SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
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AN ANALYSIS OF TWO TWO-ECHELON INVENTORY SYSTEMS

by

John A. Muckstadt
and
Paul Sasseville

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SUMMARY

In this paper we analyze the probabilistic behavior of a two-echelon inventory system. In this system primary demands occur at one of the lower echelon locations called bases. Bases are resupplied only by the upper echelon called the depot; the depot is resupplied by an external supplier. All demands occurring when a base or depot is out of stock are backordered.

We first develop the probability distribution for the number of backorders outstanding at a point in time when the bases follow continuous review (Q,r) policies and the depot follows a periodic review order-up-to-S policy. We next compare the expected number of backorders outstanding at each point in time for two systems. In the first system, bases follow a continuous review $(s-1,s)$ policy and the depot follows a periodic review order-up-to-S policy; in the second system, all locations follow a continuous review $(s-1,s)$ policy. Differences in the performance for the two systems are illustrated using Air Force data. Lastly we derive the probability distribution for the number of backorders outstanding at a particular point in time at the bases when all locations follow a periodic review order-up-to-s policy.

I. INTRODUCTION

Our main objective in this paper is to analyze the probabilistic behavior of a two-echelon, multi-item inventory system. Primary demands for any item are assumed to occur only at one of the lower echelon locations, called a base. Each base is resupplied only from the upper echelon, called the depot, --lateral resupply between bases is not allowed; the depot, in turn, is resupplied by an external supplier. Demands occurring when a base or the depot has no on-hand stock are assumed to be backordered.

In the next section we will develop the probability distribution for the number of backorders existing at a point in time at each base when the bases follow continuous review (Q,r) policies and the depot follows a periodic review order-up-to-S policy. We also assume that the demand process at each base is a Poisson process.

We will examine the important special case where the bases follow a Q = 1 or (s-1,s) continuous review policy in Section III. Specifically, we will compare a system in which bases follow a continuous review (s-1,s) policy and the depot follows an order-up-to-S policy to a system in which the base policy remains a continuous review (s-1,s) policy and the depot follows a continuous review (S-1,S) policy. Since continuous review models are often used to approximate situations in which a periodic review policy is followed, we are interested in measuring the difference in performance achieved in the two systems. To accomplish this, we measure the expected number of backorders outstanding at any point in time at a base for a sample of 68 Air Force avionics items. The example system consists of 15 bases and a depot. Thus, we will be concerned with measuring how this policy change at the depot affects performance at the bases. While it is intuitive that performance in the periodic review case should be inferior to that achieved in the continuous review case, it is of interest to see at what rate and by how much performance changes with differing review period lengths and times within the review period.

In the fourth section we derive the probability distribution for the number of backorders existing at a point in time when all locations follow a periodic review order-up-to-s policy. The development is quite general in that no assumption is made concerning the nature of the discrete demand distribution.

II. A PROBABILITY EXPRESSION FOR BACKORDERS WHEN BASES FOLLOW A CONTINUOUS REVIEW (Q,r) POLICY AND THE DEPOT FOLLOWS A PERIODIC REVIEW POLICY.

We will now develop the probability distribution for backorders at a base at an arbitrary point in time when bases follow a continuous review (Q,r) policy and the depot follows a periodic review (S-1,S) policy. The line of reasoning used here is the same as used by Muckstadt in Ref. [1] where he derived the probability distribution for backorders when all locations follow continuous review (s,S) policies. We refer the reader to this reference to obtain the details of the logic behind the development. For simplicity, we assume we are dealing with an arbitrary item.

We begin by listing the basic assumptions.

(1) All bases have the same system parameters--demand rates and resupply times -- and all bases follow the same continuous review (Q,r) policy.

(2) The depot follows a periodic review order-up-to-S policy with review period length T and initial review time x.

(3) No partial fills of base orders by the depot are permitted. All Q units of an order must be shipped simultaneously.

(4) A simple Poisson process with rate λ generates demand at each base.

(5) All unsatisfied demands at all locations are backordered.

(6) τ, the base lead time given that the depot has stock on hand, is constant and known.

(7) τ', the depot lead time from the external supplier, is constant and known.

(8) The base reorder point r is greater than or equal to -1.

(9) The number of bases m in the system is large.

(10) All demands are satisfied on a first-come, first-served basis.
A complete discussion of these assumptions and their implications is given in Ref. 1.

Before we obtain the probability distribution of base backorders at a point in time, we introduce the following nomenclature:

I^1 represents base J's inventory position at time $t-\tau-\tau'$,

I^2 represents base J's inventory position at time $t-\tau$,

G represents the number of orders placed by all bases other than base J for depot resupply during $(t-\tau-\tau',t-\tau]$,

D represents the number of demands occurring during $(t-\tau-\tau',t-\tau]$ at base J,

$\tilde{D}$ represents the number of demands occurring during $(t-\tau,t]$ at base J,

V represents the number of satisfied orders placed by base J on the depot during $(t-\tau-\tau',t-\tau]$--the orders are placed during $(t-\tau-\tau',t-\tau]$ and received at base J prior to t,

$a(x,n) = e^{-x}x^n/n!$,

$B(t)$ represents the number of backorders existing at base J at time t,

U represents the number of orders placed on the depot during $(t-\tau-\tau',t-\tau]$ by base J that are unfilled at time t,

γ represents the arrival rate of orders at the depot from all bases except base J measured in orders per day,

$N_1 = \{n: n = Q - (i-k) + \bar{n}Q,\ \bar{n} = 0,1,2,\ldots,\ \text{and}\ n \leqq S_0/Q\} \cup \{0\}$,

