w_1 + \xi_2 w_2 + \cdots + \xi_d w_d \in W$. We observe immediately $E_{\mathbf{w}}^{\mathbf{n}} \in \operatorname{Exp}(|\cdot|_{\mathbf{w}}^{\mathbf{n}})$ because $t^{E_{\mathbf{w}}^{\mathbf{n}}} w_k = t^{1/n_k} w_k$ for $k = 1, 2, \ldots, d$. An application of Proposition 4.3.2 (a basic result appearing in our background section, Section 4.3), which uses the nondegenerateness of P, gives a positive constant C for which $|\xi|_{\mathbf{w}}^{\mathbf{n}} \le C|P(\xi)|$ for all $\xi \in W$. The lemma now follows by simply noting that $|\xi|_{\mathbf{w}}^{\mathbf{n}} \to \infty$ as $\xi \to \infty$. ∎

Lemma 4.2.9. *Let P be a polynomial on W and denote by $\operatorname{Sym}(P)$ the set of $O \in$ End(W) for which $P(O\xi) = P(\xi)$ for all $\xi \in W$. If P is a nondegenerate-homogeneous* *polynomial, then* $\mathrm{Sym}(P)$*, called the symmetry group of* P*, is a compact subgroup of* *Gl*(W)*.*

Proof. Our supposition that P is a nondegenerate polynomial ensures that, for each $O \in \mathrm{Sym}(P)$, $\mathrm{Ker}(O)$ is empty and hence $O \in \mathrm{Gl}(W)$. Consequently, given O_1 and $O_2 \in \mathrm{Sym}(P)$, we observe that $P(O_1^{-1}\xi) = P(O_1 O_1^{-1}\xi) = P(\xi)$ and $P(O_1 O_2 \xi) = P(O_2\xi) = P(\xi)$ for all $\xi \in W$; therefore $\mathrm{Sym}(P)$ is a subgroup of $\mathrm{Gl}(W)$.

To see that $\mathrm{Sym}(P)$ is compact, in view of the finite-dimensionality of $\mathrm{Gl}(W)$ and the Heine-Borel theorem, it suffices to show that $\mathrm{Sym}(P)$ is closed and bounded. First, for any sequence $\{O_n\} \subseteq \mathrm{Sym}(P)$ for which $O_n \to O$ as $n \to \infty$, the continuity of P ensures that $P(O\xi) = \lim_{n\to\infty} P(O_n\xi) = \lim_{n\to\infty} P(\xi) = P(\xi)$ for each $\xi \in W$ and therefore $\mathrm{Sym}(P)$ is closed. It remains to show that $\mathrm{Sym}(P)$ is bounded; this is the only piece of the proof that makes use of the fact that P is nondegenerate-homogeneous and not simply homogeneous. Assume that, to reach a contradiction, that there exists an unbounded sequence $\{O_n\} \subseteq \mathrm{Sym}(P)$. Choosing a norm $|\cdot|$ on W, let S be the corresponding unit sphere in W. Then there exists a sequence $\{\xi_n\} \subseteq W$ for which $|\xi_n| = 1$ for all $n \in \mathbb{N}_+$ but $\lim_{n\to\infty} |O_n \xi_n| = \infty$. In view of Lemma 4.2.8,

$$\infty = \lim_{n\to\infty} |P(O_n\xi_n)| = \lim_{n\to\infty} |P(\xi_n)| \leq \sup_{\xi \in S} |P(\xi)|,$$

which cannot be true for P is necessarily bounded on S because it is continuous. $\square$

Lemma 4.2.10. *Let* Λ *be a nondegenerate-homogeneous operator. For any* $E_1, E_2 \in \mathrm{Exp}(\Lambda)$*,*

$$\mathrm{tr}\, E_1 = \mathrm{tr}\, E_2.$$

Proof. Let P be the symbol of Λ and take $E_1, E_2 \in \operatorname{Exp}(\Lambda)$. Since $E_1^*, E_2^* \in \operatorname{Exp}(P)$, $t^{E_1^*}t^{-E_2^*} \in \operatorname{Sym}(P)$ for all $t > 0$. As $\operatorname{Sym}(P)$ is a compact group in view of the previous lemma, the determinant map $\det : \mathrm{Gl}(\mathbb{V}^*) \to \mathbb{C}^*$, a Lie group homomorphism, necessarily maps $\operatorname{Sym}(P)$ into the unit circle. Consequently,

$$1 = |\det(t^{E_1^*}t^{-E_2^*})| = |\det(t^{E_1^*})\det(t^{-E_2^*}) = |t^{\operatorname{tr} E_1^*}t^{-\operatorname{tr} E_2^*}| = t^{\operatorname{tr} E_1^*}t^{-\operatorname{tr} E_2^*}$$

for all $t > 0$. Therefore, $\operatorname{tr} E_1 = \operatorname{tr} E_1^* = \operatorname{tr} E_2^* = \operatorname{tr} E_2$ as desired. □

By the above lemma, to each nondegenerate-homogenerous operator Λ, we define the *homogeneous order* of Λ to be the number

$$\mu_\Lambda = \operatorname{tr} E$$

for any $E \in \operatorname{Exp}(\Lambda)$. By an appeal to Proposition 4.2.5, $E_{\mathbf{v}}^{\mathbf{n}} \in \operatorname{Exp}(\Lambda)$ for some $\mathbf{n} \in \mathbb{N}_+$ and so we observe that

$$\mu_\Lambda = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_d}. \tag{4.8}$$

In particular, μ_Λ is a positive rational number.

4.2.1 Positive-homogeneous operators and their heat kernels

We now restrict our attention to the study of positive-homogeneous operators and their associated heat kernels. To this end, let Λ be a positive-homogeneous operator on $\mathbb{V}$ with symbol P and homogeneous order μ_Λ. The heat kernel for Λ arises naturally from the study of the following Cauchy problem for the corresponding heat equation $\partial_t + \Lambda = 0$: Given initial data $f : \mathbb{V} \to \mathbb{C}$ which is, say, bounded and continuous, find $u(t,x)$ satisfying

$$\begin{cases}(\partial_t+\Lambda)\,u=0 & \text{in } (0,\infty)\times\mathbb{V}\\ u(0,x)=f(x) & \text{for } x\in\mathbb{V}.\end{cases} \tag{4.9}$$

The initial value problem (4.9) is solved by putting

$$u(t,x)=\int_{\mathbb{V}} K_\Lambda^t(x-y)f(y)\,dy$$

where $K_\Lambda^{(\cdot)}(\cdot):(0,\infty)\times\mathbb{V}\to\mathbb{C}$ is defined by

$$K_\Lambda^t(x)=\mathcal{F}^{-1}\left(e^{-tP}\right)(x)=\int_{\mathbb{V}^*} e^{-i\xi(x)}e^{-tP(\xi)}\,d\xi$$

for $t>0$ and $x\in\mathbb{V}$; we call K_Λ *the heat kernel* associated to Λ. Equivalently, K_Λ is the integral (convolution) kernel of the continuous semigroup $\{e^{-t\Lambda}\}_{t>0}$ of bounded operators on $L^2(\mathbb{V})$ with infinitesimal generator $-\Lambda$. That is, for each $f\in L^2(\mathbb{V})$,

$$\left(e^{-t\Lambda}f\right)(x)=\int_{\mathbb{V}} K_\Lambda^t(x-y)f(y)\,dy \tag{4.10}$$

for $t>0$ and $x\in\mathbb{V}$ (see Lemma 5.3.1). Let us make some simple observations about K_Λ. First, by virtue of Lemma 4.2.8, it follows that $K_\Lambda^t\in\mathcal{S}(\mathbb{V})$ for each $t>0$. Further, for any $E\in\operatorname{Exp}(\Lambda)$,

$$\begin{aligned}K_\Lambda^t(x) &= \int_{\mathbb{V}^*} e^{-i\xi(x)}e^{-P(t^{E^*}\xi)}\,d\xi=\int_{\mathbb{V}^*} e^{-i(t^{-E^*})\xi(x)}e^{-P(\xi)}\det(t^{-E^*})\,d\xi\\ &= \frac{1}{t^{\operatorname{tr}E}}\int_{\mathbb{V}^*} e^{-i\xi(t^{-E}x)}e^{-P(\xi)}\,d\xi=\frac{1}{t^{\mu_\Lambda}}K_\Lambda^1(t^{-E}x)\end{aligned}$$

for $t>0$ and $x\in\mathbb{V}$. This computation immediately yields the so-called on-diagonal estimate for K_Λ,

$$\|e^{-t\Lambda}\|_{1\to\infty}=\|K_\Lambda^t\|_\infty=\frac{1}{t^{\mu_\Lambda}}\|K_\Lambda^1\|_\infty\le\frac{C}{t^{\mu_\Lambda}}$$

for $t>0$; this is equivalently a statement of ultracontractivity for the semigroup $e^{-t\Lambda}$. As it turns out, we can say something much stronger.

Proposition 4.2.11. *Let Λ be a positive-homogeneous operator with symbol P and homogeneous order μ_Λ. Let $R^{\#} : \mathbb{V} \to \mathbb{R}$ be the Legendre-Fenchel transform of $R = \operatorname{Re} P$ defined by*

$$R^{\#}(x) = \sup_{\xi \in \mathbb{V}^*}\{\xi(x) - R(\xi)\}$$

for $x \in \mathbb{V}$. Also, let $\mathbf{v}$ and $\mathbf{m} \in \mathbb{N}_+^d$ be as guaranteed by Proposition 4.2.5. Then, there exit positive constants C_0 and M and, for each multi-index β, a positive constant C_β such that, for all $k \in \mathbb{N}$,

$$\left|\partial_t^k D_{\mathbf{v}}^\beta K_\Lambda^t(x-y)\right| \leq \frac{C_\beta C_0^k k!}{t^{\mu_\Lambda + k + |\beta : 2\mathbf{m}|}} \exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right)\right) \tag{4.11}$$

for all $x, y \in \mathbb{V}$ and $t > 0$. In particular,

$$\left|K_\Lambda^t(x-y)\right| \leq \frac{C}{t^{\mu_\Lambda}} \exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right)\right) \tag{4.12}$$

for all $x, y \in \mathbb{V}$ and $t > 0$.

Remark 12. *In view of* (4.8)*, the exponent on the prefactor in* (4.11) *can be equivalently written, for any multi-index β and $k \in \mathbb{N}$, as $\mu_\Lambda + k + |\beta : 2\mathbf{m}| = k + |\mathbf{1} + \beta : 2\mathbf{m}| = |\mathbf{1} + 2k\mathbf{m} + \beta : 2\mathbf{m}|$ where $\mathbf{1} = (1, 1, \ldots, 1)$.*

We prove the proposition above in the Section 4.5; the remainder of this section is dedicated to discussing the result and connecting it to the existing theory. Let us first note that the estimate (4.11) is mirrored by an analogous space-time estimate, Theorem 5.3 of [72], for the convolution powers of complex-valued functions on $\mathbb{Z}^d$ satisfying certain conditions (see Section 5 of [72]). The relationship between these two results, Theorem 5.3 of [72] and Proposition 4.2.11, parallels the relationship between Gaussian off-diagonal estimates for random walks and the analogous off-diagonal estimates enjoyed by the classical heat kernel [46].

Let us first show that the estimates (4.11) and (4.12) recapture the well-known estimates of the theory of parabolic equations and systems in $\mathbb{R}^d$ – a theory in which the Laplacian operator $\Delta = \sum_{l=1}^{d} \partial_{x_l}^2$ and its integer powers play a central role. To place things into the context of this chapter, let us observe that, for each positive integer m, the partial differential operator $(-\Delta)^m$ is a positive-homogeneous operator on $\mathbb{R}^d$ with symbol $P(\xi) = |\xi|^{2m}$; here, we identify $\mathbb{R}^d$ as its own dual equipped with the dot product and Euclidean norm $|\cdot|$. Indeed, one easily observes that $P = |\cdot|^{2m}$ is a positive-definite polynomial and $E = (2m)^{-1}I \in \operatorname{Exp}((-\Delta)^m)$ where $I \in \operatorname{Gl}(\mathbb{R}^d)$ is the the identity. Consequently, the homogeneous order of $(-\Delta)^m$ is $d/2m = (2m)^{-1}\operatorname{tr}(I)$ and the Legendre-Fenchel transform of $R = \operatorname{Re} P = |\cdot|^{2m}$ is easily computed to be $R^{\#}(x) = C_m|x|^{2m/(2m-1)}$ where $C_m = (2m)^{1/(2m-1)} - (2m)^{-2m/(2m-1)} > 0$. Hence, (4.12) is the well-known estimate

$$\left|K^t_{(-\Delta)^m}(x-y)\right| \leq \frac{C}{t^{d/2m}} \exp\left(-M\frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}}\right)$$

for $x, y \in \mathbb{R}^d$ and $t > 0$; this so-called off-diagonal estimate is ubiquitous to the theory of "higher-order" elliptic and parabolic equations [20, 35, 40, 77]. To write the derivative estimate (4.11) in this context, we first observe that the basis given by Proposition 4.2.5 can be taken to be the standard Euclidean basis, $\mathbf{e} = \{e_1, e_2, \dots, e_d\}$ and further, $\mathbf{m} = (m, m, \dots, m)$ is the (isotropic) weight given by the proposition. Writing $D^\beta = D^\beta_{\mathbf{e}} = (i\partial_{x_1})^{\beta_1}(i\partial_{x_2})^{\beta_2}\cdots(i\partial_{x_d})^{\beta_d}$ and $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_d$ for each multi-index β, (4.11) takes the form

$$\left|\partial_t^k D^\beta K^t_{(-\Delta)^m}(x-y)\right| \leq \frac{C}{t^{(d+|\beta|)/2m+k}} \exp\left(-M\frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}}\right)$$

for $x, y \in \mathbb{R}^d$ and $t > 0$, c.f., [35, Property 4, p. 93].

The appearance of the 1-dimensional Legendre-Fenchel transform in heat kernel estimates was previously recognized and exploited in [9] and [10] in the context of elliptic operators. Due to the isotropic nature of elliptic operators, the 1-dimensional transform is sufficient to capture the inherent isotropic decay of corresponding heat kernels. Beyond the elliptic theory, the appearance of the full d-dimensional Legendre-Fenchel transform is remarkable because it sharply captures the general anisotropic decay of K_Λ. Consider, for instance, the particularly simple positive-homogeneous operator $\Lambda = -\partial_{x_1}^6 + \partial_{x_2}^8$ on $\mathbb{R}^2$ with symbol $P(\xi_1, \xi_2) = \xi_1^6 + \xi_2^8$. It is easily checked that the operator E with matrix representation $\operatorname{diag}(1/6, 1/8)$, in the standard Euclidean basis, is a member of the $\operatorname{Exp}(\Lambda)$ and so the homogeneous order of Λ is $\mu_\Lambda = \operatorname{tr}(\operatorname{diag}(1/6, 1/8)) = 7/24$. Here we can compute the Legendre-Fenchel transform of $R = \operatorname{Re} P = P$ directly to obtain $R^{\#}(x_1, x_2) = c_1|x_1|^{6/5} + c_2|x_2|^{8/7}$ for $(x_1, x_2) \in \mathbb{R}^2$ where c_1 and c_2 are positive constants. In this case, Proposition 4.2.11 gives positive constants M_1, M_2 and C for which

$$|K_\Lambda^t(x_1 - y_1, x_2 - y_2)| \leq \frac{C}{t^{7/24}} \exp\left(-\left(M_1 \frac{|x_1 - y_1|^{6/5}}{t^{1/5}} + M_2 \frac{|x_2 - y_2|^{8/7}}{t^{1/7}}\right)\right) \quad (4.13)$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $t > 0$. We note however that Λ is "separable" and so we can write $K_\Lambda^t(x_1, x_2) = K_{(-\Delta)^3}^t(x_1) K_{(-\Delta)^4}^t(x_2)$ where Δ is the 1-dimensional Laplacian operator. In view of Theorem 8 of [9] and its subsequent remark, the estimate (4.13) is seen to be sharp (modulo the values of M_1, M_2 and C). To further illustrate the proposition for a less simple positive-homogeneous operator, we consider the operator Λ appearing in Example 3. In this case,

$$R(\xi_1, \xi_2) = P(\xi_1, \xi_2) = \frac{1}{8}(\xi_1 + \xi_2)^2 + \frac{23}{384}(\xi_1 - \xi_2)^4$$

and one can verify directly that the $E \in \operatorname{End}(\mathbb{R}^2)$, with matrix representation

$$E_{\mathbf{e}} = \begin{pmatrix} 3/8 & 1/8 \\ 1/8 & 3/8 \end{pmatrix}$$

in the standard Euclidean basis, is a member of $\mathrm{Exp}(\Lambda)$. From this, we immediately obtain $\mu_\Lambda = \mathrm{tr}(E) = 3/4$ and one can directly compute

$$R^{\#}(x_1, x_2) = M_1|x_1 + x_2|^2 + M_2|x_1 - x_2|^{4/3}$$

for $(x_1, x_2) \in \mathbb{R}^2$ where M_1 and M_2 are positive constants. Consequently,

$$\begin{aligned}
&|K_\Lambda^t(x_1 - y_1, x_2 - y_2)| \\
&\quad \leq \frac{C}{t^{3/4}} \exp\left(-\left(M_1\frac{|(x_1 - y_1) + (x_2 - y_2)|^2}{t} + M_2\frac{|(x_1 - y_1) - (x_2 - y_2)|^{4/3}}{t^{1/3}}\right)\right)
\end{aligned}$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $t > 0$. Furthermore, $\mathbf{m} = (1, 2) \in \mathbb{N}_+^2$ and the basis $\mathbf{v} = \{v_1, v_2\}$ of $\mathbb{R}^2$ given in discussion surrounding (4.3) are precisely those guaranteed by Proposition 4.2.5. Appealing to the full strength of Proposition 4.2.11, we obtain positive constants C, M_1 and M_2 and, for each multi-index β, a positive constant C_β such that, for each $k \in \mathbb{N}$,

$$\begin{aligned}
\left|\partial_t^k D_{\mathbf{v}}^\beta K_\Lambda(x_1 - y_1, x_2 - y_2)\right| \frac{C_\beta C_0^k k!}{t^{3/4+k+|\beta:2\mathbf{m}|}} \exp\Big(-\Big(M_1&\frac{|(x_1 - y_1) + (x_2 - y_2)|^2}{t} \\
+M_2&\frac{|(x_1 - y_1) - (x_2 - y_2)|^{4/3}}{t^{1/3}}\Big)\Big)
\end{aligned}$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $t > 0$.

In the context of homogeneous groups, the off-diagonal behavior for the heat kernel of a positive Rockland operator (a positive self-adjoint operator which is homogeneous with respect to the fixed dilation structure) has been studied in [7,34,47] (see also [3]). Given a positive Rockland operator Λ on homogeneous group G, the best known estimate for the heat kernel K_Λ, due to Auscher, ter Elst and Robinson, is of the form

$$|K_\Lambda^t(h^{-1}g)| \leq \frac{C}{t^{\mu_\Lambda}} \exp\left(-M\left(\frac{\|h^{-1}g\|^{2m}}{t}\right)^{1/(2m-1)}\right) \tag{4.14}$$

where $\|\cdot\|$ is a homogeneous norm on G (consistent with Λ) and $2m$ is the highest order derivative appearing in Λ. In the context of $\mathbb{R}^d$, given a symmetric and positive-homogeneous operator Λ with symbol P, the structure $G_D = (\mathbb{R}^d, \{\delta_t^D\})$ for $D = 2mE$ where $E \in \operatorname{Exp}(\Lambda)$ is a homogeneous group on which Λ becomes a positive Rockland operator. On G_D, it is quickly verified that $\|\cdot\| = R(\cdot)^{1/2m}$ is a homogeneous norm (consistent with Λ) and so the above estimate is given in terms of $R(\cdot)^{1/(2m-1)}$ which is, in general, dominated by the Legendre-Fenchel transform of R. To see this, we need not look further than our previous and simple example in which $\Lambda = -\partial_{x_1}^6 + \partial_{x_2}^8$. Here $2m = 8$ and so $R(x_1, x_2)^{1/(2m-1)} = (|x_1|^6 + |x_2|^8)^{1/7}$. In view of (4.13), the estimate (4.14) gives the correct decay along the x_2-coordinate axis; however, the bounds decay at markedly different rates along the x_1-coordinate axis. This illustrates that the estimate (4.14) is suboptimal, at least in the context of $\mathbb{R}^d$, and thus leads to the natural question: For positive-homogeneous operators on a general homogeneous group G, what is to replace the Legendre-Fenchel transform in heat kernel estimates?

Returning to the general picture, let Λ be a positive-homogeneous operator on $\mathbb{V}$ with symbol P and homogeneous order μ_Λ. To highlight some remarkable properties about the estimates (4.11) and (4.12) in this general setting, the following proposition concerning $R^\#$ is useful; for a proof, see Section 8.3 of [72].

Proposition 4.2.12. *Let Λ be a positive-homogeneous operator with symbol P and let $R^\#$ be the Legendre-Fenchel transform of $R = \operatorname{Re} P$. Then, for any $E \in \operatorname{Exp}(\Lambda)$, $I - E \in \operatorname{Exp}(R^\#)$. Moreover $R^\#$ is continuous, positive-definite in the sense that $R^\#(x) \geq 0$ and $R^\#(x) = 0$ only when $x = 0$. Further, $R^\#$ grows superlinearly in the* *sense that, for any norm $|\cdot|$ on $\mathbb{V}$,*

$$\lim_{x\to\infty}\frac{|x|}{R^{\#}(x)}=0;$$

in particular, $R^{\#}(x)\to\infty$ as $x\to\infty$.

Let us first note that, in view of the proposition, we can easily rewrite (4.12), for any $E\in\operatorname{Exp}(\Lambda)$, as

$$\left|K_{\Lambda}^{t}(x-y)\right|\leq\frac{C}{t^{\mu_{\Lambda}}}\exp\left(-MR^{\#}\left(t^{-E}(x-y)\right)\right)$$

for $x,y\in\mathbb{V}$ and $t>0$; the analogous rewriting is true for (4.11). The fact that $R^{\#}$ is positive-definite and grows superlinearly ensures that the convolution operator $e^{-t\Lambda}$ defined by (4.10) for $t>0$ is a bounded operator from L^{p} to L^{q} for any $1\leq p,q\leq\infty$. Of course, we already knew this because K_{Λ}^{t} is a Schwartz function; however, when replacing Λ with a variable-coefficient operator H, as we will do in the sections to follow, the validity of the estimate (4.12) for the kernel of the semigroup $\{e^{-tH}\}$ initially defined on L^{2}, guarantees that the semigroup extends to a strongly continuous semigroup $\{e^{-tH_{p}}\}$ on $L^{p}(\mathbb{R}^{d})$ for all $1\leq p\leq\infty$ and, what's more, the respective infinitesimal generators $-H_{p}$ have spectra independent of p [21]. Further, the estimate (4.12) is key to establishing the boundedness of the Riesz transform, it is connected to the resolution of Kato's square root problem and it provides the appropriate starting point for uniqueness classes of solutions to $\partial_{t}+H=0$ [6, 69]. With this motivation in mind, following some background in Section 4.3, we introduce a class of variable-coefficient operators in Section 4.4 called $(2\mathbf{m},\mathbf{v})$-positive-semi-elliptic operators, each such operator H comparable to a fixed positive-homogeneous operator. In Section 4.5, under the assumption that H has Hölder continuous coefficients and this notion of comparability is uniform, we construct a fundamental solution to the heat equation $\partial_{t}+H=0$ and show the essential role played by the Legendre-Fenchel transform in this construction. As mentioned previously, in a forthcoming work we will study the semigroup $\{e^{-tH}\}$ where H is a divergence-form operator, which is comparable to a fixed positive-homogeneous operator, whose coefficients are at worst measurable. As the Legendre-Fenchel transform appears here by a complex change of variables followed by a minimization argument, in the measurable coefficient setting it appears quite naturally by an application of the so-called Davies' method, suitably adapted to the positive-homogeneous setting.

4.3 Contracting groups, Hölder continuity and the Legendre-Fenchel transform

In this section, we provide the necessary background on one-parameter contracting groups, anisotropic Hölder continuity, and the Legendre-Fenchel transform and its interplay with the two previous notions.

4.3.1 One-parameter contracting groups

In what follows, W is a d-dimensional real vector space with a norm $|\cdot|$; the corresponding operator norm on $\mathrm{Gl}(W)$ is denoted by $\|\cdot\|$. Of course, since everything is finite-dimensional, the usual topologies on W and $\mathrm{Gl}(W)$ are insensitive to the specific choice of norms.

Definition 4.3.1. *Let $\{T_t\}_{t>0} \subseteq Gl(W)$ be a continuous one-parameter group. $\{T_t\}$ is* *said to be contracting if*

$$\lim_{t\to 0}\|T_t\| = 0.$$

We easily observe that, for any diagonalizable $E \in \operatorname{End}(W)$ with strictly positive spectrum, the corresponding one-parameter group $\{t^E\}_{t>0}$ is contracting. Indeed, if there exists a basis $\mathbf{w} = \{w_1, w_2, \dots, w_d\}$ of W and a collection of positive numbers $\lambda_1, \lambda_2, \dots, \lambda_d$ for which $Ew_k = \lambda_k w_k$ for $k = 1, 2, \dots, d$, then the one parameter group $\{t^E\}_{t>0}$ has $t^E w_k = t^{\lambda_k} w_k$ for $k = 1, 2, \dots, d$ and $t > 0$. It then follows immediately that $\{t^E\}$ is contracting.

Proposition 4.3.2. *Let Q and R be continuous real-valued functions on W. If $R(w) > 0$ for all $w \neq 0$ and there exists $E \in \operatorname{Exp}(Q) \cap \operatorname{Exp}(R)$ for which $\{t^E\}$ is contracting, then, for some positive constant C, $Q(w) \leq CR(w)$ for all $w \in W$. If additionally $Q(w) > 0$ for all $w \neq 0$, then $Q \asymp R$.*

Proof. Let S denote the unit sphere in W and observe that

$$\sup_{w\in S} \frac{Q(w)}{R(w)} =: C < \infty$$

because Q and R are continuous and R is non-zero on S. Now, for any non-zero $w \in W$, the fact that t^E is contracting implies that $t^E w \in S$ for some $t > 0$ by virtue of the intermediate value theorem. Therefore, $Q(w) = Q(t^E w)/t \leq CR(t^E w)/t = CR(w)$. In view of the continuity of Q and R, this inequality must hold for all $w \in W$. When additionally $Q(w) > 0$ for all non-zero w, the conclusion that $Q \asymp R$ is obtained by reversing the roles of Q and R in the preceding argument. □

Corollary 4.3.3. *Let Λ be a positive-homogeneous operator on $\mathbb{V}$ with symbol P and let $R^{\#}$ be the Legendre-Fenchel transform of $R = \operatorname{Re} P$. Then, for any positive constant M, $R^{\#} \asymp (MR)^{\#}$.*

Proof. By virtue of Proposition 4.2.5, let $\mathbf{m} \in \mathbb{N}_+^d$ and $\mathbf{v}$ be a basis for $\mathbb{V}$ and for which $E_{\mathbf{v}}^{2\mathbf{m}} \in \operatorname{Exp}(\Lambda)$. In view of Proposition 4.2.12, $R^{\#}$ and $(MR)^{\#}$ are both continuous, positive-definite and have $F_{\mathbf{v}}^{2\mathbf{m}} := I - E_{\mathbf{v}}^{2\mathbf{m}} \in \operatorname{Exp}(R^{\#}) \cap \operatorname{Exp}((MR)^{\#})$. Upon noting that $F_{\mathbf{v}}^{2\mathbf{m}} v_k = ((2m_k - 1)/2m_k) v_k$ for $k = 1, 2, \dots, d$, we immediately conclude that $\{t^{F_{\mathbf{v}}^{2\mathbf{m}}}\}$ is contracting and so the corollary follows directly from Proposition 4.3.2. $\square$

Lemma 4.3.4. *Let P be a positive-homogeneous polynomial on W and let $\mathbf{n} = 2\mathbf{m} \in \mathbb{N}_+^d$ and $\mathbf{w}$ be a basis for W for which the conclusion of Lemma 4.2.6 holds. Let $R = \operatorname{Re} P$ and let β and γ be multi-indices such that $\gamma \leq \beta$ (in the standard partial ordering of multi-indices); we shall assume the notation of the lemma.*

1. *For any $n \in \mathbb{N}_+$ such that $|\beta : \mathbf{m}| \leq 2n$, there exist positive constants M and M' for which*
$$|\xi^\gamma \nu^{\beta-\gamma}| \leq M(R(\xi) + R(\nu))^n + M'$$
for all $\xi, \nu \in W$.

2. *If $|\beta : \mathbf{m}| = 2$, there exist positive constants M and M' for which*
$$|\xi^\gamma \nu^{\beta-\gamma}| \leq MR(\xi) + M'R(\nu)$$
for all $\nu, \xi \in W$.

3. *If $|\beta : \mathbf{m}| = 2$ and $\beta > \gamma$, then for every $\epsilon > 0$ there exists a positive constant M for which*
$$|\xi^\gamma \nu^{\beta-\gamma}| \leq \epsilon R(\xi) + MR(\nu)$$
for all $\nu, \xi \in W$.

Proof. Assuming the notation of Lemma 4.2.6, let $E = E_{\mathbf{w}}^{2\mathbf{m}} \in \operatorname{End}(W)$ and consider the contracting group $\{t^{E\oplus E}\} = \{t^E \oplus t^E\}$ on $W \oplus W$. Because R is a positive-definite polynomial, it immediately follows that $W \oplus W \ni (\xi, \nu) \mapsto R(\xi) + R(\nu)$ is positive-definite. Let $|\cdot|$ be a norm on $W \oplus W$ and respectively denote by B and S the corresponding unit ball and unit sphere in this norm.

To see Item 1, first observe that

$$\sup_{(\xi,\nu)\in S} \frac{|\xi^\gamma \nu^{\beta-\gamma}|}{(R(\xi)+R(\nu))^n} =: M < \infty$$

Now, for any $(\xi, \nu) \in W \oplus W \setminus B$, because $\{t^{E\oplus E}\}$ is contracting, it follows from the intermediate value theorem that, for some $t \geq 1$, $t^{-(E\oplus E)}(\xi, \nu) = (t^{-E}\xi, t^{-E}\nu) \in S$. Correspondingly,

$$\begin{aligned}
|\xi^\gamma \nu^{\beta-\gamma}| &= t^{|\beta:2\mathbf{m}|}|(t^{-E}\xi)^\gamma (t^{-E})^{\beta-\gamma}| \\
&\leq t^{|\beta:2\mathbf{m}|} M(R(t^{-E}\xi) + R(t^{-E}\nu))^n \\
&\leq t^{|\beta:\mathbf{m}|/2-n} M(R(\xi)+R(\nu))^n \\
&\leq M(R(\xi)+R(\nu))^n
\end{aligned}$$

because $|\beta : \mathbf{m}|/2 \leq n$. One obtains the constant M' and hence the desired inequality by simply noting that $|\xi^\gamma \nu^{\beta-\gamma}|$ is bounded for all $(\xi, \nu) \in B$.

For Item 2, we use analogous reasoning to obtain a positive constant M for which $|\xi^\gamma \nu^{\beta-\gamma}| \leq M(R(\xi)+R(\nu))$ for all $(\xi, \nu) \in S$. Now, for any non-zero $(\xi, \nu) \in W \oplus W$, the intermediate value theorem gives $t > 0$ for which $t^{E\oplus E}(\xi, \nu) = (t^E\xi, t^E\nu) \in S$ and hence

$$|\xi^\gamma \nu^{\beta-\gamma}| \leq t^{-|\beta:2\mathbf{m}|} M(R(t^E\xi) + R(t^E\nu)) = M(R(\xi)+R(\nu))$$

where we have used the fact that $|\beta : 2\mathbf{m}| = |\beta : \mathbf{m}|/2 = 1$ and that $E \in \operatorname{Exp}(R)$. As this inequality must also trivially hold at the origin, we can conclude that it holds for all $\xi, \nu \in W$, as desired.

Finally, we prove Item 3. By virtue of Item 2, for any $\xi, \nu \in W$ and $t > 0$,

$$\begin{aligned}|\xi^\gamma \nu^{\beta-\gamma}| &= |(t^E t^{-E}\xi)^\gamma \nu^{\beta-\gamma}| = t^{|\gamma:2\mathbf{m}|}|(t^{-E}\xi)^\gamma \nu^{\beta-\gamma}| \\ &\leq t^{|\gamma:2\mathbf{m}|}\left(MR(t^{-E}\xi) + M'R(\nu)\right) = Mt^{|\gamma:2\mathbf{m}|-1}R(\xi) + M't^{|\gamma:2\mathbf{m}|}R(\nu).\end{aligned}$$

Noting that $|\gamma : 2\mathbf{m}| - 1 < 0$ because $\gamma < \beta$, we can make the coefficient of $R(\xi)$ arbitrarily small by choosing t sufficiently large and thereby obtaining the desired result. □

4.3.2 Notions of regularity and Hölder continuity

Throughout the remainder of this chapter, $\mathbf{v}$ will denote a fixed basis for $\mathbb{V}$ and correspondingly we henceforth assume the notational conventions appearing in Proposition 4.2.5 and $\mathbf{n} = 2\mathbf{m}$ is fixed. For $\alpha \in \mathbb{R}_+^d$, consider the homogeneous norm $|\cdot|_{\mathbf{v}}^{\alpha}$ defined by

$$|x|_{\mathbf{v}}^{\alpha} = \sum_{i=1}^{d} |x_i|^{\alpha_i}$$

for $x \in \mathbb{V}$ where $\phi_{\mathbf{v}}(x) = (x_1, x_2, \dots, x_d)$. As one can easily check,

$$|t^{A_\alpha} x|_{\mathbf{v}}^{\alpha} = t|x|_{\mathbf{v}}^{\alpha}$$

for all $t > 0$ and $x \in \mathbb{V}$ where $A_\alpha \in \operatorname{End}(\mathbb{V})$ is represented by the matrix

$$(A_\alpha)_{\mathbf{v}} = \operatorname{diag}(\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_d^{-1})$$

with respect to the basis $\mathbf{v}$.

Definition 4.3.5. *Let $\mathbf{m} \in \mathbb{N}_+^d$. We say that $\alpha \in \mathbb{R}_+^d$ is consistent with $\mathbf{m}$ if*

$$A_\alpha = \omega(I - E) \tag{4.15}$$

for some $\omega > 0$ where A_α is as above and E is that which appears in Proposition 4.2.5.

As one can check, α is consistent with $\mathbf{m}$ if and only if $\alpha = a^{-1}\omega$ where

$$\omega = \left(\frac{2m_1}{2m_1 - 1}, \frac{2m_2}{2m_2 - 1}, \dots, \frac{2m_d}{2m_d - 1} \right). \tag{4.16}$$

Definition 4.3.6. *Let $\Omega \subseteq \Omega' \subseteq \mathbb{V}$ and let $f : \Omega' \to \mathbb{C}$. We say that f is $\mathbf{v}$-Hölder continuous on Ω if for some $\alpha \in \mathbb{I}_+^d$ and positive constant M,*

$$|f(x) - f(y)| \leq M|x - y|_{\mathbf{v}}^{\alpha} \tag{4.17}$$

for all $x, y \in \Omega$. In this case we will say that α is the $\mathbf{v}$-Hölder exponent of f. If $\Omega = \Omega'$ we will simply say that f is $\mathbf{v}$-Hölder continuous with exponent α.

