CUMULANT COMPONENT ESTIMATION IN THE BALANCED ONE-WAY POPULATION WITH FINITE SUBPOPULATIONS

BU-140-M B. M. Rao November, 1961

Introduction:

Tukey [2] has introduced an algebra of polykays in which he shows the relationships between symmetric means and polykays, which in the infinite population become cumulants and moment products. In order to work with the finite case he introduces a symbolic o-multiplication which becomes ordinary multiplication in an infinite population. Robson [1] has considered the case of an infinite population.

Using these tools an approach to cumulant component analysis in a finite population is outlined here. The situation considered here is the same as Robson [1], except that the subpopulations contain a finite number "N" of elements.

Population Cumulant Components:

Following Tukey, the υth moment (or symmetric mean) is denoted by $<\upsilon>$, and the υth cumulant (or polykay) by $[\upsilon]$.

The moment generating function is defined as

$$\phi_X(t)=M(t)=Ee^{tX}=1+<1>t+<2>\frac{t^2}{2!}+<3>\frac{t^3}{3!}+\cdots$$

$$=\sum_{\upsilon=0}^{\infty}<\upsilon>\frac{t^\upsilon}{\upsilon!}$$

Following Tukey, the polykay generating function is written as

$$\text{o-log}\ \phi_X(t)=\text{o-log}\left[1+\sum_{\upsilon=1}^{\infty}<\upsilon>\frac{t^\upsilon}{\upsilon!}\right]$$

$$=\sum_{\upsilon=1}^{\infty}<\upsilon>\frac{t^\upsilon}{\upsilon!}-\frac{1}{2}\left(\sum_{\upsilon=1}^{\infty}<\upsilon>\frac{t^\upsilon}{\upsilon!}\right)\text{o}\left(\sum_{\upsilon=1}^{\infty}<\upsilon>\frac{t^\upsilon}{\upsilon!}\right)$$

$$+\frac{1}{3}\left(\sum_{\upsilon=1}^{\infty}<\upsilon>\frac{t^\upsilon}{\upsilon!}\right)\text{o}\left(\sum_{\upsilon=1}^{\infty}<\upsilon>\frac{t^\upsilon}{\upsilon!}\right)\text{o}\left(\sum_{\upsilon=1}^{\infty}<\upsilon>\frac{t^\upsilon}{\upsilon!}\right)-\cdots$$

$$\sum_{\upsilon=1}^{\infty}[\upsilon]\frac{t^{\upsilon}}{\upsilon!} = \text{o-log}\phi_X(t) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}\left[\sum_{\upsilon=1}^{\infty}\langle\upsilon\rangle\frac{t^{\upsilon}}{\upsilon!}\right]^{\text{o-}k}$$

$$= \langle 1\rangle\frac{t}{1!} + \langle 2\rangle\frac{t^2}{2!} + \langle 3\rangle\frac{t^3}{3!} + \cdots$$

$$-\frac{1}{2}\left(\langle 1\rangle\frac{t}{1!} + \langle 2\rangle\frac{t^2}{2!} + \langle 3\rangle\frac{t^3}{3!} + \cdots\right)\text{o}\left(\langle 1\rangle\frac{t}{1!}\right.$$

$$\left.+\langle 2\rangle\frac{t^2}{2!} + \langle 3\rangle\frac{t^3}{3!} + \cdots\right)$$

$$+\frac{1}{3}\left(\langle 1\rangle\frac{t}{1!} + \langle 2\rangle\frac{t^2}{2!} + \langle 3\rangle\frac{t^3}{3!} + \cdots\right)\text{o}\left(\langle 1\rangle\frac{t}{1!}\right.$$

$$\left.+\langle 2\rangle\frac{t^2}{2!} + \langle 3\rangle\frac{t^3}{3!} + \cdots\right)\text{o}\left(\langle 1\rangle\frac{t}{1!} + \langle 2\rangle\frac{t^2}{2!}\right.$$

$$\left.+\langle 3\rangle\frac{t^3}{3!} + \cdots\right)\cdots$$

$$\sum_{\upsilon=1}^{\infty}[\upsilon]\frac{t^{\upsilon}}{\upsilon!} = \langle 1\rangle t + \frac{t^2}{2}(\langle 2\rangle - \langle 11\rangle) + \frac{t^3}{3!}(\langle 3\rangle - 3\langle 12\rangle + 2\langle 111\rangle)$$

$$+\frac{t^4}{4!}(\langle 4\rangle - 4\langle 13\rangle - 3\langle 22\rangle + 12\langle 112\rangle - 6\langle 1111\rangle) + \cdots$$

In like manner denote the υth polykay of $F(x|y)$ by $([\upsilon])_y$ defining it in terms of symmetric means in the yth population by the identity

$$\text{o-log}\ \phi_{X|y}(t) \equiv \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}\left(\sum_{\upsilon=1}^{\infty}(\langle\upsilon\rangle)_y\frac{t^{\upsilon}}{\upsilon!}\right)^{\text{o-}k}$$

o-multiplication is here defined on the elements $\langle\upsilon\rangle_y$, giving

$$\langle\upsilon_1\rangle_y \text{ o}\cdots\text{o } \langle\upsilon_r\rangle_y = \langle\upsilon_1,\cdots,\upsilon_r\rangle_y$$

$$= \frac{1}{N(N-1)\cdots(N-r+1)}\sum_{j_1\neq\cdots\neq j_r}^{N} X_{ij_1}^{\upsilon_1}\cdots X_{ij_r}^{\upsilon_r}$$

$$(1)\qquad \sum_{\upsilon=1}^{\infty}([\upsilon])_y \frac{t^{\upsilon}}{\upsilon!} \equiv \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}\Big(\sum_{\upsilon=1}^{\infty}(<\upsilon>)_y \frac{t^{\upsilon}}{\upsilon!}\Big)^{o-k}$$

$$\phi_X(t) = E_y\phi_{X|y}(t)$$

$$\log \phi_X(t) = \log E_y o\text{-}e^{o\text{-}\log \phi_{X|y}(t)}$$

$$= \log E_y o\text{-}e^{\sum_{\upsilon=1}^{\infty}([\upsilon])_y \frac{t^{\upsilon}}{\upsilon!}}$$

Writing $\frac{t^{\upsilon}}{\upsilon!}$ as s_{υ}, $Eo\text{-}e^{\sum_{\upsilon=1}^{\infty}([\upsilon])_y s_{\upsilon}}$ may be identified as the moment generating function of the joint distribution of the chance variables $([1])_y, ([2])_y, \cdots$ Then

$$\sum_{\upsilon=1}^{\infty}[\upsilon]\frac{t^{\upsilon}}{\upsilon!} = \log Eo\text{-}e^{\sum_{\upsilon=1}^{\infty}([\upsilon])_y s_{\upsilon}}$$

$$\equiv \sum_{h=1}^{\infty}(-1)^{h-1}\frac{1}{h}E\Big\{\Big(1+([1])_y s_1+\frac{([1])_y^{o-2}s_1^2}{2!}+\frac{([1])_y^{o-3}s_1^3}{3!}+\cdots\Big)$$

$$o\Big(1+([2])_y s_2+\frac{([2])_y^{o-2}s_2^2}{2!}+\frac{([2])_y^{o-3}s_2^3}{3!}+\cdots\Big)$$

$$o\Big(1+([3])_y s_3+\frac{([3])_y^{o-2}s_3^2}{2!}+\frac{([3])_y^{o-3}s_3^3}{3!}+\cdots\Big)$$

$$o(\cdots\cdots\cdots\cdots\cdots\cdots\cdots)\Big\}^h$$

The polykay generating function is then defined by the identity

$$\sum_{\upsilon=1}^{\infty}\sum_{k=1}^{\infty}\;\sum_{k_1\cdots k_\upsilon \,\big|\, \substack{\Sigma ik_i=\upsilon \\ \Sigma k_i=k}} [[\underbrace{1,\cdots,1}_{k_1},\cdots,\underbrace{\upsilon,\cdots,\upsilon}_{k_\upsilon}]]\,\frac{s_1^{k_1}s_2^{k_2}\cdots s_\upsilon^{k_\upsilon}}{k_1!k_2!\cdots k_\upsilon!}$$

(2)
$$\equiv \sum_{h=1}^{\infty}(-1)^{h-1}\frac{1}{h}\Big(\sum_{\upsilon=1}^{\infty}\sum_{k=1}^{\infty}\;\sum_{k_1\cdots k_\upsilon \,\big|\, \substack{\Sigma ik_i=\upsilon \\ \Sigma k_i=k}} E([1])_y^{o-k_1}\,o\cdots o([\upsilon])_y^{o-k_\upsilon}\,\frac{s_1^{k_1}s_2^{k_2}\cdots s_\upsilon^{k_\upsilon}}{k_1!k_2!\cdots k_\upsilon!}\Big)^h$$

Corresponding to Robson's equation (3.2).