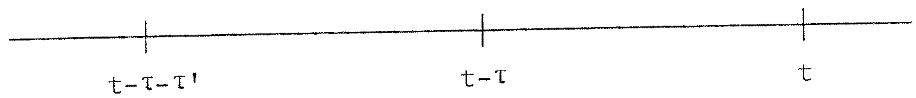
$N_2 = \{n: n = k - i + \bar{n}Q,\ \bar{n} = 0,1,2,\ldots,\ \text{and}\ n \leqq S_0/Q\}$,

$N_1' = \{n: n = Q - (i-k) + \bar{n}Q,\ \bar{n} = 0,1,2,\ldots\} \cup \{0\}$,

$N_2' = \{n: n = k - i + \bar{n}Q,\ \bar{n} = 0,1,2,\ldots\}$, and

$T(t) = (t-\tau-\tau'-x) \bmod T$.

The basic idea behind the derivation of the probability distribution for the number of backorders outstanding at a point in time at a base can be obtained from Fig. 1. When all locations follow continuous review (s,S) policies, we see



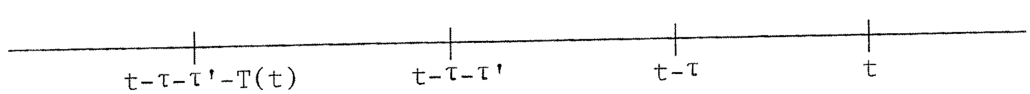
Time Sequence of Events in the Continuous Review Case

Figure 1.

that any orders placed at a particular base, say base J, prior to or at time $t-\tau-\tau'$ will be satisfied by time t. Hence, all backorders existing at time t at base J are due to demands placed on base J during the interval $(t-\tau-\tau',t]$. In Ref. 1, Muckstadt used two events to obtain the desired probability expression. Case A was defined as the event that the total depot demand during $(t-\tau-\tau',t-\tau]$ does not exceed the depot inventory position at time $t-\tau-\tau'$. Case B was defined as the event that the total depot demand during $(t-\tau-\tau',t-\tau]$ exceeds the depot inventory position at time $t-\tau-\tau'$. With these events forming a partition, he calculated the probability expression as

$$P\{B(t)=b\} = P\{B(t)=b;\ \text{Case A}\} + P\{B(t)=b;\ \text{Case B}\}.$$

In the periodic review case we use the same argument. From the way that $T(t)$ is defined, we see in Fig. 2 that $t-\tau-\tau'-T(t)$ is the closest depot review time prior to or at time $t-\tau-\tau'$. Then all base J demands occurring prior to or



Time Sequence When the Depot Follows a Periodic Review Policy

Figure 2.

at time $t-\tau-\tau'-T(t)$ will be satisfied by time t. Hence all backorders existing at time t at base J are due to demands placed on base J during the interval

$(t-\tau-\tau'-T(t),t]$. Case A is now defined as the event that the total depot demand during the interval $(t-\tau-\tau'-T(t),t-\tau]$ does not exceed the depot inventory position at time $t-\tau-\tau'-T(t)$. Case B is defined as the event that total depot demand during $(t-\tau-\tau'-T(t),t-\tau]$ exceeds the depot inventory position at time $t-\tau-\tau'-T(t)$.

The remainder of the derivation closely follows that given in Ref. 1. Consequently, we only present the final result below. However, there are two important differences between the expressions for this case and the ones given in Ref. 1. First, in the continuous review case, many terms need to be conditioned on the probability distribution of inventory position at the depot at time $t-\tau-\tau'$. In the periodic review case, the analogous time is $t-\tau-\tau'-T(t)$. However, this time is a depot review time. But, in this case the depot inventory position will always be S at time $t-\tau-\tau'-T(t)$. Hence the periodic review expression does not have summation terms involving conditioning on the depot inventory position or factors involving depot inventory position as an unknown quantity. In this sense, the periodic review expression is computationally less burdensome. The second difference between the two expressions involves stationarity. The continuous review expression is independent of t so that probability of backorders at a base remains the same throughout time. However, in the periodic review expression, this probability distribution depends both on the review period length T and the time t.

We now show how to determine the probability distribution for the number of backorders outstanding at time t at a base. Clearly, $P\{B(t)=b\} = P\{B(t)=b;\text{ Case A}\} + P\{B(t)=b;\text{ Case B}\}$. We will find $P(B(t)=b)$ by determining separately $P(B(t)=b;\text{ Case A})$ and $P(B(t)=b;\text{ Case B})$.

It is not hard to see that $P\{B(t)=b;\text{ Case A}\}$ can be stated as follows:

$$P\{B(t)=b;\text{ Case A}\} = \sum_{i=r+1}^{r+Q} P\{\tilde{D}=i+b\} \cdot P\{\text{Case A}; I^2=i\} \quad \text{when} \quad b \geq 1 \quad \text{and}$$

$$P\{B(t)=0;\text{ Case A}\} = \sum_{i=r+1}^{r+Q} \sum_{y=0}^{i} P\{\tilde{D}=y\} \cdot P\{\text{Case A}; I^2=i\}, \quad \text{where}$$