The following proposition essentially states that, for bounded functions, Hölder continuity is a local property; its proof is straightforward and is omitted.

Proposition 4.3.7. *Let $\Omega \subseteq \mathbb{V}$ be open and non-empty. If f is bounded and $\mathbf{v}$-Hölder continuous of order $\alpha \in \mathbb{I}_+^d$, then, for any $\beta < \alpha$, f is also $\mathbf{v}$-Hölder continuous of order β.*

In view of the proposition, we immediately obtain the following corollary.

Corollary 4.3.8. *Let $\Omega \subseteq \mathbb{V}$ be open and non-empty and $\mathbf{m} \in \mathbb{N}_+^d$. If f is bounded and $\mathbf{v}$-Hölder continuous on Ω of order $\beta \in \mathbb{I}_+^d$, there exists $\alpha \in \mathbb{I}_+^d$ which is consistent with $\mathbf{m}$ for which f is also $\mathbf{v}$-Hölder continuous of order α.*

Proof. The statement follows from the proposition by choosing any α, consistent with $\mathbf{m}$, such that $\alpha \leq \beta$. $\square$

The following definition captures the minimal regularity we will require of fundamental solutions to the heat equation.

Definition 4.3.9. *Let $\mathbf{n} \in \mathbb{N}_+^d$, $\mathbf{v}$ be a basis of $\mathbb{V}$ and let $\mathcal{O}$ be a non-empty open subset of $[0,T] \times \mathbb{V}$. A function $u(t,x)$ is said to be $(\mathbf{n},\mathbf{v})$-regular on $\mathcal{O}$ if on $\mathcal{O}$ it is continuously differentiable in t and has continuous (spatial) partial derivatives $D_{\mathbf{v}}^{\beta}u(t,x)$ for all multi-indices β for which $|\beta : \mathbf{n}| \leq 1$.*

4.3.3 The Legendre-Fenchel transform and its interplay with v-Hölder continuity

Throughout this section, R is the real part of the symbol P of a positive-homogeneous operator Λ on $\mathbb{V}$. We assume the notation of Proposition 4.2.12 (and hence Proposition 4.2.5) and write $E = E_{\mathbf{v}}^{2\mathbf{m}}$. Let us first record two important results which follow essentially from Proposition 4.2.12.

Corollary 4.3.10.

$$R^{\#} \asymp |\cdot|_{\mathbf{v}}^{\omega}.$$

where ω was defined in (4.16).

Proof. In view of Propositions 4.2.5 and 4.2.12, $F_{\mathbf{v}}^{2\mathbf{m}} = I - E_{\mathbf{v}}^{2\mathbf{m}} \in \operatorname{Exp}(R^{\#}) \cap \operatorname{Exp}(|\cdot|_{\mathbf{v}}^{\omega})$. After recalling that $\{t^{F_{\mathbf{v}}^{2\mathbf{m}}}\}$ is contracting, Proposition 4.3.2 yields the desired result immediately. $\square$

By virtue of Proposition 4.2.12, standard arguments immediately yield the following corollary.

Corollary 4.3.11. *For any $\epsilon > 0$ and polynomial $Q : \mathbb{V} \to \mathbb{C}$, i.e., Q is a polynomial in any coordinate system, then*

$$Q(\cdot)e^{-\epsilon R^{\#}(\cdot)} \in L^{\infty}(\mathbb{V}) \cap L^{1}(\mathbb{V}).$$

Lemma 4.3.12. *Let $\gamma = (2m_{\max} - 1)^{-1}$. Then for any $T > 0$, there exists $M > 0$ such that*

$$R^{\#}(x) \leq Mt^{\gamma} R^{\#}(t^{-E}x)$$

for all $x \in \mathbb{V}$ and $0 < t \leq T$.

Proof. In view of Corollary 4.3.10, it suffices to prove the statement

$$|t^E x|_{\mathbf{v}}^{\omega} \leq Mt^{\gamma} |x|_{\mathbf{v}}^{\omega}$$

for all $x \in \mathbb{V}$ and $0 < t \leq T$ where $M > 0$ and ω is given by (4.16). But for any $0 < t \leq T$ and $x \in \mathbb{V}$,

$$|t^E x|_{\mathbf{v}}^{\omega} = \sum_{j=1}^{d} t^{1/(2m_j - 1)} |x_j|^{\omega_j} \leq t^{\gamma} \sum_{j=1}^{d} T^{(1/(2m_j-1)-\gamma)} |x_j|^{\omega_j}$$

from which the result follows. $\square$

Lemma 4.3.13. *Let $\alpha \in \mathbb{I}_+^d$ be consistent with $\mathbf{m}$. Then there exists positive constants σ and θ such that $0 < \sigma < 1$ and for any $T > 0$ there exists $M > 0$ such that*

$$|x|_{\mathbf{v}}^{\alpha} \leq Mt^{\sigma} (R^{\#}(t^{-E}x))^{\theta}$$

for all $x \in \mathbb{V}$ and $0 < t \leq T$.

Proof. By an appeal to Corollary 4.3.10 and Lemma 4.3.12,

$$|x|_{\mathbf{v}}^{\omega} \leq Mt^{\gamma} R^{\#}(t^{-E}x)$$

for all $x \in \mathbb{V}$ and $0 < t \leq T$. Since α is consistent with $\mathbf{m}$, $\alpha = a^{-1}\omega$ where a is that of Definition 4.3.5, the desired inequality follows by setting $\sigma = \gamma/a$ and $\theta = 1/a$. Because $\alpha \in \mathbb{I}_+^d$, it is necessary that $a \geq 2m_{\min}/(2m_{\min} - 1)$ whence $0 < \sigma \leq (2m_{\min} - 1)/(2m_{\min}(2m_{\max} - 1)) < 1$. $\square$

The following corollary is an immediate application of Lemma 4.3.13.

Corollary 4.3.14. *Let $f : \mathbb{V} \to \mathbb{C}$ be $\mathbf{v}$-Hölder continuous with exponent $\alpha \in \mathbb{I}_+^d$ and suppose that α is consistent with $\mathbf{m}$. Then there exist positive constants σ and θ such that $0 < \sigma < 1$ and, for any $T > 0$, there exists $M > 0$ such that*

$$|f(x) - f(y)| \leq M t^{\sigma} (R^{\#}(t^{-E}))^{\theta}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$. In particular, this estimate holds for the coefficients of H.

4.4 On $(2\mathrm{m}, \mathrm{v})$-positive-semi-elliptic operators

In this section, we introduce a class of variable-coefficient operators on $\mathbb{V}$ whose heat equations are studied in the next section. These operators, in view of Proposition 4.2.5, generalize the class of positive-homogeneous operators. Fix a basis $\mathbf{v}$ of $\mathbb{V}$, $\mathbf{m} \in \mathbb{N}_+^d$ and, in the notation of the previous section, consider a differential operator H of the form

$$\begin{aligned} H = \sum_{|\beta:\mathbf{m}|\leq 2} a_\beta(x) D_{\mathbf{v}}^\beta &= \sum_{|\beta:\mathbf{m}|=2} a_\beta(x) D_{\mathbf{v}}^\beta + \sum_{|\beta:\mathbf{m}|<2} a_\beta(x) D_{\mathbf{v}}^\beta \\ &:= H_p + H_l \end{aligned}$$

where the coefficients $a_\beta : \mathbb{V} \to \mathbb{C}$ are bounded functions. The symbol of H, $P : \mathbb{V} \times \mathbb{V}^* \to \mathbb{C}$, is defined by

$$\begin{aligned} P(y,\xi) = \sum_{|\beta:\mathbf{m}|\leq 2} a_\beta(y) \xi^\beta &= \sum_{|\beta:\mathbf{m}|=2} a_\beta(y) \xi^\beta + \sum_{|\beta:\mathbf{m}|<2} a_\beta(y) \xi^\beta \\ &:= P_p(y,\xi) + P_l(y,\xi). \end{aligned}$$

for $y \in \mathbb{V}$ and $\xi \in \mathbb{V}^*$. We shall call H_p the principal part of H and correspondingly, P_p is its principal symbol. Let's also define $R : \mathbb{V}^* \to \mathbb{R}$ by

$$R(\xi) = \operatorname{Re} P_p(0,\xi) \tag{4.18}$$

for $\xi \in \mathbb{V}^*$. At times, we will freeze the coefficients of H and H_p at a point $y \in \mathbb{V}$ and consider the constant-coefficient operators they define, namely $H(y)$ and $H_p(y)$ (defined in the obvious way). We note that, for each $y \in \mathbb{V}$, $H_p(y)$ is homogeneous with respect to the one-parameter group $\{\delta_t^E\}_{t>0}$ where $E \in \mathrm{Gl}(\mathbb{V})$ is defined by its matrix representation

$$E_{\mathbf{v}} = \text{diag}\{(2m_1)^{-1}, (2m_2)^{-1}, \dots, (2m_d)^{-1}\}$$

in the basis $\mathbf{v}$; i.e., it is homogeneous with respect to the same one-parameter group of dilations at each point in space. This also allows us to uniquely define the *homogeneous order of* H by

$$\mu_H = \operatorname{tr} E = (2m_1)^{-1} + (2m_2)^{-1} + \cdots + (2m_d)^{-1}. \tag{4.19}$$

As in the constant-coefficient setting, $H_p(y)$ is not necessarily homogeneous with respect to a unique group of dilations, i.e., it is possible that $\operatorname{Exp}(H_p(y))$ contains members of $\mathrm{Gl}(\mathbb{V})$ distinct from E. However, we shall henceforth only work with the endomorphism E, defined above, for worrying about this non-uniqueness of dilations does not aid our understanding nor will it sharpen our results. Let us further observe that, for each $y \in \mathbb{V}$, $P_p(y, \cdot)$ and R are homogeneous with respect to $\{t^{E^*}\}_{t>0}$ where $E^* \in \mathrm{Gl}(\mathbb{V}^*)$.

Definition 4.4.1. *The operator H is called $(2\mathbf{m}, \mathbf{v})$-positive-semi-elliptic if for all $y \in \mathbb{V}$, $\operatorname{Re} P_p(y, \cdot)$ is a positive-definite polynomial. H is called uniformly $(2\mathbf{m}, \mathbf{v})$-positive-semi-elliptic if it is $(2\mathbf{m}, \mathbf{v})$-positive-semi-elliptic and there exists $\delta > 0$ for which*

$$\operatorname{Re} P_p(y, \xi) \geq \delta R(\xi)$$

for all $y \in \mathbb{V}$ and $\xi \in \mathbb{V}^*$. When the context is clear, we will simply say that H is positive-semi-elliptic and uniformly positive-semi-elliptic respectively.

In light of the above definition, a semi-elliptic operator H is one that, at every point $y \in \mathbb{V}$, its frozen-coefficient principal part $H_p(y)$, is a constant-coefficient positive-homogeneous operator which is homogeneous with respect to the same one-parameter group of dilations on $\mathbb{V}$. A uniformly positive-semi-elliptic operator is one that is semi-elliptic and is uniformly comparable to a constant-coefficient positive-homogeneous operator, namely $H_p(0)$. In this way, positive-homogeneous operators take a central role in this theory.

Remark 13. *In view of Proposition 4.2.5, the definition of R via* (4.18) *agrees with that we have given for constant-coefficient positive-homogeneous operators.*

Remark 14. *For an $(2\mathbf{m}, \mathbf{v})$-positive-semi-elliptic operator H, uniform semi-ellipticity can be formulated in terms of $\operatorname{Re} P_p(y_0, \cdot)$ for any $y_0 \in \mathbb{V}$; such a notion is equivalent in view of Proposition 4.3.2.*

4.5 The heat equation

For a uniformly positive-semi-elliptic operator H, we are interested in constructing a fundamental solution to the heat equation,

$$(\partial_t + H)u = 0 \tag{4.20}$$

on the cylinder $[0, T] \times \mathbb{V}$; here and throughout $T > 0$ is arbitrary but fixed. By definition, a fundamental solution to (4.20) on $[0, T] \times \mathbb{V}$ is a function $Z : (0, T] \times \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ satisfying the following two properties:

1. For each $y \in \mathbb{V}$, $Z(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$-regular on $(0, T) \times \mathbb{V}$ and satisfies (4.20).

2. For each $f \in C_b(\mathbb{V})$,

$$\lim_{t \downarrow 0} \int_{\mathbb{V}} Z(t,x,y) f(y) dy = f(x)$$

for all $x \in \mathbb{V}$.

Given a fundamental solution Z to (4.20), one can easily solve the Cauchy problem: Given $f \in C_b(\mathbb{V})$, find $u(t,x)$ satisfying

$$\begin{cases} (\partial_t + H)u = 0 & \text{on } (0,T) \times \mathbb{V} \\ u(0,x) = f(x) & \text{for } x \in \mathbb{V}. \end{cases}$$

This is, of course, solved by putting

$$u(t,x) = \int_{\mathbb{V}} Z(t,x,y) f(y)\, dy$$

for $x \in \mathbb{V}$ and $0 < t \leq T$ and interpreting $u(0,x)$ as that defined by the limit of $u(t,x)$ as $t \downarrow 0$. The remainder of this chapter is essentially dedicated to establishing the following result:

Theorem 4.5.1. *Let H be uniformly $(2\mathbf{m}, \mathbf{v})$-positive-semi-elliptic with bounded $\mathbf{v}$-Hölder continuous coefficients. Let R and μ_H be defined by* (4.18) *and* (4.19) *respectively and denote by $R^{\#}$ the Legendre-Fenchel transform of R. Then, for any $T > 0$, there exists a fundamental solution $Z : (0,T] \times \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ to* (4.20) *on $[0,T] \times \mathbb{V}$ such that, for some positive constants C and M,*

$$|Z(t,x,y)| \leq \frac{C}{t^{\mu_H}} \exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right)\right) \tag{4.21}$$

for $x, y \in \mathbb{V}$ and $0 < t \leq T$.

We remark that, by definition, the fundamental solution Z given by Theorem 4.5.1 is $(2\mathbf{m}, \mathbf{v})$-regular. Thus Z is necessarily continuously differentiable in t

and has continuous spatial derivatives of all orders β such that $|\beta : \mathbf{m}| \leq 2$.

As we previously mentioned, the result above is implied by the work of S. D. Eidelman for $2\vec{b}$-parabolic systems on $\mathbb{R}^d$ (where $\vec{b} = \mathbf{m}$) [36,37]. Eidelman's systems, of the form (4.2), are slightly more general than we have considered here, for their coefficients are also allowed to depend on t (but in a uniformly Hölder continuous way). Admitting this t-dependence is a relatively straightforward matter and, for simplicity of presentation, we have not included it (see Remark 15). In this slightly more general situation, stated in $\mathbb{R}^d$ and in which $\mathbf{v} = \mathbf{e}$ is the standard Euclidean basis, Theorem 2.2 (p.79) [37] guarantees the existence of a fundamental solution $Z(t,x,y)$ to (4.2), which has the same regularity appearing in Theorem 4.5.1 and satisfies

$$|Z(t,x,y)| \leq \frac{C}{t^{1/(2m_1)+1/(2m_2)+\cdots+1/(2m_d)}} \exp\left(-M\sum_{k=1}^{d} \frac{|x_k - y_k|^{2m_k/(2m_k-1)}}{t^{1/(2m_k-1)}}\right) \tag{4.22}$$

for $x, y \in \mathbb{R}^d$ and $0 < t \leq T$ where C and M are positive constants. By an appeal to Corollary 4.3.10, we have $R^{\#} \asymp |\cdot|_{\mathbf{v}}^{\omega}$ and from this we see that the estimates (4.21) and (4.22) are comparable.

In view of Corollary 4.3.8, the hypothesis of Theorem 4.5.1 concerning the coefficients of H immediately imply the following a priori stronger condition:

Hypothesis 4.1. *There exists $\alpha \in \mathbb{I}_+^d$ which is consistent with $\mathbf{m}$ and for which the coefficients of H are bounded and $\mathbf{v}$-Hölder continuous on $\mathbb{V}$ of order α.*

4.5.1 Levi's Method

In this subsection, we construct a fundamental solution to (4.20) under only the assumption that H, a uniformly $(2\mathbf{m},\mathbf{v})$-positive-semi-elliptic operator, satisfies Hypothesis 4.1. Henceforth, all statements include Hypothesis 4.1 without explicit mention. We follow the famous method of E. E. Levi, c.f., [64] as it was adopted for parabolic systems in [35] and [40]. Although well-known, Levi's method is lengthy and tedious and we will break it into three steps. Let's motivate these steps by first discussing the heuristics of the method.

We start by considering the auxiliary equation

$$\big(\partial_t + \sum_{|\beta:\mathbf{m}|=2} a_\beta(y) D^\beta_{\mathbf{v}}\big)u = (\partial_t + H_p(y))u = 0 \tag{4.23}$$

where $y \in \mathbb{V}$ is treated as a parameter. This is the so-called frozen-coefficient heat equation. As one easily checks, for each $y \in \mathbb{V}$,

$$G_p(t,x;y) := \int_{\mathbb{V}^*} e^{-i\xi(x)} e^{-tP_p(y,\xi)} d\xi \qquad (x \in \mathbb{V}, t > 0)$$

solves (4.23). By the uniform semi-ellipticity of H, it is clear that $G_p(t,\cdot;y) \in \mathcal{S}(\mathbb{V})$ for $t > 0$ and $y \in \mathbb{V}$. As we shall see, more is true: G_p is an approximate identity in the sense that

$$\lim_{t\downarrow 0} \int_{\mathbb{V}} G_p(t, x-y; y) f(y)\, dy = f(x)$$

for all $f \in C_b(\mathbb{V})$. Thus, it is reasonable to seek a fundamental solution to (4.20) of the form

$$\begin{aligned} Z(t,x,y) &= G_p(t,x-y;y) + \int_0^t \int_{\mathbb{V}} G_p(t-s, x-z; z)\phi(s,z,y) dz ds \\ &= G_p(t,x-y;y) + W(t,x,y) \end{aligned} \tag{4.24}$$

where ϕ is to be chosen to ensure that the correction term W is $(2\mathbf{m}, \mathbf{v})$-regular, accounts for the fact that G_p solves (4.23) but not (4.20), and is "small enough" as $t \to 0$ so that the approximate identity aspect of Z is inherited directly from G_p.

Assuming for the moment that W is sufficiently regular, let's apply the heat operator to (4.24) with the goal of finding an appropriate ϕ to ensure that Z is a solution to (4.20). Putting

$$K(t,x,y) = -(\partial_t + H)G_p(t, x-y; y),$$

we have formally,

$$\begin{aligned}
&(\partial_t + H)Z(t,x,y) \\
&\quad= -K(t,x,y) + (\partial_t + H)\int_0^t \int_{\mathbb{V}} G_p(t-s, x-z; z)\phi(s,z,y)\,dz\,ds \\
&\quad= -K(t,x,y) + \lim_{s\uparrow t}\int_{\mathbb{V}} G_p(t-s, x-z; z)\phi(s,z,y)\,dz \\
&\qquad\qquad - \int_0^t \int_{\mathbb{V}} -(\partial_t + H)G_p(t-s, x-z; z)\phi(s,z,y)\,dz\,ds \\
&\quad= -K(t,x,y) + \phi(t,x,y) - \int_0^t \int_{\mathbb{V}} K(t-s,x,z)\phi(s,z,y)\,dz\,ds \qquad (4.25)
\end{aligned}$$

where we have made use of Leibniz' rule and our assertion that G_p is an approximate identity. Thus, for Z to satisfy (4.20), ϕ must satisfy the integral equation

$$\begin{aligned}
K(t,x,y) &= \phi(t,x,y) - \int_0^t \int_{\mathbb{V}} K(t-s,x,z)\phi(s,z,y)\,dz\,ds \\
&= \phi(t,x,y) - L(\phi)(t,x,y). \qquad (4.26)
\end{aligned}$$

Viewing L as a linear integral operator, (4.26) is the equation $K = (I-L)\phi$ which has the solution

$$\phi = \sum_{n=0}^{\infty} L^n K \qquad (4.27)$$

provided the series converges in an appropriate sense.

Taking the above as purely formal, our construction will proceed as follows: We first establish estimates for G_p and show that G_p is an approximate identity; this is Step 1. In Step 2, we will define ϕ by (4.27) and, after deducing some subtle estimates, show that ϕ's defining series converges whence (4.26) is satisfied. Finally in Step 3, we will make use of the estimates from Steps 1 and 2 to validate the formal calculation made in (4.25). Everything will be then pieced together to show that Z, defined by (4.24), is a fundamental solution to (4.20). Our entire construction depends on obtaining precise estimates for G_p and for this we will rely heavily on the homogeneity of P_p and the Legendre-Fenchel transform of R.

Remark 15. *One can allow the coefficients of H to also depend on t in a uniformly continuous way, and Levi's method pushes though by instead taking G_p as the solution to a frozen-coefficient initial value problem [36, 37].*

Step 1. Estimates for G_p and its derivatives

The lemma below is a basic building block used in our construction of a fundamental solution to (4.20) via Levi's method and it makes essential use of the uniform semi-ellipticity of H. We note however that the precise form of the constants obtained, as they depend on k and β, are more detailed than needed for the method to work. Also, the partial differential operators $D_{\mathbf{v}}^{\beta}$ of the lemma are understood to act of the x variable of $G_p(t,x;y)$.

Lemma 4.5.2. *There exist positive constants M and C_0 and, for each multi-index β, a* *positive constant C_β such that, for any $k \in \mathbb{N}$,*

$$|\partial_t^k D_{\mathbf{v}}^\beta G_p(t,x;y)| \leq \frac{C_\beta C_0^k k!}{t^{\mu_H + k + |\beta:2\mathbf{m}|}} \exp\left(-tMR^{\#}\left(x/t\right)\right) \tag{4.28}$$

for all $x, y \in \mathbb{V}$ and $t > 0$.

Before proving the lemma, let us note that $tR^{\#}(x/t) = R^{\#}(t^{-E}x)$ for all $t > 0$ and $x \in \mathbb{V}$ in view of Proposition 4.2.12. Thus the estimate (4.28) can be written equivalently as

$$|\partial_t^k D_{\mathbf{v}}^\beta G_p(t,x;y)| \leq \frac{C_\beta C_0^k k!}{t^{\mu_H + k + |\beta:2\mathbf{m}|}} \exp(-MR^{\#}(t^{-E}x)) \tag{4.29}$$

for $x, y \in \mathbb{V}$ and $t > 0$. We will henceforth use these forms interchangeably and without explicit mention.

Proof. Let us first observe that, for each $x, y \in \mathbb{V}$ and $t > 0$,

$$\begin{aligned} \partial_t^k D_{\mathbf{v}}^\beta G_p(t,x;y) &= \int_{\mathbb{V}^*} (P_p(y,\xi))^k \xi^\beta e^{-i\xi(x)} e^{-tP_p(y,\xi)}\, d\xi \\ &= \int_{\mathbb{V}^*} (P_p(y, t^{-E^*}\xi))^k (t^{-E^*}\xi)^\beta e^{-i\xi(t^{-E}x)} e^{-P_p(y,\xi)} t^{-\operatorname{tr} E}\, d\xi \\ &= t^{-\mu_H - k - |\beta:2\mathbf{m}|} \int_{\mathbb{V}^*} (P_p(y,\xi))^k \xi^\beta e^{-i\xi(t^{-E}x)} e^{-P_p(y,\xi)} d\,\xi \end{aligned}$$

where we have used the homogeneity of P_p with respect to $\{t^{E^*}\}$ and the fact that $\mu_H = \operatorname{tr} E$. Therefore

$$t^{\mu_H + k + |\beta:2\mathbf{m}|} (\partial_t^k D_{\mathbf{v}}^\beta G_p(t, \cdot\,; y))(t^E x) = \int_{\mathbb{V}^*} (P_p(y,\xi))^k \xi^\beta e^{-i\xi(x)} e^{-P_p(y,\xi)} d\xi \tag{4.30}$$

for all $x, y \in \mathbb{V}$ and $t > 0$. Thus, to establish (4.28) (equivalently (4.29)) it suffices to estimate the right hand side of (4.30) which is independent of t.

The proof of the desired estimate requires making a complex change of variables and for this reason we will work with the complexification of $\mathbb{V}^*$, whose members are denoted by $z = \xi - i\nu$ for $\xi, \nu \in \mathbb{V}^*$; this space is isomorphic to $\mathbb{C}^d$. We claim that there are positive constants C_0, M_1, M_2 and, for each multi-index β, a positive constant C_β such that, for each $k \in \mathbb{N}$,

$$|(P_p(y, \xi - i\nu))^k (\xi - i\nu)^\beta e^{-P_p(y,\xi - i\nu)}| \leq C_\beta C_0^k k! e^{-M_1 R(\xi)} e^{M_2 R(\nu)} \tag{4.31}$$

for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$. Let us first observe that

$$P_p(y, \xi - i\nu) = P_p(y, \xi) + \sum_{|\beta:\mathbf{m}|=2} \sum_{\gamma < \beta} a_{\beta,\gamma} \xi^\gamma (-i\nu)^{\beta - \gamma}$$

for all $z, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$, where $a_{\beta,\gamma}$ are bounded functions of y arising from the coefficients of H and the coefficients of the multinomial expansion. By virtue of the uniform semi-ellipticity of H and the boundedness of the coefficients, we have

$$-\operatorname{Re} P_p(y, \xi - i\nu) \leq -\delta R(\xi) + C \sum_{|\beta:\mathbf{m}|=2} \sum_{\gamma < \beta} |\xi^\gamma \nu^{\beta - \gamma}|$$

for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$ where C is a positive constant. By applying Lemma 4.3.4 to each term $|\xi^\gamma \nu^{\beta-\gamma}|$ in the summation, we can find a positive constant M for which the entire summation is bounded above by $\delta/2R(\xi) + MR(\nu)$ for all $\xi, \nu \in \mathbb{V}^*$. By setting $M_1 = \delta/6$, we have

$$-\operatorname{Re} P_p(y, \xi - i\nu) \leq -3M_1 R(\xi) + MR(\nu) \tag{4.32}$$

for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$. By analogous reasoning (making use of item 1 of Lemma 4.3.4), there exists a positive constant C for which

$$|P_p(y, \xi - i\nu)| \leq C(R(\xi) + R(\nu))$$

for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$. Thus, for any $k \in \mathbb{N}$,

$$|P_p(y, \xi - i\nu)|^k \leq \frac{C^k k!}{M_1^k} \frac{(M_1(R(\xi) + R(\nu)))^k}{k!} \leq C_0^k k! e^{M_1(R(\xi) + R(\nu))} \tag{4.33}$$

for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$ where $C_0 = C/M_1$. Finally, for each multi-index β, another application of Lemma 4.3.4 gives $C' > 0$ for which

$$|(\xi - i\nu)^\beta| \leq |\xi^\beta| + |\nu^\beta| + \sum_{0<\gamma<\beta} c_{\gamma,\beta}|\xi^\gamma \nu^{\beta-\gamma}| \leq C'\left((R(\xi) + R(\nu))^n + 1\right)$$

for all $\xi, \nu \in \mathbb{V}^*$ where $n \in \mathbb{N}$ has been chosen to satisfy $|\beta : 2n\mathbf{m}| < 1$. Consequently, there is a positive constant C_β for which

$$|(\xi - i\nu)^\beta| \leq C_\beta e^{M_1(R(\xi)+R(\nu))} \tag{4.34}$$

for all $\xi, \nu \in \mathbb{V}^*$. Upon combining (4.32), (4.33) and (4.34), we obtain the inequality

$$\left|P_p(y, \xi - i\nu)^k (\xi - i\nu)^\beta e^{-P_p(y,\xi-i\nu)}\right| \leq C_\beta C_0^k k! e^{-M_1 R(\xi) + (M+2M_1)R(\nu)}$$

which holds for all $\xi, \nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$. Upon paying careful attention to the way in which our constants were chosen, we observe the claim is established by setting $M_2 = M + 2M_1$.

From the claim above, it follows that, for any $\nu \in \mathbb{V}^*$ and $y \in \mathbb{V}$, the following change of coordinates by means of a $\mathbb{C}^d$ contour integral is justified:

$$\begin{aligned}
&\int_{\mathbb{V}^*} (P_p(y,\xi))^k \xi^\beta e^{-i\xi(x)} e^{-P_p(y,\xi)}\, d\xi \\
&\qquad = \int_{\xi \in \mathbb{V}^*} (P_p(y, \xi - i\nu))^k (\xi - i\nu)^\beta e^{-i(\xi - i\nu)(x)} e^{-P_p(y,\xi-i\nu)}\, d\xi \\
&\qquad = e^{-\nu(x)} \int_{\xi \in \mathbb{V}^*} (P_p(y, \xi - i\nu))^k (\xi - i\nu)^\beta e^{-i\xi(x)} e^{-P_p(y,\xi-i\nu)}\, d\xi.
\end{aligned}$$

Thus, by virtue of the estimate (4.31),

$$\begin{aligned}
\left|\int_{\mathbb{V}^*} (P_p(y,\xi))^k \xi^\beta e^{-i\xi(x)} e^{-P_p(y,\xi)}\, d\xi\right| &\leq C_\beta C_0^k k! e^{-\nu(x)} e^{M_2 R(\nu)} \int_{\mathbb{V}^*} e^{-M_1 R(\xi)}\, d\xi \\
&\leq C_\beta C_0^k k! e^{-(\nu(x) - M_2 R(\nu))}
\end{aligned}$$

for all $x, y \in \mathbb{V}$ and $\nu \in \mathbb{V}^*$ where we have absorbed the integral of

$\exp(-M_1R(\xi))$ into C_β. Upon minimizing with respect to $\nu \in \mathbb{V}^*$, we have

$$\left|\int_{\mathbb{V}^*}(P_p(y,\xi))^k\xi^\beta e^{-i\xi(x)}e^{-P_p(y,\xi)}d\xi\right| \leq C_\beta C_0^k k! e^{-(M_2R)^\#(x)} \leq C_\beta C_0^k k! e^{-MR^\#(x)} \tag{4.35}$$

for all x and $y \in \mathbb{V}$ because

$$-(M_2R)^\#(x) = -\sup_\nu\{\nu(x) - M_2R(\nu)\} = \inf_\nu\{-(\nu(x) - M_2R(\nu))\};$$

in this we see the natural appearance of the Legendre-Fenchel transform. The replacement of $(M_2R)^\#(x)$ by $MR^\#(x)$ is done using Corollary 4.3.3 and, as required, the constant M is independent of k and β. Upon combining (4.30) and (4.35), we obtain the desired estimate (4.28). □

As a simple corollary to the lemma, we obtain Proposition 4.2.11.

Proof of Proposition 4.2.11. Given a positive-homogeneous operator Λ, we invoke Proposition 4.2.5 to obtain $\mathbf{v}$ and $\mathbf{m}$ for which $\Lambda = \sum_{|\beta:\mathbf{m}|=2} a_\beta D_{\mathbf{v}}^\beta$. In other words, Λ is an $(2\mathbf{m},\mathbf{v})$-positive-semi-elliptic operator which consists only of its principal part. Consequently, the heat kernel K_Λ satisfies $K_\Lambda^t(x) = G_p(t,x;0)$ for all $x \in \mathbb{V}$ and $t > 0$ and so we immediately obtain the estimate (4.11) from the lemma. □

Making use of Hypothesis 4.1, a similar argument to that given in the proof of Lemma 4.5.2 yields the following lemma.

Lemma 4.5.3. *There is a positive constant M and, to each multi-index β, a positive constant C_β such that*

$$|D_{\mathbf{v}}^\beta[G_p(t,x;y+h) - G_p(t,x;y)]| \leq C_\beta t^{-(\mu_H + |\beta:2\mathbf{m}|)}|h|_{\mathbf{v}}^\alpha \exp(-tMR^\#(x/t))$$

for all $t > 0$, $x, y, h \in \mathbb{V}$. Here, in view of Hypothesis 4.1, α is the $\mathbf{v}$-Hölder continuity exponent for the coefficients of H.

Lemma 4.5.4. *Suppose that $g \in C_b((t_0,T] \times \mathbb{V})$ where $0 \le t_0 < T < \infty$. Then, on any compact set $Q \subseteq (t_0,T] \times \mathbb{V}$,*

$$\int_{\mathbb{V}} G_p(t, x-y; y) g(s-t, y)\, dy \to g(s,x)$$

uniformly on Q as $t \to 0$. In particular, for any $f \in C_b(\mathbb{V})$,

$$\int_{\mathbb{V}} G_p(t, x-y; y) f(y)\, dy \to f(x)$$

uniformly on all compact subsets of $\mathbb{V}$ as $t \to 0$.

Proof. Let Q be a compact subset of $(t_0,T] \times \mathbb{V}$ and write

$$\begin{aligned}
&\int_{\mathbb{V}} G_p(t, x-y; y) g(s-t, y)\, dy \\
&\quad = \int_{\mathbb{V}} G_p(t, x-y; x) g(s-t, y)\, dy \\
&\qquad\qquad + \int_{\mathbb{V}} [G_p(t, x-y; y) - G_p(t, x-y; x)] g(s-t, y)\, dy \\
&\quad := I_t^{(1)}(s,x) + I_t^{(2)}(s,x).
\end{aligned}$$

Let $\epsilon > 0$ and, in view of Corollary 4.3.11, let K be a compact subset of $\mathbb{V}$ for which

$$\int_{\mathbb{V}\setminus K} \exp(-MR^{\#}(z))\, dz < \epsilon$$

where the constant M is that given in (4.28) of Lemma 4.5.2. Using the continuity of g, we have for sufficiently small $t > 0$,

$$\sup_{\substack{(s,x)\in Q \\ z \in K}} |g(s-t, x - t^E z) - g(s,x)| < \epsilon.$$

We note that, for any $t > 0$ and $x \in \mathbb{V}$,

$$\int_{\mathbb{V}} G_p(t, x-y; x)\, dy = e^{-tP_p(x,\xi)}\Big|_{\xi=0} = 1.$$

Appealing to Lemma 4.5.2 we have, for any $(s,x) \in Q$,

$$\begin{aligned} |I_t^{(1)}(s,x) - g(s,x)| &\leq \Big| \int_{\mathbb{V}} G_p(t, x-y; x)(g(s-t,y) - g(s,x))\, dy \Big| \\ &\leq \int_{\mathbb{V}} |G_p(1,z;x)(g(s-t, x - t^E z) - g(s,x))|\, dz \\ &\leq 2\|g\|_\infty C \int_{\mathbb{V}\setminus K} \exp(-MR^{\#}(z))\, dz \\ &\quad + C \int_K \exp(-MR^{\#}(z))|(g(s-t, x-t^E z) - g(s,x))|\, dz \\ &\leq \epsilon C \left(2\|g\|_\infty + \|e^{-MR^{\#}}\|_1 \right); \end{aligned}$$

here we have made the change of variables: $y \mapsto t^E(x-y)$ and used the homogeneity of P_p to see that $t^{\mu_H} G_p(t, t^E z; x) = G_p(1,z;x)$. Therefore $I_t^{(1)}(s,x) \to g(s,x)$ uniformly on Q as $t \to 0$.