A cumulant is expressed in terms of its components as

$$[\upsilon] = \upsilon!\sum_{k=1}^{\upsilon}\;\sum_{k_1\cdots k_\upsilon \,\big|\, \substack{\Sigma ik_i=\upsilon \\ \Sigma k_i=k}} [[\underbrace{1,\cdots,1}_{k_1},\cdots,\underbrace{\upsilon,\cdots,\upsilon}_{k_\upsilon}]]\,\frac{1}{\prod_{i=1}^{\upsilon} k_i!(i!)^{k_i}}$$

just as in the infinite population case.

Cumulant components may be written in terms of expected values over subpopulations from equation (2).

For the second polykay

$$[[\,2\,]] = E([2])_y$$

$$[[1,1]] = E([1])_y\,o([1])_y - E([1])_y\,o\,E([1])_y$$

Components of the third and fourth polykay may be written similarly.

Notice that in the above derivation, the whole population is considered finite. In the case where the population is composed of an infinite number of finite subpopulations, the o-multiplication of the expected values becomes ordinary multiplication, while the o-multiplication within an expected value remains the same. In effect the power o-k in definition (1) is replaced by the power k. The second case is assumed hereafter.

The second polykay components then become

$$[[2]] = E([2])_y$$

$$[[1,1]] = E([1])_y \circ ([1])_y - E([1])_y E([1])_y$$

Polykay Components in Terms of Moments:

The first four polykays in the yth subpopulation are defined in terms of moments from equation (1) as follows:

$$([1])_y = (\langle 1 \rangle)_y$$

$$([2])_y = (\langle 2 \rangle)_y - (\langle 1,1 \rangle)_y$$

$$([3])_y = (\langle 3 \rangle)_y - 3(\langle 1,2 \rangle)_y + 2(\langle 1,1,1 \rangle)_y$$

$$([4])_y = (\langle 4 \rangle)_y - 3(\langle 2,2 \rangle)_y - 4(\langle 1,3 \rangle)_y + 12(\langle 1,1,2 \rangle)_y - 6(\langle 1,1,1,1 \rangle)_y$$

To simplify notation expected values over subpopulations will be written in angle bracket form as follows:

$$E\left\{(\langle \upsilon_{11} \cdots \upsilon_{1r_1} \rangle)_y \cdots E\;(\langle \upsilon_{t_1} \cdots \upsilon_{tr_t} \rangle)_y\right\}$$

$$\equiv \langle (\langle \upsilon_{11} \cdots \upsilon_{1r_1} \rangle) \cdots (\langle \upsilon_{t_1} \cdots \upsilon_{tr_t} \rangle) \rangle$$

Combinatorial formulas for products of symmetric means are given by Robson and Tukey as

$$\langle a \rangle \langle b \rangle = \frac{1}{N} \langle a+b \rangle + \frac{N-1}{N} \langle ab \rangle$$

$$\langle a \rangle \langle bc \rangle = \frac{1}{N} \langle a+b,c \rangle + \frac{1}{N} \langle a+c,b \rangle + \frac{N-2}{N} \langle a,b,c \rangle$$

$$\langle ab \rangle \langle cd \rangle = \frac{(N-3)(N-2)}{N(N-1)} \langle abcd \rangle + \frac{(N-2)}{N(N-1)} \langle a+c,b,d \rangle$$

$$+ \frac{(N-2)}{N(N-1)} \langle a+d,b,c \rangle + \frac{(N-2)}{N(N-1)} \langle b+c,a,d \rangle$$

$$+ \frac{(N-2)}{N(N-1)} \langle b+d,a,c \rangle + \frac{1}{N(N-1)} \langle a+c,b+d \rangle$$

$$+ \frac{1}{N(N-1)} \langle a+d,b+c \rangle$$

$$\langle a \rangle \langle bcd \rangle = \frac{(N-3)}{N} \langle abcd \rangle + \frac{1}{N} \langle a+b,c,d \rangle + \frac{1}{N} \langle a+c,b,d \rangle + \frac{1}{N} \langle a+d,b,c \rangle$$

Using the above notation and relationships, components of polykays are written in moment product form. Examples are:

$$[[1,1]] = E([1])_y \circ ([1])_y - E([1])_y E([1])_y$$

$$= E(\langle 1 \rangle)_y \circ (\langle 1 \rangle)_y - E(\langle 1 \rangle)_y E(\langle 1 \rangle)_y$$

$$= E(\langle 1,1 \rangle)_y - E(\langle 1 \rangle)_y E(\langle 1 \rangle)_y$$

$$= \langle (\langle 1,1 \rangle) \rangle - \langle (\langle 1 \rangle)(\langle 1 \rangle) \rangle$$

but

$$[[1,2]] = E([1])_y \circ ([2])_y - E([1])_y E([2])_y$$

$$= E[(\langle 1 \rangle)_y] \circ [(\langle 2 \rangle)_y - (\langle 1,1 \rangle)_y]$$

$$- E(\langle 1 \rangle)_y E[(\langle 2 \rangle)_y - (\langle 1,1 \rangle)_y]$$

$$= E(\langle 1 \rangle)_y (\langle 2 \rangle)_y - E(\langle 1 \rangle)_y (\langle 1,1 \rangle)_y$$

$$- E(\langle 1 \rangle)_y E(\langle 2 \rangle)_y + E(\langle 1 \rangle)_y E(\langle 1,1 \rangle)_y$$

$$= E[\tfrac{1}{N}(\langle 3 \rangle)_y + \tfrac{(N-1)}{N}(\langle 1,2 \rangle)_y]$$

$$- E[\tfrac{2}{N}(\langle 1,2 \rangle)_y + \tfrac{(N-2)}{N}(\langle 1,1,1 \rangle)_y]$$

$$- E(\langle 1 \rangle)_y E(\langle 2 \rangle)_y + E(\langle 1 \rangle)_y E(\langle 1,1 \rangle)_y$$

$$= \frac{1}{N} <(<3>)> + \frac{(N-3)}{N} <(<1,2>)>$$

$$- \frac{(N-2)}{N} <(<1,1,1>)> - <(<1>)(<2>)> + <(<1>)(<1,1>)>$$

Also consider

$$[[1,1]] = E([1])_y([1])_y - E([1])_y E([1])_y$$

$$= E(<1>)_y(<1>)_y - E(<1>)_y E(<1>)_y$$

$$= E[\frac{1}{N}(<2>)_y + \frac{(N-1)}{N}(<1,1>)_y] - <(<1>)(<1>)>$$

$$= \frac{1}{N} <(<2>)> + \frac{(N-1)}{N} <(<1,1>)> - <(<1>)(<1>)>$$

These are the two reasonable choices. The former one is preferred because of its neater form and it is in fact identical to the form for the infinite case.

Polykay components interms of moments have been worked out for the first four polykays disregarding o-mulitplication (i.e., $\log \phi_X = \log E_y e^{\log \phi_X|_y}$) but using finite population formulas for moment products.

Estimation-Complex Symmetric Sample Means:

The estimation problem is one of finding the best estimate of

$$E\left\{(<\upsilon_{11}\cdots\upsilon_{1r_1}>)_y\right\}\cdots E\left\{(<\upsilon_{t_1}\cdots\upsilon_{tr_t}>)_y\right\}$$

$$\equiv <(<\upsilon_{11}\cdots\upsilon_{1r_1}>)\cdots(<\upsilon_{t1}\cdots\upsilon_{tr_t}>)>$$

The statistic used is

$$E\left\{(x_{11}^{\upsilon_{11}}\cdots x_{1r_1}^{\upsilon_{1r_1}})\cdots(x_{t1}^{\upsilon_{t1}}\cdots x_{tr_t}^{\upsilon_{tr_t}})\Big| t(x)\right\}$$

$$\equiv <(<\upsilon_{11}\cdots\upsilon_{1r_1}>)\cdots(<\upsilon_{t1}\cdots\upsilon_{tr_t}>)>'$$

This is exactly Robson's equation (4.2) and is appropriate, since $E<\upsilon_1\cdots\upsilon_r>' = <\upsilon_1\cdots\upsilon_r>$.