$P\{\tilde{D}=i+b\} = a(\lambda\tau, i+b)$, $P\{\tilde{D}=y\} = a(\lambda\tau, y)$, and

$$P\{\text{Case A}; I^2=i\} = \sum_{k=r+1}^{i} \sum_{n\in N_1} \sum_{m=0}^{S/Q-[(n-k+Q+r)/Q]} 1/Q \cdot a(\gamma(\tau'+T(t)),m) \cdot a(\lambda(\tau'+T(t)),n)$$

$$+ \sum_{k=i+1}^{r+Q} \sum_{n\in N_2} \sum_{m=0}^{S/Q-[(n-k+Q+r)/Q]} 1/Q \cdot a(\gamma(\tau'+T(t)),m) \cdot a(\lambda(\tau'+T(t)),n).$$

Furthermore, one can see that

$$P\{B(t)=b; \text{Case B}\} = \sum_{i=r+1}^{r+Q} \sum_{u=0}^{[(i+b)/Q]} P\{\tilde{D}=i+b-uQ\} \cdot P\{U=u | \text{Case B}; I^2=i\}$$

$$\cdot P\{I^2=i; \text{Case B}\} \quad \text{when} \quad b \geqq 1 \quad \text{and}$$

$$P\{B(t)=0; \text{Case B}\} = \sum_{i=r+1}^{r+Q} \sum_{u=0}^{[(i/Q)]} \sum_{y=0}^{i-uQ} P\{\tilde{D}=y\} \cdot P\{U=u | \text{Case B}; I^2=i\}$$

$$\cdot P\{I^2=i; \text{Case B}\}.$$

In this case

$$P\{\tilde{D}=i+b-uQ\} = a(\lambda\tau, i+b-uQ), \quad P\{\tilde{D}=y\} = a(\lambda\tau, y),$$

$$P\{U=u \mid \text{Case B};\ I^2=i\} = \sum_{k=r+1}^{i} \sum_{d\in N_1'} \sum_{g=(S/Q - [(d+Q+r-k)/Q]+1)^+}^{\infty}$$

$$P\{V=[(d+Q+r-k)/Q]-u \mid D=d;\ G=g;\ I^1=k;\ I^2=i;\ \text{Case B}\}$$

$$\cdot\ P\{D=d;\ G=g;\ I^1=k \mid \text{Case B};\ I^2=i\}$$

$$+ \sum_{k=i+1}^{r+Q} \sum_{d\in N_2'} \sum_{g=(S/Q - [(d+Q+r-k)/Q]+1)^+}^{\infty}$$

$$P\{V=[(d+Q+r-k)/Q]-u \mid D=d;\ G=g;\ I^1=k;\ I^2=i;\ \text{Case B}\}$$

$$\cdot\ P\{D=d;\ G=g;\ I^1=k \mid \text{Case B};\ I^2=i\},$$

$$P\{V=v \mid D=d;\ G=g;\ I^1=k;\ I^2=i;\ \text{Case B}\}$$

$$= \sum_{w=0}^{Q-1} \frac{\binom{d}{vQ-(Q-k+r)+w}\binom{g}{S/Q - v-1}}{\binom{d+g}{vQ-(Q-k+r)+w+S/Q - v-1}} \cdot \frac{g-(S/Q - v-1)}{(g+d-(vQ-(Q-k+r)+w+S/Q - v-1)}$$

$$+ \frac{\binom{d}{vQ-(Q-k+r)-1}\binom{g}{S/Q - v}}{\binom{d+g}{vQ-(Q-k+r) + S/Q - v-1}} \cdot \frac{d-(vQ-(Q-k+r)-1)}{g+d-(vQ-(Q-k+r) + S/Q - v-1)}, \quad \text{and}$$

$$P\{D=d;\ G=g;\ I^1=k \mid \text{Case B};\ I^2=i\}$$

$$= \begin{cases} 0, & \text{if } [(d+Q+r-k)/Q]+g \leqq S/Q \\ & \text{or } d \notin N_1' \text{ when } i \geqq k \\ & \text{or } d \notin N_2' \text{ when } i < k \\ a(\lambda(\tau'+T(t)),d) \cdot a(\gamma(\tau'+T(t)),g) \cdot 1/Q \cdot \dfrac{1}{P\{\text{Case B};\ I^2=i\}}, & \text{o.w.} \end{cases}$$

Finally,

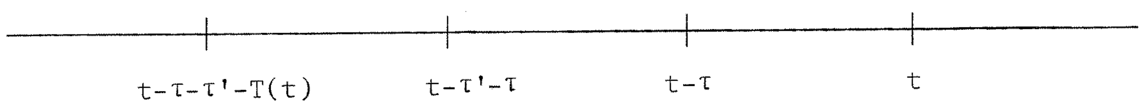
$$P\{\text{Case B};\ I^2=i\} = \sum_{k=r+1}^{i} \sum_{n \in N_1'} \sum_{\bar{m}=(S/Q-[(n+Q+r-k)/Q]+1)^+}^{\infty} 1/Q \cdot a(\gamma(\tau'+T(t)),\bar{m}) \cdot a(\lambda(\tau'+T(t)),n)$$

$$+ \sum_{k=i+1}^{r+Q} \sum_{n \in N_2'} \sum_{\bar{m}=(S/Q-[(n+Q+r-k)/Q]+1)^+}^{\infty} 1/Q \cdot a(\gamma(\tau'+T(t)),\bar{m}) \cdot a(\lambda(\tau'+T(t)),n).$$

Thus we have demonstrated how $P(B(t)=b)$ may be calculated in the situation where the m identical bases follow a continuous review (Q,r) policy and the depot follows a periodic review order-up-to-S policy.