Let us now consider $I^{(2)}$. With the help of Lemmas 4.3.13 and 4.5.3 and by making similar arguments to those above we have

$$\begin{aligned} |I_t^{(2)}(s,x)| &\leq C\|g\|_\infty \int_{\mathbb{V}} t^{-\mu_H} |x-y|_{\mathbf{v}}^\alpha \exp(-MR^{\#}(t^{-E}(x-y))\, dy \\ &\leq \|g\|_\infty C t^\sigma \int_{\mathbb{V}} t^{-\operatorname{tr} E} (R^{\#}(t^{-E}(x-y)))^\theta \exp(-MR^{\#}(t^{-E}(x-y)))\, dy \\ &\leq \|g\|_\infty C t^\sigma \int_{\mathbb{V}} (R^{\#}(x))^\theta \exp(-MR^{\#}(z))\, dz \leq \|g\|_\infty C' t^\sigma \end{aligned}$$

for all $s \in (t_0, T]$, $0 < t < s - t_0$ and $x \in \mathbb{V}$; here $0 < \sigma < 1$. Consequently, $I_t^{(2)}(s,x) \to 0$ uniformly on Q as $t \to 0$ and the lemma is proved. □

Combining the results of Lemmas 4.5.2 and 4.5.4 yields at once:

Corollary 4.5.5. *For each $y \in \mathbb{V}$, $G_p(\cdot, \cdot - y; y)$ is a fundamental solution to* (4.23).

Step 2. Construction of ϕ and the integral equation

For $t > 0$ and $x, y \in \mathbb{V}$, put

$$\begin{aligned} K(t,x,y) &= -(\partial_t + H)G_p(t, x-y; y) \\ &= \big(H_p(y) - H\big)G_p(t, x-y; y) \\ &= \int_{\mathbb{V}^*} e^{-i\xi(x-y)}\big(P_p(y,\xi) - P(x,\xi)\big)e^{-tP_p(y,\xi)}\,d\xi \end{aligned}$$

and iteratively define

$$K_{n+1}(t,x,y) = \int_0^t \int_{\mathbb{V}} K_1(t-s,x,z)K_n(s,z,y)\,dzds$$

where $K_1 = K$. In the sense of (4.27), note that $K_{n+1} = L^n K$.

We claim that for some $0 < \rho < 1$,

$$|K(t,x,y)| \le Ct^{-(\mu_H + 1 - \rho)} \exp(-MR^{\#}(t^{-E}(x-y))) \tag{4.36}$$

for all $x, y \in \mathbb{V}$ and $0 < t \le T$ where M and C are positive constants. Indeed, observe that

$$|K(t,x,y)| \le \sum_{|\beta:\mathbf{m}|=2} |a_\beta(y) - a_\beta(x)||D_{\mathbf{v}}^\beta G_p(t, x-y; y)| + C \sum_{|\beta:\mathbf{m}|<2} |D_{\mathbf{v}}^\beta G_p(t, x-y; y)|$$

for all $x, y \in \mathbb{V}$ and $t > 0$ where we have used the fact that the coefficients of H are bounded. In view of Lemma 4.5.2, we have

$$\begin{aligned} |K(t,x,y)| \le \sum_{|\beta:\mathbf{m}|=2} |a_\beta(y) - a_\beta(x)|Ct^{-(\mu_H+1)} \exp(-MR^{\#}(t^{-E}(x-y))) \\ +Ct^{-(\mu_H+\eta)} \exp(-MR^{\#}(t^{-E}(x-y))) \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \le T$ where

$$\eta = \max\{|\beta : 2\mathbf{m}| : |\beta : \mathbf{m}| \neq 2 \text{ and } a_\beta \neq 0\} < 1.$$

Using Hypothesis 4.1, an appeal to Corollary 4.3.14 ensures that

$$\begin{aligned} |K(t,x,y)| &\le Ct^{\sigma-(\mu_H+1)}(R^{\#}(t^{-E}(x-y)))^\theta \exp(-MR^{\#}(t^{-E}(x-y))) \\ &\qquad +Ct^{-(\mu_H+\eta)} \exp(-MR^{\#}(t^{-E}(x-y))) \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \le T$ where θ is positive and $0 < \sigma < 1$. Our claim is then justified by choosing $\rho = \max\{\sigma, 1-\eta\}$ and adjusting the constants C and M appropriately.

Taking cues from our heuristic discussion, we will soon form a series whose summands are the functions K_n for $n \ge 1$. In order to talk about the convergence of this series, our next task is to estimate these functions and in doing this we will observe two separate behaviors: a finite number of terms will exhibit singularities in t at the origin; the remainder of the terms will be absent of such singularities and will be estimated with the help of the Gamma function. We first address the terms with the singularities.

Lemma 4.5.6. *Let $0 < \rho < 1$ and $M > 0$ be as above. For any positive natural number n such that $\rho(n-1) \le \mu_H + 1$ and $\epsilon > 0$ for which $\epsilon n < 1$, there is a constant $C_n(\epsilon) \ge 1$ such that*

$$|K_n(t,x,y)| \le C_n(\epsilon) t^{-(\mu_H+1-n\rho)} \exp(-M(1-\epsilon n) R^{\#}(t^{-E}(x-y)))$$

for all $x, y \in \mathbb{V}$ and $0 < t \le T$.

Proof. In view (4.36), it is clear that the estimate holds when $n = 1$. Let us assume the estimate holds for $n \ge 1$ such that $\rho n < 1 + \mu_H$ and $\epsilon > 0$ for which $\epsilon n < \epsilon(n+1) < 1$. Then

$$\begin{aligned} &|K_{n+1}(t,x,y)| \\ &\le \int_0^t \int_{\mathbb{V}} C_n(\epsilon)(t-s)^{-(\mu_H+1-n\rho)} C_1(\epsilon) s^{-(\mu_H+1-\rho)} \\ &\qquad \times \exp(-M_{\epsilon,n} R^{\#}((t-s)^{-E}(x-z))) \exp(-M R^{\#}(s^{-E}(z-y)))\, dz\, ds \end{aligned} \tag{4.37}$$

for $x, y \in \mathbb{V}$ and $0 < t \leq T$ where we have set $M_{\epsilon,n} = M(1-\epsilon n)$. Observe that

$$\begin{aligned}
R^{\#}(t^{-E}(x-y)) &= \sup\{\xi(x-y) - tR(\xi)\} \\
&= \sup\{\xi(x-z) - (t-s)R(\xi) + \xi(z-y) - sR(\xi)\} \\
&\leq R^{\#}((t-s)^{-E}(x-z)) + R^{\#}(s^{-E}(z-y)) \qquad (4.38)
\end{aligned}$$

for all $x, y, z \in \mathbb{V}$ and $0 < s \leq t$ and therefore

$$\begin{aligned}
&\exp(-M_{\epsilon,n}R^{\#}((t-s)^{-E}(x-z)))\exp(-MR^{\#}(s^{-E}(z-y))) \\
&\qquad \leq \exp(-M_{\epsilon,n+1}R^{\#}(t^{-E}(x-y))) \\
&\qquad\qquad \times \exp(-\epsilon n M(R^{\#}((t-s)^{-E}(x-z) + R^{\#}(s^{-E}(z-y))). \quad (4.39)
\end{aligned}$$

Combining (4.37), (4.38) and (4.39) yields

$$\begin{aligned}
&|K_{n+1}(t,x,y)| \\
&\leq C_1(\epsilon)C_n(\epsilon)\exp(-M_{\epsilon,n+1}R^{\#}(t^{-E}(x-y)))\int_0^t\int_{\mathbb{V}}(t-s)^{-(\mu_H+1-n\rho)} \\
&\quad \times s^{-(\mu_H+1-\rho)}\exp(-\epsilon n M(R^{\#}((t-s)^{-E}(x-z) + R^{\#}(s^{-E}(z-y)))\,dz\,ds \\
&\leq C_1(\epsilon)C_n(\epsilon)\exp(-M_{\epsilon,n+1}R^{\#}(t^{-E}(x-y))) \\
&\qquad \times\Big[(t/2)^{-(\mu_H+1-n\rho)}\int_0^{t/2}\int_{\mathbb{V}} s^{-(\mu_H+1-\rho)} \\
&\qquad\qquad\qquad \times\exp(-\epsilon n M R^{\#}(s^{-E}(z-y)))\,dz\,ds \\
&\qquad +(t/2)^{-(\mu_H+1+\rho)}\int_{t/2}^t\int_{\mathbb{V}}(t-s)^{-(\mu_H+1-n\rho)} \\
&\qquad\qquad\qquad \exp(-\epsilon n M R^{\#}((t-s)^{-E}(x-z)))\,dz\,ds\Big] \\
&\leq C_1(\epsilon)M_n(\epsilon)\exp(-M_{\epsilon,n+1}R^{\#}(t^{-E}(x-y))) \\
&\quad \times\Big[(t/2)^{-(\mu_H+1-n\rho)}\int_0^{t/2} s^{-(1-\rho)}\,ds\int_{\mathbb{V}}\exp(-\epsilon n M R^{\#}(z))\,dz \\
&\qquad\qquad +(t/2)^{-(\mu_H+1+\rho)}\int_0^{t/2} s^{-(1-n)\rho}ds\int_{\mathbb{V}}\exp(-\epsilon n M R^{\#}(z))\,dz\Big] \\
&\leq C_{n+1}(\epsilon)t^{-(\mu_H+1-(n+1)\rho)}\exp(-M_{\epsilon,n+1}R^{\#}(t^{-E}(x-y))
\end{aligned}$$

for all $x, y \in \mathbb{V}$ and $t > 0$ where we have put

$$C_{n+1}(\epsilon) = C_1(\epsilon)C_n(\epsilon)\frac{n+1}{n\rho}2^{\mu_H+(1-(n+1)\rho)}\int_{\mathbb{V}}\exp(-\epsilon n M R^{\#}(z))\,dz$$

and made use of Corollary 4.3.11. $\square$

Remark 16. *The estimate* (4.38) *is an important one and will be used again. In the context of elliptic operators, i.e., where* $R^{\#}(x) = C_m|x|^{2m/(2m-1)}$*, the analogous result is captured in Lemma 5.1 of [35]. It is interesting to note that S. D. Eidelman worked somewhat harder to prove it. Perhaps this is because the appearance of the Legendre-Fenchel transform wasn't noticed.*

It is clear from the previous lemma that for sufficiently large n, K_n is bounded by a positive power of t. The first such n is $\bar{n} := \lceil \rho^{-1}(\operatorname{tr} E + 1)\rceil$. In view of the previous lemma,

$$|K_{\bar{n}}(t,x,y)| \leq C_{\bar{n}}(\epsilon)\exp(-M(1-\epsilon\bar{n})R^{\#}(t^{-E}(x-y)))$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where we have adjusted $C_{\bar{n}}(\epsilon)$ to account for this positive power of t. Let $\delta < 1/2$ and set

$$\epsilon = \frac{\delta}{\bar{n}}, \quad M_1 = M(1-\delta) \quad \text{and} \quad C_0 = \max_{1\leq n\leq \bar{n}} C_n(\epsilon).$$

Upon combining proceeding estimate with the estimates(4.36) and (4.38), we have

$$\begin{aligned}
&|K_{\bar{n}+1}(t,x,y)| \\
&\quad \leq \quad C_0^2 \int_0^t \int_{\mathbb{V}} (t-s)^{-(\mu_H+(1-\rho))} \\
&\qquad\qquad \times \exp(-MR^{\#}((t-s)^{-E}(x-z))\exp(-M(1-\epsilon\bar{n})R^{\#}(s^{-E}(z-y)))\,ds\,dz \\
&\quad \leq \quad C_0^2 \exp(-M_1R^{\#}(t^{-E}(x-y)))\int_0^t\int_{\mathbb{V}}(t-s)^{-(\mu_H+(1-\rho))} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \exp(-C\delta R^{\#}((t-s)^{-E}(z)))\,dz\,ds \\
&\quad \leq \quad C_0(C_0F)\frac{t^{\rho}}{\rho}\exp(-M_1R^{\#}(t^{-E}(x-y)))
\end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where

$$F = \int_{\mathbb{V}} \exp(-M\delta R^{\#}(z))\,dz < \infty.$$

Let us take this a little further.

Lemma 4.5.7. *For every $k \in \mathbb{N}_+$,*

$$|K_{\bar{n}+k}(t,x,y)| \le \frac{C_0}{\Gamma(\rho)} \frac{(C_0 F \Gamma(\rho))^k}{k!} t^{\rho k} \exp(-M_1 R^{\#}(t^{-E}(x-y))) \tag{4.40}$$

for all $x, y \in \mathbb{V}$ and $0 < t \le T$. Here $\Gamma(\cdot)$ denotes the Gamma function.

Proof. The Euler-Beta function $B(\cdot,\cdot)$ satisfies the well-known identity $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Using this identity, one quickly obtains the estimate

$$\prod_{j=1}^{k-1} B(\rho, 1+j\rho) = \frac{\Gamma(\rho)^{k-1}}{\Gamma(1+k\rho)} \le \frac{\Gamma(\rho)^{k-1}}{k!}.$$

It therefore suffices to prove that

$$|K_{\bar{n}+k}(t,x,y)| \le C_0 (C_0 F)^k \prod_{j=0}^{k-1} B(\rho, 1+j\rho) t^{k\rho} \exp(-M_1 R^{\#}(t^{-E}(x-y))) \tag{4.41}$$

for all $x, y \in \mathbb{V}$ and $0 < t \le T$.

We first note that $B(\rho, 1) = \rho^{-1}$ and so, for $k = 1$, (4.41) follows directly from the calculation proceeding the lemma. We shall induct on k. By another application of (4.36) and (4.38), we have

$$\begin{aligned}
&J_{k+1}(t,x,y) \\
&\quad := \left[C_0^2 (C_0 F)^k \prod_{j=0}^{k-1} B(\rho, 1+j\rho) \right]^{-1} |K_{\bar{n}+k+1}(t,x,y)| \\
&\quad \le \int_0^t \int_{\mathbb{V}} (t-s)^{-(\mu_H + (1-\rho))} s^{-k\rho} \exp(-M R^{\#}((t-s)^{-E}(x-z))) \\
&\qquad\qquad \times \exp(-M_1 R^{\#}(s^{-E}(z-y)))\, dz\, ds \\
&\quad \le \exp(-M_1 R^{\#}(t^{-E}(x-y))) \\
&\qquad \times \int_0^t \int_{\mathbb{V}} (t-s)^{-(\mu_H + (1-\rho))} s^{-k\rho} \exp(-M \delta R^{\#}((t-s)^{-E}(x-z)))\, dz\, ds
\end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$. Upon making the changes of variables $z \to (t-s)^{-E}(x-z)$ followed by $s \to s/t$, we have

$$\begin{aligned} J_{k+1}(t,x,y) &\leq \exp(-M_1 R^{\#}(t^{-E}(x-y)))F \int_0^1 (t-st)^{\rho-1}(st)^{k\rho} t\, ds \\ &\leq \exp(-M_1 R^{\#}(t^{-E}(x-y)))F t^{(k+1)\rho} B(\rho, 1+k\rho) \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$. Therefore (4.41) holds for $k+1$ as required. □

Proposition 4.5.8. *Let $\phi : (0,T] \times \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ be defined by*

$$\phi = \sum_{n=1}^{\infty} K_n.$$

This series converges uniformly for $x, y \in \mathbb{V}$ and $t_0 \leq t \leq T$ where t_0 is any positive constant. There exists $C \geq 1$ for which

$$|\phi(t,x,y)| \leq \frac{C}{t^{\mu_H + (1-\rho)}} \exp(-M_1 R^{\#}(t^{-E}(x-y))) \tag{4.42}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where M_1 and ρ are as in the previous lemmas. Moreover, the identity

$$\phi(t,x,y) = K(t,x,y) + \int_0^t \int_{\mathbb{V}} K(t-s,x,z)\phi(s,z,y)\, dz\, ds \tag{4.43}$$

holds for all $x, y \in \mathbb{V}$ and $0 < t \leq T$.

Proof. Using Lemmas 4.5.6 and 4.5.7 we see that

$$\begin{aligned} &\sum_{k=1}^{\infty} |K_n(t,x,y)| \\ &\leq \; C_0 \Big[\sum_{n=1}^{\bar{n}} t^{-(\mu_H + (1-n\rho))} + \frac{1}{\Gamma(\rho)} \sum_{k=1}^{\infty} \frac{(C_0 F \Gamma(\rho))^k}{k!} t^{k\rho} \Big] \exp(-M_1 R^{\#}(t^{-E}(x-y))) \end{aligned}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ from which (4.42) and our assertion concerning uniform convergence follow. A similar calculation and an application of

Tonelli's theorem justify the following use of Fubini's theorem: For $x, y \in \mathbb{V}$ and $0 < t \leq T$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{V}} K(t-s,x,z)\phi(s,z,y)\,ds\,dz \\
&\qquad = \sum_{n=1}^{\infty} \int_0^t \int_{\mathbb{V}} K(t-s,x,z)K_n(s,z,y)\,dz\,ds = \sum_{n=1}^{\infty} K_{n+1}(t,z,y) \\
&\qquad = \phi(t,x,y) - K(t,x,y)
\end{aligned}$$

as desired. □

The following Hölder continuity estimate for ϕ is obtained by first showing the analogous estimate for K and then deducing the desired result from the integral formula (4.43). As the proof is similar in character to those of the preceding two lemmas, we omit it. A full proof can be found in [37, p.80]. We also note here that the result is stronger than is required for our purposes (see its use in the proof of Lemma 4.5.11). All that is really required is that $\phi(\cdot,\cdot,y)$ satisfies the hypotheses (for f) in Lemma 4.5.10 for each $y \in \mathbb{V}$.

Lemma 4.5.9. *There exists $\alpha \in \mathbb{I}_+^d$ which is consistent with $\mathbf{m}$, $0 < \eta < 1$ and $C \geq 1$ such that*

$$|\phi(t,x+h,y) - \phi(t,x,y)| \leq \frac{C}{t^{\mu_H + (1-\eta)}} |h|_{\mathbf{v}}^{\alpha} \exp(-M_1 R^{\#}(t^{-E}(x-y)))$$

for all $x, y, h \in \mathbb{V}$ and $0 < t \leq T$.

Step 3. Verifying that Z is a fundamental solution to (4.20)

Lemma 4.5.10. *Let $\alpha \in \mathbb{I}_+^d$ be consistent with $\mathbf{m}$ and, for $t_0 > 0$, let $f : [t_0, T] \times \mathbb{V} \to \mathbb{C}$ be bounded and continuous. Moreover, suppose that f is uniformly $\mathbf{v}$-Hölder continuous in x on $[t_0,T] \times \mathbb{V}$ of order α, by which we mean that there is a constant $C > 0$ such that*

$$\sup_{t \in [t_0,T]} |f(t,x) - f(t,y)| \leq C|x-y|_{\mathbf{v}}^{\alpha}$$

for all $x, y \in \mathbb{V}$. Then $u : [t_0, T] \times \mathbb{V} \to \mathbb{C}$ defined by

$$u(t,x) = \int_{t_0}^{t} \int_{\mathbb{V}} G_p(t-s, x-z; z) f(s,z)\, ds\, dz$$

is $(2\mathbf{m}, \mathbf{v})$-regular on $(t_0, T) \times \mathbb{V}$. Moreover,

$$\partial_t u(t,x) = f(t,x) + \lim_{h \downarrow 0} \int_{t_0}^{t-h} \int_{\mathbb{V}} \partial_t G_p(t-s, x-z; z) f(s,z)\, dz\, ds \tag{4.44}$$

and for any β such that $|\beta : \mathbf{m}| \leq 2$, we have

$$D_{\mathbf{v}}^{\beta} u(t,x) = \lim_{h \downarrow 0} \int_{t_0}^{t-h} \int_{\mathbb{V}} D_{\mathbf{v}}^{\beta} G(t-s, x-z; z) f(s,z)\, dz\, ds \tag{4.45}$$

for $x \in \mathbb{V}$ and $t_0 < t < T$.

Before starting the proof, let us observe that, for each multi-index β,

$$\begin{aligned} &|D_{\mathbf{v}}^{\beta} G_p(t-s, x-z; z) f(s,z)| \\ &\qquad \leq \; C(t-s)^{-(\mu_H + |\beta:2\mathbf{m}|)} \exp(-MR^{\#}((t-s)^{-E}(x-z)))|f(s,z)|. \end{aligned}$$

Using the assumption that f is bounded, we observe that

$$\begin{aligned} &\int_{t_0}^{t} \int_{\mathbb{V}} |D_{\mathbf{v}}^{\beta} G_p(t-s, x-z; z) f(s,z)|\, dz\, ds \\ &\qquad \leq \; C \int_{t_0}^{t} \int_{\mathbb{V}} (t-s)^{-\mu_H + |\beta:2\mathbf{m}|} \exp(-MR^{\#}((t-s)^{-E}(x-z)))\, dz\, ds \\ &\qquad \leq \; C \int_{t_0}^{t} \int_{\mathbb{V}} (t-s)^{-|\beta:2\mathbf{m}|} \exp(-MR^{\#}(z))\, dz\, ds \\ &\qquad \leq \; C \int_{t_0}^{t} (t-s)^{-|\beta:2\mathbf{m}|}\, ds \end{aligned}$$

for all $t_0 \leq t \leq T$ and $x \in \mathbb{V}$. When $|\beta : \mathbf{m}| < 2$,

$$\int_{t_0}^{t} (t-s)^{-|\beta:2\mathbf{m}|}\, ds \tag{4.46}$$

converges and consequently

$$D_{\mathbf{v}}^{\beta} u(t,x) = \int_{t_0}^{t} \int_{\mathbb{V}} D_{\mathbf{v}}^{\beta} G_p(t-s, z-x; z) f(s,z)\, dz\, ds$$

for all $t_0 \le t \le T$ and $x \in \mathbb{V}$. From this it follows that $D_{\mathbf{v}}^{\beta}u(t,x)$ is continuous on $(t_0,T) \times \mathbb{V}$ and moreover (4.45) holds for such an β in view of Lebesgue's dominated convergence theorem. When $|\beta : \mathbf{m}| = 2$, (4.46) does not converge and hence the above argument fails. The main issue in the proof below centers around using $\mathbf{v}$-Hölder continuity to remove this obstacle.

Proof. Our argument proceeds in two steps. The fist step deals with the spatial derivatives of u. Therein, we prove the asserted x-regularity and show that the formula (4.45) holds. In fact, we only need to consider the case where $|\beta : \mathbf{m}| = 2$; the case where $|\beta : \mathbf{m}| < 2$ was already treated in the paragraph proceeding the proof. In the second step, we address the time derivative of u. As we will see, (4.44) and the asserted t-regularity are partial consequences of the results proved in Step 1; this is, in part, due to the fact that the time derivative of G_p can be exchanged for spatial derivatives. The regularity shown in the two steps together will automatically ensure that u is $(2\mathbf{m},\mathbf{v})$-regular on $(t_0,T) \times \mathbb{V}$.

Step 1. Let β be such that $|\beta : \mathbf{m}| = 2$. For $h > 0$ write

$$u_h(t,x) = \int_{t_0}^{t-h} \int_{\mathbb{V}} G_p(t-s, x-z; z) f(s,z)\, dz\, ds$$

and observe that

$$D_{\mathbf{v}}^{\beta} u_h(t,x) = \int_{t_0}^{t-h} \int_{\mathbb{V}} D_{\mathbf{v}}^{\beta} G_p(t-s, x-z; z) f(s,z)\, dz\, ds$$

for all $t_0 \le t-h < t \le T$ and $x \in \mathbb{V}$; it is clear that $D_{\mathbf{v}}^{\beta}u_h(t,x)$ is continuous in t and x. The fact that we can differentiate under the integral sign is justified because t has been replaced by $t-h$ and hence the singularity in (4.46) is avoided in the upper limit. We will show that $D_{\mathbf{v}}^{\beta}u_h(t,x)$ converges uniformly on all compact subsets of $(t_0,T)\times\mathbb{V}$ as $h\to 0$. This, of course, guarantees that $D^\beta_{\mathbf{v}}u(x,t)$ exists, satisfies (4.45) and is continuous on $(t_0,T)\times\mathbb{V}$. To this end, let us write

$$\begin{aligned} D^\beta_{\mathbf{v}}u_h(t,x) &= \int_{t_0}^{t-h}\int_{\mathbb{V}} D^\beta_{\mathbf{v}}G_p(t-s,x-z;z)(f(s,z)-f(s,x))\,dz\,ds \\ &\qquad + \int_{t_0}^{t-h}\int_{\mathbb{V}} D^\beta_{\mathbf{v}}G_p(t-s,x-z;z)f(s,x)\,dz\,ds \\ &=: I_h^{(1)}(t,x)+I_h^{(2)}(t,x). \end{aligned}$$

Using our hypotheses, Corollary 4.3.8 and Lemma 4.3.13, for some $0<\sigma<1$ and $\theta>0$, there is $M>0$ such that

$$|f(s,z)-f(s,x)|\le C(t-s)^\sigma(R^\#((t-s)^{-E}(x-z)))^\theta$$

for all $x,z\in\mathbb{V}$, $t\in[t_0,T]$ and $s\in[t_0,t]$; consequently

$$\begin{aligned} &|D^\beta_{\mathbf{v}}G_p(t-s,x-z;z)(f(s,z)-f(s,x))| \\ &\qquad \le C(t-s)^{-(\mu_H+1)}t^\sigma(R^\#(t^{-E}(x-z)))^\theta\exp(-MR^\#((t-s)^{-E}(x-z))) \\ &\qquad \le C(t-s)^{-(\mu_H+(1-\sigma))}\exp(-MR^\#(t-s)^{-E}(x-z)) \end{aligned}$$

for all $x,z\in\mathbb{V}$, $t\in[t_0,T]$ and $s\in[t_0,t]$. This estimate guarantees that

$$I^{(1)}(t,x):=\int_{t_0}^{t}\int_{\mathbb{V}} D^\beta_{\mathbf{v}}G_p(t-s,x-z;z)(f(s,z)-f(s,x))\,dz\,ds$$

exists for each $t\in[t_0,T]$ and $x\in\mathbb{V}$. Moreover, for all $t_0\le t-h<t\le T$ and $x\in\mathbb{V}$,

$$\begin{aligned} |I^{(1)}(t,x)-I_h^{(1)}(t,x)| &\le \int_{t-h}^{t}\int_{\mathbb{V}}|D^\beta_{\mathbf{v}}G_p(t-s,x-z;z)(f(s,z)-f(s,x))|\,dz\,ds \\ &\le C\int_{t-h}^{t}\int_{\mathbb{V}}(t-s)^{\sigma-1}\exp(-MR^\#(z))\,dz\,ds\le Ch^\sigma. \end{aligned}$$

From this we see that $\lim_{h\downarrow 0}I_h^{(1)}(t,x)$ converges uniformly on all compact subsets of $(t_0,T)\times\mathbb{V}$.

We claim that for some $0 < \rho < 1$, there exists $C > 0$ such that

$$\Big| \int_{\mathbb{V}} D_{\mathbf{v}}^{\beta} G_p(t-s, x-z; z)\, dz \Big| \leq C(t-s)^{-(1-\rho)} \tag{4.47}$$

for all $x \in \mathbb{V}$ and $s \in [t_0, t]$. Indeed,

$$\begin{aligned}
&\int_{\mathbb{V}} D_{\mathbf{v}}^{\beta} G_p(t-s, x-z; z)\, dz \\
&\quad = \int_{\mathbb{V}} D_{\mathbf{v}}^{\beta}[G_p(t-s, x-z; z) - G_p(t-s, x-z; y)]\big|_{y=x}\, dz \\
&\qquad\qquad + \big[D_{\mathbf{v}}^{\beta} \int_{\mathbb{V}} G_p(t-s, x-z; y)\, dz\big]\big|_{y=x}.
\end{aligned}$$

The first term above is estimate with the help of Lemma 4.5.3 and by making arguments analogous to those in the previous paragraph; the appearance of ρ follows from an obvious application of Lemma 4.3.13. This term is bounded by $C(t-s)^{-(1-\rho)}$. The second term is clearly zero and so our claim is justified.

By essentially repeating the arguments made for $I_h^{(1)}$ and making use of (4.47), we see that

$$\lim_{h \downarrow 0} I_h^{(2)}(t, x) = I^{(2)}(t, x) =: \int_{t_0}^{t} \int_{\mathbb{V}} D_{\mathbf{v}}^{\beta} G_p(t-s, x-z; z) f(s, x)\, dz\, ds$$

where this limit converges uniformly on all compact subsets of $(t_0, T) \times \mathbb{V}$.

Step 2. It follows from Leibnitz' rule that

$$\begin{aligned}
\partial_t u_h(x, t) &= \int_{\mathbb{V}} G_p(h, x-z; z) f(t-h, z)\, dz \\
&\qquad + \int_{t_0}^{t-h} \int_{\mathbb{V}} \partial_t G_p(t-s, x-z; z) f(s, z)\, dz\, ds \\
&=: J_h^{(1)}(t, x) + J_h^{(2)}(t, x)
\end{aligned}$$

for all $t_0 < t - h < t < T$ and $x \in \mathbb{V}$. Now, in view of Lemma 4.5.4 and our hypotheses concerning f,

$$\lim_{h \downarrow 0} J_h^{(1)}(t, x) = f(t, x)$$

where this limit converges uniformly on all compact subsets of $(t_0, T) \times \mathbb{V}$.

Using the fact that $\partial_t G_p(t-s, x-z; z) = -H_p(z) G_p(t-s, x-z; z)$, we see that

$$\begin{aligned}
&\lim_{h\downarrow 0} J_h^{(2)}(t,x) \\
&\quad = \lim_{h\downarrow 0} \int_0^{t-h} \int_{\mathbb{V}} \Big(- \sum_{|\beta:\mathbf{m}|=2} a_\beta(z) D_{\mathbf{v}}^\beta \Big) G_p(t-s, x-z; z) f(s,z)\, dz\, ds \\
&\quad = - \sum_{|\beta:\mathbf{m}|=2} \lim_{h\downarrow 0} \int_0^{t-h} \int_{\mathbb{V}} D_{\mathbf{v}}^\beta G_p(t-s, x-z; z)(a_\beta(z) f(s,z))\, dz\, ds
\end{aligned}$$

for all $t \in (t_0, T)$ and $x \in \mathbb{V}$. Because the coefficients of H are $\mathbf{v}$-Hölder continuous and bounded, for each β, $a_\beta(z) f(s,z)$ satisfies the same condition we have required for f and so, in view of Step 1, it follows that $J_h^{(2)}(t,x)$ converges uniformly on all compact subsets of $(t_0, T) \times \mathbb{V}$ as $h \to 0$. We thus conclude that $\partial_t u(t,x)$ exists, is continuous on $(t_0, T) \times \mathbb{V}$ and satisfies (4.44). $\square$

Lemma 4.5.11. *Let $W : (0, T] \times \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ be defined by*

$$W(t,x,y) = \int_0^t \int_{\mathbb{V}} G_p(t-s, x-z; z) \phi(s,z,y)\, dz\, ds,$$

for $x, y \in \mathbb{V}$ and $0 < t \leq T$. Then, for each $y \in \mathbb{V}$, $W(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$-regular on $(0, T) \times \mathbb{V}$ and satisfies

$$(\partial_t + H) W(t,x,y) = K(t,x,y). \tag{4.48}$$

for all $x, y \in \mathbb{V}$ and $t \in (0, T)$. Moreover, there are positive constants C and M for which

$$|W(t,x,y)| \leq C t^{-\mu_H + \rho} \exp(-M R^{\#}(t^{-E}(x-y))) \tag{4.49}$$

for all $x, y \in \mathbb{V}$ and $0 < t \leq T$ where ρ is that which appears in Lemma 4.5.6.

Proof. The estimate (4.49) follows from (4.28) and (4.42) by an analogous computation to that done in the proof of Lemma 4.5.6. It remains to show that, for each $y \in \mathbb{V}$, $W(\cdot,\cdot,y)$ is $(2\mathbf{m},\mathbf{v})$-regular and satisfies (4.48) on $(0,T)\times\mathbb{V}$. These are both local properties and, as such, it suffices to examine them on $(t_0,T)\times\mathbb{V}$ for an arbitrary but fixed $t_0>0$. Let us write

$$\begin{aligned} W(t,x,y) &= \int_{t_0}^{t}\int_{\mathbb{V}} G_p(t-s,x-z;z)\phi(s,z,y)\,dz\,ds \\ &\qquad\qquad + \int_0^{t_0}\int_{\mathbb{V}} G_p(t-s,x-z;z)\phi(s,z,y)\,dz\,ds \\ &=: W_1(t,x,y)+W_2(t,x,y) \end{aligned}$$

for $x,y\in\mathbb{V}$ and $t_0<t<T$. In view of Lemmas 4.5.9 and 4.5.10, for each $y\in\mathbb{V}$, $W_1(\cdot,\cdot,y)$ is $(2\mathbf{m},\mathbf{v})$-regular on $(t_0,T)\times\mathbb{V}$ and

$$\begin{aligned} &(\partial_t+H)W_1(t,x,y) \\ &= \partial_t W_1(t,x,y)+\sum_{|\beta:\mathbf{m}|\le 2} a_\beta(x)D_{\mathbf{v}}^{\beta}W_1(t,x,y) \\ &= \phi(t,x,y)+\lim_{h\downarrow 0}\int_{t_0}^{t-h}\int_{\mathbb{V}}\partial_t G_p(t-s,x-z;z)\phi(s,z,y)\,dz\,dy \\ &\quad +\lim_{h\downarrow 0}\int_{t_0}^{t-h}\int_{\mathbb{V}}\sum_{|\beta:\mathbf{m}|\le 2} a_\beta(x)D_{\mathbf{v}}^{\beta}G_p(t-s,x-z;z)\phi(s,z,y)\,dz\,ds \\ &= \phi(t,x,y)+\lim_{h\downarrow 0}\int_{t_0}^{t-h}\int_{\mathbb{V}}(\partial_t+H)G_p(t-s,x-z;z)\phi(s,z,y)\,dz\,ds \\ &= \phi(t,x,y)-\lim_{h\downarrow 0}\int_{t_0}^{t-h}\int_{\mathbb{V}}K(t-s,x,z)\phi(s,z,y)\,dz\,ds \end{aligned} \tag{4.50}$$

for all $x\in\mathbb{V}$ and $t_0<t<T$; here we have used the fact that

$$(\partial_t+H)G_p(t-s,x-z;z)=-K(t-s,x,z).$$

Treating W_2 is easier because $G_p(t-s,x-z,z)$ and its derivatives remain bounded for $z,x\in\mathbb{V}$ and $0<s\le t_0$. Consequently, derivatives may be taken under the integral sign and so it follows that, for each $y\in\mathbb{V}$, $W_2(\cdot,\cdot,y)$ is $(2\mathbf{m},\mathbf{v})$-regular on $(t_0,T)\times\mathbb{V}$ and

$$(\partial_t+H)W_2(t,x,y)=-\int_0^{t_0}\int_{\mathbb{V}}K(t-s,x,z)\phi(s,z,y)\,dz\,ds \tag{4.51}$$

for $x \in \mathbb{V}$ and $t_0 < t < T$. We can thus conclude that, for each $y \in \mathbb{V}$, $W(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$-regular on $(t_0, T) \times \mathbb{V}$ and, by combining (4.50) and (4.51),

$$(\partial_t + H)W(t,x,y) = \phi(t,x,y) - \lim_{h \downarrow 0} \int_0^{t-h} \int_{\mathbb{V}} K(t-s,x,z)\phi(s,z,y)\,dz\,ds$$

for $x \in \mathbb{V}$ and $t_0 < t < T$. By (4.36), Proposition 4.5.8 and the Dominated Convergence Theorem,

$$\begin{aligned}
\lim_{h \downarrow 0} \int_0^{t-h} \int_{\mathbb{V}} K(t-s,x,z)\phi(s,z,y)\,dz\,ds &= \int_0^{t} \int_{\mathbb{V}} K(t-s,x,z)\phi(s,z,y)\,dz\,ds \\
&= \phi(t,x,y) - K(t,x,y)
\end{aligned}$$

and therefore

$$(\partial_t + H)W(t,x,y) = K(t,x,y)$$

for all $x, y \in \mathbb{V}$ and $t_0 < t < T$. □

The theorem below is our main result. It is a more refined version of Theorem 4.5.1 because it gives an explicit formula for the fundamental solution Z; in particular, Theorem 4.5.1 is an immediate consequence of the result below.