Since complex symmetric means are unbiased, expressions for component estimates in terms of symmetric means may be obtained by putting primes on the population formulas. This should be checked. Therefore, from

$$[[2]] = <(<2>)> - <(<1,1>)>$$

and

$$[[1,1]] = \frac{1}{N} <(<2>)> + \frac{(N-1)}{N} <(<1,1>)> - <(<1>)(<1>)>$$

We have for the second cumulant

$$[[2]]' = <(<2>)>' - <(<1,1>)>'$$

and

$$[[1,1]]' = \frac{1}{N} <(<2>)>' + \frac{(N-1)}{N} <(<1,1>)>' - <(<1>)(<1>)>'$$

for estimates of the second polykay components.

Using the analysis of variance method of estimation we find that the statistics computed in the analysis of variance are $[[2]']'$ and $[[1]',[1]']'$.

The expected value of $[[2]']'$ is denoted as $[[2]']$ and that of $[[1]'[1]']'$ as $[[1]'[1]']$.

The expression for $[[1]'[1]']$ in terms of population components is derived as follows:

$$[[1]'[1]'] = E[1]'_y[1]'_y - E[1]'_y E[1]'_y$$

$$= E<1>'_y<1>'_y - E[1]'_y E[1]'_y$$

$$= E\left\{\frac{1}{n}<2>'_y + \frac{(n-1)}{n}<1,1>'_y\right\} - E[1]'_y E[1]'_y$$

$$= E\left\{\frac{1}{n}<2>_y + \frac{(n-1)}{n}<1,1>_y\right\} - E[1]_y E[1]_y \quad \text{(by unbiasedness)}$$

$$<1,1>_y = -\frac{1}{N}\left\{<2>_y - <1,1>_y\right\} + <1>_y<1>_y$$

so

$$[[1]'[1]'] = E\left[\frac{1}{n}\left\{<2>_y - <11>_y\right\} - \frac{1}{N}\left\{<2>_y - <11>_y\right\}\right] + E<1>_y<1>_y$$
$$- E[1]_y E[1]_y$$

$$= \left(\frac{1}{n} - \frac{1}{N}\right)E\left\{<2>_y - <11>_y\right\} + [[1,1]]$$

$$= \frac{1}{n}\left\{\frac{(N-n)}{N}[[2]] + n\,[[1,1]]\right\} \quad \text{as expected.}$$

Therefore

$$[[1]'[1]']' = \frac{1}{n}\left\{\frac{(N-n)}{N}[[2]]' + n\,[[1,1]]'\right\}$$

Written in terms of moments of the form

$$[[1]'[1]'] = \frac{1}{n}E<2>_y + \frac{(n-1)}{n}E<11>_y - E<1>_y E<1>_y$$

$$= \frac{1}{n}<(<2>)> + \frac{(n-1)}{n}<(<11>)> - <(<1>)(<1>)>$$

then

$$[[1]'[1]']' = \frac{1}{n}<(<2>)>' + \frac{(n-1)}{n}<(<11>)>' - <(<1>)(<1>)>'$$

Now

$$[[2]]' = <(<2>)>' - <(<11>)>'$$

so

$$[[11]]' = [[1]'[1]']' - \frac{(N-n)}{Nn}[[2]]'$$

$$= \frac{1}{n}<(<2>)>' + \frac{(n-1)}{n}<(<11>)>' - <(<1>)(<1>)>'$$

$$- \frac{(N-n)}{Nn}[<(<2>)>' - <(<11>)>']$$

$$= \frac{1}{N}<(<2>)>' + \frac{(N-1)}{N}<(<11>)>' - <(<1>)(<1>)>'$$

There are several assumptions made about the unbiasedness of several of the statistics used above which should be checked to see if they are actually true.

However, using the above method, the expected values for the analysis of third and fourth polykay statistics have been worked out.

$$[[3]] = \langle(\langle 3\rangle)\rangle - 3\langle(\langle 12\rangle)\rangle + 2\langle(\langle 111\rangle)\rangle$$

so

$$[[3]]' = \langle(\langle 3\rangle)\rangle' - 3\langle(\langle 12\rangle)\rangle' + 2\langle(\langle 111\rangle)\rangle'$$

$$[[1]'[2]'] = \frac{1}{n}\frac{(N-n)}{N}[[3]] + [[1,2]]$$

so

$$[[1,2]]' = [[1]'[2]']' - \frac{(N-n)}{nN}[[3]]'$$

$$[[1]'[2]'] = \frac{1}{n}\langle(\langle 3\rangle)\rangle + \frac{(n-3)}{n}\langle(\langle 12\rangle)\rangle - \frac{(n-2)}{n}\langle(\langle 111\rangle)\rangle$$

$$-\langle(\langle 1\rangle)(\langle 2\rangle)\rangle + \langle(\langle 1\rangle)(\langle 11\rangle)\rangle$$

so

$$[[1]'[2]']' = \frac{1}{n}\langle(\langle 3\rangle)\rangle' + \frac{(n-3)}{n}\langle(\langle 12\rangle)\rangle' - \frac{(n-2)}{n}\langle(\langle 111\rangle)\rangle'$$

$$-\langle(\langle 1\rangle)(\langle 2\rangle)\rangle' + \langle(\langle 1\rangle)(\langle 11\rangle)\rangle'$$

Hence

$$[[1,2]]' = \frac{1}{n}\langle(\langle 3\rangle)\rangle' + \frac{(n-3)}{n}\langle(\langle 1,2\rangle)\rangle' - \frac{(n-2)}{n}\langle(\langle 111\rangle)\rangle'$$

$$-\langle(\langle 1\rangle)(\langle 2\rangle)\rangle' - \frac{(N-n)}{Nn}\Big\{\langle(\langle 3\rangle)\rangle' - 3\langle(\langle 1,2\rangle)\rangle'$$

$$+2\langle(\langle 111\rangle)\rangle'\Big\} + \langle(\langle 1\rangle)(\langle 11\rangle)\rangle'$$

$$= \frac{1}{N}\langle(\langle 3\rangle)\rangle' + \frac{(N-3)}{N}\langle(\langle 12\rangle)\rangle' - \frac{(N-2)}{N}\langle(\langle 111\rangle)\rangle'$$

$$-\langle(\langle 1\rangle)(\langle 2\rangle)\rangle' + \langle(\langle 1\rangle)(\langle 11\rangle)\rangle'$$

$$[[1]'[1]'[1]'] = \left(\frac{N^2-n^2}{N^2n^2}\right)[[3]] + \frac{3(N-n)}{nN}[[1,2]] + [[111]]$$

so

$$[[111]]' = [[1]'[1]'[1]']' - \frac{(N^2-n^2)}{N^2n^2}[[3]]' - \frac{3(N-n)}{nN}[[12]]'$$

$$[[3]]' = <(<3>)>' - 3<(<12>)>' + 2<(<111>)>'$$

$$[[1,2]]' = \frac{1}{N}<(<3>)>' + \frac{(N-3)}{N}<(<12>)>' - \frac{(N-2)}{N}<(<111>)>'$$

$$-<(<1>)(<2>)>' + <(<1>)(<11>)>'$$

$$[[1]'[1]'[1]']' = \frac{1}{n^2}<(<3>)>' + \frac{3(n-1)}{n^2}<(<12>)>'$$

$$+\frac{(n-1)(n-2)}{n^2}<(<111>)>' - \frac{3}{n}<(<1>)(<2>)>'$$

$$-\frac{3(n-1)}{n}<(<1>)(<11>)>'$$

$$+2<(<1>)(<1>)(<1>)>'$$

$$[[111]]' = \frac{1}{n^2}<(<3>)>' + \frac{3(n-1)}{n^2}<(<12>)>'$$

$$+\frac{(n-1)(n-2)}{n^2}<(<111>)>'$$

$$-\frac{3}{n}<(<1>)(<2>)>' - \frac{3(n-1)}{n}<(<1>)(<11>)>'$$

$$+2<(<1>)(<1>)(<1>)>'$$

$$-\frac{(N^2-n^2)}{N^2n^2}\Big\{<(<3>)>' - 3<(<12>)>'$$

$$+2<(<111>)>'\Big\} - \frac{3(N-n)}{nN}\Big\{\frac{1}{N}<(<3>)>'$$

$$+\frac{(N-3)}{N}<(<12>)>' - \frac{(N-2)}{N}<(<111>)>'$$

$$-<(<1>)(<2>)>' + <(<1>)(<11>)>'\Big\}$$

$$= \frac{(4n-3N)}{nN^2}<(<3>)>' + \frac{3(3N+nN-4n)}{nN^2}<(<12>)>'$$

$$+\frac{(N^2n-3nN-6N+8n)}{nN^2}<(<111>)>'$$

$$-\frac{3}{N}<(<1>)(<2>)>'$$

$$-\frac{3(N-1)}{N}<(<1>)(<11>)>'+2<(<1>)(<1>)(<1>)>'$$

Conversion Formulas: Complex Means to Simple:

The formulas to convert complex symmetric means into simple symmetric means are those given by Robson in two stages.