III. A COMPARISON OF PERFORMANCE WHEN THE DEPOT FOLLOWS DIFFERENT POLICIES

A major problem in comparing the continuous review model to the periodic review model developed in Section II is determining what the review period length T should be and when to examine the system's behavior. Since the number of expected base backorders in the continuous review system is independent of time, we have available a lower bound on expected system backorders for the periodic review model at any time t. However, in the periodic review model, expected base backorders depend both on T and the time t at which we examine the system. To examine this situation, we must again refer to the variable $T(t) = (t-\tau-\tau'-x) \bmod T$ and Figure 3. First note that $T(t)$ incorporates both T and t. Also, by examining the probability expression derived in Section II, we see that



Time Sequence For the Periodic Review Model
Figure 3.

this single quantity is all that is needed (in addition to normal system parameters) to determine a backorder distribution at any arbitrary time. In fact, expected base backorders in the periodic review case monotonically increase as a function of $T(t)$ with $0 \leq T(t) < T$. If $T(t) = 0$, then time $t-\tau-\tau'-T(t)$ coincides with time $t-\tau-\tau'$ and expected base backorders is the same for both the continuous and periodic review models. However, as $T(t)$ becomes larger, the length of time for which backorders may exist becomes longer in the periodic review model and we should expect a rise in expected base backorders. This now solves the problem of choosing values of T and t. Rather than choosing these values individually, we calculate expected backorders by varying $T(t)$ with $T(t) \geq 0$. For any given T and t, $T(t)$ can be computed and system performance can be easily found.

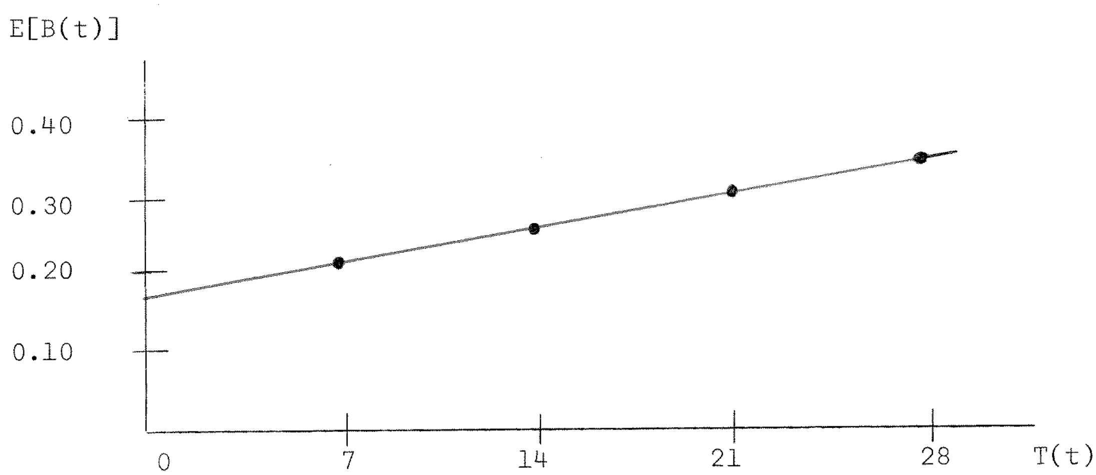
We now present some results obtained when the continuous and periodic review models were tested on a 68 item 15 base inventory system. As stated earlier, all locations are assumed to follow an $(s-1,s)$ policy (note that the depot's order-up-to-S policy is the same as an $(S-1,S)$ policy). Our goal is to examine the change in expected backorders for the periodic review model with differing values of $T(t)$. No attempt is made to quantify rigorously the various factors causing the differences in performance among the items. Rather, we will be content to obtain observations that our empirical evidence indicates holds for all items in the system.

Initially, stock levels were computed for each of the 68 items using the algorithm described in Ref. 2 for the case where all locations follow a continuous review policy. The same system data and stock levels were then used for the case where the depot follows a periodic review policy. The number of expected base backorders was then computed for each item for values of $T(t) = 0,7,14,21$ and 28 days. The results obtained show that expected base backorders for the periodic review model are monotone in $T(t)$. Furthermore, this relationship appears to be convex; the rate of change in performance was very nearly constant over the region tested for all items. To illustrate this observation, we present graphs of four representative items. The data for the four items is given in Table I. Figures four through seven display the affect of time on expected base backorders. Since the demand rate and stock level are the only parameters that change among the items, the differing rates of change in performance are due solely to these factors. As would be expected, the slope is sharper for high demand, low stock level items. In general, the results show that when the review period length at the depot is long (e.g. a month or more) the continuous review model $(T(t) = 0)$ provides a poor approximation to the periodic review model. Backorders are severely underestimated in many cases as $T(t)$ increases. However, when the review period is short (e.g. a week) the approximation appears to be satisfactory for most items.

Base Daily Demand Rate	Base Transportation Time (in Days)	Average Base Resupply Time (in Days)	Depot Lead Time (in Days)	Base Stock Level	Depot Stock Level
0.0408	12	15.3282	41	1	25
0.0341	12	17.7592	41	0	19
0.0077	12	22.8011	41	0	4
0.0096	12	15.7013	41	0	7