Theorem 4.5.12. *Let H be a uniformly $(2\mathbf{m}, \mathbf{v})$-positive-semi-elliptic operator. If H satisfies Hypothesis 4.1 then $Z : (0,T] \times \mathbb{V} \times \mathbb{V} \to \mathbb{C}$, defined by*

$$Z(t,x,y) = G_p(t, x-y; y) + W(t,x,y) \tag{4.52}$$

for $x, y \in \mathbb{V}$ and $0 < t \le T$, is a fundamental solution to (4.20). *Moreover, there are positive constants C and M for which*

$$|Z(t,x,y)| \le \frac{C}{t^{\mu_H}} \exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right)\right) \tag{4.53}$$

for all $x, y \in \mathbb{V}$ and $0 < t \le T$.

Proof. As $0 < \rho < 1$, (4.49) and Lemma 4.5.2 imply the estimate (4.53). In view of Lemma 4.5.11 and Corollary 4.5.5, for each $y \in \mathbb{V}$, $Z(\cdot, \cdot, y)$ is $(2\mathbf{m}, \mathbf{v})$-regular on $(0, T) \times \mathbb{V}$ and

$$\begin{aligned}(\partial_t + H)Z(t, x, y) &= (\partial_t + H)G_p(t, x - y, y) + (\partial_t + H)W(t, x, y) \\ &= -K(t, x, y) + K(t, x, y) = 0\end{aligned}$$

for all $x \in \mathbb{V}$ and $0 < t < T$. It remains to show that for any $f \in C_b(\mathbb{V})$,

$$\lim_{t \to 0} \int_{\mathbb{V}} Z(t, x, y) f(y)\, dy = f(x)$$

for all $x \in \mathbb{V}$. Indeed, let $f \in C_b(\mathbb{V})$ and, in view of (4.49), observe that

$$\begin{aligned}\left| \int_{\mathbb{V}} W(t, x, y) f(y) \right| &\leq Ct^{\rho} \|f\|_{\infty} \int_{\mathbb{V}} t^{-\mu_H} \exp(-MR^{\#}(t^{-E}(x - y)))dy \\ &\leq Ct^{\rho} \|f\|_{\infty} \int_{\mathbb{V}} \exp(-MR^{\#}(y))\, dy \leq Ct^{\rho} \|f\|_{\infty}\end{aligned}$$

for all $x \in \mathbb{V}$ and $0 < t \leq T$. An appeal to Lemma 4.5.4 gives, for each $x \in \mathbb{V}$,

$$\begin{aligned}\lim_{t \to 0} \int_{\mathbb{V}} Z(t, x, y) f(y) dy &= \lim_{t \to 0} \int_{\mathbb{V}} G_p(t, x - y; y) f(y)\, dy \\ &\qquad + \lim_{t \to 0} \int_{\mathbb{V}} W(t, x, y) f(y)\, dy \\ &= f(x) + 0 = f(x)\end{aligned}$$

as required. In fact, the above argument guarantees that this convergence happens uniformly on all compact subsets of $\mathbb{V}$. □

We remind the reader that implicit in the definition of fundamental solution to (4.20) is the condition that Z is $(2\mathbf{m}, \mathbf{v})$-regular. In fact, one can further deduce estimates for the spatial derivatives of Z, $D_{\mathbf{v}}^{\beta} Z$, of the form (4.11) for all β such that $|\beta : 2\mathbf{m}| \leq 1$ (see [37, p. 92]). Using the fact that Z satisfies (4.20) and H's coefficients are bounded, an analogous estimate is obtained for a single t derivative of Z.

CHAPTER 5

UNIFORMLY POSITIVE-HOMOGENEOUS OPERATORS WITH MEASURABLE COEFFICIENTS AND HEAT KERNEL ESTIMATES

5.1 Introduction

Throughout this chapter $\mathbb{V}$ is a d-dimensional real vector space equipped with the standard smooth structure; we henceforth assume the notation of Chapter 4. Taking our motivation from Theorem 4.5.1, given a self-adjoint partial differential operator H on $L^2(\mathbb{V})$, which we will take to be uniformly comparable to a positive-homogeneous operator Λ with symbol P and homogeneous order μ_Λ (in the sense that it satisfies a Gårding inequality), we are interested in the validity of heat kernel estimates for H in terms of the Legendre-Fenchel transform of $R = \operatorname{Re} P$. Specifically, we ask: Under what conditions on H does the semigroup $\{e^{-tH}\}$ have an integral kernel $Z : (0,\infty) \times \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ satisfying

$$|Z(t,x,y)| \leq \frac{C}{t^{\mu_\Lambda}} \exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right)\right)$$

for $x, y \in \mathbb{V}$ and $0 < t \leq T < \infty$ where $C = C(H,T)$ and $M = M(H,T)$ are positive constants? We recall that Z is an integral kernel for $\{e^{-tH}\}$ provided, for each $f \in L^2(\mathbb{V})$,

$$\left(e^{-tH} f\right)(x) = \int_{\mathbb{V}} Z(t,x,y) f(y)\, dy$$

for $t > 0$ and almost every $x \in \mathbb{V}$. Of course, when Z is sufficiently regular, this notion coincides with the definition of fundamental solution to the heat equation given in Section 4.5. Consequently, Theorem 4.5.1 can be seen as an affirmative answer to the above question in the case that H has Hölder coefficients (and is uniformly positive-semi-elliptic). In this chapter, we adapt the functional analytic method of E. B. Davies (presented in [21]) to the positive-homogeneous setting and show that, in particular, the above question also has an affirmative answer when the coefficients of H are only bounded and measurable, provided $\mu_\Lambda < 1$ (see Theorem 5.8.4). In this adaptation of Davies' elegant method, we will see the natural appearance of the full d-dimensional Legendre-Fenchel transform in heat kernels estimates for (uniformly) positive-homogeneous operators. We recall that the 1-dimensional Legendre-Fenchel transform was previously observed (and exploited) in [9] and [10] for elliptic operators; in the elliptic setting, the anisotropic character of the full d-dimensional transform isn't needed. For further discussion on the Legendre-Fenchel transform in heat kernel estimates (and its virtues), we refer the reader to Subsection 4.2.1.

5.2 Sobolev spaces, uniformly positive-homogeneous self-adjoint operators and their quadratic forms

In the first part of this section, we define a family of Sobolev spaces on $\mathbb{V}$. These spaces, which include those of the classical elliptic theory, were also discussed in the context of $\mathbb{R}^d$ in [58]. Then, given a formally self-adjoint positive-homogeneous operator Λ on $\mathbb{V}$, we study the quadratic form Q_Λ it defines. Because Λ is symmetric, its symbol is necessarily real and it is henceforth denoted by R. We then realize Λ as a self-adjoint operator on L^2 whose domain and form domain are characterized by the previously defined Sobolev spaces; everything here relies on the semi-elliptic representation of positive-homogeneous operators given in Proposition 4.2.5.

Let $1 \le p < \infty$, $\mathbf{m} \in \mathbb{N}_+^d$ and let $\mathbf{v}$ be a basis for $\mathbb{V}$. For any non-empty open set $\Omega \subseteq \mathbb{V}$ define

$$W_{\mathbf{v}}^{\mathbf{m},p}(\Omega) = \{f \in L^p(\Omega) : D_{\mathbf{v}}^{\alpha} f \in L^p(\Omega)\ \forall\, \alpha \text{ with } |\alpha : \mathbf{m}| \le 1\}$$

where $D_{\mathbf{v}}^{\alpha} = (i\partial_1)^{\alpha_1}(i\partial_2)^{\alpha_2}\cdots(i\partial_d)^{\alpha_d}$. By the symbol, ∂_i, we mean the restriction of the derivation $\partial_i = \partial_{v_i}$ onto the open set Ω. As usual, each derivative is to be understood in the distributional sense. For any $f \in W_{\mathbf{v}}^{\mathbf{m},p}(\Omega)$ let

$$\|f\|_{W_{\mathbf{v}}^{\mathbf{m},p}(\Omega)} = \left[\sum_{|\alpha:\mathbf{m}|\le 1} \int_{\Omega} |D_{\mathbf{v}}^{\alpha} f|^p dx\right]^{1/p}.$$

Clearly, $\|\cdot\|_{W_{\mathbf{v}}^{\mathbf{m},p}(\Omega)}$ is a norm on $W_{\mathbf{v}}^{\mathbf{m},p}(\Omega)$ and the usual arguments show that $W_{\mathbf{v}}^{\mathbf{m},p}(\Omega)$ is a Banach space in this norm. Naturally, we will call these spaces *Sobolev spaces*; in the context of $\mathbb{R}^d$, these spaces were previously studied in [24] and [58]. Notice that when $\mathbb{V} = \mathbb{R}^d$, $\mathbf{v} = \mathbf{e}$ and $\mathbf{m} = (m, m, \dots, m)$, our definition coincides with that of $W^{m,p}(\Omega)$, the standard Sobolev spaces of $\mathbb{R}^d$. In fact, the basis $\mathbf{e}$ is immaterial in this setting. Let us also denote by $W_{\mathbf{v},0}^{\mathbf{m},p}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in the $\|\cdot\|_{W_{\mathbf{v}}^{\mathbf{m},p}}(\Omega)$ norm.

Temporarily, we restrict our attention to the case where $\Omega = \mathbb{V}$ and $p = 2$. As one can check by the use of smooth cut-off functions and mollification, $C_0^{\infty}(\mathbb{V})$ is dense in $W_{\mathbf{v}}^{\mathbf{m},p}(\mathbb{V})$. The following result follows by the standard method, c.f., [66]; its proof is omitted.

Lemma 5.2.1. *Let $\mathbf{m} \in \mathbb{N}^d$, $\mathbf{v}$ be a basis of $\mathbb{V}$ and $\mathbf{v}^*$ be the corresponding dual basis. Then*

$$W_{\mathbf{v}}^{\mathbf{m},2}(\mathbb{V}) = \left\{f \in L^2(\mathbb{V}) : \xi^{\alpha}\hat{f}(\xi) \in L^2(\mathbb{V}^*)\ \forall\, \alpha \textit{ with } |\alpha : \mathbf{m}| \le 1\right\} \tag{5.1}$$

and

$$\|f\|^2_{W_{\mathbf{v}}^{\mathbf{m},2}(\mathbb{V})} = \sum_{|\alpha:\mathbf{m}|\le 1} \|\xi^{\alpha}\hat{f}(\xi)\|^2_{2^*}.$$

Lemma 5.2.2. *Let Λ be a symmetric positive-homogeneous operator with symbol R and, in view of Proposition 4.2.5, let $\mathbf{m} \in \mathbb{N}_+^d$ and $\mathbf{v}$ be a basis of $\mathbb{V}$ as guaranteed by the proposition. Then*

$$W_{\mathbf{v}}^{\mathbf{m},2}(\mathbb{V}) = \left\{ f \in L^2(\mathbb{V}) : \int_{\mathbb{V}^*} R(\xi)|\hat{f}(\xi)|^2 d\xi < \infty \right\}$$

and moreover, the norms

$$\|f\|' := \left(\|f\|_2^2 + \int_{\mathbb{V}^*} R(\xi)|\hat{f}(\xi)|^2 d\xi \right)^{1/2}$$

and $\|\cdot\|_{W_{\mathbf{v}}^{\mathbf{m},2}(\mathbb{V})}$ are equivalent.

Proof. By virtue of Proposition 4.3.2 (working in the coordinates defined by $\mathbf{v}$), there are positive constants C and C' for which

$$C(1 + R(\xi)) \leq \sum_{|\alpha:\mathbf{m}|\leq 1} \xi^{2\alpha} \leq C'(1 + R(\xi)).$$

for all $\xi \in \mathbb{V}^*$. With this estimate, the result follows directly from Lemma 5.2.1 using the Fourier transform. $\square$

Returning to the general situation, let $\Omega \subseteq \mathbb{V}$ be a non-empty open set. For $f \in L^2(\Omega)$ define $f_* \in L^2(\mathbb{V})$ by

$$f_*(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Of course, $\|f\|_{L^2(\Omega)} = \|f_*\|_{L^2(\mathbb{V})}$. The following lemma shows that $W_{\mathbf{v},0}^{\mathbf{m},2}(\Omega)$ is continuously embedded in $W_{\mathbf{v}}^{\mathbf{m},2}(\mathbb{V})$:

Lemma 5.2.3. *For any $f \in W_{\mathbf{v},0}^{\mathbf{m},2}(\Omega)$, $f_* \in W_{\mathbf{v}}^{\mathbf{m},2}(\mathbb{V})$ and*

$$\|f\|_{W_{\mathbf{v}}^{\mathbf{m},2}(\Omega)} = \|f_*\|_{W_{\mathbf{v}}^{\mathbf{m},2}(\mathbb{V})}.$$

Proof. Let $f \in W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$ and let $\{f_n\} \subseteq C_0^\infty(\Omega)$ for which $\|f_n - f\|_{W^{\mathbf{m},2}_{\mathbf{v}}(\Omega)} \to 0$ as $n \to \infty$. Then for any $\phi \in C_0^\infty(\mathbb{V})$ and multi-index α for which $|\alpha : \mathbf{m}| \leq 1$,

$$\begin{aligned}\int_{\mathbb{V}} f_*(D^\alpha_{\mathbf{v}}\phi)dx = \int_\Omega f(D^\alpha_{\mathbf{v}}\phi)dx = \lim_{n\to\infty}\int_\Omega f_n(D^\alpha_{\mathbf{v}}\phi)dx \\ = \lim_{n\to\infty}(-1)^{|\alpha|}\int_\Omega (D^\alpha_{\mathbf{v}} f_n)\phi dx = (-1)^{|\alpha|}\int_\Omega (D^\alpha_{\mathbf{v}} f)\phi dx \\ = (-1)^{|\alpha|}\int_{\mathbb{V}} (D^\alpha_{\mathbf{v}} f)_*\phi dx\end{aligned}$$

where we used the fact that each f_n has compact support in Ω and thus partial integration produces no boundary terms. Thus for each such α, $D^\alpha_{\mathbf{v}} f_* = (D^\alpha_{\mathbf{v}} f)_* \in L^2(\mathbb{V})$ and $\|D^\alpha_{\mathbf{v}} f\|_{L^2(\Omega)} = \|D^\alpha_{\mathbf{v}} f_*\|_{L^2(\mathbb{V})}$ from which the result follows. $\square$

We now turn to positive-homogeneous operators, viewed in the L^2 setting and their quadratic forms. Let $\Omega \subseteq \mathbb{V}$ be a non-empty open set and let Λ be a positive-homogeneous operator on $\mathbb{V}$ with symbol R and let $\mathbf{m} \in \mathbb{N}^d$ and $\mathbf{v}$ be the basis of $\mathbb{V}$ guaranteed by Proposition 4.2.5. Define

$$\mathrm{Dom}(Q_{\Lambda_\Omega}) = W^{\mathbf{m},2}_{0,\mathbf{v}}(\Omega)$$

and for each $f, g \in \mathrm{Dom}(Q_{\Lambda_\Omega})$, put

$$Q_{\Lambda_\Omega}(f,g) = \int_{\mathbb{V}^*} P(\xi)\widehat{f_*}(\xi)\overline{\widehat{g_*}(\xi)}d\xi.$$

Proposition 5.2.4. *Then the restriction $\Lambda|_{C_0^\infty(\Omega)}$ extends to a non-negative self-adjoint operator on $L^2(\Omega)$, denoted by Λ_Ω. Its associated quadratic form is Q_{Λ_Ω} has domain Dom$(Q_{\Lambda_\Omega}) = W^{\mathbf{m},2}_{\mathbf{v},0} =$ Dom$(\Lambda^{1/2})$ and moreover $C_0^\infty(\Omega)$ is a core for Q_{Λ_Ω}.*

Remark 17. *The self-adjoint operator Λ_Ω is the Dirichlet operator on Ω, i.e., the operator associated with the Dirichlet problem.*

Remark 18. *One can show that Dom$(\Lambda_\Omega) = W^{2\mathbf{m},2}_{\mathbf{v},0}(\Omega)$. This fact however isn't needed for our development.*

Proof of Proposition 5.2.4. In view of Lemma 5.2.2, there are constants $C, C' > 0$ for which

$$C\|f\|_{W^{\mathbf{m},2}_{\mathbf{v}}(\mathbb{V})} \leq \left(\|f\|^2_{L^2(\mathbb{V})} + \int_{\mathbb{V}^*} R(\xi)|\hat{f}(\xi)|^2 d\xi\right)^{1/2} \leq C'\|f\|_{W^{\mathbf{m},2}_{\mathbf{v}}(\mathbb{V})}$$

for all $f \in W^{\mathbf{m},2}_{\mathbf{v}}(\mathbb{V})$. Thus by Lemma 5.2.3,

$$C\|f\|_{W^{\mathbf{m},2}_{\mathbf{v}}(\Omega)} \leq \left(\|f\|^2_{L^2(\Omega)} + Q_{\Lambda_\Omega}(f)\right)^{1/2} \leq C'\|f\|_{W^{\mathbf{m},2}_{\mathbf{v}}(\Omega)}$$

for all $f \in W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$. It follows that

$$\|f\|'_\Omega := \left(\|f\|^2_{L^2(\Omega)} + Q_{\Lambda_\Omega}(f)\right)^{1/2}$$

defines a norm on $W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$, equivalent to the norm $\|\cdot\|_{W^{\mathbf{m},2}_{\mathbf{v}}(\Omega)}$. From this we can also conclude that Q_{Λ_Ω} is a *bona fide* quadratic form.

It is easy to see that Q_{Λ_Ω} is symmetric, positive-definite (in the sense of forms) and densely defined; these assertions follow because R is positive-definite and $C_0^\infty(\Omega) \subseteq W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega) \subseteq L^2(\Omega)$. We claim that Q_{Λ_Ω} is closed. Indeed, let $\{f_n\} \subseteq W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$ be a Q_{Λ_Ω}-Cauchy sequence and such that $f_n \to f$ in $L^2(\Omega)$ for some $f \in L^2(\Omega)$. Because the norms $\|\cdot\|'_\Omega$ and $\|\cdot\|_{W^{\mathbf{m},2}_{\mathbf{v}}(\Omega)}$ are equivalent, we know that $\{f_n\}$ is also a Cauchy sequence in $W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$ and so it converges. Moreover, as the topology on $W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$ is finer than the topology induced by the $L^2(\Omega)$ norm, we can conclude that $f \in W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$ and $f_n \to f$ in $W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$. By again appealing to the equivalence of norms, it follows that Q_{Λ_Ω} is closed. It is now evident that $C_0^\infty(\Omega)$ is a core for Q_{Λ_Ω}.

In view of the theory of quadratic forms, Q_{Λ_Ω} has a unique associated non-negative self-adjoint operator Λ_Ω with $\mathrm{Dom}(\Lambda_\Omega^{1/2}) = \mathrm{Dom}(Q_{\Lambda_\Omega})$. Also, because

$$\langle \Lambda f, g\rangle_\Omega = \langle \Lambda f_*, g_*\rangle = \int_{\mathbb{V}^*} P(\xi)\hat{f}_*(\xi)\overline{\hat{g}_*(\xi)}d\xi = Q_{\Lambda_\Omega}(f,g) = \langle f, \Lambda g\rangle_\Omega$$

for all $f, g \in C_0^\infty(\Omega)$, Λ_Ω must be a self-adjoint extension of $\Lambda|_{C_0^\infty(\Omega)}$. $\square$

5.3 Ultracontractivity and Sobolev-type inequalities

In this section we show that (self-adjoint) positive-homogeneous operators have many desirable properties shared by elliptic operators. In particular, for a self-adjoint positive-homogeneous operator Λ, we will prove corresponding Nash and Gagliardo-Nirenberg inequalities.

Let Λ be a self-adjoint positive-homogeneous operator on $\mathbb{V}$ with symbol R and homogeneous order μ_Λ. In view of Proposition 5.2.4, Λ determines a self-adjoint positive-homogeneous operator on $L^2(\mathbb{V})$, $\Lambda_{\mathbb{V}}$. By an abuse of notation we shall write $\Lambda = \Lambda_{\mathbb{V}}$ and $Q_{\Lambda_{\mathbb{V}}} = Q_\Lambda$. Using the spectral calculus, define semigroup $\{e^{-t\Lambda}\}$; this is a C_0-contraction semigroup of self-adjoint operators on $L^2(\mathbb{V})$. In view of our discussion in Chapter 4, it should be no surprise that the semigroup $e^{-t\Lambda}$, defined here by the spectral calculus coincides with that given by the Fourier transform; this, in particular, is verified by the following lemma.

Lemma 5.3.1. *For $f \in L^2(\mathbb{V})$ and $t > 0$,*

$$\left(e^{-t\Lambda} f\right)(x) = \int_{\mathbb{V}} K_\Lambda(t, x-y) f(y) dy \tag{5.2}$$

almost everywhere, where $K_\Lambda(t,x) = (e^{-tR})^\vee(x) \in \mathcal{S}(\mathbb{V})$. For each $t > 0$, this formula extends $\{e^{-t\Lambda}\}$ to a bounded operator from $L^p(\mathbb{V})$ to $L^q(\mathbb{V})$ for any $1 \leq p, q \leq \infty$. Furthermore, for each $1 \leq p, q \leq \infty$, there exists $C_{p,q} > 0$ such that

$$\|e^{-t\Lambda}\|_{p\to q} \leq \frac{C_{p,q}}{t^{\mu_\Lambda(1/p-1/q)}}$$

for all $t > 0$. In particular, the semigroup is ultracontractive with

$$\|e^{-t\Lambda}\|_{2\to\infty} \leq \frac{C_{2,\infty}}{t^{\mu_\Lambda/2}}$$

for all $t > 0$.

Remark 19. *Given a C_0-semigroup $\{T_t\}$ of self-adjoint operators on L^2, we say that the semigroup is ultracontractive if, for each $t > 0$, T_t is a bounded operator from L^2 to L^∞. We note that this condition immediately implies (by duality) that, for each $t > 0$, T_t is a bounded operator from L^1 to L^∞ and this is often (though not exclusively, e.g., [43]) taken to be the definition of ultracontractivity, see [18]. Our terminology is not meant to imply (as it does in the case of Markovian semigroups) that the semigroup is contractive on L^p for any p; it usually isn't.*

Proof of Lemma 5.3.1. We first verify the representation formula (5.2). Using the Fourier transform, one sees easily that convolution by K_Λ defines a C_0-contraction semigroup on $L^2(\mathbb{V})$ of self-adjoint operators. Denote this semigroup and its corresponding generator by T_t and A respectively and note that A is necessarily self-adjoint. For each $f \in C_0^\infty(\mathbb{V})$, observe that

$$\lim_{t\to 0}\left\|t^{-1}\left(T_t f - f\right) + \Lambda f\right\|_2 = \lim_{t\to 0}\left\|\left(t^{-1}(e^{-tR(\xi)} - 1) + R(\xi)\right)\hat{f}(\xi)\right\|_{2^*} = 0$$

where we have appealed to the dominated convergence theorem and the fact that $\mathcal{F}(\Lambda f) = R\hat{f}$. Consequently, $C_0^\infty(\mathbb{V}) \subseteq \mathrm{Dom}(A)$ and $Af = -\Lambda f$ for all $f \in C_0^\infty(\mathbb{V})$. Our aim is to show that $\Lambda|_{C_0^\infty(\mathbb{V})}$ is essentially self-adjoint for then $A = -\Lambda$ and so necessarily, $T_t = e^{-\Lambda t}$ as claimed.

Let $f \in \mathrm{Ran}(\Lambda|_{C_0^\infty(\mathbb{V})} \pm i)^\perp$. By the unitarity of the Fourier transform,

$$0 = \langle f, (\Lambda \pm i)g\rangle = \langle \hat{f}, (R \pm i)\hat{g}\rangle_* = \langle (R \pm i)\hat{f}, \hat{g}\rangle_*$$

for all $g \in C_0^\infty(\mathbb{V})$. We know that $\mathcal{F}(C_0^\infty(\mathbb{V}))$ is dense in $L^2(\mathbb{V}^*)$ and so it follows that $(R(\xi) \pm i)\hat{f}(\xi)) = 0$ almost everywhere. Using the fact that R is real-valued, we conclude that $f = 0$ and so $\mathrm{Ran}(\Lambda|_{C_0^\infty(\mathbb{V})} \pm i)^\perp = \{0\}$. This implies that $\mathrm{Ran}(\Lambda|_{C_0^\infty(\mathbb{V})} \pm i)$ is dense in $L^2(\mathbb{V})$ and thus the proof is complete in view of von Neumann's criteria for essential self-adjointness.

The asserted $L^p \to L^q$ estimates for $\{e^{-t\Lambda}\}$ are now established using the Fourier transform. In fact, we already established the $L^1 \to L^\infty$ estimate in the paragraph proceeding Proposition 4.2.12; we leave the remaining estimate to the reader. $\square$

Remark 20. *It should be pointed out that $\Lambda|_{C_0^\infty(\Omega)}$ is not generally essentially self-adjoint; for instance one can consider the Dirichlet and Neumann operators when Ω is, say, a bounded open non-empty subset of $\mathbb{V}$. In this case the above argument fails because $\mathcal{F}(C_0^\infty(\Omega))$ isn't dense in $L^2(\mathbb{V}^*)$.*

Proposition 5.3.2 (Nash's inequality)**.** *Let Ω be a non-empty open subset of $\mathbb{V}$ and let Λ be a positive-homogeneous operator with homogeneous order μ_Λ. We consider the self-adjoint operator Λ_Ω and its quadratic form Q_{Λ_Ω}. There exists $C > 0$ such that*

$$\|f\|_{L^2(\Omega)}^{1+1/\mu_\Lambda} \leq C Q_{\Lambda_\Omega}(f)^{1/2} \|f\|_{L^1(\Omega)}^{1/\mu_\Lambda}$$

for all $f \in C_0^\infty(\Omega)$.

Proof. It suffices to prove the estimate when $\Omega = \mathbb{V}$, for the general result follows from the isometric embedding of $W_{\mathbf{v},0}^{\mathbf{m},2}(\Omega)$ into $W_{\mathbf{v}}^{\mathbf{m},2}(\mathbb{V})$, c.f., Lemma 5.2.3, and that of $L^1(\Omega)$ into $L^1(\mathbb{V})$. Again, we will denote $\Lambda_{\mathbb{V}}$ and $Q_{\Lambda_{\mathbb{V}}}$ by Λ and Q_Λ respectively. In view of Lemma 5.3.1, the self-adjointness of Λ and duality give $C' > 0$ such that

$$\|e^{-t\Lambda}\|_{1\to 2} \leq \frac{C'}{t^{\mu_\Lambda/2}}$$

for all $t > 0$. Thus for any $f \in C_0^\infty(\mathbb{V})$,

$$\begin{aligned} \|f\|_2 &\leq \|e^{-t\Lambda}f - f\|_2 + \|e^{-t\Lambda}f\|_2 \\ &\leq \left\| \int_0^t \frac{d}{ds} e^{-s\Lambda} f ds \right\|_2 + \frac{C'}{t^{\mu_\Lambda/2}} \|f\|_1 \\ &\leq \int_0^t \|\Lambda^{1/2} e^{-s\Lambda} \Lambda^{1/2} f\|_2 ds + \frac{C'}{t^{\mu_\Lambda/2}} \|f\|_1 \\ &\leq \int_0^t \|\Lambda^{1/2} e^{-s\Lambda}\|_{2\to 2} ds Q_\Lambda(f)^{1/2} + \frac{C'}{t^{\mu_\Lambda/2}} \|f\|_1 \end{aligned} \tag{5.3}$$

for all $t > 0$. Using spectral theory we see that

$$\|\Lambda^{1/2} e^{-s\Lambda}\|_{2\to 2} \leq \sup_{\lambda > 0} |\lambda^{1/2} e^{-s\lambda}| \leq \frac{C''}{s^{1/2}}$$

for all $s > 0$ and therefore

$$\|f\|_2 \leq 2C'' t^{1/2} Q_\Lambda(f)^{1/2} + \frac{C'}{t^{\mu_\Lambda}} \|f\|_1$$

for all $t > 0$. The result follows by optimizing the above inequality and noting that $\mu_\Lambda > 0$. □

Suppose additionally that $\mu_\Lambda < 1$. Using ultracontractivity directly, a calculation analogous to (5.3) yields

$$\begin{aligned} \|f\|_\infty &\leq \int_0^t \|e^{-s\Lambda/2}\|_{2\to\infty} \|\lambda^{1/2} e^{-s\Lambda/2}\|_{2\to 2} ds Q_\Lambda(f)^{1/2} + \frac{C}{t^{\mu_\Lambda/2}} \|f\|_2 \\ &\leq C' t^{(1-\mu_\Lambda)/2} Q_\Lambda(f)^{1/2} + \frac{C}{t^{\mu_\Lambda/2}} \|f\|_2 \end{aligned}$$

for $f \in C_0^\infty(\mathbb{V})$ and $t > 0$. Upon optimizing with respect to t and using the density of $C_0^\infty(\mathbb{V})$ in $W_{\mathbf{v}}^{\mathbf{m},2}(\mathbb{V})$, we obtain the following lemma:

Lemma 5.3.3. *If $\mu_\Lambda < 1$ then there is $C > 0$ such that for all $f \in W_{\mathbf{v},0}^{\mathbf{m},2}(\Omega)$,*

$$\|f\|_{L^\infty(\Omega)} \leq C Q_{\Lambda_\Omega}(f)^{\mu_\Lambda/2} \|f\|_{L^2(\Omega)}^{1-\mu_\Lambda}.$$

Lemma 5.3.3 is the analog of the Gagliardo-Nirenberg inequality in our setting.

5.4 Fundamental Hypotheses

In this section, we consider arbitrary self-adjoint operators on $L^2(\Omega)$ where Ω is a non-empty open subset of $\mathbb{V}$; herein and in the next three sections $\|\cdot\|_2$ denotes the $L^2(\Omega)$ norm, $\langle\cdot,\cdot\rangle$ denotes its inner product and all mentions of a positive-homogeneous operator Λ refer to the self-adjoint operator Λ_Ω of Proposition 5.2.4. Correspondingly, Q_{Λ_Ω} is denoted by Q_Λ. We will state three hypotheses for such self-adjoint operators under which one can deduce the existence of heat kernels and prove corresponding off-diagonal estimates. Our construction is based on E.B. Davies' article [21], wherein a general class of higher order self-adjoint uniformly elliptic operators on $\mathbb{R}^d$ is studied.

Let's consider a self-adjoint operator H, bounded below, with domain $\mathrm{Dom}(H) \subseteq L^2(\Omega)$ and its corresponding quadratic form Q with domain $\mathrm{Dom}(Q)$. We require that $C_0^\infty(\Omega) \subseteq \mathrm{Dom}(Q)$. The first of three fundamental hypotheses concerning H and Q is as follows:

Hypothesis 5.1. *Let H and Q be as above. There exists a self-adjoint positive-homogeneous operator Λ with corresponding quadratic form Q_Λ such that*

$$\frac{1}{2}Q_\Lambda(f) \leq Q(f) \leq C(Q_\Lambda(f) + ||f||_2^2) \tag{5.4}$$

for all $f \in C_0^\infty(\Omega)$ where $C \geq 1$. We shall call Λ a reference operator for H.

Hypothesis 5.1 is a comparability statement between H and the positive-homogeneous operator Λ. In this way, (5.4) is analogous to Gårding's inequality in that the latter compares second order elliptic operators to the Laplacian.

When Hypothesis 5.1 holds, the inequality (5.4) ensures that $\mathrm{Dom}(Q) = \mathrm{Dom}(Q_\Lambda)$ and that $H \geq 0$. In view of Proposition 5.2.4, there exist $\mathbf{m} \in \mathbb{N}^d$ and a basis $\mathbf{v}$ of $\mathbb{V}$ such that

$$\mathrm{Dom}(Q) = \mathrm{Dom}(Q_\Lambda) = W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$$

and, because $C_0^\infty(\Omega)$ is dense in $W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$, (5.4) holds for all f in this common domain. These remarks are summarized in the following lemma:

Lemma 5.4.1. *Let H be a self-adjoint operator satisfying Assuption 5.1. Then $H \geq 0$ and*

$$\mathit{Dom}(Q) = W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega)$$

where $\mathbf{m}$ and $\mathbf{v}$ are those associated with Λ via Proposition 5.2.4. Moreover, (5.4) *holds for all f in this common domain.*

As in [21], we avoid identification of $\mathrm{Dom}(H)$ as it isn't necessary. By virtue of Lemma 5.4.1 and Theorem 1.53 of [69], $-H$ generates a strongly continuous semigroup $T_t = e^{-tH}$ on $L^2(\Omega)$ which is a bounded holomorphic semigroup on a non-trivial sector of $\mathbb{C}$. The main goal of this chapter is to show that the semigroup T_t has an integral kernel Z satisfying off-diagonal estimates in terms of the Legendre-Fenchel transform of R. Under the hypotheses given in this section, we obtain these off-diagonal estimates by means of Davies' perturbation method, suitably adapted to our naturally anisotropic setting. Specifically, we study perturbations of the semigroup T_t formed by conjugating T_t by "nice" operators. To this end, set

$$C^\infty_\infty(\Omega,\Omega) = \{\phi \in C^\infty(\Omega,\Omega) : \partial_v^k(\lambda(\phi)) \in L^\infty(\Omega) \text{ for all } v \in \mathbb{V}, \lambda \in \mathbb{V}^* \text{ and } k \geq 0\}$$

and, for $\phi \in C_\infty^\infty(\Omega, \Omega)$ and $\lambda \in \mathbb{V}^*$, we consider the smooth functions $e^{\lambda(\phi)}$ and $e^{-\lambda(\phi)}$; these will act as bounded and real-valued multiplication operators on $L^2(\Omega)$. For each such λ and ϕ, we define the *twisted* semigroup $T_t^{\lambda,\phi}$ on $L^2(\Omega)$ by

$$T_t^{\lambda,\phi} = e^{\lambda(\phi)} T_t e^{-\lambda(\phi)}$$

for $t > 0$. For any $f \in L^2(\Omega)$ such that $e^{-\lambda(\phi)} f \in \mathrm{Dom}(H)$, observe that

$$\begin{aligned} e^{\lambda(\phi)}(-H)e^{-\lambda(\phi)} f &= e^{\lambda(\phi)} \lim_{t \to 0} \frac{T_t(e^{-\lambda(\phi)} f) - (e^{-\lambda(\phi)} f)}{t} \\ &= \lim_{t \to 0} \frac{T_t^{\lambda,\phi} f - f}{t} \end{aligned}$$

where we have used the fact that $e^{\lambda(\phi)}$ acts as a bounded multiplication operator on $L^2(\Omega)$. Upon pushing this argument a little further one sees that $T_t^{\lambda,\phi}$ has infinitesimal generator $-H_{\lambda,\phi} = -e^{\lambda(\phi)} H e^{-\lambda(\phi)} = e^{\lambda(\phi)}(-H)e^{-\lambda(\phi)}$ and

$$\mathrm{Dom}(H_{\lambda,\phi}) = \left\{ f \in L^2(\Omega) : e^{-\lambda(\phi)} f \in \mathrm{Dom}(H) \right\}.$$

We also note that, in view of the resolvent characterization of bounded holomorphic semigroups, e.g., Theorem 1.45 of [69], it is straightforward to verify that $\{T_t^{\lambda,\phi}\}$ is a bounded holomorphic semigroup on $L^2(\Omega)$.