For $[[1]'[1]']'=\frac{1}{n}<(<2>)>'+\frac{(n-1)}{n}<(<11>)>'-<(<1>)(<1>)>'$ the computing formula is $\frac{c}{c-1}<<1>'<1>'>'-\frac{c}{c-1}<<1>'>'<<1>'>'$ which is seen to be the same as in the infinite case. $[[1]'[1]'[1]']'$ has also been derived as

$$\frac{c^2}{(c-1)(c-2)}<<1>'<1>'<1>'>'-\frac{3c^2}{(c-1)(c-2)}<<1>'>'<<1>'<1>'>$$

$$+\frac{2c^2}{(c-1)(c-2)}<<1>'>'<<1>'>'<<1>'>'$$

again the same as for the infinite case.

First Stage:

$$<(<\upsilon_{11}\cdots\upsilon_{1r_1}>)>'=<<\upsilon_{11}\cdots\upsilon_{1r_1}>'>'$$

$$<(<\upsilon_{11}\cdots\upsilon_{1r_1}>)(<\upsilon_{21}\cdots\upsilon_{2r_2}>)>'$$

$$=-\frac{1}{c-1}<<\upsilon_{11}\cdots\upsilon_{1r_1}>'<\upsilon_{21}\cdots\upsilon_{2r_2}>'>'$$

$$+\frac{c}{c-1}<<\upsilon_{11}\cdots\upsilon_{1r_1}>'>'<<\upsilon_{21}\cdots\upsilon_{2r_2}>'>'$$

$$<(<\upsilon_{11}\cdots\upsilon_{1r_1}>)(<\upsilon_{21}\cdots\upsilon_{2r_2}>)(<\upsilon_{31}\cdots\upsilon_{3r_3}>)>'$$

$$=\frac{2}{(c-1)(c-2)}<<\upsilon_{11}\cdots\upsilon_{1r_1}>'<\upsilon_{21}\cdots\upsilon_{2r_2}>'<\upsilon_{31}\cdots\upsilon_{3r_3}>'>'$$

$$-\frac{c}{(c-1)(c-2)}\Big\{<<\upsilon_{11}\cdots\upsilon_{1r_1}>'>'<<\upsilon_{21}\cdots\upsilon_{2r_2}>'<\upsilon_{31}\cdots\upsilon_{3r_3}>'>'$$

$$+<<\upsilon_{21}\cdots\upsilon_{2r_2}>'>'<<\upsilon_{11}\cdots\upsilon_{1r_1}>'<\upsilon_{31}\cdots\upsilon_{3r_3}>'>'$$

$$+<<\upsilon_{31}\cdots\upsilon_{3r_3}>'>'<<\upsilon_{11}\cdots\upsilon_{1r_1}>'<\upsilon_{21}\cdots\upsilon_{2r_2}>'>'\Big\}$$

$$+\frac{c^2}{(c-1)(c-2)} << \upsilon_{11}\cdots\upsilon_{1r_1}>'>' << \upsilon_{21}\cdots\upsilon_{2r_2}>'>'<< \upsilon_{31}\cdots\upsilon_{3r_3}>'>'$$

Second Stage:

$$<\upsilon_{21},\upsilon_{22}>' = \frac{n}{(n-1)}<\upsilon_{21}>'<\upsilon_{22}>' - \frac{1}{(n-1)}<\upsilon_{21}+\upsilon_{22}>'$$

$$<\upsilon_{31},\upsilon_{32},\upsilon_{33}>' = \frac{1}{n(n-1)(n-2)}\Big[(-1)^2 n 2!<\upsilon_{31}+\upsilon_{32}+\upsilon_{33}>'$$

$$+(-1)^1 n^2\Big\{<\upsilon_{31}>'<\upsilon_{32}+\upsilon_{33}>'+<\upsilon_{32}>'<\upsilon_{31}+\upsilon_{33}>'$$

$$+<\upsilon_{33}>'<\upsilon_{31}+\upsilon_{32}>'\Big\}+(-1)^0 n^3<\upsilon_{31}>'<\upsilon_{32}>'<\upsilon_{33}>'\Big]$$

$$= \frac{2}{(n-1)(n-2)}<\upsilon_{31}+\upsilon_{32}+\upsilon_{33}>' - \frac{n}{(n-1)(n-2)}\Big\{<\upsilon_{31}>'<\upsilon_{32}+\upsilon_{33}>'$$

$$+<\upsilon_{32}>'<\upsilon_{31}+\upsilon_{33}>'+<\upsilon_{33}>'<\upsilon_{31}+\upsilon_{32}>'\Big\}$$

$$+\frac{n^2}{(n-1)(n-2)}<\upsilon_{31}>'<\upsilon_{32}>'<\upsilon_{33}>'$$

These refer directly to Tukey's formulas.

$$<\upsilon_{i1},\upsilon_{i2},\upsilon_{i3},\upsilon_{i4}>' = \frac{1}{n(n-1)(n-2)(n-3)}\Big[(-1)^3 n(4-1)!<\upsilon_{i1}+\upsilon_{i2}+\upsilon_{i3}+\upsilon_{i4}>'$$

$$+(-1)^2 n^2\Big\{(3-1)!(1-1)!\big[<\upsilon_{i1}>'<\upsilon_{i2}+\upsilon_{i3}+\upsilon_{i4}>'$$

$$+<\upsilon_{i2}>'<\upsilon_{i1}+\upsilon_{i3}+\upsilon_{i4}>'+<\upsilon_{i3}>'<\upsilon_{i1}+\upsilon_{i2}+\upsilon_{i4}>'$$

$$+<\upsilon_{i4}>'<\upsilon_{i1}+\upsilon_{i2}+\upsilon_{i3}>'\big]\Big\}$$

$$+(-1)^1 n^3(2-1)!(1-1)!^2\Big\{<\upsilon_{i1}>'<\upsilon_{i2}>'<\upsilon_{i3}+\upsilon_{i4}>'$$

$$+<\upsilon_{i1}>'<\upsilon_{i3}>'<\upsilon_{i2}+\upsilon_{i4}>'+<\upsilon_{i1}>'<\upsilon_{i4}>'<\upsilon_{i2}+\upsilon_{i3}>'$$

$$+<\upsilon_{i2}>'<\upsilon_{i3}>'<\upsilon_{i1}+\upsilon_{i4}>'+<\upsilon_{i2}>'<\upsilon_{i4}>'<\upsilon_{i1}+\upsilon_{i3}>'$$

$$+<\upsilon_{i3}>'<\upsilon_{i4}>'<\upsilon_{i1}+\upsilon_{i2}>'\Big\} +n^2\Big\{<\upsilon_{i1}+\upsilon_{i2}>'<\upsilon_{i3}+\upsilon_{i4}>'$$

$$+\langle \upsilon_{i1}+\upsilon_{i3}\rangle'\langle \upsilon_{i2}\ \upsilon_{i4}\rangle' + \langle \upsilon_{i1}+\upsilon_{i4}\rangle'\langle \upsilon_{i2}+\upsilon_{i3}\rangle' \}$$

$$+(-1)^{0}n^{4}\mu!\,\langle \upsilon_{i1}\rangle'\langle \upsilon_{i2}\rangle'\langle \upsilon_{i3}\rangle'\langle \upsilon_{i4}\rangle']$$

$$= \frac{-6}{(n-1)(n-2)(n-3)}\langle \upsilon_{i1}+\upsilon_{i2}+\upsilon_{i3}+\upsilon_{i4}\rangle'$$

$$+ \frac{2n}{(n-1)(n-2)(n-3)}[\langle \upsilon_{i1}\rangle'\langle \upsilon_{i2}+\upsilon_{i3}+\upsilon_{i4}\rangle'$$

$$+\langle \upsilon_{i2}\rangle'\langle \upsilon_{i1}+\upsilon_{i3}+\upsilon_{i4}\rangle' + \langle \upsilon_{i3}\rangle'\langle \upsilon_{i1}+\upsilon_{i2}+\upsilon_{i4}\rangle'$$

$$+\langle \upsilon_{i4}\rangle'\langle \upsilon_{i1}+\upsilon_{i2}+\upsilon_{i3}\rangle']$$

$$- \frac{n^2}{(n-1)(n-2)(n-3)}[\langle \upsilon_{i1}\rangle'\langle \upsilon_{i2}\rangle'\langle \upsilon_{i3}+\upsilon_{i4}\rangle'$$