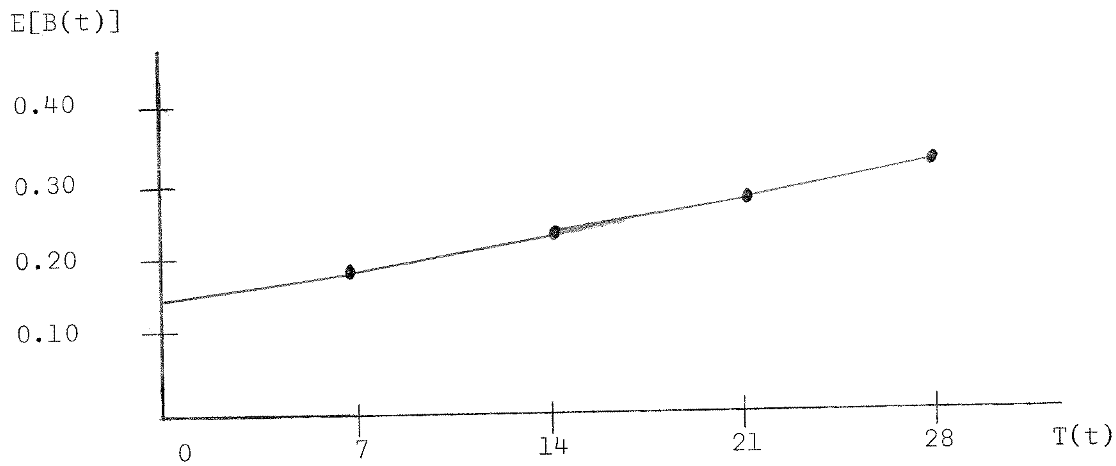
Data For Four Items

Table I



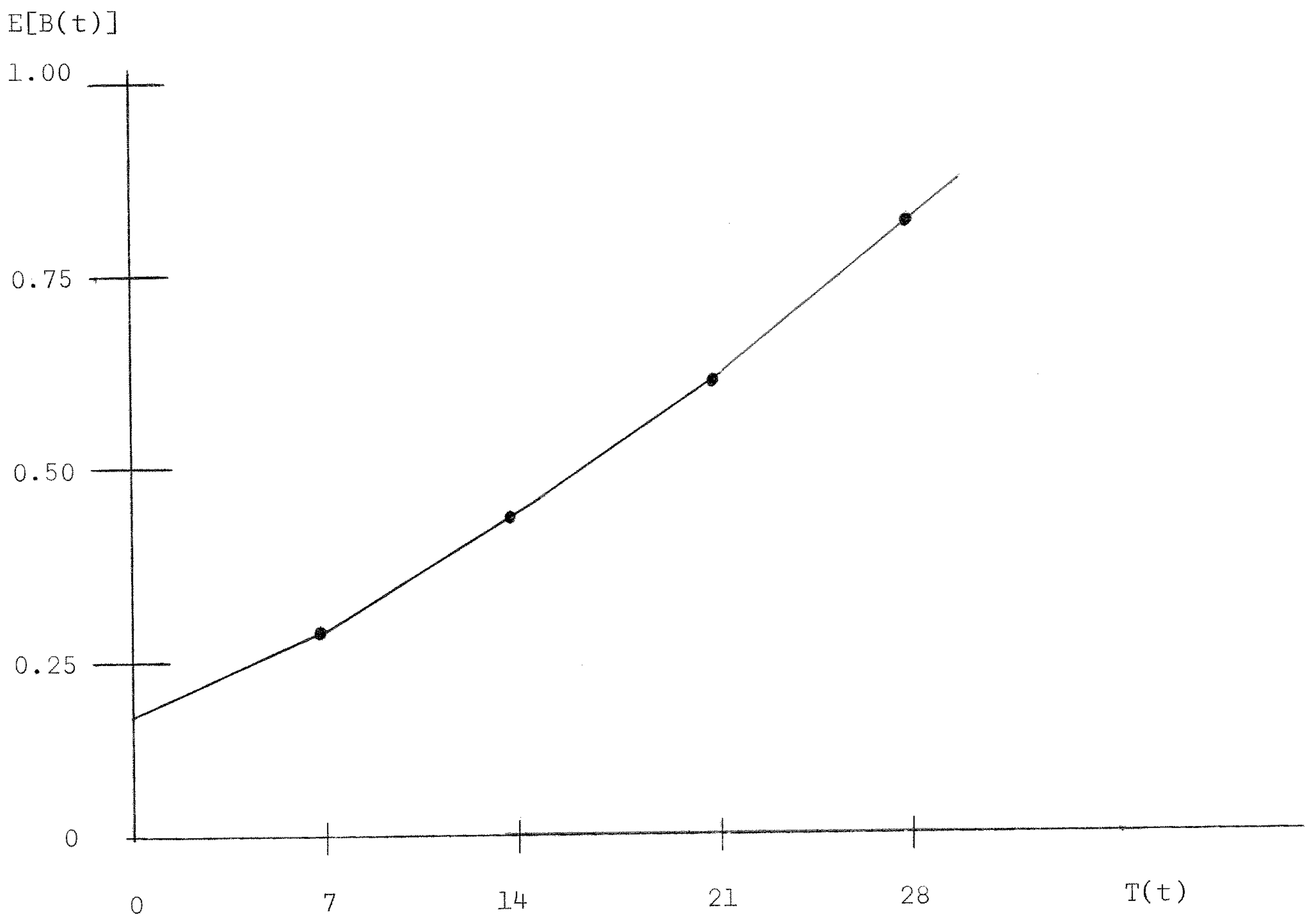
Expected Base Backorders as a Function of $T(t)$ for Item 1

Figure 4.



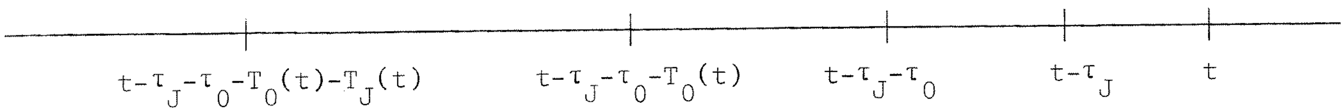
Expected Base Backorders as a Function of T(t) for Item 2

Figure 5.