Remark 21. *This construction for $T_t^{\lambda,\phi}$ is similar to that done in [21]. The difference being that λ for us is a "multi-parameter" whereas in [21] it is a scalar. This construction is the basis behind the suitable adaptation of Davies' method for positive-homogeneous operators, discussed in the introductory section of this chapter.*

In the same spirit, define *twisted* form $Q_{\lambda,\phi}$ by

$$Q_{\lambda,\phi}(f) = Q(e^{-\lambda(\phi)} f, e^{\lambda(\phi)} f)$$

for all $f \in \mathrm{Dom}(Q_{\lambda,\phi}) := \mathrm{Dom}(Q)$. One can easily check that multiplication by $e^{\lambda(\phi)}$ for $\phi \in C_\infty^\infty(\Omega, \Omega)$ is an isomorphism of $W_{\mathbf{v},0}^{\mathbf{m},2}(\Omega)$ and so necessarily $Q_{\lambda,\phi}$ is densely defined and closed. It generally isn't symmetric or real-valued. As the next lemma shows, $H_{\lambda,\phi}$ corresponds to $Q_{\lambda,\phi}$ in the usual sense.

Lemma 5.4.2. *For any $\lambda \in \mathbb{V}^*$ and $\phi \in C_\infty^\infty(\Omega, \Omega)$,*

$$\mathit{Dom}(H_{\lambda,\phi}) \subseteq \mathit{Dom}(Q_{\lambda,\phi}) = \mathit{Dom}(Q)$$

and

$$Q_{\lambda,\phi}(f) = \langle H_{\lambda,\phi} f, f \rangle$$

for all $f \in \mathit{Dom}(H_{\lambda,\phi})$.

Proof. For $f \in \mathrm{Dom}(H_{\lambda,\phi})$,

$$e^{-\lambda(\phi)} f \in \mathrm{Dom}(H) \subseteq \mathrm{Dom}(Q) = W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega).$$

Because $\phi \in C_\infty^\infty(\Omega, \Omega)$, $\partial_i^k e^{\lambda(\phi)} \in L^\infty(\Omega)$ for all $i = 1, 2, \ldots, d$ and $k \geq 0$. Using the Leibniz rule it follows that

$$f = e^{\lambda(\phi)}(e^{-\lambda(\phi)} f) \in W^{\mathbf{m},2}_{\mathbf{v},0}(\Omega) = \mathrm{Dom}(Q_{\lambda,\phi}).$$

We see that,

$$\langle H_{\lambda,\phi} f, f \rangle = \langle H(e^{-\lambda(\phi)} f), e^{\lambda(\phi)} f \rangle = Q(e^{-\lambda(\phi)} f, e^{\lambda(\phi)} f) = Q_{\lambda,\phi}(f)$$

as desired. □

Our second fundamental hypothesis is as follows:

Hypothesis 5.2. *Let H and Q satisfy Hypothesis 5.1 with associated reference operator Λ. There exist $\mathcal{E} \subseteq C_\infty^\infty(\Omega, \Omega)$ and $M > 0$ such that:*

i) For each pair $x, y \in \Omega$, there is $\phi \in \mathcal{E}$ for which $\phi(x) - \phi(y) = x - y$.

ii) For all $\phi \in \mathcal{E}$, $\lambda \in \mathbb{V}^*$ and $f \in Dom(Q)$,

$$|Q_{\lambda,\phi}(f) - Q(f)| \leq \frac{1}{4}(Q(f) + M(1 + R(\lambda))\|f\|_2^2) \tag{5.5}$$

where R is the symbol of Λ. We will call (5.5) *the form comparison inequality.*

Our next lemma follows immediately from Lemma 5.4.2 and Hypothesis 5.2. Its proof is omitted.

Lemma 5.4.3. *Let $\phi \in \mathcal{E}$ and $\lambda \in \mathbb{V}^*$. If Hypothesis 5.2 holds,*

$$2\operatorname{Re}[Q_{\lambda,\phi}(f)] = 2\operatorname{Re}[(H_{\lambda,\phi}f, f)] \geq -\frac{M}{2}(1 + R(\lambda))\|f\|_2^2 \tag{5.6}$$

for all $f \in Dom(H_{\lambda,\phi})$.

Hypothesis 5.3. *Let H satisfy Hypotheses 5.1 and 5.2 and let Λ be the associated self-adjoint positive-homogeneous operator with symbol R and homogeneous order μ_Λ. Set $\kappa = \min\{n \in \mathbb{N} : \mu_\Lambda/n < 1\}$ and denote by Q_{Λ^κ} the quadratic form corresponding to Λ^κ. There is $C > 0$ such that, for any $\phi \in \mathcal{E}$ and $\lambda \in \mathbb{V}^*$,*

$$Dom(H_{\lambda,\phi}^\kappa) \subseteq Dom(Q_{\Lambda^\kappa})$$

and

$$Q_{\Lambda^\kappa}(f) \leq C(|\langle H_{\lambda,\phi}^\kappa f, f\rangle| + (1 + R(\lambda))^\kappa \|f\|_2^2)$$

for all $f \in Dom(H_{\lambda,\phi}^\kappa)$.

In [21], the self-adjoint operators considered are required to satisfy Hypothesis 5.1 in the special case that $\Lambda = (-\Delta)^m$ on $\mathbb{R}^d$ for some $m \in \mathbb{N}$. The theory in [21] proceeds under only two hypotheses which are paralleled by Hypotheses 5.1 and 5.2 above respectively. Incidentally, off-diagonal estimates are only shown in the case that $2m < d$ which corresponds to $\mu_\Lambda < 1$ in our setting (See Example 2.8.2). As the proposition below shows, when $\mu_\Lambda < 1$, Hypothesis 5.3 is superfluous.

Proposition 5.4.4. *Let H be a self-adjoint operator satisfying Hypotheses 5.1 and 5.2. Let Λ be the associated positive-homogeneous operator with symbol R and homogeneous order μ_Λ. If $\mu_\Lambda < 1$, i.e., $\kappa = 1$, then Hypothesis 5.3 holds.*

Proof. The assertion that $\mathrm{Dom}(H_{\lambda,\phi}) \subseteq \mathrm{Dom}(Q_\Lambda)$ for all $\phi \in \mathcal{E}$ and $\lambda \in \mathbb{V}^*$ is a consequence of Lemma 5.4.2. Using (5.4) and (5.5), we have

$$\begin{aligned} Q_\Lambda(f) \leq 2Q(f) \quad &\leq \quad C(\mathrm{Re}(Q_{\lambda,\phi}(f)) + (1 + R(\lambda))\|f\|_2^2) \\ &\leq \quad C(|Q_{\lambda,\phi}(f)| + (1 + R(\lambda))\|f\|_2^2) \end{aligned}$$

for all $f \in \mathrm{Dom}(Q)$, $\phi \in \mathcal{E}$ and $\lambda \in \mathbb{V}^*$. In view of Lemma 5.4.2, the proof is complete. $\square$

5.5 The L^2 theory

We now return to the general theory. Throughout this section all hypotheses are to include Hypotheses 5.1 and 5.2 without explicit mention. Except for Lemma 5.5.3, all statements mirror those in [21] and their proofs follow with little or no change. We will keep track of certain constants and to this end, any mention of $M > 0$ refers to that which is specified in Hypothesis 5.2. As usual, positive constants denoted by C will change from line to line.

Lemma 5.5.1. *For any $\lambda \in \mathbb{V}^*$ and $\phi \in \mathcal{E}$,*

$$\|T_t^{\lambda,\phi}\|_{2\to 2} \leq \exp(M(1 + R(\lambda))t/4)$$

for all $t > 0$.

Proof. For $f \in L^2(\Omega)$, put $f_t = T_t^{\lambda,\phi} f$. By Lemma 5.4.3,

$$\frac{d}{dt}\|f_t\|_2^2 = -2\,\mathrm{Re}[(H_{\lambda,\phi} f_t, f_t)] \leq \frac{M}{2}(1 + R(\lambda))\|f_t\|_2^2.$$

The result now follows from Grönwall's lemma. □

Lemma 5.5.2. *There exists* $C > 0$ *such that*

$$\|H_{\lambda,\phi} T_t^{\lambda,\phi}\|_{2\to 2} \leq \frac{C}{t}\exp\left(\frac{M}{2}(1 + R(\lambda))t\right)$$

for all $t > 0$, $\lambda \in \mathbb{V}^*$ *and* $\phi \in \mathcal{E}$.

Proof. Our argument uses the theory of bounded holomorphic semigroups, c.f. [19]. For $f \in L^2(\Omega)$, $r > 0$ and $|\theta| \leq \pi/3$ put

$$f_r = \exp[-re^{i\theta} H_{\lambda,\phi}] f.$$

It follows that $f_r \in \mathrm{Dom}(H_{\lambda,\phi})$ and

$$\begin{aligned}
\frac{d}{dr}\|f_r\|_2^2 &= -e^{i\theta}(H_{\lambda,\phi} f_r, f_r) - e^{-i\theta}(f_r, H_{\lambda,\phi} f_r)\\
&= -e^{i\theta} Q_{\lambda,\phi}(f_r) - e^{-i\theta}\overline{Q_{\lambda,\phi}(f_r)}\\
&= -(e^{i\theta} + e^{-i\theta}) Q(f_r) + D_r
\end{aligned}$$

where

$$D_r = -e^{i\theta}[Q_{\lambda,\phi}(f_r) - Q(f_r)] - e^{-i\theta}[\overline{Q_{\lambda,\phi}(f_r)} - Q(f_r)].$$

By Hypothesis 5.2,

$$|D_r| \leq (Q(f_r) + M(1 + R(\lambda))\|f\|_2^2)/2$$

and so with the observation that $e^{i\theta} + e^{-i\theta} \geq 1$ for all $|\theta| \leq \pi/3$,

$$\frac{d}{dr}\|f_r\|_2^2 \leq \frac{M}{2}(1 + R(\lambda))\|f\|_2^2.$$

Hence,

$$\|f_r\|_2 \leq \exp(M(1+R(\lambda))r/4)\|f\|_2$$

in view of Grönwall's lemma. From the above estimate we have

$$\begin{aligned}
&\|\exp[-zH_{\lambda,\phi} - M(1+R(\lambda))z]\|_{2\to 2} \\
&\qquad \leq \quad \exp(M(1+R(\lambda))r/4)\exp(-M(1+R(\lambda))\,\mathrm{Re}(z)/2) \leq 1
\end{aligned}$$

for all $z = re^{i\theta}$ for $r > 0$ and $|\theta| \leq \pi/3$ because $2\,\mathrm{Re}(z) \geq r$. Theorem 8.4.6 of [19] yields

$$\|(H_{\lambda,\phi} + M(1+R(\lambda))/2)\exp[-tH_{\lambda,\phi} - M(1+R(\lambda))t/2]\|_{2\to 2} \leq \frac{C'}{t}$$

for all $t > 0$. It now follows that

$$\|H_{\lambda,\phi}T_t^{\lambda,\phi}\|_{2\to 2} \leq \frac{C}{t}\exp(M(1+R(\lambda))t/2)$$

for all $t > 0$ where we have put $C = C' + 2$. □

Lemma 5.5.3. *For any $k \in \mathbb{N}$, there is $C > 0$ such that*

$$\|H_{\lambda,\phi}^k e^{-tH_{\lambda,\phi}}\|_{2\to 2} \leq \frac{C}{t^k}\exp(M(1+R(\lambda))t/2)$$

for all $t > 0$, $\phi \in \mathcal{E}$ and $\lambda \in \mathbb{V}^*$.

Proof. As $-H_{\lambda,\phi}$ is the generator of the semigroup $e^{-tH_{\lambda,\phi}}$, for any $t > 0$ and $f \in L^2(\Omega)$, $e^{-tH_{\lambda,\phi}}f \in \mathrm{Dom}(H_{\lambda,\phi}^k)$. We have

$$H_{\lambda,\phi}^k e^{-tH_{\lambda,\phi}} = \left(H_{\lambda,\phi}e^{-(t/k)H_{\lambda,\phi}}\right)^k$$

and so by the previous lemma

$$\|H_{\lambda,\phi}^k e^{-tH_{\lambda,\phi}}\|_{2\to 2} \leq \left(\frac{C}{t}\exp(M(1+R(\lambda)t/2k)\right)^k$$

from which the result follows. □

5.6 Off-diagonal estimates

In this section we prove off-diagonal estimates for the semigroup $T_t = e^{-tH}$. Let R be the symbol of the positive-homogeneous reference operator Λ for H and let μ_Λ be its homogeneous order.

Lemma 5.6.1. *Let H be a self-adjoint operator satisfying Hypotheses 5.1 and 5.2 with reference operator Λ. Let $T_t = e^{-tH}$ be the associated semigroup. If the twisted semigroup $T_t^{\lambda,\phi}$ satisfies the ultracontractive estimate*

$$\|T_t^{\lambda,\phi}\|_{2\to\infty} \leq \frac{C}{t^{\mu_\Lambda/2}} \exp[M(R(\lambda)+1)t/2] \tag{5.7}$$

for all $\lambda \in \mathbb{V}^*$, $\phi \in \mathcal{E}$ and $t > 0$ where $C, M > 0$, then T_t has integral kernel $Z(t,x,y) = Z(t,x,\cdot) \in L^1(\Omega)$ satisfying the off-diagonal bound

$$|Z(t,x,y)| \leq \frac{C}{t^{\mu_\Lambda}} \exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right) + Mt\right)$$

for all $x, y \in \mathbb{V}$ and $t > 0$.

Proof. It is clear that the adjoint of $T_t^{\lambda,\phi}$ is $T_t^{-\lambda,\phi}$ and so by duality and (5.7),

$$\|T_t^{\lambda,\phi}\|_{1\to 2} \leq \frac{C}{t^{\mu_\Lambda/2}} \exp[M(R(\lambda)+1)t/2]$$

for $t > 0$ where we have replaced $MR(-\lambda)$ by $MR(\lambda)$ in view of Proposition 4.3.2. Thus for all $t > 0$, $\lambda \in \mathbb{V}^*$ and $\phi \in \mathcal{E}$,

$$\begin{aligned}
\|T_t^{\lambda,\phi}\|_{1\to\infty} &\leq \|T_t^{\lambda,\phi}\|_{1\to 2}\|T_t^{\lambda,\phi}\|_{2\to\infty} \\
&\leq \frac{C}{t^{\mu_\Lambda/2}} \exp[M(R(\lambda)+1)t/2] \frac{C}{t^{\mu_\Lambda/2}} \exp[M(R(\lambda)+1)t/2] \\
&\leq \frac{C}{t^{\mu_\Lambda}} \exp[Mt(R(\lambda)+1)].
\end{aligned}$$

The above estimate guarantees that $T_t^{\lambda,\phi}$ has integral kernel $Z^{\lambda,\phi}(t,x,y)$ satisfying the same bound (see Theorem 2.27 of [19]). By construction, we also have

$$Z^{\lambda,\phi}(t,x,y) = e^{-\lambda(\phi(x))} Z(t,x,y) e^{\lambda(\phi(y))}$$

where $Z = Z^{0,\phi}$ is the integral kernel of $T_t = T_t^{0,\phi}$. Therefore

$$|e^{-\lambda(\phi(x))}Z(t,x,y)e^{\lambda(\phi(y))}| \leq \frac{C}{t^{\mu_\Lambda}}\exp(Mt(R(\lambda)+1))$$

or equivalently

$$|Z(t,x,y)| \leq \frac{C}{t^{\mu_\Lambda}}\exp\left(\lambda(\phi(y)-\phi(x))+Mt(R(\lambda)+1)\right)$$

for all $t>0$, $x,y\in\Omega$, $\lambda\in\mathbb{V}^*$ and $\phi\in\mathcal{E}$. In view of Hypothesis 5.2, for any x and $y\in\Omega$ there is $\phi\in\mathcal{E}$ for which $\phi(x)=x$ and $\phi(y)=y$. Consequently, we have that for all $x,y\in\Omega$, $\lambda\in\mathbb{V}^*$ and $t>0$,

$$|Z(t,x,y| \leq \frac{C}{t^{\mu_\Lambda}}\exp\left(\lambda(y-x)+Mt(R(\lambda)+1)\right).$$

The proof of the lemma will be complete upon minimizing the above bound with respect to $\lambda\in\mathbb{V}^*$. In this process, we shall see how the Legendre-Fenchel transform appears naturally. For any $x,y\in\Omega$ and $t>0$, we have

$$\begin{aligned}
|Z(t,x,y)| &\leq \frac{C}{t^{\mu_\Lambda}}\inf_{\lambda}\{\exp\left\{\lambda(y-x)+Mt(R(\lambda)+1)\right\}\}\\
&\leq \frac{C}{t^{\mu_\Lambda}}\exp\left(-t\sup_{\lambda}\left\{\lambda\left(\frac{x-y}{t}\right)-MR(\lambda)\right\}\right)\exp(Mt)\\
&\leq \frac{C}{t^{\mu_\Lambda}}\exp\left(-t(MR)^{\#}\left(\frac{x-y}{t}\right)+Mt\right)\\
&\leq \frac{C}{t^{\mu_\Lambda}}\exp\left(-tR)^{\#}\left(\frac{x-y}{t}\right)+Mt\right)
\end{aligned}$$

where we replaced $(MR)^{\#}$ by $MR^{\#}$ in view of Corollary 4.3.3. $\square$

Theorem 5.6.2. *Let H be a self-adjoint operator satisfying Hypotheses 5.1, 5.2 and 5.3 where Λ is the associated positive-homogeneous reference operator with symbol R and homogeneous order μ_Λ. Then the cooresponding semigroup $T_t = e^{-tH}$ has integral kernel $Z:(0,\infty)\times\Omega\times\Omega\to\mathbb{C}$ satisfying*

$$|Z(t,x,y)| \leq \frac{C}{t^{\mu_\Lambda}}\exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right)+Mt\right)$$

for all $x,y\in\Omega$ and $t>0$ where C is some positive constant.

Proof. Take κ as in Hypothesis 5.3. We note that for all $f \in \mathrm{Dom}(\Lambda^\kappa)$,

$$\|f\|_\infty \leq C Q_{\Lambda^\kappa}(f)^{\mu_\Lambda/2\kappa} \|f\|_2^{1-\mu_\Lambda/\kappa}$$

in view of Lemma 5.3.3. The application of the lemma is justified because Λ^κ is positive-homogeneous with $\kappa^{-1}\,\mathrm{Exp}(\Lambda^\kappa) = \mathrm{Exp}(\Lambda)$ and, as required, $\mu_\Lambda/\kappa < 1$. For $f \in L^2(\Omega)$, set $f_t = T_t^{\lambda,\phi} f$. In view of Hypothesis 5.3 and Lemmas 5.5.1 and 5.5.2, we have

$$\begin{aligned}
\|f_t\|_\infty &\leq Q_{\Lambda^\kappa}(f_t)^{\mu_\Lambda/2\kappa} \|f_t\|_2^{1-\mu_\Lambda/\kappa} \\
&\leq C\left(|\langle H^\kappa_{\lambda,\phi} f_t, f_t\rangle| + (1+R(\lambda))^\kappa \|f_t\|_2^2\right)^{\mu_\Lambda/2\kappa} \|f_t\|_2^{1-\mu_\Lambda/\kappa} \\
&\leq C\left(\|H^\kappa_{\lambda,\phi} f_t\|_2 \|f_t\|_2 + (1+R(\lambda))^\kappa \|f_t\|_2^2\right)^{\mu_\Lambda/2\kappa} \|f_t\|_2^{1-\mu_\Lambda/\kappa} \\
&\leq C\left(\frac{\exp(M(1+R(\lambda))t/4)}{t^\kappa} + (1+R(\lambda))^k\right)^{\mu_\Lambda/2\kappa} \\
&\qquad\qquad \times \exp(M(1+R(\lambda))t/4)\|f\|_2 \\
&\leq \frac{C}{t^{\mu_\Lambda/2}} \exp(M(1+R(\lambda))t/2)\|f\|_2
\end{aligned}$$

for all $\phi \in \mathcal{E}$ and $\lambda \in \mathbb{V}^*$. In view of Lemma 5.6.1, the theorem is proved. □

5.7 Homogeneous Operators

In this short section, we show that the term Mt in heat kernel estimate of Theorem 5.6.2 can be removed when H is "homogeneous" in the sense given by Definition 5.7.1 below. For simplicity, we work on full vector space, i.e., $\Omega = \mathbb{V}$. Our arguments in this section follow closely to the work of G. Barbatis and E. B. Davies [9]. Given a self-adjoint operator H on $L^2(\mathbb{V})$ satisfying Hypotheses 5.1 and 5.2 with quadratic form Q and associated positive-homogeneous operator Λ. For any $E \in \mathrm{Exp}(\Lambda)$, observe that

$$(U_s f)(x) = s^{\mu_\Lambda/2} f(s^E x)$$

defines a unitary operator U_s on $L^2(\mathbb{V})$ for each $s > 0$ with $U_s^* = U_{1/s}$. For each $s > 0$, set

$$H_s = s^{-1} U_s^* H U_s.$$

and note that H_s is a self-adjoint operator on $L^2(\mathbb{V})$. It is easily verified that the quadratic form Q^s associated to H_s has

$$Q^s(f) = s^{-1} Q(U_s f)$$

for all f in the common domain $\text{Dom}(Q^s) = \text{Dom}(Q) = Dom(\Lambda^{1/2})$. Being a rescaled version of the operator H, it is clear the H_s will satisfy the Hypotheses 5.1 and 5.2. Let us isolate the following special situation:

Definition 5.7.1. *Assuming the notation above, we say that H is homogeneous provided that H_s satisfies Hypothesis 5.1 and 5.2 with the same constants as H for all $s > 0$. In other words, H_s (and so Q^s) satisfies the estimates* (5.4) *and* (5.5) *uniformly for $s > 0$.*

We note that a positive-homogeneous operator Λ is homogeneous in the above sense, for our defining property of homogeneous constant coefficient operators can be written equivalently as $\Lambda_s = \Lambda$ for all $s > 0$. In the example section below, we will see that the replacement of H_s by H amounts to a rescaling of the arguments of the coefficients in the case that H consists of only terms of "principal order".

Theorem 5.7.2. *Let H be a self-adjoint operator satisfying the hypotheses of Theorem 5.6.2 with associated reference operator Λ with symbol R and order $\mu_\Lambda < 1$. If, additionally, H is homogeneous, then its heat kernel Z satisfies the estimate*

$$|Z(t,x,y)| \leq \frac{C}{t^{\mu_\Lambda}} \exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right)\right)$$

for all $x, y \in \mathbb{V}$ and $t > 0$, where C and M are positive constants.

Proof. Using the fact that U_s is unitary for each $s > 0$, it follows that

$$e^{-tH_s} = e^{-ts^{-1}U_{1/s}HU_s} = U_{1/s}e^{-(t/s)H}U_s$$

for $s, t > 0$. Consequently, for $f \in L^2(\mathbb{V})$,

$$\begin{aligned} \left(e^{-tH_s}f\right)(x) &= \int_{\mathbb{V}} s^{-\mu_\Lambda} K(t/s, s^{-E}x, y) s^{\mu_\Lambda} f(s^E y)\, dy \\ &= s^{-\mu_\Lambda} \int_{\mathbb{V}} Z(t/s, s^{-E}x, s^{-E}y) f(y)\, dy \end{aligned}$$

for $s, t > 0$ and almost every $x \in \mathbb{V}$. Thus, e^{-tH_s} has an integral kernel $Z^s : (0,\infty) \times \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ satisfying

$$Z^s(t, x, y) = s^{-\mu_\Lambda} Z(t/s, s^{-E}x, s^{-E}y)$$

for $x, y \in \mathbb{V}$. Equivalently,

$$Z(t, x, y) = s^{\mu_\Lambda} Z^s(st, s^E x, s^E y)$$

for $t, s > 0$ and $x, y \in \mathbb{V}$. We now apply the same sequence of arguments to the self-adjoint operators H_s and the semigroups e^{-tH_s}. Under the hypothesis that H is homogeneous, a careful study reveals that each estimate in the sequence of lemmas preceding Theorem 5.6.2 and the estimates in the proof of Theorem 5.6.2 are independent of s. From this, we obtain positive constants C and M for which

$$|Z^s(t, x, y)| \le \frac{C}{t^{\mu_\Lambda}} \exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right) + Mt\right)$$

for all $t > 0$ and $x, y \in \mathbb{V}$ and this holds uniformly for $s > 0$. Consequently,

$$\begin{aligned} |Z(t, x, y)| &\le \frac{s^{\mu_\Lambda} C}{(st)^{\mu_\Lambda}} \exp\left(-(st)MR^{\#}\left(\frac{s^E(x-y)}{st}\right) + Mst\right) \\ &\le \frac{C}{t^{\mu_\Lambda}} \exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right) + Mst\right) \end{aligned}$$

for all $s, t > 0$ and$x, y \in \mathbb{V}$ where we have used the fact that $I - E \in \operatorname{Exp}(R^{\#})$. The desired estimate follows by letting $s \to 0$. $\square$

5.8 Regularity of Z when $\mu_\Lambda < 1$

In this section, we show that the heat kernel Z is particularly nice when $\mu_\Lambda < 1$.

Lemma 5.8.1. *Let Λ be a positive-homogeneous operator with real symbol R and homogeneous order μ_Λ. If $\mu_\Lambda < 1$, then*

$$\int_{\mathbb{V}^*} \frac{1}{(1+R(\xi))^{1-\epsilon}} d\xi < \infty$$

where $\epsilon = (1-\mu_\Lambda)/2$. In particular, $(1+R)^{-1} \in L^1(\mathbb{V}^*)$.

Proof. For any Borel set B, write $m(B) = \int_B d\xi$. It suffices to prove that

$$\sum_{l=0}^{\infty} \frac{m(F_l)}{2^l} < \infty$$

where $F_l := \{\xi \in \mathbb{V}^* : 2^l \leq R(\xi)^{1-\epsilon} \leq 2^{l+1}\}$. To this end, fix $E \in \operatorname{Exp}(R)$ and observe that, for any $l \geq 1$,

$$\begin{aligned} F_l &= \left\{\xi : 2^{l-1} \leq (t^{-1}R(\xi))^{1-\epsilon} \leq 2^l\right\} \\ &= \left\{\xi : 2^{l-1} \leq R(t^{-E}\xi)^{1-\epsilon} \leq 2^l\right\} \\ &= \{t^E\xi : 2^{l-1} \leq R(\xi)^{1-\epsilon} \leq 2^l\} = t^E F_{l-1} \end{aligned}$$

where we have set $t = 2^{1/(1-\epsilon)}$. Continuing inductively we see that $F_l = t^{lE}F_0$ for all $l \in \mathbb{N}$ and so it follows that

$$m(F_l) = \int_{t^{lE}F_0} d\xi = \int_{F_0} \det(t^{lE}) d\xi = (t^{l \operatorname{tr} E}) m(F_0) = t^{l\mu_\Lambda} m(F_0).$$

where we have used the fact that $\mu_\Lambda = \operatorname{tr} E^* = \operatorname{tr} E$ because $E^* \in \operatorname{Exp}(\Lambda)$. Consequently,

$$\sum_{l=0}^{\infty} 2^{-l} m(F_l) = m(F_0) \sum_{l=0}^{\infty} 2^{-l}(t^{l\mu_\Lambda}) = \sum_{l=0}^{\infty} \left(2^{-1}t^{\mu_\Lambda}\right)^l < \infty$$

because $2^{-1}t^{\mu_\Lambda} = 2^{(\mu_\Lambda/(1-\epsilon)-1)} < 1$. □

Lemma 5.8.2. *Let $|\cdot|$ be a norm on $\mathbb{V}$ and suppose that $\mu_\Lambda < 1$. There exists $C > 0$ such that*

$$\int_{\mathbb{V}*} \frac{|e^{i\xi(x)} - e^{i\xi(y)}|^2}{1 + R(\xi)} d\xi \leq C|x - y|^{(1-\mu_\Lambda)}$$

for all $x, y \in \mathbb{V}$.

Proof. In view of the preceding lemma,

$$\frac{|e^{i\xi(x)} - e^{i\xi(y)}|^2}{(1 + R(\xi)} \leq 4(1 + R(\xi))^{-1} \in L^1(\mathbb{V}^*)$$

for all $x, y \in \mathbb{V}$. Consequently, it suffices to treat only the case in which $0 < |x - y| \leq 1$. In this case, let $E \in \operatorname{Exp}(R)$ be that given by Lemma 4.2.6, set $t = |x - y|^{-1}$ and observe that

$$\begin{aligned}
\int_{\mathbb{V}^*} \frac{|e^{i\xi(x)} - e^{i\xi(y)}|^2}{(1 + R(\xi))} d\xi &= \int_{t \leq R(\xi)} \frac{|e^{i\xi(x)} - e^{i\xi(y)}|^2}{(1 + R(\xi))} d\xi + \int_{t > R(\xi)} \frac{|e^{i\xi(x)} - e^{i\xi(y)}|^2}{(1 + R(\xi))} d\xi \\
&\leq \int_{t \leq R(\xi)} \frac{4}{R(\xi)} d\xi + \int_{t > R(\xi)} |e^{i\xi(x)} - e^{i\xi(y)}|^2 d\xi \\
&\leq \int_{1 \leq R(\xi)} \frac{4}{R(t^{E^*}\xi)} t^{\mu_\Lambda} d\xi + \int_{1 > R(\xi)} |e^{i\xi(t^E x)} - e^{i\xi(t^E y)}|^2 t^{\mu_\Lambda} d\xi \\
&\leq t^{\mu_\Lambda - 1} \int_{1 \leq R(\xi)} \frac{4}{R(\xi)} d\xi + t^{\mu_\Lambda} |t^E(x - y)|^2 \int_{1 > R(\xi)} 4|\xi|_*^2 d\xi
\end{aligned}$$

where $|\cdot|_*$ is the corresponding dual norm on $\mathbb{V}^*$. Using Lemma 5.8.1 and the fact that $|\xi|_*^2$ is bounded on the bounded set $\{1 > R(\xi)\}$, it follows that

$$\int_{\mathbb{V}^*} \frac{|e^{i\xi(x)} - e^{i\xi(y)}|^2}{(1 + R(\xi))} d\xi \leq C \left(t^{\mu_\Lambda - 1} + t^{\mu_\Lambda} |t^E(x - y)|^2 \right)$$

for some $C > 0$. In view of Lemma 4.2.6 and Lemma A.1.2, $\|t^E\| \leq C' t^{1/2}$ for some $C' > 0$ because $t \geq 1$. Consequently,

$$\int_{\mathbb{V}^*} \frac{|e^{i\xi(x)} - e^{i\xi(y)}|^2}{(1 + R(\xi))} d\xi \leq C \left(t^{\mu_\Lambda - 1} + t^{\mu_\Lambda + 1} |x - y|^2 \right) = 2C|x - y|^{(1-\mu_\Lambda)}.$$

□

The following lemma is analogous to Lemma 14 of [21].

Lemma 5.8.3. *Let H be a self-adjoint operator satisfying Assuption 5.1 with reference operator Λ and suppose that $\mu_\Lambda < 1$. There exists a uniformly bounded function $\phi : \mathbb{V} \to L^2(\mathbb{V})$ such that for every $f \in L^2(\mathbb{V})$,*

$$\{(H+1)^{-1/2} f\}(x) = (f, \phi(x)) \tag{5.8}$$

for almost every $x \in \mathbb{V}$. Moreover, ϕ is Hölder continuous of order $\alpha = (1-\mu_\Lambda)/2$. In particular, $(H+1)^{-1/2}$ is a bounded operator from $L^2(\mathbb{V})$ into $L^\infty(\mathbb{V})$ and for each $f \in L^2(\mathbb{V})$, there is a version of $(H+1)^{-1/2} f$ which is bounded and Hölder continuous of order α.

Proof. In view of (5.4),

$$\int_{\mathbb{V}^*} (1+R(\xi))|\hat{g}(\xi)|^2 d\xi \leq c\|(1+H)^{1/2} g\|_2^2$$

for all $g \in W^{\mathbf{m},2}_{\mathbf{v}}(\mathbb{V})$. Also by the Cauchy-Schwarz inequality

$$\int_{\mathbb{V}^*} (1+R(\xi))^{\epsilon/2}|\hat{g}(\xi)| d\xi \leq C \left(\int_{\mathbb{V}^*} (1+R(\xi))|\hat{g}(\xi)|^2 d\xi \right)^{1/2}$$

where

$$C^2 = \int_{\mathbb{V}^*} \frac{(1+R(\xi))^\epsilon}{(1+R(\xi))} d\xi < \infty$$

in view of Lemma 5.8.1. Consequently, for all $g \in W^{\mathbf{m},2}_{\mathbf{v}}(\mathbb{V})$, $\hat{g} \in L^1(\mathbb{V}^*)$ and

$$\|g\|_\infty \leq \int_{\mathbb{V}} (1+R(\xi))^{\epsilon/2}|\hat{g}(\xi)| d\xi \leq C\|(1+H)^{1/2} g\|_2. \tag{5.9}$$

So $(H+1)^{1/2}$ is an injective self-adjoint operator and therefore has dense range in $L^2(\mathbb{V})$. We can therefore consider $(H+1)^{-1/2}$, which by (5.9) is a bounded operator from $L^2(\mathbb{V})$ into $L^\infty(\mathbb{V})$.