$$+\langle \upsilon_{i1}\rangle'\langle \upsilon_{i3}\rangle'\langle \upsilon_{i2}+\upsilon_{i4}\rangle' + \langle \upsilon_{i1}\rangle'\langle \upsilon_{i4}\rangle'\langle \upsilon_{i2}+\upsilon_{i3}\rangle'$$

$$+\langle \upsilon_{i2}\rangle'\langle \upsilon_{i3}\rangle'\langle \upsilon_{i1}+\upsilon_{i4}\rangle' + \langle \upsilon_{i2}\rangle'\langle \upsilon_{i4}\rangle'\langle \upsilon_{i1}+\upsilon_{i3}\rangle'$$

$$+\langle \upsilon_{i3}\rangle'\langle \upsilon_{i4}\rangle'\langle \upsilon_{i1}+\upsilon_{i2}\rangle']$$

$$+ \frac{n}{(n-1)(n-2)(n-3)}[\langle \upsilon_{i1}+\upsilon_{i2}\rangle'\langle \upsilon_{i3}+\upsilon_{i4}\rangle'$$

$$+\langle \upsilon_{i1}+\upsilon_{i3}\rangle'\langle \upsilon_{i2}+\upsilon_{i4}\rangle' + \langle \upsilon_{i1}+\upsilon_{i4}\rangle'\langle \upsilon_{i2}+\upsilon_{i3}\rangle']$$

$$+ \frac{n^3}{(n-1)(n-2)(n-3)}\langle \upsilon_{i1}\rangle'\langle \upsilon_{i2}\rangle'\langle \upsilon_{i3}\rangle'\langle \upsilon_{i4}\rangle'$$

From Tukey's ab formula

$$\langle ab\rangle = \frac{N}{(N-1)}\langle a\rangle\langle b\rangle - \frac{1}{(N-1)}\langle a+b\rangle$$

$$\langle abc\rangle = \frac{N}{(N-2)}\langle a\rangle\langle bc\rangle - \frac{1}{(N-2)}\langle a+b,c\rangle - \frac{1}{(N-2)}\langle a+c,b\rangle$$

applying the above expression to each term

$$< abc > = \frac{N}{(N-2)} < a > \left[\frac{N}{(N-1)} < b > < c > - \frac{1}{(N-1)} < b+c > \right]$$

$$- \frac{1}{(N-2)} \left[\frac{N}{(N-1)} < a+b > < c > - \frac{1}{(N-1)} < a+b+c > \right]$$

$$- \frac{1}{N-2} \left[\frac{N}{N-1} < a+c > < b > - \frac{1}{(N-1)} < a+b+c > \right]$$

$$= \frac{N^2}{(N-1)(N-2)} < a > < b > < c > - \frac{N}{(N-1)(N-2)} < a > < b+c >$$

$$- \frac{N}{(N-1)(N-2)} < b > < a+c > - \frac{N}{(N-1)(N-2)} < c > < a+b >$$

$$+ \frac{2}{(N-1)(N-2)} < a+b+c >$$

$$< abcd > = \frac{N^3}{(N-1)(N-2)(N-3)} < a > < b > < c > < d >$$

$$+ \frac{2N}{(N-1)(N-2)(N-3)} [< a > < b+c+d > + < b > < a+c+d > + < c > < a+b+d >$$

$$+ < d > < a+b+c >] + \frac{N}{(N-1)(N-2)(N-3)} [< a+b > < c+d > + < a+c > < b+d >$$

$$+ < a+d > < b+c >] - \frac{N^2}{(N-1)(N-2)(N-3)} [< a > < b > < c+d >$$

$$+ < a > < c > < b+d > + < a > < d > < b+c > + < a+b > < c > < d >$$

$$+ < a+c > < b > < d > + < a+d > < b > < c >] - \frac{6}{(N-1)(N-2)(N-3)} < a+b+c+d >$$

From this it is known that υ formulas and Tukey's formulas are the same.

Derivation of Conversion Formulas:

$$[[1]'[1]']' = \left(\frac{1}{n} - \frac{1}{N}\right) [[2]]' + [[11]]'$$

$$[[2]]' = < (<2>) >' - < (<11>) >'$$

$$[[11]]' = \frac{1}{N} < (<2>) >' + \frac{(N-1)}{N} < (<11>) >' - < (<1>)(<1>) >'$$

so

$$[[1]'[1]']' = \left(\frac{1}{n} - \frac{1}{N}\right) \left\{ < (<2>) >' - < (<11>) >' \right\}$$

$$+\frac{1}{N}<(<2>)>'+\frac{(N-1)}{N}<(<11>)>'-<(<1>)(<1>)>'$$

$$=\frac{1}{n}<(<2>)>'+\frac{(n-1)}{n}<(<11>)>'-<(<1>)(<1>)>'$$

$$=\frac{1}{n}<<2>'>'+\frac{(n-1)}{n}<<11>'>'-\frac{c}{c-1}<<1>'>'<<1>'>'$$

$$+\frac{1}{c-1}<<1>'<1>'>'$$

$$=\frac{1}{n}<<2>'>'+\frac{(n-1)}{n}\left(\frac{n}{(n-1)}<<1>'<1>'>'-\frac{1}{(n-1)}<<2>'>'\right)$$

$$-\frac{c}{c-1}<<1>'>'<<1>'>'+\frac{1}{c-1}<<1>'<1>'>'$$

$$=\frac{c}{(c-1)}<<1>'<1>'>'-\frac{c}{c-1}<<1>'>'<<1>'>'$$

For

$$[[2]]' = <(<2>)>'-<(<11>)>'$$

$$=<<2>'>'-\frac{n}{(n-1)}<<1>'<1>'>'+\frac{1}{(n-1)}<<2>'>'$$

$$=\frac{n}{(n-1)}<<2>'>'-\frac{n}{(n-1)}<<1>'<1>'>' .$$

For

$$[[1]'[1]'[1]']'=\left(\frac{1}{n^2}-\frac{1}{N^2}\right)[[3]]'+\frac{3(N-n)}{nN}[[12]]'+[[111]]'$$

$$[[3]]' = <(<3>)>'-3<(<12>)>'+2<(<111>)>'$$

$$[[12]]' = \frac{1}{N}<(<3>)>'+\frac{(N-3)}{N}<(<12>)>'-\frac{(N-2)}{N}<(<111>)>'$$

$$-<(<1>)(<2>)>'+<(<1>)(<11>)>'$$

$$[[111]]' = \frac{(4n-3N)}{nN^2}<(<3>)>'+\frac{3(3N+nN-4n)}{nN^2}<(<12>)>'$$

$$+\frac{(N^2n-3nN-6N+8n)}{nN^2}<(<111>)>'-\frac{3}{N}<(<1>)(<2>)>'$$

$$-\frac{3(N-1)}{N}<(<1>)(<11>)>'+2<(<1>)(<1>)(<1>)>'$$

so

$$[[1]'[1]'[1]']' = \left(\frac{1}{n^2} - \frac{1}{N^2}\right)\left\{\langle(\langle 3\rangle)\rangle' - 3\langle(\langle 12\rangle)\rangle' + 2\langle(\langle 111\rangle)\rangle'\right\}$$

$$+ \frac{3(N-n)}{nN}\left\{\frac{1}{N}\langle(\langle 3\rangle)\rangle' + \frac{(N-3)}{N}\langle(\langle 12\rangle)\rangle' - \frac{(N-2)}{N}\langle(\langle 111\rangle)\rangle'\right.$$

$$\left. - \langle(\langle 1\rangle)(\langle 2\rangle)\rangle' + \langle(\langle 1\rangle)(\langle 11\rangle)\rangle'\right\}$$

$$+ \frac{(4n-3N)}{nN^2}\langle(\langle 3\rangle)\rangle' + \frac{3(3N+nN-4n)}{nN^2}\langle(\langle 12\rangle)\rangle'$$

$$+ \frac{(N^2n-3nN-6N+8n)}{nN^2}\langle(\langle 111\rangle)\rangle' - \frac{3}{N}\langle(\langle 1\rangle)(\langle 2\rangle)\rangle'$$

$$- \frac{3(N-1)}{N}\langle(\langle 1\rangle)(\langle 11\rangle)\rangle' + 2\langle(\langle 1\rangle)(\langle 1\rangle)(\langle 1\rangle)\rangle'$$

$$= \frac{1}{n^2}\langle(\langle 3\rangle)\rangle' + \frac{3(n-1)}{n^2}\langle(\langle 12\rangle)\rangle' + \frac{(n-1)(n-2)}{n^2}\langle(\langle 111\rangle)\rangle'$$