Expected Base Backorders as a Function of T(t) for Item 3

Figure 6



Time Sequence of Events When Both the Depot and Bases Follow A Periodic Review Policy

Figure 8.

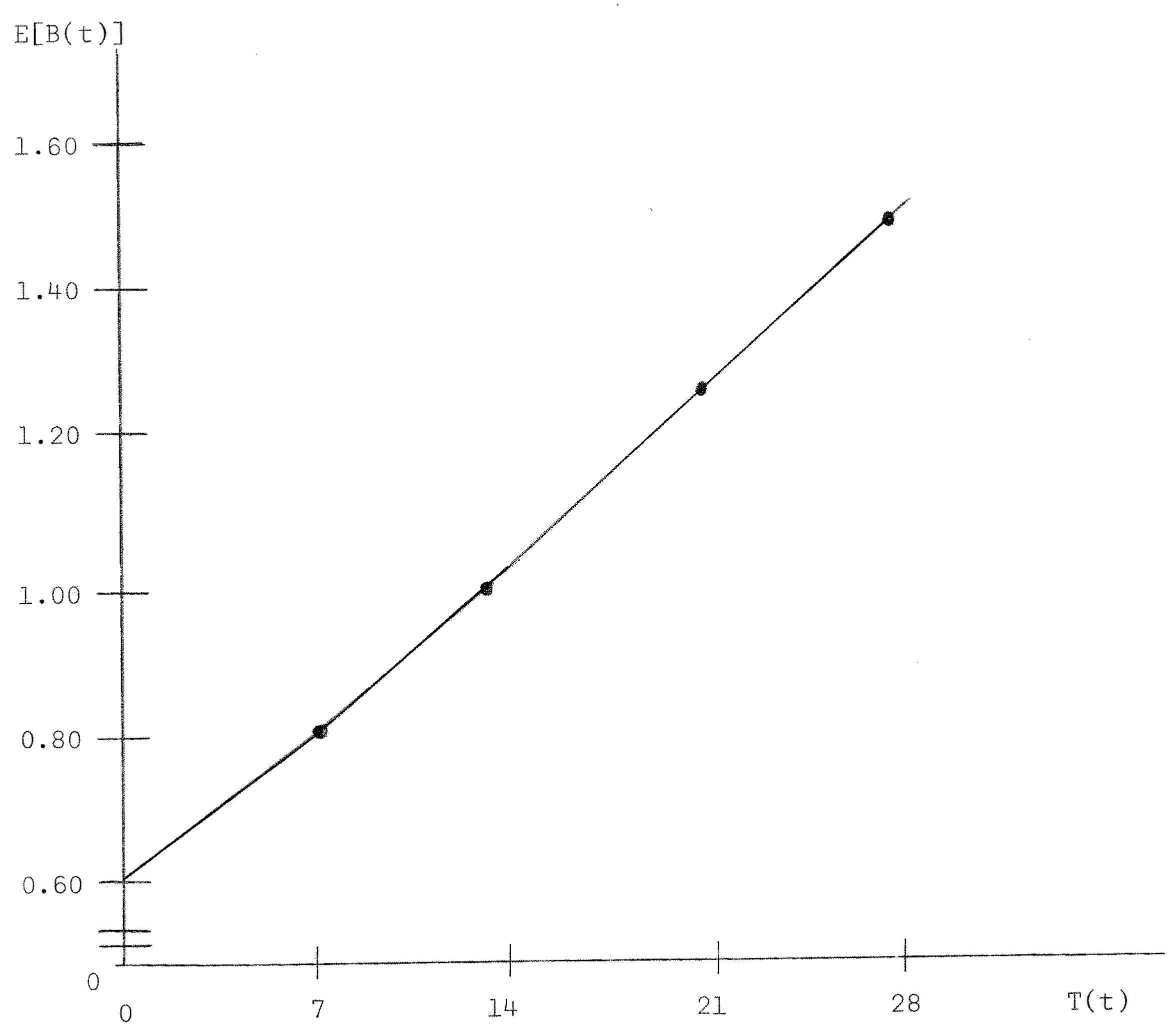
Observe from Fig. 8 that $t-\tau_J-\tau_0-T_0(t)$ is the last depot review time prior to or at time $t-\tau_J-\tau_0$. Similarly, $t-\tau_J-\tau_0-T_0(t)-T_J(t)$ is the last base J review time prior to or at time $t-\tau_J-\tau_0-T_0(t)$. Any base J demand occurring prior to time $t-\tau_J-\tau_0-T_0(t)-T_J(t)$ will be satisfied by time t. Hence, any backorders existing at base J at time t are due to demands placed at this location in the interval $(\tau-\tau_J-\tau_0-T_0(t)-T_J(t),t]$. Also note that base J orders placed in the interval $(t-\tau_J,t]$ will not help in satisfying demand by time t. This is also the case for any depot orders placed in the interval $(t-\tau_J-\tau_0,t]$. Next, let

$$X_i = \{v_i+nT_i \mid n = 0,1,2,\ldots\}, \quad i = 1,\ldots,m,$$

$$Y(t) = \bigcup_{i=1}^{m} (X_i \cap (t-\tau_J-\tau_0-T_0(t),t-\tau_J])$$

$$Z(t) = Y(t) \cup \{t-\tau_0-\tau_J-T_0(t)-T_J(t)\}.$$

$Z(t)$ contains all base review times in the interval $(t-\tau_J-\tau_0-T_0(t),t-\tau_J]$ plus the base J review time $t-\tau_0-\tau_J-T_0-T_J(t)$. Let $Z(t) = \{Z_0,Z_1,\ldots,Z_e\}$ with $Z_0 \prec \ldots \prec Z_e$ and $Z_0 = t-\tau_J-\tau_0-T_0(t)-T_J(t)$, where $Z_i \prec Z_j$ implies order i arrives prior to order j in the arrival sequence. Note that although Z_i may