Let $|\cdot|$ be a norm on $\mathbb{V}$ and for $f \in L^2(\mathbb{V})$ set $g = (H+1)^{-1/2}f$. For almost every $x, y \in \mathbb{V}$ we have

$$\begin{aligned}
|g(x)-g(y)| &\leq \int_{\mathbb{V}^*} |e^{i\xi(x)} - e^{i\xi(y)}||\hat{g}(\xi)|d\xi \\
&\leq \left(\int_{\mathbb{V}^*} (1+R(\xi))|\hat{g}(\xi)|^2 d\xi\right)^{1/2} \left(\int_{\mathbb{V}^*} \frac{|e^{i\xi(x)} - e^{i\xi(y)}|^2}{(1+R(\xi))} d\xi\right)^{1/2} \\
&\leq c\|f\|_2 \left(\int_{\mathbb{V}^*} \frac{|e^{i\xi(x)} - e^{i\xi(y)}|^2}{(1+R(\xi))} d\xi\right)^{1/2} \leq C\|f\|_2|x-y|^{\alpha} \qquad (5.10)
\end{aligned}$$

in view of the previous lemma. It follows from (5.9) that for almost every $x \in \mathbb{V}$, there exists $\phi(x) \in L^2(\mathbb{V})$ such that

$$(H+1)^{-1/2}f(x) = (f, \phi(x)).$$

By putting $f = \phi(x)$, another application of (5.9) shows that $\|\phi(x)\| \leq C$. Moreover, (5.10) guarantees that

$$|(f, \phi(x) - \phi(y))| \leq C\|f\|_2|x-y|^{\alpha}$$

from which it follows that $\|\phi(x)-\phi(y)\|_2 \leq C|x-y|^{\alpha}$ almost everywhere. Finally, redefine ϕ, so that all of the above statements hold on all of $\mathbb{V}$. $\square$

Theorem 5.8.4. *Let H be a self-adjoint operator satisfying Hypotheses 5.1 and 5.2. Let Λ be the associated positive-homogeneous operator with symbol R and homogeneous order μ_Λ. If $\mu_\Lambda < 1$ then, there exists $Z : \mathbb{C}^+ \times \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ such that*

$$\left(e^{-zH}f\right)(x) = \int_{\mathbb{V}} Z(z,x,y)f(y)dy$$

for all $f \in L^1(\mathbb{V}) \cap L^2(\mathbb{V})$. For fixed $z \in \mathbb{C}^+$, $Z(z,\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ is Hölder continuous of order $\alpha = (1-\mu_\Lambda)/2$. Moreover for each $x, y \in \mathbb{V}$, $\mathbb{C}^+ \ni z \mapsto Z(z,x,y)$ is analytic. Finally, there exists $C > 0$ such that

$$|Z(t,x,y)| \leq \frac{C}{t^{\mu_\Lambda}} \exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right) + Mt\right)$$

for all $x, y \in \mathbb{V}$ and $t > 0$.

Proof. In view of Proposition 5.4.4, the final conclusion follows from Theorem 5.6.2. The fact that e^{-zH} is a bounded holomorphic semigroup ensures that $B(z) = (1+H)e^{-zH}$ is a bounded holomorphic function on $L^2(\mathbb{V})$ for $z \in \mathbb{C}^+$. For $x, y \in \mathbb{V}$, $z \in \mathbb{C}^+$ define

$$Z(z,x,y) := (B(z)\phi(y), \phi(x)).$$

It follows that $\mathbb{C}^+ \ni z \mapsto Z(z,x,y)$ is analytic for any $x, y \in \mathbb{V}$. Now for fixed $z \in \mathbb{C}^+$, $Z(z,\cdot,\cdot)$ is Hölder continuous of order α. Indeed, let $|\cdot|$ be a norm on $\mathbb{V}$. With the help of Lemma 5.8.3, observe that for $z \in \mathbb{C}^+$,

$$\begin{aligned} |Z(z,x,y) - Z(z,x',y')| &\leq |Z(z,x,y) - Z(z,x',y)| + |Z(z,x',y) - Z(z,x',y')| \\ &\leq C\|B(z)\|_{2\to 2}\left(\|\phi(x)-\phi(x')\|_2 + \|\phi(y)-\phi(y')\|\right) \\ &\leq C\|B(z)\|_{2\to 2}\left(|x-x'|^{2(\alpha/2)} + |y-y'|^{2(\alpha/2)}\right) \\ &\leq C\|B(z)\|_{2\to 2}\left(|x-x'|^2 + |y-y'|^2\right)^{\alpha/2} \end{aligned}$$

for all $(x,y), (x',y') \in \mathbb{V} \times \mathbb{V}$ as claimed. It remains to show that $Z(z,x,y)$ is the integral kernel of e^{-zH}.

Again by Lemma 5.8.3, $(H+1)^{-1/2} : L^2(\mathbb{V}) \to L^\infty(\mathbb{V})$ is bounded and so $(H+1)^{-1/2} : L^1(\mathbb{V}) \to L^2(\mathbb{V})$ is also bounded by duality. More is true: Using the self-adjointness of H one can check that

$$\phi_x(y) = \overline{\phi_y(x)}$$

for almost every $x, y \in \mathbb{V}$. Here, the variable of integration is that which appears in the subscript. So for $f \in L^1(\mathbb{V}) \cap L^2(\mathbb{V})$,

$$\begin{aligned}
\left(e^{-Hz} f\right)(x) &= ((H+1)^{-1/2} B(z)(H+1)^{-1/2} f)(x) \\
&= \int_{\mathbb{V}} (B(z)(H+1)^{-1/2} f)(w) \overline{\phi_w(x)} dw \\
&= \int_{\mathbb{V}} (f, \phi(w)) \overline{(B(z)\phi(x))}(w) dw \\
&= \int_{\mathbb{V}} \int_{\mathbb{V}} f(y) \overline{\phi_y(w)(B(z)\phi(x))}(w) dw dy \\
&= \int_{\mathbb{V}} \int_{\mathbb{V}} f(y) \phi_w(y) \overline{(B(z)\phi(x))}(w) dw dy \\
&= \int_{\mathbb{V}} \int_{\mathbb{V}} (B(z)\phi(y))(w) \overline{\phi_w(x)} dw f(y) dy
\end{aligned}$$

as desired. □

5.9 Super-semi-elliptic operators

In this section, we consider a class of operators to which we apply the theory of the preceding sections. We call this class of operators super-semi-elliptic operators, a term motivated by the super-elliptic operators of E. B. Davies [21] (see also [9,85]). Naturally, the class of super-semi-elliptic operators include the class of super-elliptic operators.

Let $\mathbf{m} \in \mathbb{N}_+^d$ be such that

$$|\mathbf{1} : 2\mathbf{m}| = (2m_1)^{-1} + (2m_2)^{-1} + \cdots + (2m_d)^{-1} < 1,$$

let $\mathbf{v} = \{v_1, v_2, \ldots, v_d\}$ be a basis of $\mathbb{V}$ and let $E \in \mathrm{Gl}(\mathbb{V})$ be given by $Ev_k = (2m_k)^{-1} v_k$ for $k = 1, 2, \ldots, d$. Herein, we consider a class of operators written in divergence form whose coefficients are only required to be bounded and measurable. In the notation of Chapter 4, consider the partial differential operator

H given formally by

$$H = \sum_{\substack{|\alpha:\mathbf{m}|\leq 1 \\ |\beta:\mathbf{m}|\leq 1}} D_{\mathbf{v}}^{\alpha}\{a_{\alpha,\beta}(x) D_{\mathbf{v}}^{\beta}\} \tag{5.11}$$

where we require the following conditions for the functions $a_{\alpha,\beta}$:

- The collection

  $$\{a_{\alpha,\beta}(\cdot)\}_{\substack{|\alpha:\mathbf{m}|\leq 1 \\ |\beta:\mathbf{m}|\leq 1}} \subseteq L^{\infty}(\mathbb{V})$$

  and we shall put

  $$\nu = \max_{\substack{|\alpha:\mathbf{m}|\leq 1 \\ |\beta:\mathbf{m}|\leq 1}} \|a_{\alpha,\beta}\|_{\infty}.$$

- For each $x \in \mathbb{V}$, the matrix

  $$\{a_{\alpha,\beta}(x)\}_{\substack{|\alpha:\mathbf{m}|\leq 1 \\ |\beta:\mathbf{m}|\leq 1}}$$

  is Hermitian.

- There exists $\{A_{\alpha,\beta} : |\alpha : \mathbf{m}| = 1, |\beta : \mathbf{m}| = 1\} \subseteq \mathbb{R}$ such that

  $$\Lambda := \sum_{\substack{|\alpha:\mathbf{m}|=1 \\ |\beta:\mathbf{m}|=1}} A_{\alpha,\beta} D_{\mathbf{v}}^{\alpha+\beta}$$

  has positive-definite symbol R (and so it is a positive-homogeneous operator with $E \in \mathrm{Exp}(\Lambda)$ and $\mu_{\Lambda} = |\mathbf{1} : \mathbf{m}| < 1$) and, for some $C \geq 1$,

  $$\frac{3}{4} \sum_{\substack{|\alpha:\mathbf{m}|=1 \\ |\beta:\mathbf{m}|=1}} A_{\alpha,\beta}\eta_{\alpha}\overline{\eta}_{\beta} \leq \sum_{\substack{|\alpha:\mathbf{m}|=1 \\ |\beta:\mathbf{m}|=1}} a_{\alpha,\beta}(x)\eta_{\alpha}\overline{\eta}_{\beta} \leq C \sum_{\substack{|\alpha:\mathbf{m}|=1 \\ |\beta:\mathbf{m}|=1}} A_{\alpha,\beta}\eta_{\alpha}\overline{\eta}_{\beta}$$

  for all $\eta \in \oplus_{|\alpha:\mathbf{m}|=1}\mathbb{C}$ and almost every $x \in \mathbb{V}$.

We will call such operators *super-semi-elliptic*.

For a general super-semi-elliptic operator H, its quadratic form Q is given by

$$Q(f,g) = \sum_{\substack{|\alpha:\mathbf{m}|\leq 1\\ |\beta:\mathbf{m}|\leq 1}} \int_{\mathbb{V}} a_{\alpha,\beta}(x) D_{\mathbf{v}}^{\alpha} f(x) \overline{D_{\mathbf{v}}^{\beta} g(x)} dx$$

for $f, g \in C_0^\infty(\mathbb{V})$ (in fact, this defines H). Q is clearly symmetric. Taking $C_0^\infty(\mathbb{V})$ as a form core for Q, we see that Q extends to a closed quadratic form on $L^2(\mathbb{V})$, also denoted by Q with domain $\mathrm{Dom}(Q) = W^{2,\mathbf{m}}(\mathbb{V})$ and H extends to a self-adjoint operator with $\mathrm{Dom}(H) \subseteq \mathrm{Dom}(Q)$.

Proposition 5.9.1. *There exists a positive constant C for which $H + C$ satisfies Hypothesis 5.1. If H consists only of its principal terms, i.e,*

$$H = \sum_{\substack{|\alpha:\mathbf{m}|=1\\ |\beta:\mathbf{m}|=1}} D_{\mathbf{v}}^{\alpha}\{a_{\alpha,\beta}(x) D_{\mathbf{v}}^{\beta}\},$$

then H satisfies the Hypotheses 5.1.

Proof. For $f \in C_0^\infty(\mathbb{V})$, observe that

$$\begin{aligned}
&\frac{3}{4} Q_\Lambda(f) + \sum_{|\alpha+\beta:\mathbf{m}|<2} \int_{\mathbb{V}} a_{\alpha,\beta} D_{\mathbf{v}}^{\alpha} f \overline{D_{\mathbf{v}}^{\beta} f} dx \\
&\qquad = \frac{3}{4} \sum_{\substack{|\alpha:\mathbf{m}|=1\\ |\beta:\mathbf{m}|=1}} \int_{\mathbb{V}} A_{\alpha,\beta} D_{\mathbf{v}}^{\alpha} f \overline{D_{\mathbf{v}}^{\beta} f} dx + \sum_{|\alpha+\beta:\mathbf{m}|<2} \int_{\mathbb{V}} a_{\alpha,\beta}(x) D_{\mathbf{v}}^{\alpha} f \overline{D_{\mathbf{v}}^{\beta} f} dx \\
&\qquad \leq \sum_{\substack{|\alpha:\mathbf{m}|\leq 1\\ |\beta:\mathbf{m}|\leq 1}} \int_{\mathbb{V}} a_{\alpha,\beta} D_{\mathbf{v}}^{\alpha} f \overline{D_{\mathbf{v}}^{\beta} f} dx = Q(f) \\
&\qquad \leq C \sum_{\substack{|\alpha:\mathbf{m}|=1\\ |\beta:\mathbf{m}|=1}} \int_{\mathbb{V}} A_{\alpha,\beta} D_{\mathbf{v}}^{\alpha} f \overline{D_{\mathbf{v}}^{\beta} f} dx + \sum_{|\alpha+\beta:\mathbf{m}|<2} \int_{\mathbb{V}} a_{\alpha,\beta} D_{\mathbf{v}}^{\alpha} f \overline{D_{\mathbf{v}}^{\beta} f} dx \\
&\qquad \leq C Q_\Lambda(f) + \sum_{|\alpha+\beta:\mathbf{m}|<2} \int_{\mathbb{V}} a_{\alpha,\beta} D_{\mathbf{v}}^{\alpha} f \overline{D_{\mathbf{v}}^{\beta} f} dx.
\end{aligned}$$

Thus

$$\frac{3}{4}Q_\Lambda(f) + L(f) \leq Q(f) \leq CQ_\Lambda(f) + L(f) \tag{5.12}$$

where we have put

$$L(f) = \sum_{|\alpha+\beta:\mathbf{m}|<2} \int_{\mathbb{V}} a_{\alpha,\beta} D_{\mathbf{v}}^{\alpha} f \overline{D_{\mathbf{v}}^{\beta} f} dx.$$

Using uniform bound on the coefficients $a_{\alpha,\beta}$ and Cauchy-Schwarz inequality we see that

$$|L(f)| \leq C \sum_{|\alpha+\beta:\mathbf{m}|<2} \int_{\mathbb{V}} |D_{\mathbf{v}}^{\alpha} f||D_{\mathbf{v}}^{\beta} f| dx \leq C \sum_{|\alpha+\beta:\mathbf{m}|<2} \|D_{\mathbf{v}}^{\alpha} f\|_2 \|D_{\mathbf{v}}^{\beta} f\|_2$$

for some $C > 0$. For each multi-index γ such that $|\gamma : \mathbf{m}| < 1$, it follows from Item 3 of Lemma 4.3.4 (where $\beta > \gamma$ and $\nu = v_1 + v_2 + \cdots + v_d$) that

$$\|D_{\mathbf{v}}^{\gamma} f\|_2^2 = \int_{\mathbb{V}^*} |\xi^{2\gamma}||\hat{f}(\xi)|^2 d\xi \leq \int_{\mathbb{V}^*} (\epsilon R(\xi) + M_\epsilon |\hat{f}(\xi)|^2 d\xi = \epsilon Q_\Lambda(f) + M_\epsilon \|f\|_2^2$$

where ϵ can be taken arbitrarily small. Taking into account all possible multi-indices appearing in L, we can produce a positive constant M for which

$$|L(f)| \leq \frac{1}{4} Q_\Lambda(f) + M\|f\|_2^2. \tag{5.13}$$

By combining (5.12) and (5.13), we obtain

$$\begin{aligned}
\frac{1}{2} Q_\Lambda(f) &= \frac{3}{4} Q_\Lambda(f) - \frac{1}{4} Q_\Lambda(f) \\
&\leq Q(f) - L(f) - \frac{1}{4} Q_\Lambda(f) \\
&\leq Q(f) + C\|f\|_2^2 \\
&\leq C_1 Q_\Lambda(f) + C_2 \|f\|_2^2
\end{aligned}$$

from which the first assertion follows immediately. In the case that H consists only of its pricipal terms, L is identically 0 and so the second assertion follows from (5.12) at once. $\square$

To address Hypothesis 5.2 we need to first introduce an appropriate class $\mathcal{E}$. For any integer $l \geq 1$, put

$$\mathcal{F}_l = \left\{ \psi \in C_0^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} \left| \frac{d^j \psi}{dx^j}(x) \right| \leq 1 \text{ for all } j = 1, 2, \dots, l \right\}.$$

We will take $\mathcal{E}$ to be the set of $\phi \in C_\infty^\infty(\mathbb{V}, \mathbb{V})$ for which there are $\psi_1, \psi_2, \dots, \psi_d \in \mathcal{F}_l$ such that

$$(\theta_{\mathbf{v}} \circ \phi \circ \theta_{\mathbf{v}}^{-1})(x_1, x_2, \dots, x_d) = (\psi_1(x_1), \psi_2(x_2), \dots, \psi_3(x_3)) \tag{5.14}$$

for all $(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$.

Remark 22. *What is important for us is that the j^{th}-coordinate function of $\theta_{\mathbf{v}} \circ \phi \circ \theta_{\mathbf{v}}^{-1}$ only depends on x_j for each $j = 1, 2, \dots, d$.*

Lemma 5.9.2. *For each multi-index $\alpha > 0$, there exists $C_\alpha > 0$ such that for all $f \in Dom(Q)$, $\phi \in \mathcal{E}$ and $\lambda \in \mathbb{V}^*$,*

$$|e^{-\lambda(\phi(x))} D_{\mathbf{v}}^\alpha (e^{\lambda(\phi)} f)(x) - D_{\mathbf{v}}^\alpha f(x)| \leq C_\alpha \sum_{0 < \beta \leq \alpha} \sum_{0 < \gamma \leq \beta} |\lambda^\gamma| |D_{\mathbf{v}}^{\alpha - \beta} f(x)| \tag{5.15}$$

for almost every $x \in \mathbb{V}$.

Proof. In view of the coordinate charts $(\mathbb{V}, \theta_{\mathbf{v}})$ and $(\mathbb{V}^*, \theta_{\mathbf{v}^*})$, we have

$$\lambda(\phi(x)) = (\lambda_1, \lambda_2, \dots, \lambda_d) \cdot (\psi_1(x_1), \psi_2(x_2), \dots, \psi_d(x_d))$$

for $x \in \mathbb{V}$ and $\lambda \in \mathbb{V}^*$ where $\theta_{\mathbf{v}}(x) = (x_1, x_2, \dots, x_d)$ and $\theta_{\mathbf{v}^*}(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_d)$. So for any multi-index $\beta > 0$,

$$\begin{aligned} D_{\mathbf{v}}^\beta (e^{\lambda(\phi)}) &= \left(i \frac{\partial}{\partial x_1} \right)^{\beta_1} \left(i \frac{\partial}{\partial x_2} \right)^{\beta_2} \cdots \left(i \frac{\partial}{\partial x_d} \right)^{\beta_d} \left(e^{(\lambda_1, \lambda_2, \dots, \lambda_d) \cdot (\psi_1, \psi_2, \dots, \psi_d)} \right) \\ &= \left(i^{\beta_1} \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} e^{\lambda_1 \psi_1} \right) \left(i^{\beta_2} \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}} e^{\lambda_2 \psi_2} \right) \cdots \left(i^{\beta_d} \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}} e^{\lambda_d \psi_d} \right). \end{aligned}$$

Using the properties we have required for each ψ_j, it follows that

$$|e^{-\lambda(\phi)} D_{\mathbf{v}}^{\beta}(e^{\lambda(\phi)})| \leq C_\beta \prod_{\beta_j \neq 0} \left(\sum_{l=1}^{\beta_j} |\lambda^l| \right) \leq C_\beta \sum_{0<\gamma\leq\beta} |\lambda^\gamma|$$

where $C_\beta > 0$ is independent of ϕ and λ. In view of the Leibniz rule,

$$\begin{aligned}
&\left| e^{-\lambda(\phi(x))} D_{\mathbf{v}}^{\alpha} \left(e^{\lambda(\phi)} f \right)(x) - D_{\mathbf{v}}^{\alpha} f(x) \right| \\
&\qquad = \left| \sum_{0<\beta\leq\alpha} C_{\alpha,\beta} e^{-\lambda(\phi(x))} D_{\mathbf{v}}^{\beta} \left(e^{\lambda(\phi)} \right)(x) D_{\mathbf{v}}^{\alpha-\beta} f(x) \right| \\
&\qquad\qquad \leq C_\alpha \sum_{0<\beta\leq\alpha} \sum_{0<\gamma\leq\beta} |\lambda^\gamma| |D_{\mathbf{v}}^{\alpha-\beta} f(x)|.
\end{aligned}$$

for almost every $x \in \mathbb{V}$ where C_α is independent of λ and ϕ. The constants $C_{\alpha,\beta}$ appearing in the penultimate line are the standard multi-index combinations. $\square$

Proposition 5.9.3. *With respect to the class $\mathcal{E}$ above, H (and so $H + C$ satisfies Hypothesis 5.2). Furthermore, $H + C$ satisfies Hypothesis 5.3.*

Proof. Let $x, y \in \mathbb{V}$ and set $(x_1, x_2, \dots, x_d) = \theta_{\mathbf{v}}(x)$ and $(y_1, y_2, \dots, y_d) = \theta_{\mathbf{v}}(y)$. For each pair $x_i, y_i \in \mathbb{R}$ there is $\psi_i \in \mathcal{F}_l$ for which $\psi_i(x_i) = x_i$ and $\psi_i(y_i) = y_i$; such functions can be found by smoothly cutting off the identity while keeping derivatives bounded appropriately. Using this collection of ψ_i's, we define ϕ as in (5.14) and note that

$$\begin{aligned}
\phi(x) - \phi(y) &= \theta_{\mathbf{v}}^{-1}(\psi_1(x_1), \psi_2(x_2), \dots, \psi_d(x_d)) - \theta_{\mathbf{v}}^{-1}(\psi_1(y_1), \psi_2(y_2), \dots, \psi_d(y_d)) \\
&= \theta_{\mathbf{v}}^{-1}(x_1, x_2, \dots, x_d) - \theta_{\mathbf{v}}^{-1}(y_1, y_2, \dots, y_d) \\
&= x - y
\end{aligned}$$

as required.

For any $\lambda \in \mathbb{V}^*$, $\phi \in \mathcal{E}$ and $f \in \mathrm{Dom}(Q)$,

$$Q_{\lambda,\phi}(f) = \sum_{\substack{|\alpha:\mathbf{m}|\leq 1\\|\beta:\mathbf{m}|\leq 1}} \int_{\mathbb{V}} a_{\alpha,\beta}(x) D_{\mathbf{v}}^{\alpha}(e^{-\lambda(\phi)}f)(x)\overline{D_{\mathbf{v}}^{\beta}(e^{\lambda(\phi)}f)(x)}dx.$$

Using the uniform boundedness of the collection $\{a_{\alpha,\beta}\}$, we have

$$\begin{aligned}
&|Q_{\lambda,\phi}(f) - Q(f)| \\
&\quad= \Big| \sum_{\substack{0<|\alpha:\mathbf{m}|\leq 1\\0<|\beta:\mathbf{m}|\leq 1}} \int_{\mathbb{V}} a_{\alpha,\beta}\Big[e^{\lambda(\phi)}D_{\mathbf{v}}^{\alpha}(e^{-\lambda(\phi)}f)\overline{e^{-\lambda(\phi)}D_{\mathbf{v}}^{\beta}(e^{\lambda(\phi)}f)} - D_{\mathbf{v}}^{\alpha}f\overline{D_{\mathbf{v}}^{\beta}f}\Big]dx\Big| \\
&\quad= \Big| \sum_{\substack{0<|\alpha:\mathbf{m}|\leq 1\\0<|\beta:\mathbf{m}|\leq 1}} \int_{\mathbb{V}} a_{\alpha,\beta}\Big[\left(e^{\lambda(\phi)}D_{\mathbf{v}}^{\alpha}(e^{-\lambda(\phi)}f) - D_{\mathbf{v}}^{\alpha}f\right)\overline{e^{-\lambda(\phi)}D_{\mathbf{v}}^{\beta}(e^{\lambda(\phi)}f)} \\
&\qquad\qquad + D_{\mathbf{v}}^{\alpha}f\left(\overline{e^{-\lambda(\phi)}D_{\mathbf{v}}^{\beta}(e^{\lambda(\phi)}f) - D_{\mathbf{v}}^{\beta}f}\right)\Big]dx\Big| \\
&\quad\leq C \sum_{\substack{0<|\alpha:\mathbf{m}|\leq 1\\0<|\beta:\mathbf{m}|\leq 1}} \int_{\mathbb{V}} |e^{\lambda(\phi)}D_{\mathbf{v}}^{\alpha}(e^{-\lambda(\phi)}f) - D_{\mathbf{v}}^{\alpha}f||e^{-\lambda(\phi)}D_{\mathbf{v}}^{\beta}(e^{\lambda(\phi)}f)| \\
&\qquad\qquad + |D_{\mathbf{v}}^{\alpha}f||e^{-\lambda(\phi)}D_{\mathbf{v}}^{\beta}(e^{\lambda(\phi)}f) - D_{\mathbf{v}}^{\beta}f|dx \\
&\quad\leq C \sum_{\substack{0<|\alpha:\mathbf{m}|\leq 1\\0<|\beta:\mathbf{m}|\leq 1}} \int_{\mathbb{V}} |e^{\lambda(\phi)}D_{\mathbf{v}}^{\alpha}(e^{-\lambda(\phi)}f) - D_{\mathbf{v}}^{\alpha}f||e^{-\lambda(\phi)}D_{\mathbf{v}}^{\beta}(e^{\lambda(\phi)}f) - D_{\mathbf{v}}^{\beta}f| \\
&\qquad\qquad + |D_{\mathbf{v}}^{\alpha}f||e^{-\lambda(\phi)}D_{\mathbf{v}}^{\beta}(e^{\lambda(\phi)}f) - D_{\mathbf{v}}^{\beta}f|dx.
\end{aligned}$$

With the help of Lemma 5.9.2,

$$\begin{aligned}
|Q_{\lambda,\phi}(f) - Q(f)| &\leq C \sum_{\substack{0<|\alpha:\mathbf{m}|\leq 1\\0<|\beta:\mathbf{m}|\leq 1}} \sum_{\substack{0<\gamma_\alpha\leq\alpha\\0<\gamma_\beta\leq\beta}} \sum_{\substack{0<\eta_\alpha\leq\gamma_\alpha\\0<\eta_\beta\leq\gamma_\beta}} \int_{\mathbb{V}} |\lambda^{\eta_\alpha}||D_{\mathbf{v}}^{\alpha-\gamma_\alpha}f||\lambda^{\eta_\beta}||D_{\mathbf{v}}^{\beta-\gamma_\beta}f|dx \\
&\quad + C \sum_{\substack{0<|\alpha:\mathbf{m}|\leq 1\\0<|\beta:\mathbf{m}|\leq 1}} \sum_{0<\gamma_\beta\leq\beta} \sum_{0<\eta_\beta\leq\gamma_\beta} \int_{\mathbb{V}} |D_{\mathbf{v}}^{\alpha}f||\lambda^{\eta_\beta}||D_{\mathbf{v}}^{\beta-\gamma_\beta}f|dx \\
&\leq C \sum_{\substack{0<|\alpha:\mathbf{m}|\leq 1\\0<|\beta:\mathbf{m}|\leq 1}} \sum_{\substack{0\leq\gamma_\alpha\leq\alpha\\0<\gamma_\beta\leq\beta}} \sum_{\substack{0\leq\eta_\alpha\leq\gamma_\alpha\\0<\eta_\beta\leq\gamma_\beta}} \int_{\mathbb{V}} |\lambda^{\eta_\alpha}D_{\mathbf{v}}^{\alpha-\gamma_\alpha}f||\lambda^{\eta_\beta}D_{\mathbf{v}}^{\beta-\gamma_\beta}f|dx
\end{aligned}$$

where $C > 0$ is independent of ϕ, λ and f. Thus by the Cauchy-Schwarz inequality,

$$|Q_{\lambda,\phi}(f) - Q(f)| \leq C \sum_{\substack{0<|\alpha:\mathbf{m}|\leq 1 \\ 0<|\beta:\mathbf{m}|\leq 1}} \sum_{\substack{0\leq\gamma_\alpha\leq\alpha \\ 0<\gamma_\beta\leq\beta}} \sum_{\substack{0\leq\eta_\alpha\leq\gamma_\alpha \\ 0<\eta_\beta\leq\gamma_\beta}} \|\lambda^{\eta_\alpha} D_{\mathbf{v}}^{\alpha-\gamma_\alpha} f\|_2 \|\lambda^{\eta_\beta} D_{\mathbf{v}}^{\beta-\gamma_\beta} f\|_2. \quad (5.16)$$

It is important to note that for no such summand is $|\beta - \gamma_\beta : \mathbf{m}| = 1$. In view of Lemma 4.3.4 and Proposition 5.9.1 it follows that for all such β, γ_β and η_β,

$$\begin{aligned}
\|\lambda^{\eta_\beta} D_{\mathbf{v}}^{\beta-\gamma_\beta} f\|_2^2 &= \int_{\mathbb{V}^*} |\lambda^{2\eta_\beta} \xi^{2(\beta-\gamma_\beta)}||\hat{f}(\xi)|^2 d\xi \\
&\leq \epsilon \int_{\mathbb{V}^*} R(\xi)|\hat{f}(\xi)|^2 d\xi + M_\epsilon(1+R(\lambda))\|f\|_2^2 \\
&\leq \epsilon Q_\Lambda(f) + M_\epsilon(1+R(\lambda))\|f\|_2^2 \\
&\leq \epsilon Q(f) + M(1+R(\lambda))\|f\|_2^2
\end{aligned}$$

where ϵ can be taken arbitrarily small. For all admissible α, γ_α and η_α, a similar calculation (making use of Lemma 4.3.4 and Proposition 5.9.1) shows that

$$\|\lambda^{\eta_\alpha} D_{\mathbf{v}}^{\alpha-\gamma_\alpha} f\|_2^2 \leq M(Q(f) + (1+R(\lambda))\|f\|_2^2)$$

for some $M > 0$. Thus for any $\epsilon > 0$, each summand in (5.16) satisfies

$$\begin{aligned}
&\|\lambda^{\eta_\alpha} D_{\mathbf{v}}^{\alpha-\gamma_\alpha} f\|_2 \|\lambda^{\eta_\beta} D_{\mathbf{v}}^{\beta-\gamma_\beta} f\|_2 \\
&\quad\leq (M(Q(f) + (1+R(\lambda))\|f\|_2^2))^{1/2} (\epsilon Q(f) + M(1+R(\lambda))\|f\|_2^2)^{1/2} \\
&\quad\leq (\epsilon M)^{1/2} Q(f) + \frac{M^{3/2}}{\epsilon^{1/2}}(1+R(\lambda))\|f\|_2^2.
\end{aligned}$$

The result now follows by choosing ϵ appropriately and combining these estimates. $\square$

In view of the preceding Proposition, $\mu_\Lambda < 1$, $H + C$ necessarily satisfied Hypothesis 5.3 and the results of Section 5.8. Upon noting that the semigroup generated by $-H$ and that generated by $-(H+C)$ are related by $e^{-t(H+C)} = e^{-tC}e^{-tH}$, we immediately obtain the following result.

Proposition 5.9.4. *Let H be a super-semi-elliptic operator with associated reference operator Λ, defined above. Let μ_Λ be the homogeneous order of Λ and let R be its symbol. Then the semigroup $T_z = e^{-zH}$ has integral kernel $Z : \mathbb{C}^+ \times \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ for which*

$$\left(e^{-zH} f\right)(x) = \int_{\mathbb{V}} Z(z,x,y) f(y)\, dy$$

for all $f \in L^1(\mathbb{V}) \cap L^2(\mathbb{V})$. For fixed z, $Z(z,\cdot,\cdot)$ is jointly Hölder continuous of order $\alpha = (1-\mu_\Lambda)/2$. For fixed $x, y \in \mathbb{V}$, $z \mapsto Z(z,x,y)$ is analytic. Finally,

$$|Z(t,x,y)| \leq \frac{C}{t^{\mu_\Lambda}} exp\left(-tMR^{\#}\left(\frac{x-y}{t}\right) + Mt\right)$$

for all $x, y \in \mathbb{V}$ and $t > 0$.

Homogeneous super-semi-elliptic operators

If a super-semi-elliptic operator H has the special form

$$H = \sum_{\substack{|\alpha:\mathbf{m}|=1 \\ |\beta:\mathbf{m}|=1}} D_{\mathbf{v}}^{\alpha}\{a_{\alpha,\beta}(x) D_{\mathbf{v}}^{\beta}\}$$

or, more precisely, its quadratic form is given by

$$Q(f,g) = \sum_{\substack{|\alpha:\mathbf{m}|=1 \\ |\beta:\mathbf{m}|=1}} \int_{\mathbb{V}} a_{\alpha,\beta}(x) D_{\mathbf{v}}^{\alpha} f \overline{D_{\mathbf{v}}^{\beta} g}\, dx$$

we will call it a *homogeneous super-semi-elliptic operator*. By the work of the last section, H satisfies Hypothesis 5.1, 5.2 and 5.3. In the notation of Section 5.7, we observe that

$$Q^s(f,g) = s^{-1} Q(U_s f, U_s g) = s^{-1} s^{\mu_\Lambda} \sum_{\substack{|\alpha:\mathbf{m}|=1 \\ |\beta:\mathbf{m}|=1}} \int_{\mathbb{V}} a_{\alpha,\beta}(x) D_{\mathbf{v}}^{\alpha}(f_s)(x) \overline{D_{\mathbf{v}}^{\alpha}(f_s)(x)}\, dx$$

for $f, g \in W^{2,\mathbf{m}}(\mathbb{V})$ where $f_s(x) = f(s^E x)$ for $s > 0$ and $x \in \mathbb{V}$. Noting the definition of E at the the beginning of the subsection, for each multi-index γ

such that $|\gamma : \mathbf{m}| = 1$,

$$D_{\mathbf{v}}^{\gamma} f_s(x) = s^{|\gamma:2\mathbf{m}|}(D_{\mathbf{v}}^{\gamma} f)(s^E x) = s^{1/2}(D_{\mathbf{v}}^{\gamma} f)(s^E x).$$

Consequently,

$$\begin{aligned} Q^s(f,g) &= s^{\mu_\Lambda} \sum_{\substack{|\alpha:\mathbf{m}|=1 \\ |\beta:\mathbf{m}|=1}} \int_{\mathbb{V}} a_{\alpha,\beta}(x) D_{\mathbf{v}}^{\alpha} f(s^E x) \overline{D_{\mathbf{v}}^{\alpha} f(s^E x)}\, dx \\ &= \sum_{\substack{|\alpha:\mathbf{m}|=1 \\ |\beta:\mathbf{m}|=1}} \int_{\mathbb{V}} a_{\alpha,\beta}(s^{-E} x) D_{\mathbf{v}}^{\alpha} f(x) \overline{D_{\mathbf{v}}^{\alpha} f(x)}\, dx. \end{aligned}$$

In other words, we have formally

$$H_s = \sum_{\substack{|\alpha:\mathbf{m}|=1 \\ |\beta:\mathbf{m}|=1}} D_{\mathbf{v}}^{\alpha}\{a_{\alpha,\beta}(s^{-E}x) D_{\mathbf{v}}^{\beta}\}$$

for $s > 0$. Thus, H_s is H with coefficients whose arguments are dilated by s^{-E}. Thus, under the assumption that H super-semi-elliptic and so all estimates concerning $a_{\alpha,\beta}$ hold uniformly for $x \in \mathbb{V}$, we see immediately that H is homogeneous in the sense of Section 5.7 and therefore ripe for the application of Theorem 5.7.2. In other words, one can take $M = 0$ in Proposition 5.9.4

Writing semi-elliptic operators in divergence form

The uniformly positive semi-elliptic operators of Section 4.4 are of the form

$$H = \sum_{|\alpha:\mathbf{m}| \le 2} a_\alpha D_{\mathbf{v}}^{\alpha} \tag{5.17}$$

whereas the super-semi-elliptic operators considered in the preceding subsection are of the form

$$H = \sum_{\substack{|\alpha:\mathbf{m}|\le 1 \\ |\beta:\mathbf{m}|\le 1}} D_{\mathbf{v}}^{\alpha}\{a_{\alpha,\beta} D_{\mathbf{v}}^{\beta}\}. \tag{5.18}$$

An operator H given by the expression (5.18) is said to be in *divergence form*. We note that many of the calculations in the preceding section used the fact that the operators were given in divergence form and, in particular, this allowed us to carry out our analysis in terms of the quadratic form Q. For this reason, one readily asks the question: Can each semi-elliptic operator H, given in the form 5.17, be written in divergence form provided its coefficients are sufficiently smooth. A moment's thought shows that this is equivalent to the following number-theoretic question:

Question 5.9.5. Given $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_+^d$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$ for which $|\alpha : \mathbf{m}| = 2$. Does there exists $\beta \in \mathbb{N}^d$ with $\beta \leq \alpha$ such that $|\beta : \mathbf{m}| = 1$?