$$- \frac{3}{n}\langle(\langle 1\rangle)(\langle 2\rangle)\rangle' - \frac{3(n-1)}{n}\langle(\langle 1\rangle)(\langle 11\rangle)\rangle'$$

$$+2\langle(\langle 1\rangle)(\langle 1\rangle)(\langle 1\rangle)\rangle'$$

$$= \frac{1}{n^2}\langle\langle 3\rangle'\rangle' + \frac{3(n-1)}{n^2}\langle\langle 12\rangle'\rangle' + \frac{(n-1)(n-2)}{n^2}\langle\langle 111\rangle'\rangle'$$

$$+ \frac{3}{n(c-1)}\langle\langle 1\rangle'\langle 2\rangle'\rangle' - \frac{3c}{n(c-1)}\langle\langle 1\rangle'\rangle'\langle\langle 2\rangle'\rangle'$$

$$+ \frac{3(n-1)}{n(c-1)}\langle\langle 1\rangle'\langle 11\rangle'\rangle' - \frac{3(n-1)c}{n(c-1)}\langle\langle 1\rangle'\rangle'\langle\langle 11\rangle'\rangle'$$

$$+ \frac{4}{(c-1)(c-2)}\langle\langle 1\rangle'\langle 1\rangle'\langle 1\rangle'\rangle'$$

$$- \frac{6c}{(c-1)(c-2)}\langle\langle 1\rangle'\rangle'\langle\langle 1\rangle'\langle 1\rangle'\rangle'$$

$$+ \frac{2c^2}{(c-1)(c-2)}\langle\langle 1\rangle'\rangle'\langle\langle 1\rangle'\rangle'\langle\langle 1\rangle'\rangle'$$

$$= \frac{1}{n^2}\langle\langle 3\rangle'\rangle' + \frac{3(n-1)}{n^2}\left\{\frac{n}{(n-1)}\langle\langle 1\rangle'\langle 2\rangle'\rangle' - \frac{1}{(n-1)}\langle\langle 3\rangle'\rangle'\right\}$$

$$+ \frac{(n-1)(n-2)}{n^2}\left\{\frac{2}{(n-1)(n-2)}\langle\langle 3\rangle'\rangle' - \frac{3n}{(n-1)(n-2)}\langle\langle 1\rangle'\langle 2\rangle'\rangle'\right.$$

$$
\begin{aligned}
& + \frac{n^2}{(n-1)(n-2)} \langle\langle 1 \rangle' \langle 1 \rangle' \langle 1 \rangle' \rangle' \Big\} \\
& + \frac{3}{n(c-1)} \langle\langle 1 \rangle' \langle 2 \rangle' \rangle' - \frac{3c}{n(c-1)} \langle\langle 1 \rangle' \rangle' \langle\langle 2 \rangle' \rangle' \\
& + \frac{3(n-1)}{n(c-1)} \left\{ \frac{n}{(n-1)} \langle\langle 1 \rangle' \langle 1 \rangle' \langle 1 \rangle' \rangle' - \frac{1}{(n-1)} \langle\langle 1 \rangle' \langle 2 \rangle' \rangle' \right\} \\
& - \frac{3c(n-1)}{n(c-1)} \langle\langle 1 \rangle' \rangle' \Big\{ \frac{n}{(n-1)} \langle\langle 1 \rangle' \langle 1 \rangle' \rangle' \\
& - \frac{1}{(n-1)} \langle\langle 2 \rangle' \rangle' \Big\} + \frac{4}{(c-1)(c-2)} \langle\langle 1 \rangle' \langle 1 \rangle' \langle 1 \rangle' \rangle' \\
& - \frac{6c}{(c-1)(c-2)} \langle\langle 1 \rangle' \rangle' \langle\langle 1 \rangle' \langle 1 \rangle' \rangle' \\
& + \frac{2c^2}{(c-1)(c-2)} \langle\langle 1 \rangle' \rangle' \langle\langle 1 \rangle' \rangle' \langle\langle 1 \rangle' \rangle' \\
= {} & \frac{c^2}{(c-1)(c-2)} \langle\langle 1 \rangle' \langle 1 \rangle' \langle 1 \rangle' \rangle' \\
& - \frac{3c^2}{(c-1)(c-2)} \langle\langle 1 \rangle' \rangle' \langle\langle 1 \rangle' \langle 1 \rangle' \rangle' \\
& + \frac{2c^2}{(c-1)(c-2)} \langle\langle 1 \rangle' \rangle' \langle\langle 1 \rangle' \rangle' \langle\langle 1 \rangle' \rangle' \ .
\end{aligned}
$$

Note that all means to powers higher than one cancel out, so that we only need to be concerned with those means which contain powers no higher than one.

Expected values of some statistics.

$$[2] = [[2]] + [[11]]$$

$$[[1]'[1]'] = E[1]'[1]' - E[1]'E[1]'$$

$$= E<1>'<1>' - E[1]'E[1]'$$

$$= E\frac{1}{n}\left\{<2> + (n-1)<11>\right\} - E[1]'E[1]'$$

$$= \frac{1}{n}\left\{<<2>> - <<11>>\right\} + E<11> - E[1]'E[1]'$$

$$= \frac{1}{n}[[2]] + \frac{1}{(N-1)}E\left\{N<1><1> - <2>\right\} - E[1]'E[1]'$$

$$= \frac{1}{n}[[2]] - \frac{1}{(N-1)}<<2>> + \frac{1}{(N-1)}E<1><1> + [[1,1]]$$

$$= \frac{1}{n}[[2]] - \frac{1}{(N-1)}\left\{<<2>> - \frac{1}{N}<<2>> - \frac{(N-1)}{N}<<11>>\right\} + [[11]]$$

$$= \frac{1}{n}[[2]] - \frac{1}{(N-1)}\frac{(N-1)}{N}[[2]] + [[11]]$$

$$= \left(\frac{1}{n} - \frac{1}{N}\right)[[2]] + [[11]].$$

$$[3] = [[3]] + 3[[1,2]] + [[1,1,1]]$$

$$[[1]'[2]'] = E[1]'[2]' - E[1]'E[2]'$$

$$= E<1>'\left\{<2>' - <11>'\right\} - E[1]'E[2]'$$

$$= E<1>'<2>' - E<1>'<11>' - E[1]'E[2]'$$

$$= E\left\{\frac{1}{n}<3> + \frac{(n-1)}{n}<1,2> - \frac{(n-2)}{n}<111> - \frac{2}{n}<12>\right\} - E[1]'E[2]'$$

$$= \frac{1}{n}E\left\{<3> + (n-3)<1,2> - (n-2)<111>\right\} - E[1]'E[2]'$$

$$= \frac{1}{n}[[3]] + <<12>> - <<111>> - E[1]'E[2]'$$

$$= \frac{1}{n}[[3]] + \frac{N}{(N-1)} < \left\{ <1><2> - \frac{1}{N}<3> \right\} > - < \left\{ \frac{N}{(N-2)} <1><11> \right.$$

$$\left. - \frac{2}{(N-2)} <12> \right\} > - E[1]'E[2]'$$

$$= \frac{1}{n}[[3]] - \frac{1}{(N-1)} <<3>> + < \left\{ \frac{1}{(N-1)} <1><2> - \frac{2}{(N-2)} <1><11> \right.$$

$$\left. + \frac{2}{(N-2)} <12> \right\} > + [[1,2]]$$

$$= \frac{1}{n}[[3]] - \frac{1}{(N-1)} <<3>> + \frac{1}{(N-1)} < \left\{ \frac{(N-1)}{N} <12> + \frac{1}{N}<3> \right\} >$$

$$- \frac{2}{(N-2)} < \left\{ \frac{N-2}{N} <111> + \frac{2}{N}<12> \right\} > + \frac{2}{N-2} <<12>> + [[12]]$$

$$= \frac{1}{n}[[3]] - \frac{1}{(N-1)} <<3>> + \frac{1}{N(N-1)} <<3>> + \left\{ \frac{1}{N} - \frac{4}{N(N-2)} \right.$$

$$\left. + \frac{2}{(N-2)} \right\} <<12>> - \frac{2}{N} <<111>> + [[12]]$$

$$= \frac{1}{n}[[3]] - \frac{1}{N} \left\{ <<3>> - 3 <<12>> + 2 <<111>> \right\} + [[12]]$$