Expected Base Backorders as a Function of T(t) for Item 4

Figure 7

IV. A PROBABILITY EXPRESSION FOR BACKORDERS WHEN ALL LOCATIONS FOLLOW (s-1,s) PERIODIC REVIEW POLICIES.

In this section we derive the probability expression for backorders at a point in time for a two-echelon inventory system in which all locations follow periodic review (s-1,s) policies. The number of restrictions and assumptions made is small in order to obtain a general expression capable of being used in many situations. Specifically, we make the following assumptions for the model developed in this section:

(1) An (s-1,s) periodic review policy is followed at all locations. At location i, we use an (s_i-1,s_i) policy with review period T_i. The initial review time is v_i. (Subsequently, the subscript $i = 0$ refers to the depot while $i = 1,\dots,m$ refers to the bases.)

(2) Partial fills of base orders by the depot are in effect.

(3) The sequence of base requisitions is totally ordered.

(4) Orders placed on the depot are satisfied on a first-come, first-served basis according to the arrival sequence.

(5) τ_i, the lead time for location i, is constant and known.

(6) All unsatisfied demand at all locations is backordered.

(7) The demand distribution at each base is an arbitrary discrete distribution.

As before, our objective is to compute $P\{B(t)=b\}$. To carry out the derivation we first define

$$T_0(t) = (t-\tau_J-\tau_0-v_0) \bmod T_0 \quad \text{and}$$

$$T_J(t) = (t-\tau_J-\tau_0-T_0(t)-v_J) \bmod T_J \;.$$

equal Z_j we have assumed--Assumption 3--an ordering of the arrivals is predetermined. Also let $\{Z_{n_0},\ldots,Z_{n_k}\}$ be the subsequence of $Z(t)$ that consists of all base J review times. Note that $Z_0 = Z_{n_0}$ in all cases. Before proceeding with the calculation of $P\{B(t)=b\}$, we must compute the probabilities for the following events.

Let

E_q be the event that Z_q is the first time in the interval $(t-\tau_J-\tau_0-T_0(t),t-\tau_J]$ that $\sum_{i=1}^{m} \sum_{x\in X_i\cap(t-\tau_J-\tau_0-T_0(t),Z_q]} D_i(x-T_i,x) > s_0$, where $D_i(\alpha,\beta)$ is the demand occurring at base i in the interval $(\alpha,\beta]$, $q = 1,\ldots,e$,

and

E_{e+1} be the event that $\sum_{i=1}^{m} \sum_{x\in X_i\cap(t-\tau_J-\tau_0-T_0(t),Z_e]} D_i(x-T_i,x) \leqq s_0$.

E_q is the event that cumulative depot demand in the interval $(t-\tau_J-\tau_0-T_0(t),t-\tau_J]$ first exceeds s_0 at time Z_q, $q = 1,\ldots,e$. E_{e+1} is the event that cumulative depot demand in the interval $(t-\tau_J-\tau_0-T_0(t),t-\tau_J]$ does not exceed s_0.

Also, let

F_p be the event that base J orders placed at times $Z_{n_0},Z_{n_1},\ldots,Z_{n_p}$ are all completely satisfied by time t, $p = 0,1,\ldots,k$,

G_p be the event that Z_{n_p} is the first base J order placed in $(t-\tau_J-\tau_0-T_0(t),t-\tau_J]$ which will not be completely satisfied by time t, $p = 1,\ldots,k$, and

G_{k+1} be the event that all base J orders placed in $(t-\tau_J-\tau_0-T_0(t),t-\tau_J]$ will be completely satisfied by time t.

One can easily see that the probabilities for the events $E_1, E_{e+1},$ and F_0 are given as follows:

$$P\{E_1\} = P\{\sum_{i=1}^{m} \sum_{x \in X_i \cap (t-\tau_J-\tau_0-T_0(t), Z_1]} D_i(x-T_i,x) > s_0\},$$

$$P\{E_{e+1}\} = P\{\sum_{i=1}^{m} \sum_{x \in X_i \cap (t-\tau_J-\tau_0-T_0(t), Z_e]} D_i(x-T_i,x) \leqq s_0\}, \quad \text{and}$$

$$P\{F_0\} = 1.$$

Next we find $P\{E_q\}$, for $q = 2,\ldots,e$. It is not hard to see that

$$P\{E_q\} = \sum_{d=0}^{\infty} P\{E_q \mid \sum_{i=1}^{m} \sum_{x \in X_i \cap (t-\tau_J-\tau_0-T_0(t), Z_{q-1}]} D_i(x-T_i,x) = d\}$$

$$\cdot\, P\{\sum_{i=1}^{m} \sum_{x \in X_i \cap (t-\tau_J-\tau_0-T_0(t), Z_{q-1}]} D_i(x-T_i,x) = d\},$$

where

$$P\{E_q \mid \sum_{i=1}^{m} \sum_{x \in X_i \cap (t-\tau_J-\tau_0-T_0(t), Z_{q-1}]} D_i(x-T_i,x) = d\}$$

$$= \begin{cases} 0, & d > s_0 \\ P\{\sum_{i=1}^{m} \sum_{x \in X_i \cap (Z_{q-1}, Z_q]} D_i(x-T_i,x) > s_0 - d\}, & d \leqq s_0. \end{cases}$$