In general, the answer to this question is no and can be seen by considering $\mathbf{m} = (2, 3, 5, 30) \in \mathbb{N}_+^4$ and $\alpha = (1, 2, 4, 1) \in \mathbb{N}^4$. With this example in mind, it is easy to see that the answer to the above question is no whenever $d \geq 4$. However, the question does have an affirmative answer in dimensions $1, 2,$ and 3. This is captured by the following result due to Robert Kesler of Cornell University [59].

Proposition 5.9.6. *Suppose that $d \leq 3$ and $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbf{N}_+^d$. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$ is such that $|\alpha : \mathbf{m}| = 2$, then there exists $\beta \in \mathbb{N}^d$ such that $\beta \leq \alpha$ and $|\beta : \mathbf{m}| = 1$.*

Proof. The assertion is obvious when $d = 1$. In the case that $d = 2$, it is necessary that $\alpha_1 \geq m_1$ or $\alpha_2 \geq m_2$ for otherwise $|\alpha : \mathbf{m}| < 2$. If $\alpha_1 \geq m_1$, then $\beta := (m_1, 0)$ is such that $\beta \leq \alpha$ and $|\beta : \mathbf{m}| = 1$. Similarly, if $\alpha_2 \geq m_2$, then $\beta = (0, m_2)$ does the trick. It remains to prove the assertion when $d = 3$.

We can and do assume without loss of generality that

$$\frac{\alpha_1}{m_1} \geq \frac{\alpha_2}{m_2} \geq \frac{\alpha_3}{m_3}$$

and that $\alpha_i < m_i$ for $i = 1, 2, 3$. Set

$$S = \left\{ 0 \leq n \leq \alpha_2 : \text{there is an integer } k \text{ for which } \frac{n}{m_2} = \frac{k}{m_1} \right\}$$

and put $\beta_2 = \sup S$. Now, the equation

$$|\alpha : \mathbf{m}| = \frac{\alpha_1}{m_1} + \frac{\alpha_2}{m_2} + \frac{\alpha_3}{m_3} = 2$$

guarantees that $m_2 | m_1 \cdot m_2$ and so $m_2 = m_{2,1} m_{2,3}$ where $m_{2,1} | m_1$ and $m_{2,3} | m_3$. Observe that, for some integer l,

$$\frac{\beta_2 + m_{2,3}}{m_2} = \frac{l}{m_1} + \frac{M_{2,3}}{m_2} = \frac{l}{m_1} + \frac{1}{m_{12}}.$$

But since, $m_1 = k m_{1,2}$ for some integer k, we obtain

$$\frac{\beta_2 + m_{2,3}}{m_2} = \frac{l + k}{m_1}.$$

It now follows immediately that $\alpha_2 > \beta_2 + m_{2,3}$ for otherwise, $\beta_2 + m_{2,3} \in S$ contrary to the definition of β_2. Therefore

$$\frac{\beta_2}{m_2} \geq \frac{\alpha_2}{m_2} - \frac{m_{2,3}}{m_2}$$

and so

$$\frac{\alpha_1}{m_1} + \frac{\beta_2}{m_2} \geq \frac{\alpha_1}{m_1} + \frac{\alpha_2}{m_2} - \frac{m_{2,3}}{m_2}.$$

If $m_{2,3}/m_2 \leq 1/3$ then

$$\frac{\alpha_1}{m_1} + \frac{\beta_2}{m_2} \geq 1$$

and using the definition of β_2 it follows that we can fine $\beta_1 \leq \alpha_1$ for which

$$\frac{\beta_1}{m_1} + \frac{\beta_2}{m_2} = 1$$

which gives the result with $\beta = (\beta_1, \beta_2, 0)$. If instead $m_{2,3}/m_2 = 1/m_{2,1} > 1/3$, we only have two options, $m_{2,1} = 1$ and $m_{2,1} = 2$. In the first case, $m_2 = m_{2,3} | m_3$ and, given the fact that

$$\frac{\alpha_2}{m_2} + \frac{\alpha_3}{m_3} \geq 1,$$

we may easily produce $\beta \leq \alpha$ which does the trick. In the final case, $m_2 = 2m_{2,3}$ and so it follows that m_2 and m_1 are necessarily even. Consequently,

$$\frac{\alpha_1}{2\tilde{m}_1} + \frac{\alpha_2}{2\tilde{m}_2} + \frac{\alpha_3}{m_3} = 2$$

for $\tilde{m}_2, \tilde{m}_2 \in \mathbb{N}_+$. However, by our initial constraints, $\alpha_1/2\tilde{m}_1, \alpha_2/2\tilde{m}_2 \geq 1/2$ and so $\alpha_1 \geq \tilde{m}_1$ and $\alpha_2 \geq \tilde{m}_2$. In this case, it is easily seen $\beta = (\tilde{m}_1, \tilde{m}_2, 0)$ has $\beta \leq \alpha$ and

$$|\beta : \mathbf{m}| = \frac{\tilde{m}_1}{2\tilde{m}_1} + \frac{\tilde{m}_2}{2\tilde{m}_2} = 1.$$

$\square$

APPENDIX A

APPENDIX

A.1 Properties of contracting one-parameter groups

The following proposition is standard [45, Section 1.1].

Proposition A.1.1. *Let $E, G \in M_d(\mathbb{R})$ and $A \in Gl_d(\mathbb{R})$. Also, let $E^* \in M_d(\mathbb{R})$ denote the adjoint of E. Then for all $t, s > 0$, the following statements hold:*

- $1^E = I$
- $t^{E^*} = (t^E)^*$
- *If $EG = GE$, then $t^E t^G = t^{E+G}$*
- $(st)^E = s^E t^E$
- $At^E A^{-1} = t^{AEA^{-1}}$
- $\det(t^E) = t^{\operatorname{tr} E}$
- $(t^E)^{-1} = t^{-E}$

Lemma A.1.2. *Let $\{T_t\} \subseteq Gl_d(\mathbb{R})$ be a continuous one-parameter contracting group. Then, for some $E \in Gl_d(\mathbb{R})$, $T_t = t^E$ for all $t > 0$. Moreover, there exists a positive constant C for which*

$$\|T_t\| \leq C + t^{\|E\|}$$

for all $t > 0$.

Proof. The representation $T_t = t^E$ for some $E \in \mathrm{M}_d(\mathbb{R})$ follows from the Hille-Yosida construction. If for some non-zero vector η, $E\eta = 0$, then $t^E\eta = \eta$ for all $t > 0$ and this would contradict our assumption that $\{T_t\}$ is contracting. Hence $E \in \mathrm{Gl}_d(\mathbb{R})$ and, in particular, $\|E\| > 0$. From the representation $T_t = t^E$ it follows immediately that $\|T_t\| \leq t^{\|E\|}$ for all $t \geq 1$ and so it remains to estimate

$\|T_t\|$ for $t < 1$. Given that $\{T_t\}$ is continuous and contracting, the map $t \mapsto \|T_t\|$ is continuous and approaches 0 as $t \to 0$ and so it is necessarily bounded for $0 < t \leq 1$. $\square$

Lemma A.1.3. *Let $E \in Gl_d(\mathbb{R})$ be diagonalizable with strictly positive spectrum. Then $\{t^E\}$ is a continuous one-parameter contracting group. Moreover, there is a positive constant C such that*

$$\|t^E\| \leq Ct^{\lambda_{max}}$$

for all $t \geq 1$ and

$$\|t^E\| \leq Ct^{\lambda_{min}}$$

for all $0 < t < 1$, where $\lambda_{max} = \max(\operatorname{Spec}(E))$ and $\lambda_{min} = \min(\operatorname{Spec}(E))$.

Proof. Let $A \in \mathrm{Gl}_d(\mathbb{R})$ be such that $A^{-1}EA = D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ where necessarily $\operatorname{Spec}(E) = \operatorname{Spec}(D) = \{\lambda_1, \lambda_2, \ldots, \lambda_d\} \subseteq (0, \infty)$. It follows from the spectral mapping theorem that $\operatorname{Spec}(t^D) = \{t^{\lambda_1}, t^{\lambda_2}, \ldots, t^{\lambda_d}\}$ for all $t > 0$ and moreover, because t^D is symmetric,

$$\|t^D\| \leq \max(\{t^{\lambda_1}, t^{\lambda_2}, \ldots, t^{\lambda_d}\}) = \begin{cases} t^{\lambda_{\text{max}}} & \text{if } t \geq 1 \\ t^{\lambda_{\text{min}}} & \text{if } t < 1. \end{cases}$$

By virtue of Proposition A.1.1, we have

$$\|t^E\| = \|At^DA^{-1}\| \leq \|A\|\|t^D\|\|A^{-1}\| \leq C\|t^D\| = C \times \begin{cases} t^{\lambda_{\text{max}}} & \text{if } t \geq 1 \\ t^{\lambda_{\text{min}}} & \text{if } t < 1 \end{cases}$$

for $t > 0$ where $C = \|A\|\|A^{-1}\|$; in particular, $\{t^E\}$ is contracting because $\lambda_{\text{min}} > 0$. $\square$

Proposition A.1.4. *Let $\{T_t\}_{t>0} \subseteq Gl_d(\mathbb{R})$ be a continuous one-parameter contracting group. Then, for all non-zero $\xi \in \mathbb{R}^d$,*

$$\lim_{t \to 0} |T_t\xi| = 0 \quad \textit{and} \quad \lim_{t \to \infty} |T_t\xi| = \infty.$$

Proof. The validity of the first limit is clear. Upon noting that $|\xi| = |T_{1/t}T_t\xi| \leq \|T_{1/t}\||T_t\xi|$ for all $t > 0$, the second limit follows at once. $\square$

Proposition A.1.5. *Let $\{T_t\}_{t>0} \subseteq Gl_d(\mathbb{R})$ be a continuous one-parameter contracting group. There holds the following:*

a) For each non-zero $\xi \in \mathbb{R}^d$, there exists $t > 0$ and $\eta \in S$ for which $T_t\eta = \xi$. Equivalently,

$$\mathbb{R}^d \setminus \{0\} = \{T_t\eta : t > 0 \text{ and } \eta \in S\}.$$

b) For each sequence $\{\xi_n\} \subseteq \mathbb{R}^d$ such that $\lim_n |\xi_n| = \infty$, $\xi_n = T_{t_n}\eta_n$ for each n, where $\{\eta_n\} \subseteq S$ and $t_n \to \infty$ as $n \to \infty$.

c) For each sequence $\{\xi_n\} \subseteq \mathbb{R}^d$ such that $\lim_n |\xi_n| = 0$, $\xi_n = T_{t_n}\eta_n$ for each n, where $\{\eta_n\} \subseteq S$ and $t_n \to 0$ as $n \to \infty$.

Proof. In view of Proposition A.1.4, the assertion a) is a straightforward application of the intermediate value theorem. For b), suppose that $\{\xi_n\} \subseteq \mathbb{R}^d$ is such that $|\xi_n| \to \infty$ as $n \to \infty$. In view of a), take $\{\eta_n\} \subseteq S$ and $\{t_n\} \subseteq (0, \infty)$ for which $\xi_n = T_{t_n}\eta_n$ for each n. In view of Lemma A.1.2,

$$\infty = \liminf_n |\xi_n| \leq \liminf_n \left(C + t_n^M\right)|\eta_n| \leq C + \liminf_n t_n^M,$$

where $C, M > 0$, and therefore $t_n \to \infty$. If instead $\lim_n \xi_n = 0$,

$$\infty = \lim_{n\to\infty} \frac{|\eta_n|}{|\xi_n|} = \lim_{n\to\infty} \frac{|T_{1/t_n}\xi_n|}{|\xi_n|} \leq \limsup_n \|T_{1/t_n}\| \leq \limsup_n (C + (1/t_n)^M)$$

from which we see that $t_n \to 0$, thus proving c). $\square$

Proposition A.1.6. *Let $\{T_t\}$ be a continuous contracting one-parameter group. Then for any open neighborhood $\mathcal{O} \subseteq \mathbb{R}^d$ of the origin and any compact set $K \subseteq \mathbb{R}^d$, $K \subseteq T_t(\mathcal{O})$ for sufficiently large t.*

Proof. Assume, to reach a contradiction, that there are sequences $\{\xi_n\} \subseteq K$ and $t_n \to \infty$ for which $\xi_n \notin T_{t_n}(\mathcal{O})$ for all n. Because K is compact, $\{\xi_n\}$ has a subsequential limit and so by relabeling, let us take sequences $\{\zeta_k\} \subseteq K$ and $\{r_k\} \subseteq (0,\infty)$ for which $\zeta_k \to \zeta$, $r_k \to \infty$ and $\zeta_k \notin T_{r_k}(\mathcal{O})$ for all k. Setting $s_k = 1/r_k$ and using the fact that $\{T_t\}$ is a one-parameter group, we have $T_{s_k}\zeta_k \notin \mathcal{O}$ for all k and so $\liminf_k |T_{s_k}\zeta_k| > 0$, where $s_k \to 0$. This is however impossible because $\{T_t\}$ is contracting and so

$$\lim_{k\to\infty} |T_{s_k}\zeta_k| \leq \lim_{k\to\infty} |T_{s_k}(\zeta_k - \zeta)| + \lim_{k\to\infty} |T_{s_k}\zeta| \leq C \lim_{k\to\infty} |\zeta_k - \zeta| + 0 = 0$$

in view of Lemma A.1.2. □

A.2 Properties of homogeneous functions on $\mathbb{R}^d$

Proposition A.2.1. *Let $\{T_t\} \subseteq Gl_d(\mathbb{R})$ be a contracting one-parameter group and let $R, Q : \mathbb{R}^d \to \mathbb{R}$ be continuous and homogeneous with respect to $\{T_t\}$. If R is positive definite, then there exists $C > 0$ such that*

$$|Q(\xi)| \leq CR(\xi) \tag{A.1}$$

for all $\xi \in \mathbb{R}^d$. If both Q and R are positive definite, then

$$Q \asymp R. \tag{A.2}$$

Proof. Upon reversing the roles of R and Q, it is clear that the (A.2) follows from (A.1) and so it suffices to prove (A.1). Because R is continuous and positive definite, it is strictly positive on S and so, given that Q is also continuous,

$$C := \sup_{\eta\in S} \frac{|Q(\eta)|}{R(\eta)} < \infty.$$

For any non-zero $\xi \in \mathbb{R}^d$, let $t > 0$ be such that $\xi = T_t\nu$ for $\nu \in S$ in view of Proposition A.1.5 and observe that

$$|Q(\xi)| = |Q(T_t\eta)| = t|Q(\eta)| \leq tCR(\eta) = CR(T_t\eta) = CR(\xi).$$

By invoking the continuity of R and Q, the above estimate must also hold for $\xi = 0$. $\square$

Proposition A.2.2. *Let $E \in Gl_d(\mathbb{R})$ be diagonalizable with strictly positive spectrum and suppose that $R : \mathbb{R}^d \to \mathbb{R}$ is continuous, positive definite, and homogeneous with respect to $\{t^E\}$. Then, for any $\gamma > (\min(\mathrm{Spec}(E)))^{-1}$,*

$$|\xi|^\gamma = o(R(\xi)) \textit{ as } \xi \to 0.$$

Proof. By virtue of Lemma A.1.3, we know that $\{t^E\}$ is contracting and $\|t^E\| \leq Ct^\lambda$ for all $t < 1$ where $\lambda = \min(\mathrm{Spec}(E))$ and $C > 0$. Let $\{\xi_n\}$ be such that $\lim_n \xi_n \to 0$ and, in view of Proposition A.1.5, let $\{\eta_n\} \subseteq S$ and $t_n \to 0$ be such that $\xi_n = t_n^E \eta_n$. Then

$$\limsup_n \frac{|\xi_n|^\gamma}{R(\xi_n)} = \limsup_n \frac{|t_n^E \eta_n|^\gamma}{t_n R(\eta_n)} \leq \limsup_n \frac{(Ct_n^\lambda |\eta_n|)^\gamma}{t_n R(\eta_n)} \leq M \lim_n t_n^{\gamma\lambda - 1} = 0,$$

where

$$M := \sup_{\eta \in S} \frac{C^\gamma |\eta|^\gamma}{R(\eta)}$$

is finite because R is continuous and positive definite. $\square$

Lemma A.2.3. *Let $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_+^d$, $D = \mathrm{diag}(m_1^{-1}, m_2^{-1}, \ldots, m_d^{-1}) \in Gl_d(\mathbb{R})$ and suppose that $R : \mathbb{R}^d \to \mathbb{R}$ is continuous, positive definite and homogeneous with respect to $\{t^D\}$. Then for any multi-index β such that $|\beta : \mathbf{m}| > 1$,*

$$\xi^\beta = o(R(\xi)) \qquad \textit{as } \xi \to 0.$$

Proof. Put $\gamma = |\beta : \mathbf{m}| - 1 > 0$ observe that

$$\sup_{\eta \in S} \frac{|\eta^\beta|}{R(\eta)} := M < \infty$$

because R is continuous and non-zero on S. Let $\{\xi_n\} \subseteq \mathbb{R}^d$ be such that $|\xi_n| \to 0$ as $n \to \infty$. By virtue of Proposition A.1.5, there are sequences $\{\eta_n\} \subseteq S$ and $\{t_n\} \subseteq (0, \infty)$ for which $t_n \to 0$ and $\xi_n = t_n^D \eta_n$ for all n. We see that

$$\xi_n^\beta = (t_n^D \eta_n)^\beta = \left(t_n^{(m_1)^{-1}} \eta_1\right)^{\beta_1} \left(t_n^{(m_2)^{-1}} \eta_2\right)^{\beta_2} \cdots \left(t_n^{(m_d)^{-1}} \eta_d\right)^{\beta_d} = t^{|\beta:\mathbf{m}|} \eta_n^\beta$$

for each n. Therefore

$$\limsup_n \frac{|\xi_n^\beta|}{R(\xi_n)} = \limsup_n \frac{t^{|\beta:\mathbf{m}|} |\eta_n^\beta|}{t R(\eta_n)} \leq \limsup_n M t_n^\gamma = 0$$

as desired. $\square$

For the remainder of this appendix, P is a positive homogeneous polynomial and $R = \operatorname{Re} P$.

Proposition A.2.4. *For any compact set K, there are positive constants M and M' such that*

$$M R(\xi) \leq R(\xi + \zeta) + M'$$

for all $\xi \in \mathbb{R}^d$ and $\zeta \in K$.

Proof. Set $U = \overline{B}_{3/2} \setminus B_{1/2} = \{u \in \mathbb{R}^d : 1/2 \leq |u| \leq 3/2\}$ and

$$M = \inf_{\eta \in S, u \in U} \frac{R(u)}{R(\eta)};$$

M is necessarily positive because R is continuous and positive definite. For $E \in \operatorname{Exp}(P)$, $\{t^E\}$ is contracting and so it follows that for some $T > 0$, $\left(\eta + t^{-E}\zeta\right) \in U$ for all $\eta \in S$, $\zeta \in K$ and $t > T$. Consider the set $V = \{\xi = t^E \eta \in \mathbb{R}^d : t > T, \eta \in$

$S\}$ and observe that for any $\xi \in V$ and $\zeta \in K$, $t^{-E}(\xi+\zeta) = \eta + t^{-E}\zeta \in U$ for some $t > T$ and consequently

$$\frac{R(\xi+\zeta)}{R(\xi)} = \frac{tR(\eta+t^{-E}\zeta)}{tR(\eta)} \geq M.$$

We have shown that $MR(\xi) \leq R(\xi+\zeta)$ for all $\xi \in V$ and $\zeta \in K$. To complete the proof, it remains to show that R is bounded on $V^{\mathsf{C}} = \mathbb{R}^d \setminus V$; however, given the continuity of R, we need only verify that the set V^{C} is bounded. By virtue of Proposition A.1.5, we can write

$$V^{\mathsf{C}} = \{0\} \cup \left\{\xi = t^E \eta : t \leq T, \eta \in S\right\}.$$

and so, by an appeal to Lemma A.1.2, we see that $|\xi| \leq C + T^{\|E\|}$ for all $\xi \in V^{\mathsf{C}}$. $\square$

Our final three results in this subsection concern estimates for P and R regarded as functions on $\mathbb{C}^d$. In what follows, $|\cdot|$ denotes the standard euclidean norm on $\mathbb{C}^d = \mathbb{R}^{2d}$ and S denotes the $2d$-sphere.

Proposition A.2.5. *For any $M, M' > 0$, there exists $C > 0$, for which*

$$|z| \leq C + MR(\xi) + M'R(\nu).$$

for all $z = \xi - i\nu \in \mathbb{C}^d$.

Proof. Define $Q(\xi,\nu) = MR(\xi) + M'R(\nu)$ for $(\xi,\nu) = z \in \mathbb{R}^{2d}$ and observe that Q is positive definite. It suffices to show that there exists a set V with bounded complement $V^{\mathsf{C}} = \mathbb{R}^{2d} \setminus V$ such that

$$|z| = |(\xi,\nu)| \leq Q(\xi,\nu) \tag{A.3}$$

for all $(\xi,\nu) \in V$. To this end, set

$$N = \sup_{(\eta,\zeta)\in S} \frac{|(\eta,\zeta)|}{Q(\eta,\zeta)}$$

which is finite because Q is strictly positive on S. Let $E \in \operatorname{Exp}(P)$ have real spectrum and recall that E is diagonalizable with $\lambda := \max(\operatorname{Spec}(E)) < 1$ in view of Proposition 3.2.2. An appeal to Lemma A.1.3 gives $C > 0$ for which $\|t^E\| \leq Ct^\lambda$ for all $t \geq 1$; the lemma also guarantees that $\{t^{E\oplus E}\} \subseteq \mathrm{Gl}_{2d}(\mathbb{R})$ is contracting. Set $T = \max(\{1, (CN)^{1/(1-\lambda)}\})$ and consider the set $V = \{(\xi,\nu) = t^{E\oplus E}(\eta,\zeta) \in \mathbb{R}^{2d} : t > T, (\eta,\zeta) \in S\}$. For any $(\xi,\nu) \in V$, we have

$$\frac{|(\xi,\nu)|}{Q(\xi,\nu)} = \frac{|(t^E\eta, t^E\zeta)|}{Q(t^{E\oplus E}(\eta,\zeta))} \leq \frac{Ct^\lambda|(\eta,\zeta)|}{tQ(\eta,\zeta)} \leq Ct^{\lambda-1}N < N^{-1}N = 1$$

and therefore (A.3) is satisfied. To see that V^{C} is bounded, one simply repeats the argument given in the proof of Proposition A.2.5 where, in this case, Proposition A.1.5 and Lemma A.1.2 are applied to the one-parameter contracting group $\{t^{E\oplus E}\}$. $\square$

By considering only real arguments $\xi \in \mathbb{R}^d$, Proposition A.2.5 ensures that, for some constant $C_1 > 0$, $|\xi| \leq C_1 + R(\xi)$ for all $\xi \in \mathbb{R}^d$. Upon noting that R is strictly positive on any sphere of radius δ, one easily obtains the following corollary.

Corollary A.2.6. *For each $C, \delta > 0$, there exists $M > 0$ for which*

$$|\xi| + C \leq MR(\xi)$$

for all $|\xi| > \delta$.

Proposition A.2.7. *Let P be a positive homogeneous polynomial with $R = \operatorname{Re} P$. There exist $\epsilon > 0$ and $M > 0$ such that*

$$-\operatorname{Re} P(z) \leq -\epsilon R(\xi) + MR(\nu) \tag{A.4}$$

and

$$|P(z)| \leq M(R(\xi) + R(\nu)) \tag{A.5}$$

for all $z = \xi - i\nu \in \mathbb{C}^d$.

Proof. Let $E \in \operatorname{Exp}(P)$ have strictly real spectrum and, by virtue of Proposition 3.2.2, let A be such that $D = A^{-1}EA = \operatorname{diag}((2m_1)^{-1}, (2m_2)^{-1}, \dots, (2m_d)^{-1})$ and

$$P_A(\xi) := (P \circ L_A)(\xi) = \sum_{|\alpha:\mathbf{m}|=2} a_\alpha \xi^\alpha,$$

where $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_+^d$. Because $A \in \mathrm{Gl}_d(\mathbb{R}) \subseteq \mathrm{Gl}_d(\mathbb{C})$, it suffices to verify the estimates (A.4) and (A.5) for P_A and $R_A = \operatorname{Re} P_A$. As in the proof of the previous proposition, we identify $\mathbb{C}^d = \mathbb{R}^{2d}$ by $z = (\xi, \nu)$, and observe that $\{t^{D\oplus D}\} \subseteq \mathrm{Gl}_{2d}(\mathbb{R})$ is contracting. Consequently, by considering $T_t = t^{D\oplus D}$, the estimate (A.5) follows directly from Proposition A.2.1.

An appeal to the multivariate binomial theorem shows that for all $z = \xi - i\nu \in \mathbb{C}^d$,

$$P_A(\xi - i\nu) = P_A(\xi) + Q(\xi, \nu), \tag{A.6}$$

where

$$Q(\xi,\nu) = \sum_{|\alpha:\mathbf{m}|=2} a_\alpha \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \xi^\gamma (-i\nu)^{\alpha-\gamma} = \sum_{\substack{|\alpha:\mathbf{m}|=2\\ \gamma<\alpha}} b_{\alpha,\gamma} \xi^\gamma \nu^{\alpha-\gamma};$$

here, $\{b_{\alpha,\gamma}\} \subseteq \mathbb{C}$. We claim that for each $\delta > 0$, there exists $M > 0$ such that

$$|Q(\xi,\nu)| \le \delta R_A(\xi) + M R_A(\nu) \tag{A.7}$$

for all $\xi, \nu \in \mathbb{R}^d$. For the moment, let us accept the validity of the claim. By choosing $\delta < 1$, a combination of (A.6) and (A.7) yields

$$-\operatorname{Re}(P_A(\xi - i\nu)) + R_A(\xi) \le \delta R_A(\xi) + M R_A(\nu)$$

for all $\xi, \nu \in \mathbb{R}^d$ and from this we see that (A.4) is satisfied with $\epsilon = 1 - \delta$. It therefore suffices to prove (A.7).

For any multi-indices β and γ for which $|\beta : \mathbf{m}| = 2$ and $\gamma < \beta$, it is straightforward to see that

$$(t^D\xi)^\gamma (t^D\nu)^{\beta-\gamma} = t^{|\gamma:2\mathbf{m}|} t^{|\beta-\gamma:2\mathbf{m}|} \xi^\gamma \nu^{\beta-\gamma} = t^{|\beta:2\mathbf{m}|} \xi^\gamma \nu^{\beta-\gamma} = t \xi^\gamma \nu^{\beta-\gamma}$$

for all $\xi, \nu \in \mathbb{R}^d$ and so the map $(\xi, \nu) \mapsto \xi^\gamma \nu^{\beta-\gamma}$ is homogeneous with respect to the contracting group $\{t^{D \oplus D}\} \subseteq \mathrm{Gl}_{2d}(\mathbb{R})$. Consequently, an application of Proposition A.2.1 gives $C > 0$ for which

$$|\xi^\gamma \nu^{\beta-\gamma}| \leq C(R_A(\xi) + R_A(\nu))$$

for all $\xi, \nu \in \mathbb{R}^d$. By invoking the homogeneity of ξ^γ and $R_A(\xi)$ with respect to $\{t^D\} \subseteq \mathrm{Gl}_d(\mathbb{R})$, the above estimate ensures that, for all $t > 0$,

$$\begin{aligned} |\xi^\gamma \nu^{\beta-\gamma}| &= |t^{|\gamma:\mathbf{2m}|}(t^{-D}\xi)^\gamma \nu^{\beta-\gamma}| \\ &\leq t^{|\gamma:\mathbf{2m}|} C(R_A(t^{-D}\xi) + R_A(\nu)) \\ &= Ct^{|\gamma:\mathbf{2m}|-1} R_A(\xi) + Ct^{|\gamma:\mathbf{2m}|} R_A(\nu) \end{aligned}$$

for all $\xi, \nu \in \mathbb{R}^d$. Noting that $|\gamma : \mathbf{2m}| - 1 < 0$ because $\gamma < \beta$, we can make the coefficient of $R_A(\xi)$ in the above estimate arbitrarily small by choosing t sufficiently large. Consequently, for any $\delta > 0$ there exists $M > 0$ for which

$$|\xi^\gamma \nu^{\beta-\gamma}| \leq \delta R_A(\xi) + M R_A(\nu)$$

for all $\xi, \nu \in \mathbb{R}^d$. The claim (A.7) now follows by a simple application of the triangle inequality. $\square$

A.3 Properties of the Legendre-Fenchel transform of a positive-homogeneous polynomial

Lemma A.3.1. *Let P be a positive homogeneous polynomial and let $R = \operatorname{Re} P$. For $E \in \operatorname{Exp}(P)$ with real spectrum let $\lambda_{max} = \max(\operatorname{Spec}(E))$ and $\lambda_{min} = \min(\operatorname{Spec}(E))$*

(note that $0 < \lambda_{min}, \lambda_{max} \leq 1/2$ by Proposition 3.2.2) and set

$$\mathcal{N}_E(x) = \begin{cases} |x|^{1/(1-\lambda_{max})} & \text{if } |x| \geq 1 \\ |x|^{1/(1-\lambda_{min})} & \text{if } |x| < 1 \end{cases}$$

for $x \in \mathbb{R}^d$. There are positive constants M, M' for which

$$|x| - M \leq R^{\#}(x) \leq M'\mathcal{N}_E(x) \tag{A.8}$$

for all $x \in \mathbb{R}^d$.

Proof. Set $M = \sup_{\xi \in S} R(\xi)$ and observe that, for any non-zero $x \in \mathbb{R}^d$,

$$R^{\#}(x) = \sup_{\xi \in \mathbb{R}^d}\{x \cdot \xi - R(\xi)\} \geq x \cdot \frac{x}{|x|} - R\left(\frac{x}{|x|}\right) \geq |x| - M.$$

The lower estimate in (A.8) now follows from the observation that $R^{\#}(0) = 0$ which is true because R is positive definite. We now focus on the upper estimate. In view of Lemma A.1.3 and Proposition 3.2.2, let $C \geq 1$ be such that $\|t^E\| \leq Ct^{\lambda_{\max}}$ for $t \geq 1$ and $\|t^E\| \leq Ct^{\lambda_{\min}}$ for $t \leq 1$. An appeal to Proposition A.2.5 gives $M' > 0$ for which $C|\xi| \leq R(\xi) + M'$ for all $\xi \in \mathbb{R}^d$. In the case that $|x| \geq 1$, we set $t = |x|^{1/(1-\lambda_{\max})}$ and observe that

$$\begin{aligned} x \cdot \xi &\leq |x||t^E t^{-E}\xi| \\ &\leq |x|\|t^E\||t^{-E}\xi| \\ &\leq |x|t^{\lambda_{\max}}C|t^{-E}\xi| \\ &\leq |x|t^{\lambda_{\max}}\left(R(t^{-E}\xi) + M'\right) \\ &= |x|t^{\lambda_{\max}-1}R(\xi) + M'|x|t^{\lambda_{\max}} \\ &= R(\xi) + M'|x|^{1/(1-\lambda_{\max})} \end{aligned}$$

for all $\xi \in \mathbb{R}^d$ and therefore

$$R^{\#}(x) = \sup_{\xi \in \mathbb{R}^d}\{x \cdot \xi - R(\xi)\} \leq M'|x|^{1/(1-\lambda_{\max})} = M'\mathcal{N}_E(x).$$

When $|x| \leq 1$, we repeat the argument above to find that $R^{\#}(x) \leq M'|x|^{1/(1-\lambda_{\min})} = M'\mathcal{N}_E(x)$ as desired. □

Proposition A.3.2. *Let P be a positive homogeneous polynomial with $R = \operatorname{Re} P$. Then $R^{\#}$ is continuous, positive definite, and for any $E \in \operatorname{Exp}(P)$, $F = (I-E)^* \in \operatorname{Exp}(R^{\#})$.*

Proof. Since $R^{\#}$ is the Legendre-Fenchel transform of $R : \mathbb{R}^d \to \mathbb{R}$ it is convex (and lower semi-continuous). Furthermore, the upper estimate in Lemma A.3.1 guarantees that $R^{\#}$ is finite on $\mathbb{R}^d$ and therefore continuous [78, Corollary 10.1.1].

Given that R is positive definite and homogeneous with respect to $\{t^E\}$, it follows directly from the definition of the Legendre-Fenchel transform that $R^{\#}$ is non-negative, homogeneous with respect to $\{t^F\}$ where $F = (I-E)^*$ and has $R^{\#}(0) = 0$. To complete the proof, it remains to show that $R^{\#}(x) \neq 0$ for all non-zero $x \in \mathbb{R}^d$. Using the lower estimate in Lemma A.3.1, we have

$$\lim_{x \to \infty} R^{\#}(x) = \infty. \tag{A.9}$$

By virtue of Proposition 3.2.2, F is diagonalizable with $\operatorname{Spec}(F) \subseteq [1/2, 1)$; in particular, $\{t^F\}$ is contracting in view of Lemma A.1.3. Now if for some non-zero $x \in \mathbb{R}^d$, $R^{\#}(x) = 0$,

$$0 = \lim_{t \to \infty} tR^{\#}(x) = \lim_{t \to \infty} R^{\#}(t^F x),$$

which is impossible in view of Proposition A.1.4 and (A.9). □

Corollary A.3.3. *Let P be a positive homogeneous polynomial of the form* (3.18) *for $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_+^d$ and $\{a_\beta\} \subseteq \mathbb{C}$. That is, the conclusion of Proposition*

3.2.2 holds where $A = I \in Gl_d(\mathbb{R})$. Let $R = \operatorname{Re} P$, let $R^{\#}$ be the Legendre-Fenchel transform of R and define $|\cdot|_{\mathbf{m}} : \mathbb{R}^d \to [0,\infty)$ by (3.88) *for $x \in \mathbb{R}^d$. Then*

$$R^{\#}(x) \asymp |x|_{\mathbf{m}}.$$

Proof. We note that $|\cdot|_{\mathbf{m}}$ is continuous, positive definite and homogeneous with respect to the one-parameter contracting group $\{t^F\}$ where $F = \operatorname{diag}((2m_1-1)/(2m_1), (2m_2-1)/(2m_2), \dots, (2m_d-1)/(2m_d))$. Because $E = \operatorname{diag}\left((2m_1)^{-1}, (2m_2)^{-1}, \dots, (2m_d)^{-1}\right) \in \operatorname{Exp}(R)$, Proposition A.3.2 ensures that $R^{\#}$ is continuous, positive definite and has $F = (I-E)^* \in \operatorname{Exp}(R^{\#})$. The desired result follows directly by an appeal to Proposition A.2.1. $\square$

Another application of Proposition A.3.2 and A.2.1 yields the following corollary.