$$= \left(\frac{1}{n} - \frac{1}{N}\right)[[3]] + [[1,2]] .$$

$$[[1]'[1]'[1]'] = E[1]'[1]'[1]' - 3E[1]'E[1]'[1]' + 2E[1]'E[1]'E[1]'$$

$$= E<1>'<1>'<1>' - 3E<1>'E<1>'<1>' + 2E[1]'E[1]'E[1]'$$

$$= E\left\{ \frac{(n-1)(n-2)}{n^2} <111> + \frac{3(n-1)}{n^2} <12> + \frac{1}{n^2}<3> \right\} - 3E<1>E\left\{ \frac{(n-1)}{n} <11> \right.$$

$$\left. + \frac{1}{n}<2> \right\} + 2E[1]'E[1]'E[1]'$$

$$= \frac{1}{n^2}[[3]] + E\left\{ \frac{(n-3)}{n} <111> + \frac{3}{n}<12> \right\} - \frac{3(n-1)}{n} E<1>E<11>$$

$$- \frac{3}{n} E<1>E<2> + 2E[1]'E[1]'E[1]'$$

$$= \frac{1}{n^2}[[3]] + \frac{(n-3)}{n} E\left\{\frac{N^2}{(N-1)(N-2)}<1><1><1> - \frac{3N}{(N-1)(N-2)}<1><2>\right.$$

$$\left. + \frac{2}{(N-1)(N-2)}<3>\right\} + \frac{3}{n}\frac{N}{(N-1)} E\left\{<1><2> - \frac{1}{N}<3>\right\}$$

$$- \frac{3(n-1)}{n} E<1>E\left\{\frac{N}{(N-1)}<1><1> - \frac{1}{(N-1)}<2>\right\} - \frac{3}{n} E<1>E<2>$$

$$+2E[1]'E[1]'E[1]'$$

$$= \frac{1}{n^2}[[3]] + \frac{(2n-3N)}{n(N-1)(N-2)} <<3>> + \frac{(3nN-2n-3N^2)}{n(N-1)(N-2)} E<1><1><1>$$

$$- \frac{3N(n-N-1)}{n(N-1)(N-2)} E<1><2> + \frac{3(N-n)}{n(N-1)} E<1>E<1><1>$$

$$+ \frac{3(n-N)}{n(N-1)} E<1>E<2> + [[111]]$$

$$= \frac{1}{n^2}[[3]] + \frac{(2n-3N)}{n(N-1)(N-2)} <<3>> + \frac{(3nN-2n-3N^2)}{n(N-1)(N-2)} E\left\{\frac{(N-1)(N-2)}{N^2}<111>\right.$$

$$\left. + \frac{3(N-1)}{N^2}<12> + \frac{1}{N^2}<3>\right\} - \frac{3N(n-N-1)}{n(N-1)(N-2)} E\left\{\frac{(N-1)}{N}<12> + \frac{1}{N}<3>\right\}$$

$$+ \frac{3(N-n)}{n(N-1)} E<1>E\left\{\frac{(N-1)}{N}<11> + \frac{1}{N}<2>\right\} + \frac{3(n-N)}{n(N-1)} E<1>E<2>$$

$$+ [[111]]$$

$$= \frac{1}{n^2}[[3]] - \frac{1}{N^2}<<3>> + \frac{3(3nN-2n-3N^2-nN^2+N^3+N^2)}{nN^2(N-2)}<<12>>$$

$$- \frac{3(N-n)}{nN}<<1><2>> + \frac{3(N-n)}{nN}<<1><11>> - \left\{\frac{2}{N^2}\right.$$

$$\left. + \frac{3(N-n)}{Nn}\right\}<<111>> + [[111]]$$

$$= \frac{1}{n^2}[[3]] - \frac{1}{N^2}[[3]] + \frac{3(N-n)}{nN}[[12]] + [[111]]$$

$$= \left(\frac{1}{n^2} - \frac{1}{N^2}\right)[[3]] + 3\left(\frac{1}{n} - \frac{1}{N}\right)[[12]] + [[111]]$$

[4] $$= [[4]] + 3[[22]] + 4[[13]] + 6[[112]] + [[1111]]$$

$$[[2]'[2]'] = E[2]'[2]' - E[2]'E[2]'$$

$$= E\left\{ \langle 2\rangle' - \langle 11\rangle' \right\}^2 - E[2]'E[2]'$$

$$= E\langle 2\rangle\langle 2\rangle - 2E\langle 2\rangle\langle 11\rangle + E\langle 11\rangle\langle 11\rangle - E[2]'E[2]'$$

$$= E\left\{ \frac{(n-1)}{n}\langle 22\rangle + \frac{1}{n}\langle 4\rangle \right\} - 2E\left\{ \frac{(n-2)}{n}\langle 112\rangle + \frac{2}{n}\langle 13\rangle \right\}$$

$$+ E\left\{ \frac{(n-2)(n-3)}{n(n-1)}\langle 1111\rangle + \frac{4(n-2)}{n(n-1)}\langle 112\rangle + \frac{2}{n(n-1)}\langle 22\rangle \right\} - E[2]'E[2]'$$

$$= \frac{1}{n}\langle\langle 4\rangle\rangle - \frac{3}{n}\langle\langle 22\rangle\rangle - \frac{4}{n}\langle\langle 13\rangle\rangle + \frac{12}{n}\langle\langle 112\rangle\rangle - \frac{6}{n}\langle\langle 1111\rangle\rangle$$

$$+ \frac{(n+1)}{(n-1)} E\langle 22\rangle - \frac{2(n+1)}{(n-1)} E\langle 112\rangle + \frac{(n+1)}{(n-1)} E\langle 1111\rangle - E[2]'E[2]'$$

$$= \frac{1}{n}[[4]] + \frac{(n+1)}{(n-1)}\left\{ E\langle 22\rangle - 2E\langle 112\rangle + E\langle 1111\rangle \right\} - E[2]'E[2]'$$

$$= \frac{1}{n}[[4]] + \frac{(n+1)}{(n-1)}\Big[\frac{N}{(N-1)} E\left\{ \langle 2\rangle\langle 2\rangle - \frac{1}{N}\langle 4\rangle \right\}$$

$$- \frac{2N}{(N-2)} E\left\{ \langle 2\rangle\langle 11\rangle - \frac{2}{N}\langle 13\rangle \right\} + \frac{N(N-1)}{(N-2)(N-3)} E\Big\{ \langle 11\rangle\langle 11\rangle$$

$$- \frac{4(N-2)}{N(N-1)}\langle 112\rangle - \frac{2}{N(N-1)}\langle 22\rangle \Big\} \Big] - E[2]'E[2]'$$

$$= \frac{1}{n}[[4]] - \frac{(n+1)}{(N-1)(n-1)}\langle\langle 4\rangle\rangle + \left\{ \frac{(n+1)N}{(n-1)(N-1)} - 1 \right\} E\langle 2\rangle\langle 2\rangle$$

$$+ \left\{ 2 - \frac{2N(n+1)}{(N-2)(n-1)} \right\} E\langle 2\rangle\langle 11\rangle + \frac{4(n+1)}{(N-2)(n-1)} E\langle 13\rangle$$

$$+ \left\{ \frac{(n+1)N(N-1)}{(n-1)(N-2)(N-3)} - 1 \right\} E\langle 11\rangle\langle 11\rangle - \frac{4(n+1)}{(n-1)(N-3)} E\langle 112\rangle$$

$$- \frac{2(n+1)}{(n-1)(N-2)(N-3)} E\langle 22\rangle + [[22]]$$

$$= \frac{1}{n}[[4]] - \frac{(n+1)}{(N-1)(n-1)}\langle\langle 4\rangle\rangle + \frac{(2N+n-1)}{(n-1)(N-1)} E\langle 2\rangle\langle 2\rangle$$

$$+ \frac{4(1-n-N)}{(N-2)(n-1)} E\langle 2\rangle\langle 11\rangle + \frac{4(n+1)}{(N-2)(n-1)} E\langle 13\rangle$$

$$+ \frac{(4nN+2N^2-6N-6n+6)}{(n-1)(N-2)(N-3)} E\langle 11\rangle\langle 11\rangle - \frac{4(n+1)}{(n-1)(N-3)} E\langle 112\rangle$$

$$- \frac{2(n+1)}{(n-1)(N-2)(N-3)} E <22> + [[22]]$$

$$= \frac{1}{n}[[4]] - \frac{(n+1)}{(N-1)(n-1)} \ll 4 \gg + \frac{(2N+n-1)}{(n-1)(N-1)} E\left\{\frac{(N-1)}{N} <22> + \frac{1}{N}<4>\right\}$$