With $P\{E_q\}$ known for $q = 1,\ldots,e+1$, we can now proceed to obtain $P\{F_p\}$ for $p = 1,\ldots,k$. Clearly $P\{F_p\} = \sum_{q=1}^{e+1} P\{F_p \mid E_q\}P\{E_q\}$. We can evaluate this expression by recognizing that

that

$$P\{F_p|E_q\} = \begin{cases} 1, & n_p < q\,, \\ 0, & q \in \{n_1,\ldots,n_p\}, \\ P\{D_J(r(q),Z_{n_p}) = 0\}, & n_p > q \text{ and } q \notin \{n_1,\ldots,n_p\}, \end{cases}$$

where $r(q) = \max\{Z_{n_s}|Z_{n_s} < q,\ s = 1,\ldots,k\}$.

Next, let us calculate $P\{G_p\}$. Observe that

$$G_p = F_{p-1} \cap F_p^c\,,\ p = 1,\ldots,k.$$

Then

$$P\{G_p\} = P\{F_{p-1} \cap F_p^c\} = P\{F_p^c|F_{p-1}\}P\{F_{p-1}\}.$$

Furthermore,

$$P\{F_p^c|F_{p-1}\} = \sum_{q=1}^{e+1} P\{F_p^c|E_q;F_{p-1}\}P\{E_q|F_{p-1}\}$$

and

$$P\{F_p^c|E_q;F_{p-1}\} = \begin{cases} P\{D_J(Z_{n_{p-1}},Z_{n_p}) > 0\}, & q < n_{p-1}, \\ 0, & q = n_{p-1}, \\ P\{D_J(Z_{n_{p-1}},Z_{n_p}) > 0\}, & n_{p-1} < q < n_p\,, \\ 1, & q = n_p\,, \\ 0, & q > n_p\,. \end{cases}$$

Also,

$$P\{E_q | F_{p-1}\} = \frac{P\{F_{p-1} | E_q\} P\{E_q\}}{P\{F_{p-1}\}},$$

where

$$P\{F_{p-1} | E_q\} = \begin{cases} 1, & n_{p-1} < q, \\ 0, & q \in \{n_1,\ldots,n_{p-1}\}, \\ P\{D_J(r(q), Z_{n_{p-1}}) = 0\}, & n_{p-1} > q \text{ and } q \notin \{n_1,\ldots,n_{p-1}\}. \end{cases}$$

Since $P\{E_q\}$ and $P\{F_{p-1}\}$ have already been determined, we have found $P\{G_p\}$ for $p = 1,\ldots,k$. Finally, note that $P\{G_{k+1}\} = P\{F_k\}$.

Clearly

$$P\{B(t)=b\} = \sum_{p=1}^{k+1} P\{B(t)=b | G_p\} P\{G_p\} \quad \text{for} \quad b \geqq 1.$$

To find $P\{B(t)=b | G_p\}$ we require an additional definition. Let H_{up} be the event that u (≥ 1) units of the order placed by base J at time Z_{n_p} cannot be filled. Then

$$P\{B(t)=b | G_p\} = \sum_{u=1}^{\infty} P\{B(t)=b | H_{up}; G_p\} P\{H_{up} | G_p\} \quad \text{for} \quad p = 1,\ldots,k.$$

But

$$P\{B(t)=b | H_{up}; G_p\} = \begin{cases} 0, & 1 \leqq b \leqq u, \\ P\{D_J(Z_{n_p}, t) = s_J + b - u\}, & b \geqq u \geqq 1. \end{cases}$$

Also,

$$P\{H_{up}|G_p\} = \sum_{q=1}^{e+1} P\{H_{up}|E_q;G_p\}P\{E_q|G_p\},$$

where

$$P\{H_{up}|E_q;G_p\} = \begin{cases} 0, & n_p < q, \\ \sum_{d\le s_0} P\{\sum_{i=1}^{m} \sum_{x\in X_i\cap(t-\tau_j-\tau_0-T_0(t),Z_{q-1}]} D_i(x-T_i,x) = d; & \\ \qquad D_J(Z_{n_{p-1}},Z_{n_p} = u+s_0-d)\}, & n_p = q, \\ P\{D_J(Z_{n_{p-1}},Z_{n_p}) = u\}, & n_p > q, \end{cases}$$

$$P\{E_q|G_p\} = \frac{P\{G_p|E_q\}P\{E_q\}}{P\{G_p\}}, \quad \text{and}$$

$$P\{G_p|E_q\} = \begin{cases} 0, & n_p < q, \\ 1, & n_p = q, \\ P\{D_J(r(q),Z_{n_{p-1}}) = 0;\ D_J(Z_{n_{p-1}},Z_{n_p}) > 0\}, & n_p > q. \end{cases}$$

Also,

$$P\{B(t)=b|G_{k+1}\} = P\{D_J(Z_{n_k},t) = s_J+b\}, \quad b \geqq 1.$$

Combining these results we have shown how to find $P(B(t)=b)$, for $b \geq 1$.

Furthermore,

$$P\{B(t)=0\} = 1 - \sum_{b\geq 1} P\{B(t)=b\} .$$

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