Corollary A.3.4. *Let P be a positive homogeneous polynomial with $R = \operatorname{Re} P$. For any constant $M > 0$, $(MR)^{\#} \asymp R^{\#}$.*

A.4 The proof of Proposition 3.3.3

Proof of Proposition 3.3.3. $(a \Rightarrow b)$ Let $P = P_{\xi_0}$, take $E \in \operatorname{Exp}(P_{\xi_0})$ with strictly real spectrum and set $m = \max_{i=1,2\dots,d} 2m_i$ in view of Proposition 3.2.2. Noting that E is diagonalizable, $m+1 > (\min(\operatorname{Spec}(E)))^{-1}$ and $Q_{\xi_0}^m(\xi) + O(|\xi|^{m+1}) = P_{\xi_0}(\xi) + \Upsilon_{\xi_0}(\xi)$ for ξ sufficiently close to 0, our result follows from Proposition A.2.2.

$(b \Rightarrow c)$ Let $E \in \operatorname{Exp}(P)$ have real spectrum and observe that, for all $n \in \mathbb{N}_+$,

$$C^{-1}R(\xi) \leq n \operatorname{Re} Q_{\xi_0}^m(n^{-E}\xi) \leq CR(\xi) \quad \text{and} \quad |n \operatorname{Im} Q_{\xi_0}^m(n^{-E}\xi)| \leq CR(\xi) \tag{A.10}$$

for $\xi \in \overline{B_r}$. It follows that the sequence $\{\rho_n\} \subseteq C(\overline{B_r})$ of degree m polynomials, defined by $\rho_n(\xi) = nQ_{\xi_0}^m(n^{-E}\xi)$ for all $n \in \mathbb{N}_+$ and $\xi \in \overline{B_r}$, is bounded. As the subspace of degree m polynomials is a finite dimensional subspace of $C(\overline{B_r})$, $\{\rho_n\}$ must have a convergent subsequence. Moreover, because $R(\xi)$ is positive definite, (A.10) ensures that the subsequential limit has positive real part on S_r.

$(c \Rightarrow a)$ The proof of this implication is lengthy and will be shown using a sequence of lemmas. We fix $E \in \mathrm{M}_d(\mathbb{R})$ with real spectrum and for which the condition (3.33) is satisfied. As the characteristic polynomial of E completely factors over $\mathbb{R}$, the Jordan-Chevally decomposition for E gives $A \in \mathrm{Gl}_d(\mathbb{R})$ for which $F := A^{-1}EA = D+N$ where D is diagonal, N is nilpotent and $DN = ND$. Upon setting $Q_A = Q_{\xi_0}^m \circ L_A$, it follows that

$$Q_A(\xi) = \sum_{1<|\beta|\leq m} a_\beta \xi^\beta$$

for $\xi \in \mathbb{R}^d$ where $\{a_\beta\} \subseteq \mathbb{C}$. Define $\rho_A : (0,\infty) \times \mathbb{R}^d \to \mathbb{C}$ by $\rho_A(t,\xi) = tQ_A(t^{-F}\xi)$ for $t > 0$ and $\xi \in \mathbb{R}^d$. The hypotheses (3.33) ensures that, for each $\xi \in A^{-1}\overline{B_r}$,

$$P_A(\xi) := \lim_{n\to\infty} \rho_A(t_n, \xi) \tag{A.11}$$

exists and is such that $\operatorname{Re} P_A(\xi) > 0$ whenever $\xi \in A^{-1}S_r$.

Lemma A.4.1. *Under the hypotheses* (3.33), $\lim_{t\to\infty} \rho_A(t,\xi)$ *exists for all* $\xi \in \mathbb{R}^d$ *and the convergence is uniform on all compact sets of* $\mathbb{R}^d$. *In particular,* P_A *extends uniquely to* $\mathbb{R}^d$ *(which we also denote by* P_A*) by*

$$P_A(\xi) = \lim_{t\to\infty} \rho_A(t,\xi) = \lim_{n\to\infty} \rho_A(t_n,\xi) \tag{A.12}$$

for all $\xi \in \mathbb{R}^d$. Moreover, $P_A : \mathbb{R}^d \to \mathbb{C}$ is a positive homogeneous polynomial with the representation

$$P_A(\xi) = \sum_{|\beta:\mathbf{m}|=2} a_\beta \xi^\beta \tag{A.13}$$

for some $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_+^d$ where $m \geq 2m_i$ for $i = 1, 2, \ldots d$ and

$$F = D = \operatorname{diag}((2m_1)^{-1}, (2m_2)^{-1}, \ldots, (2m_d)^{-1}) \in \operatorname{Exp}(P_A). \tag{A.14}$$

Furthermore

$$Q_A(\xi) = \sum_{|\beta:2\mathbf{m}|\geq 1} a_\beta \xi^\beta = P_A(\xi) + \sum_{|\beta:2\mathbf{m}|>1} a_\beta \xi^\beta \tag{A.15}$$

for $\xi \in \mathbb{R}^d$.

Proof of Lemma A.4.1. Our proof is broken into three steps. In the first step we show that the representation (A.13) is valid on $A^{-1}\overline{B_r}$ and the first equality in (A.15) holds on $\mathbb{R}^d$. The first step also ensures the validity of the second equality in (A.14). In the second step, we define $P_A : \mathbb{R}^d \to \mathbb{C}$ by the right hand side of (A.13) and check that P_A is a positive homogeneous polynomial with $D \in \operatorname{Exp}(P_A)$. In the third step we show that $N = 0$ and hence $F = D$ and in the fourth step we show that the limit (A.12) converges uniformly on any compact set $K \subseteq \mathbb{R}^d$. The second inequality in (A.15) follows directly by combining the results.

Step 1. Write $D = \operatorname{diag}(\gamma_1, \gamma_2, \ldots, \gamma_d)$ and put $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d) \in \mathbb{R}^d$. We fix $\xi \in A^{-1}\overline{B_r}$ and observe that

$$\begin{aligned}
\rho_A(t,\xi) &= \sum_{1<|\beta|\leq m} a_\beta t \Big(t^{-D} \Big(I + \log t N\xi + \cdots + \frac{(\log t)^k}{k!} N^k \xi \Big) \Big)^\beta \\
&= \sum_{1<|\beta|\leq m} a_\beta t^{1-\gamma\cdot\beta} \xi^\beta + \sum_{j=1}^{l} b_j t^{\omega_j} (\log t)^j
\end{aligned} \tag{A.16}$$

for all $t > 0$ where, by invoking the multinomial theorem, we have simplified the expression so that $\omega_1, \omega_2, \dots, \omega_l \in \mathbb{R}$ are distinct and $b_j = b_j(\xi; N) \in \mathbb{C}$ for $j = 1, 2, \dots l = km$. Considering the sum

$$\sum_{j=1}^{l} b_j t^{\omega_j} (\log t)^j \tag{A.17}$$

we see that, as $t \to \infty$, the summands must either converge to 0 or diverge to ∞ in absolute value. Moreover, the distinctness of the collection $\{\omega_1, \omega_2, \dots, \omega_l\}$ and the presence of positive powers of $\log t$ ensure that this convergence or divergence happens at distinct rates. Consequently, as $t_n \to \infty$ the divergence of even a single summand would force the non-existence of the limit (A.11). Consequently, the expression (A.17) converges to 0 as $t \to \infty$ and so

$$P_A(\xi) = \lim_{n \to \infty} \rho_A(t_n, \xi) = \lim_{t \to \infty} \rho_A(t, \xi) = \lim_{t \to \infty} \sum_{1 < |\beta| \leq m} a_\beta t^{1 - \gamma \cdot \beta} \xi^\beta. \tag{A.18}$$

Since ξ was arbitrary, (A.18) must hold for all $\xi \in A^{-1}\overline{B_r}$. This is the only part of the argument in which the subsequence $\{t_n\}$ appears.

We claim that, for all multi-indices β for which $a_\beta \neq 0$, $\beta \cdot \gamma = \beta_1 \gamma_1 + \beta_2 \gamma_2 + \cdots + \beta_d \gamma_d \geq 1$. Indeed, fix $\kappa = \min(\{\beta \cdot \gamma : a_\beta \neq 0\})$, set $\mathcal{I}_\kappa = \{\beta : a_\beta \neq 0 \text{ and } \beta \cdot \gamma = \kappa\}$ and define $B_\kappa : \mathbb{R}^d \to \mathbb{C}$ by

$$B_\kappa(\xi) = \sum_{\beta \in \mathcal{I}_\kappa} a_\beta \xi^\beta$$

for $\xi \in \mathbb{R}^d$. The linear independence of the monomials $\{\xi^\beta\}_{\beta \in \mathcal{I}_\kappa}$ ensures that $B_\kappa(\xi') \neq 0$ for some $\xi' \in A^{-1}\overline{B_r}$. It follows from (A.18) that $\lim_{t \to \infty} \rho_A(t, \xi') = \lim_{t \to \infty} t^{1-\kappa} B_\kappa(\xi')$ from which we conclude that $\kappa = 1$; the hypotheses that P_A has positive real part on $A^{-1}S_r$ rules out the possibility that $\kappa > 1$.

From the claim it is now evident that

$$P_A(\xi) = \sum_{\beta \cdot \gamma = 1} a_\beta \xi^\beta \tag{A.19}$$

for $\xi \in A^{-1}\overline{B_r}$ and

$$Q_A(\xi) = \sum_{\beta\cdot\gamma \geq 1} a_\beta \xi^\beta \tag{A.20}$$

for $\xi \in \mathbb{R}^d$.

It is straightforward to see that the set $A^{-1}S_r$ intersects each coordinate axis at exactly two antipodal points. That is, for each $j = 1, 2, \dots d$, there exists $x_j > 0$ for which $\{\pm x_j e_j\} = A^{-1}S_r \cap \{xe_j : x \in \mathbb{R}\}$. Upon evaluating $\mathrm{Re}(P_A)$ at such points and recalling that $\mathrm{Re}\, P_A > 0$ on $A^{-1}S_r$, one sees by the same argument given in *Step 2* of the proof of Proposition 3.2.2 that $1/\gamma_j$ is an even natural number which cannot be greater than m. Therefore, for each $j = 1, 2, \dots, d$, $1/\gamma_j = 2m_j \geq m$ for some $m_j \in \mathbb{N}_+$. The representation (A.13) on $A^{-1}\overline{B_r}$ and the first equality in (A.15) now follow from (A.19) (A.20) and the observation that $\beta \cdot \gamma = \sum_{j=1}^d \beta_j / 2m_j = |\beta : 2\mathbf{m}|$. Moreover,

$$D = \mathrm{diag}((2m_1)^{-1}, (2m_2)^{-1}, \dots, (2m_d)^{-1}). \tag{A.21}$$

Step 2. We define $P_A : \mathbb{R}^d \to \mathbb{C}$ by the right hand side of (A.13). It is clear that $D \in \mathrm{Exp}(P_A)$ and so, to prove that P_A is positive homogeneous, it suffices to show that that $R_A(\xi) = \mathrm{Re}\, P_A(\xi) > 0$ whenever $\xi \neq 0$. To this end, let $\xi \in \mathbb{R}^d$ be non-zero and find $t > 0$ for which $t^D\xi \in A^{-1}S_r$; this can be done because $\{t^D\}$ is contracting in view of (A.21). From the previous step we know that (A.13) holds on $A^{-1}S_r$ and thus by invoking (A.11), we find that $R_A(\xi) = t^{-1}\,\mathrm{Re}\, P_A(t^D\xi) > 0$ as claimed.

Step 3. We now show that $F \in \mathrm{Exp}(P_A)$ and use it to conclude that $N = 0$. As we will see, this assertion relies on P_A being originally defined via a "scaling" limit. Indeed, for any $\xi \in \mathbb{R}^d$ and $t > 0$, find $u > 0$ for which both $u^{-D}\xi$ and

$u^{-D}t^F\xi$ belong to $A^{-1}\overline{B_r}$; this can be done because $A^{-1}\overline{B_r}$ necessarily contains an open neighborhood of 0. In view of (A.18),

$$\begin{aligned}
tP_A(\xi) &= tuP_A(u^{-D}\xi)\\
&= ut\lim_{s\to\infty} s\rho_A(s,u^{-D}\xi)\\
&= ut\lim_{s\to\infty} sQ_A(s^{-F}u^{-D}\xi)\\
&= u\lim_{s\to\infty} stQ_A(s^{-F}t^{-F}t^Fu^{-D}\xi)\\
&= u\lim_{(st)\to\infty} (st)Q_A((st)^{-F}u^{-D}t^F\xi)\\
&= u\lim_{v\to\infty} \rho_A(v,u^{-D}t^F\xi)\\
&= (uP_A(u^{-D}t^F\xi)\\
&= P_A(t^F\xi)
\end{aligned}$$

where we have used Proposition A.1.1 and the fact that $D\in\mathrm{Exp}(P_A)$. Consequently $F\in\mathrm{Exp}(P_A)$ and since P_A is a positive homogeneous polynomial, the same argument given in *Step 3* of the proof of Proposition 3.2.2 ensures that $N=0$.

Step 4. Let $K\subseteq\mathbb{R}^d$ be compact and note that $t^{-F}K\subseteq A^{-1}\overline{B_r}$ for sufficiently large t by virtue of Proposition A.1.6. Thus by invoking (A.13), which we know to be valid on $A^{-1}\overline{B_r}$, we have

$$\begin{aligned}
|\rho_A(t,\xi)-P_A(\xi)| &= |tQ_A(t^{-F}\xi)-tP_A(t^{-F}\xi)|\\
&= \left|t\sum_{|\beta:2\mathbf{m}|>1} a_\beta(t^{-F}\xi)^\beta\right|\\
&\leq \sum_{|\beta:2\mathbf{m}|>1} t^{1-|\beta:2\mathbf{m}|}|a_\beta\xi^\beta\\
&\leq t^\omega\sum_{|\beta:2\mathbf{m}|>1}|a_\beta\xi^\beta|
\end{aligned}$$

for all $\xi\in K$ and sufficiently large t where $\omega<0$ is independent of K. The assertion concerning the uniform limit follows at once because $\sum_{|\beta:2\mathbf{m}|>1}|a_\beta\xi^\beta|$ is necessarily bounded on K. //

We shall henceforth abandon using the symbol D and write

$$F = A^{-1}EA = \operatorname{diag}((2m_1)^{-1},(2m_2)^{-1},\dots,(2m_d)^{-1}) \in \operatorname{Exp}(P_A).$$

Lemma A.4.2. *Under the hypotheses of Lemma A.4.1,* $Q_A(\xi)-P_A(\xi)=o(R_A(\xi))$ *as* $\xi\to 0$.

Proof of Lemma A.4.2. In view of Lemma A.4.1,

$$|Q_A(\xi)-P_A(\xi)| \le \sum_{|\beta:2\mathbf{m}|>1}|a_\beta\xi^\beta|$$

for all $\xi\in\mathbb{R}^d$. The desired result now follows directly from Lemma A.2.3. //

We now define $P_{\xi_0}:\mathbb{R}^d\to\mathbb{C}$ by $P_{\xi_0}=P_A\circ L_{A^{-1}}$. By virtue of our results above, it is clear that P_{ξ_0} is positive homogeneous with $E\in\operatorname{Exp}(P_{\xi_0})$. We have

$$\Upsilon_{\xi_0}(\xi)=\Gamma_{\xi_0}(\xi)-i\alpha_{\xi_0}\cdot\xi+P_{\xi_0}(\xi)=P_{\xi_0}(\xi)-Q^m_{\xi_0}(\xi)+O(|\xi|^{(m+1)})$$

as $\xi\to 0$. Because $R_{\xi_0}=\operatorname{Re}P_{\xi_0}=R_A\circ L_{A^{-1}}$, it follows from Lemma A.4.2 that $P_{\xi_0}(\xi)-Q_{\xi_0}(\xi)=o(R_{\xi_0}(\xi))$ as $\xi\to 0$. Moreover, because E is diagonalizable and $m+1>2m_i\ge(\min(\operatorname{Spec}(E)))^{-1}$, $|\xi|^{(m+1)}=o(R_{\xi_0}(\xi))$ as $\xi\to 0$ by virtue of Proposition A.2.2. Therefore

$$\Gamma_{\xi_0}(\xi)=i\alpha_{\xi_0}-P_{\xi_0}(\xi)+\Upsilon_{\xi_0}(\xi)$$

where $\Upsilon_{\xi_0}=o(R_{\xi_0})$ as $\xi\to 0$ and thus completing the proof of the implication $(c\Rightarrow a)$.

To finish the proof of Proposition 3.3.3, it remains to prove that, for any $m' \geq m$,

$$P_{\xi_0}(\xi) = \lim_{t\to\infty} tQ_{\xi_0}^{m'}(t^{-E}\xi)$$

for all $\xi \in \mathbb{R}^d$ and this limit is uniform on all compact subsets of $\mathbb{R}^d$. Indeed, Let $K \subseteq \mathbb{R}^d$ be compact. By virtue of Lemma A.4.1,

$$\begin{aligned} P_{\xi_0}(\xi) &= P_A(A^{-1}\xi) \\ &= \lim_{t\to\infty} \rho_A(t, A^{-1}\xi) \\ &= \lim_{t\to\infty} tQ_A(A^{-1}t^{-E}\xi) \\ &= \lim_{t\to\infty} tQ_{\xi_0}^{m}(t^{-E}\xi) \end{aligned} \tag{A.22}$$

uniformly for $\xi \in K$. Observe that for any $m' > m$, there exists $M > 0$ for which

$$\begin{aligned} \left|tQ_{\xi_0}^{m'}(t^{-E}\xi) - tQ_{\xi_0}^{m}(t^{-E}\xi)\right| &\leq \sum_{m<|\beta|\leq m'} t\left|c_\beta(t^{-E}\xi)^\beta\right| \\ &= \sum_{m<|\beta|\leq m'} t\left|c_\beta(At^{-F}A^{-1}\xi)^\beta\right| \\ &\leq M \sum_{m<|\gamma|\leq m'} t\left|(t^{-F}A^{-1}\xi)^\gamma\right| \\ &= \sum_{m<|\gamma|\leq m'} t^{1-|\gamma:\mathbf{2m}|}\left|(A^{-1}\xi)^\gamma\right| \end{aligned}$$

for all $\xi \in \mathbb{R}^d$ and $t > 0$. Noting that $|\gamma : 2\mathbf{m}| > 1$ whenever $m < |\gamma| \leq m'$, by repeating the argument given in *Step 4* of Lemma A.4.1, we observe that

$$\lim_{t\to\infty} \left(tQ_{\xi_0}^{m'}(t^{-E}\xi) - tQ_{\xi_0}^{m}(t^{-E}\xi)\right) = 0 \tag{A.23}$$

uniformly for $\xi \in K$. The desired result follows by combining (A.22) and (A.23).

□

BIBLIOGRAPHY

[1] Thomas Apel, Thomas G. Flaig, and Serge Nicaise. A Priori Error Estimates for Finite Element Methods for H (2,1) -Elliptic Equations. *Numer. Funct. Anal. Optim.*, 35(2):153–176, feb 2014.

[2] R.A. Artino. Completely semielliptic boundary value problems. *Port. Math.*, 50(2), 1993.

[3] R.A. Artino and J. Barros-Neto. Semielliptic Pseudodifferential Operators. *J. Funct. Anal.*, 129(2):471–496, may 1995.

[4] Ralph A Artino. On semielliptic boundary value problems. *J. Math. Anal. Appl.*, 42(3):610–626, jun 1973.

[5] Ralph A Artino and J Barros-Neto. Regular semielliptic boundary value problems. *J. Math. Anal. Appl.*, 61(1):40–57, nov 1977.

[6] Pascal Auscher, Steve Hofmann, Alan McIntosh, and Philippe Tchamitchian. The Kato square root problem for higher order elliptic operators and systems on $\mathbb{R}^n$. *J. Evol. Equations*, 1(4):361–385, dec 2001.

[7] Pascal Auscher, A. ter Elst, and Derek Robinson. On positive Rockland operators. *Colloq. Math.*, 67(2):197–216, 1994.

[8] N. G. Bakhoom. Asymptotic Expansions of the Function $F_k(x) = \int_0^\infty e^{-u^k+xu}du$. *Proc. London Math. Soc.*, s2-35(1):83–100, jan 1933.

[9] G. Barbatis and E.B. Davies. Sharp bounds on heat kernels of higher order uniformly elliptic operators. *J. Oper. Theory*, 36(1):179–198, 1996.

[10] S. Blunck and P. C. Kunstmann. Generalized Gaussian estimates and the legendre transform. *J. Oper. Theory*, 53(2):351–365, 2005.

[11] L. N. Bondar′. Conditions for the solvability of boundary value problems for quasi-elliptic systems in a half-space. *Differ. Equations*, 48(3):343–353, mar 2012.

[12] L. N. Bondar′. Solvability of boundary value problems for quasielliptic systems in weighted Sobolev spaces. *J. Math. Sci.*, 186(3):364–378, oct 2012.

[13] L. N. Bondar and G. V. Demidenko. Boundary value problems for quasielliptic systems. *Sib. Math. J.*, 49(2):202–217, mar 2008.

[14] P. Brenner, V. Thomee, and L. Wahlbin. *Besov Spaces and Applications to Difference Methods for Initial Value Problems*, volume 434 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, 1975.

[15] Felix E Browder. The asymptotic distribution of eigenfunctions and eigenvalues for semi-elliptic differential operators. *Proc. Nat. Acad. Sci. U.S.A.*, 43(3):270–273, 1957.

[16] W. R. Burwell. Asymptotic Expansions of Generalized Hyper-Geometric Functions. *Proc. London Math. Soc.*, s2-22(1):57–72, jan 1924.

[17] Angelo Cavallucci. Sulle proprietà differenziali delle soluzioni delle equazioni quasi-ellittiche. *Ann. di Mat. Pura ed Appl. Ser. 4*, 67(1):143–167, dec 1965.

[18] Thierry Coulhon. Ultracontractivity and Nash Type Inequalities. *J. Funct. Anal.*, 539(0140):510–539, 1996.

[19] E. B. Davies. *One-Parameter Semigroups*. Academic Press, London, 1980.

[20] E. B. Davies. Lp Spectral Theory of Higher-Order Elliptic Differential Operators. *Bull. London Math. Soc.*, 29(5):513–546, 1997.

[21] E.B. Davies. Uniformly Elliptic Operators with Measurable Coefficients. *J. Funct. Anal.*, 132(1):141–169, aug 1995.

[22] Edward T. Dean. Dedekind's treatment of Galois theory in the Vorlesungen. Technical Report Technical Report No. CMU-PHIL-184, Carnegie Mellon, 2009.

[23] G. V. Demidenko. Correct solvability of boundary-value problems in a halfspace for quasielliptic equations. *Sib. Math. J.*, 29(4):555–567, 1989.

[24] G. V. Demidenko. Integral operators determined by quasielliptic equations. I. *Sib. Math. J.*, 34(6):1044–1058, 1993.

[25] G. V. Demidenko. Integral operators determined by quasielliptic equations. II. *Sib. Math. J.*, 35(1):37–61, jan 1994.

[26] G. V. Demidenko. On quasielliptic operators in Rn. *Sib. Math. J.*, 39(5):884–893, oct 1998.

[27] G V Demidenko. Isomorphic properties of one class of differential operators and their applications. *Sib. Math. J.*, 42(5):865–883, 2001.

[28] G V Demidenko. Quasielliptic operators and Sobolev type equations. *Sib. Math. J.*, 49(5):842–851, sep 2008.

[29] G V Demidenko. Quasielliptic operators and Sobolev type equations. II. *Sib. Math. J.*, 50(5):838–845, sep 2009.

[30] Gennadii V. Demidenko. Mapping properties of quasielliptic operators and applications. *Int. J. Dyn. Syst. Differ. Equations*, 1(1):58, 2007.

[31] Persi Diaconis and Laurent Saloff-Coste. Convolution powers of complex functions on Z. *Math. Nachrichten*, 287(10):1106–1130, 2014.

[32] Peter Gustav Lejeune Dirichlet and Richard Dedekind. *Volesungen uber Zahlentheorie*. Friedrich Vieweg und Sohn, 4th edition, 1894.

[33] Nick Dungey. Higher order operators and gaussian bounds on Lie groups of polynomial growth. *J. Oper. Theory*, 46(1):45–61, 2001.

[34] Jacek Dziubański and Andrzej Hulanicki. On semigroups generated by left-invariant positive differential operators on nilpotent Lie groups. *Stud. Math.*, 94(1):81–95, 1989.

[35] S. D. Eidelman. *Parabolic Systems*. North-Holland Publishing Company, Amsterdam and Wolters-Nordhoff, 1969.

[36] Samuil D. Eidelman. On a class of parabolic systems. *Sov. Math. Dokl.*, 1:85–818, 1960.

[37] Samuil D. Eidelman, Anatoly N. Kochubei, and Stepan D. Ivasyshen. *Analytic Methods in the Theory of Differential and Pseudo-Differential Equations of Parabolic Type*, volume 1. Birkhäuser Basel, Basel, 2004.

[38] M.V. Fedoryuk. On the stability in c of the Cauchy problem for difference and partial differential equations. *USSR Comput. Math. Math. Phys.*, 7(3):48–89, jan 1967.

[39] Gerald B. Folland and Elias M. Stein. *Hardy spaces on homogeneous groups*. Princeton University Press, Princeton, NJ, 1982.

[40] A Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Englewood Cliffs, N.J., 1964.

[41] Enrico Giusti. Equazioni quasi ellittiche e spazi $L^{p,\theta}(\Omega, \delta)$ (I). *Ann. di Mat. Pura ed Appl. Ser. 4*, 75(1):313–353, dec 1967.

[42] T. N. E. Greville. On Stability of Linear Smoothing Formulas. *SIAM J. Numer. Anal.*, 3(1):157–170, mar 1966.

[43] Leonard Gross. Logarithmic Sobolev Inequalities and Contractivity Properties of Semigroups. *Lect. Notes Math.*, 1563:54–88, 1993.

[44] G. H. Hardy and J. E. Littlewood. Some problems of 'Partitio numerorum'; I: A new solution of Waring's problem. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math. Klasse*, pages 33–54, 1920.

[45] Wilfried Hazod and Eberhard Siebert. *Stable Probability Measures on Euclidean Spaces and on Locally Compact Groups*. Springer Netherlands, Dordrecht, 2001.

[46] W. Hebisch and L. Saloff-Coste. Gaussian Estimates for Markov Chains and Random Walks on Groups. *Ann. Probab.*, 21(2):673–709, apr 1993.

[47] Waldemar Hebisch. Sharp pointwise estimates for the kernels of the semigroup generated by sum of even powers of vector fields on homogeneous groups, 1989.

[48] Reuben Hersh. A Class of "Central Limit Theorems" For Convolution Products of Generalized Functions. *Trans. Am. Math. Soc.*, 140:71, jun 1969.

[49] G. N. Hile. Fundamental solutions and mapping properties of semielliptic operators. *Math. Nachrichten*, 279(13-14):1538–1564, oct 2006.

[50] G.N. Hile, C.P. Mawata, and Chiping Zhou. A Priori Bounds for Semielliptic Operators. *J. Differ. Equ.*, 176(1):29–64, 2001.

[51] Kenneth J. Hochberg. A Signed Measure on Path Space Related to Wiener Measure. *Ann. Probab.*, 6(3):433–458, jun 1978.

[52] Kenneth J. Hochberg. Central Limit Theorem for Signed Distributions. *Proc. Am. Math. Soc.*, 79(2):298, jun 1980.

[53] Kenneth J. Hochberg and Enzo Orsingher. Composition of stochastic processes governed by higher-order parabolic and hyperbolic equations. *J. Theor. Probab.*, 9(2):511–532, apr 1996.

[54] Lars Hörmander. *Linear partial differential operators*. Springer-Verlag Berlin Heidelberg, Berlin, 1963.

[55] Lars Hörmander. *The Analysis of Linear Partial Differential Operators II*. Springer-Verlag Berlin Heidelberg, Berlin, 1983.

[56] Fritz John. On integration of parabolic equations by difference methods: I. Linear and quasi-linear equations for the infinite interval. *Commun. Pure Appl. Math.*, 5(2):155–211, may 1952.

[57] E. Kaniuth, A. T. Lau, and A. Ülger. Power boundedness in Fourier and Fourier-Stieltjes algebras and other commutative Banach algebras. *J. Funct. Anal.*, 260(8):2366–2386, 2011.

[58] Yakar Kannai. On the asymptotic behavior of resolvent kernels, spectral functions and eigenvalues of semi-elliptic systems. *Ann. della Sc. Norm. Super. di Pisa - Cl. di Sci.*, 23(4):563–634, 1969.

[59] Robert Kesler. Private Communication, 2014.

[60] V. Yu. Krylov. V. Yu. Krylov. Some properties of the distributions corresponding to the equation $\partial u/\partial t = (-1)^{q+1}\partial^{2q}u/\partial x^{2q}$. *Sov. Math. Dokl.*, 1:760–763, 1960.

[61] Aimé Lachal. First hitting time and place, monopoles and multipoles for pseudo-processes driven by the equation $\frac{\partial}{\partial t} = \pm\frac{\partial^N}{\partial x^N}$. *Electron. J. Probab. [electronic only]*, 12:300–353, 2007.

[62] Aimé Lachal. A Survey on the Pseudo-process Driven by the High-order Heat-type Equation $\partial/\partial t = \pm\partial^N/\partial x^N$ Concerning the Hitting and Sojourn Times. *Methodol. Comput. Appl. Probab.*, 14(3):549–566, sep 2012.

[63] Gregory F Lawler and Vlada Limic. *Random Walk: A Modern Introduction*, volume 123. Cambridge University Press, Cambridge, 2010.

[64] Eugenio Elia Levi. Sulle equazioni lineari totalmente ellittiche alle derivate parziali. *Rend. del Circ. Mat. di Palermo*, 24(1):275–317, dec 1907.

[65] Daniel Levin and Terry Lyons. A signed measure on rough paths associated to a PDE of high order: results and conjectures. *Rev. Matemática Iberoam.*, pages 971–994, 2009.

[66] Elliott H. Lieb and Michael Loss. *Analysis (GTM 14)*. American Mathematical Society, Providence, RI, 2nd edition, 2001.

[67] Tadato Matsuzawa. On quasi-elliptic boundary problems. *Trans. Am. Math. Soc.*, 133(1):241–241, jan 1968.

[68] Kunio Nishioka. Monopoles and dipoles in biharmonic pseudo process. *Proc. Japan Acad. Ser. A, Math. Sci.*, 72(3):47–50, 1996.

[69] El-Maati Ouhabaz. *Analysis of heat equations on domains (LMS-31)*. Princeton University Press, Princeton, NJ, 2009.

[70] Bruno Pini. Proprietà locali delle soluzioni di una classe di equazioni ipoellittiche. *Rend. del Semin. Mat. della Univ. di Padova*, 32:221–238, 1962.

[71] Georg Pólya. Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz. *Math. Ann.*, 84(1-2):149–160, 1921.

[72] Evan Randles and Laurent Saloff-Coste. Convolution powers of complex functions on $\mathbb{Z}^d$. *(to Appear Rev. Mat. Iberoam.)*, 2015.

[73] Evan Randles and Laurent Saloff-Coste. On the Convolution Powers of Complex Functions on $\mathbb{Z}$. *J. Fourier Anal. Appl.*, 21(4):754–798, 2015.

[74] Evan Randles and Laurent Saloff-Coste. Positive-homogeneous operators, heat kernel estimates and the Legendre-Fenchel transform. pages 1–47, feb 2016.

[75] Robert D. Richtmyer and K. W. Morton. *Difference Methods for Initial-value Problems*. Wiley, New York, NY, 2nd edition, 1967.

[76] Derek W. Robinson. Elliptic differential operators on Lie groups. *J. Funct. Anal.*, 97(2):373–402, 1991.

[77] Derek W. Robinson. *Elliptic operators and Lie groups*. Oxford University Press, Oxford, 1991.

[78] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, NJ, 1970.

[79] Sadao Sato. An approach to the Biharmonic pseudo process by a Random walk. *J. Math. Kyoto Univ.*, 42(3):403–422, 2002.

[80] Isaac Jacob Schoenberg. On smoothing operations and their generating functions. *Bull. Am. Math. Soc.*, 59(3):199–231, may 1953.

[81] Bertram M Schreiber. Measures with bounded convolution powers. *Trans. Am. Math. Soc.*, 151(2):405–405, feb 1970.

[82] S.I. Serdyukova. On the stability in a uniform metric of sets of difference equations. *USSR Comput. Math. Math. Phys.*, 7(3):30–47, jan 1967.

[83] Frank Spitzer. *Principles of Random Walk*, volume 34 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1964.

[84] Gilbert Strang. Polynomial Approximation of Bernstein Type. *Trans. Am. Math. Soc.*, 105(3):525, dec 1962.

[85] A. F. M. ter Elst and Derek W Robinson. High order divergence-form elliptic operators on Lie groups. *Bull. Aust. Math. Soc.*, 55(02):335, apr 1997.

[86] V. Thomee. Stability theory for partial difference operators. *SIAM Rev.*, 11(2):152–195, 1969.

[87] Vidar Thomée. Stability of difference schemes in the maximum-norm. *J. Differ. Equ.*, 1(3):273–292, jul 1965.

[88] Hugo Touchette. Legendre-Fenchel transforms in a nutshell. Technical report, Rockefeller University, New York, 2007.

[89] Lloyd N. Trefethen. *Finite Difference and Spectral Methods for Ordinary and Partial Differential Equations*. unpublished text, 1996.

[90] Hans Triebel. A priori estimates and boundary vatue problems for semielliptic differenttial equations: A model case. *Commun. Partial Differ. Equations*, 8(15):1621–1664, jan 1983.

[91] Mario Troisi. Problemi al contorno con condizioni omogenee per le equazioni quasi-ellittiche. *Ann. di Mat. Pura ed Appl. Ser. 4*, 90(1):331–412, dec 1971.

[92] Akira Tsutsumi. On the asymptotic behavior of resolvent kernels and spectral functions for some class of hypoelliptic operators. *J. Differ. Equ.*, 18(2):366–385, jul 1975.

[93] L. R. Volevich. A class of hypoelliptic systems. *Sov. Math. Dokl.*, (1):1990–1193, 1960.

[94] L. R. Volevich. Local properties of solutions of quasi-elliptic systems. *Mat. Sb.*, 59(101):3–52, 1962.

[95] Olof B. Widlund. On the Stability of Parabolic Difference Schemes. *Math. Comput.*, 19(89):1, apr 1965.

[96] Olof B. Widlund. Stability of parabolic difference schemes in the maximum norm. *Numer. Math.*, 8(2):186–202, apr 1966.

[97] A. I. Zhukov. A limit theorem for difference operators. *Uspekhi Mat. Nauk*, 14(3(87)):129–136, 1959.