$$+ \frac{4(1-n-N)}{(n-1)(N-2)} E\left\{\frac{(N-2)}{N}<112> + \frac{2}{N}<13>\right\} + \frac{4(n+1)}{(n-1)(N-2)} E<13>$$

$$+ \frac{(4nN+2N^2-6N-6n+6)}{(n-1)(N-2)(N-3)} E\left\{\frac{(N-2)(N-3)}{N(N-1)}<1111> + \frac{4(N-2)}{N(N-1)}<112>\right.$$

$$\left. + \frac{2}{N(N-1)}<22>\right\} - \frac{4(n+1)}{(n-1)(N-3)} E<112> - \frac{2(n+1)}{(n-1)(N-2)(N-3)} E<22>$$

$$+ [[22]]$$

$$= \frac{1}{n}[[4]] - \frac{1}{N}\ll 4\gg + \frac{3}{N}\ll 22\gg + \frac{4}{N}\ll 13\gg - \frac{12}{N}\ll 112\gg + \frac{6}{N}\ll 1111\gg$$

$$+ \frac{2(N-n)}{(N-1)(n-1)} E\left\{<22> -2<112> + <1111>\right\} + [[22]]$$

$$= \frac{1}{n}[[4]] - \frac{1}{N}[[4]] + [[22]] + \frac{2(N-n)}{(N-1)(n-1)} E[2]_y[2]_y$$

$$= (\frac{1}{n} - \frac{1}{N})[[4]] + [[22]] + \frac{2(N-n)}{(N-1)(n-1)} E[2]'[2]'$$

$$= (\frac{1}{n} - \frac{1}{N})[[4]] + [[22]] + (\frac{2n}{(n-1)} - \frac{2N}{(N-1)})E[2]'[2]'$$

$$[[1]'[3]'] = E[1]'[3]' - E[1]'E[3]'$$

$$= E<1>'\left\{<3>' - 3<12>' + 2<111>'\right\} - E[1]'E[3]'$$

$$= E<1><3> - 3E<1><12> + 2E<1><111> - E[1]'E[3]'$$

$$= \frac{(n-1)}{n} E<13> + \frac{1}{n}\ll 4\gg - 3E\left\{\frac{(n-2)}{n}<112> + \frac{1}{n}<22> + \frac{1}{n}<13>\right\}$$

$$+2E\left\{\frac{(n-3)}{n}<1111> + \frac{3}{n}<112>\right\} - E[1]'E[3]'$$

$$= \frac{1}{n}\ll 4\gg - \frac{3}{n}\ll 22\gg - \frac{4}{n}\ll 13\gg + \frac{12}{n}\ll 112\gg - \frac{6}{n}\ll 1111\gg + E<13>$$

$$-3E<112> + 2E<1111> - E[1]'E[3]'$$

$$= \frac{1}{n}\,[[4]] + E<13> - 3E<112> + 2E<1111> - E[1]'E[3]'$$

$$= \frac{1}{n}\,[[4]] + \frac{N}{(N-1)}\,E\left\{<1><3> - \frac{1}{N}<4>\right\} - 3E\left\{\frac{N^2}{(N-1)(N-2)}<1><1><2>\right.$$

$$\left. - \frac{2N}{(N-1)(N-2)}<1><3> - \frac{N}{(N-1)(N-2)}<2><2> + \frac{2}{(N-1)(N-2)}<4>\right\}$$

$$+2E\left\{\frac{N^3}{(N-1)(N-2)(N-3)}<1><1><1><1> + \frac{8N}{(N-1)(N-2)(N-3)}<1><3>\right.$$

$$+ \frac{3N}{(N-1)(N-2)(N-3)}<2><2> - \frac{6N^2}{(N-1)(N-2)(N-3)}<1><1><2>$$

$$\left. - \frac{6}{(N-1)(N-2)(N-3)}<4>\right\} - E[1]'E[3]'$$

$$= \frac{1}{n}\,[[4]] - \frac{(N^2+N)}{(N-1)(N-2)(N-3)}<<4>> + \frac{N(N^2+N+4)}{(N-1)(N-2)(N-3)}\,E<1><3>$$

$$- \frac{N^2(3N+3)}{(N-1)(N-2)(N-3)}\,E<1><1><2> + \frac{3N}{(N-2)(N-3)}\,E<2><2>$$

$$+ \frac{2N^3}{(N-1)(N-2)(N-3)}\,E<1><1><1><1> - E<1><3>$$

$$+3E<1><12> - 2E<1><111> + E<1><3> - 3E<1><12>$$

$$+2E<1><111> - E[1]'E[3]'$$

$$= \frac{1}{n}\,[[4]] - \frac{(N^2+N)}{(N-1)(N-2)(N-3)}<<4>> + \frac{(7N^2-7N+6)}{(N-1)(N-2)(N-3)}\,E\left\{\frac{(N-1)}{N}<13>\right.$$

$$\left. + \frac{1}{N}<4>\right\} + 3E\left\{\frac{(N-2)}{N}<112> + \frac{1}{N}<22> + \frac{1}{N}<13>\right\}$$

$$-2E\left\{\frac{(N-3)}{N}<1111> + \frac{3}{N}<112>\right\} - \frac{N^2(3N+3)}{(N-1)(N-2)(N-3)}\,E\left\{\frac{1}{N^2}<4>\right.$$

$$\left. + \frac{(N-1)(N-2)}{N^2}<112> + \frac{2(N-1)}{N^2}<13> + \frac{(N-1)}{N^2}<22>\right\}$$

$$+ \frac{3N}{(N-2)(N-3)}\,E\left\{\frac{(N-1)}{N}<22> + \frac{1}{N}<4>\right\}$$

$$+ \frac{2N^3}{(N-1)(N-2)(N-3)}\,E\left\{\frac{(N-1)(N-2)(N-3)}{N^3}<1111> + \frac{6(N-1)(N-2)}{N^3}<112>\right.$$

$$\left. + \frac{3(N-1)}{N^3}<22> + \frac{4(N-1)}{N^3}<13> + \frac{1}{N^3}<4>\right\} + [[13]]$$

$$= \frac{1}{n}[[4]] - \frac{1}{N}\langle\langle 4\rangle\rangle + \frac{3}{N}\langle\langle 22\rangle\rangle + \frac{4}{N}\langle\langle 13\rangle\rangle - \frac{12}{N}\langle\langle 112\rangle\rangle$$

$$+ \frac{6}{N}\langle\langle 1111\rangle\rangle + [[13]]$$

$$= \left(\frac{1}{n} - \frac{1}{N}\right)[[4]] + [[13]]$$

$$[[1]'[1]'[2]'] = E[1]'[1]'[2]' - 2E[1]'E[1]'[2]' - E[1]'[1]'E[2]' + 2E[1]'E[1]'E[2]'$$

$$= E\langle 1\rangle\langle 1\rangle[\langle 2\rangle - \langle 11\rangle] - 2E\langle 1\rangle E\langle 1\rangle[\langle 2\rangle - \langle 11\rangle]$$

$$- E\langle 1\rangle\langle 1\rangle E[2]' + 2E[1]'E[1]'E[2]'$$

$$= E\left\{\frac{(n-1)(n-2)}{n^2}\langle 112\rangle + \frac{2(n-1)}{n^2}\langle 13\rangle + \frac{(n-1)}{n^2}\langle 22\rangle + \frac{1}{n^2}\langle 4\rangle\right\}$$

$$- E\left\{\frac{(n-2)(n-3)}{n^2}\langle 1111\rangle + \frac{5(n-2)}{n^2}\langle 112\rangle + \frac{2}{n^2}\langle 22\rangle + \frac{2}{n^2}\langle 13\rangle\right\}$$

$$- 2E\langle 1\rangle E\left\{\frac{(n-1)}{n}\langle 12\rangle + \frac{1}{n}\langle 3\rangle\right\} + 2E\langle 1\rangle E\left\{\frac{(n-2)}{n}\langle 111\rangle\right.$$

$$\left. + \frac{2}{n}\langle 12\rangle\right\} - E[2]'E\left\{\frac{(n-1)}{n}\langle 11\rangle + \frac{1}{n}\langle 2\rangle\right\} + 2E[1]'E[1]'E[2]'$$

$$= \frac{1}{n^2}\langle\langle 4\rangle\rangle - \frac{3}{n^2}\langle\langle 22\rangle\rangle - \frac{4}{n^2}\langle\langle 13\rangle\rangle + \frac{12}{n^2}\langle\langle 112\rangle\rangle - \frac{6}{n^2}\langle\langle 1111\rangle\rangle$$

$$+ \frac{1}{n}E\langle 22\rangle + \frac{2}{n}E\langle 13\rangle + \frac{(n-8)}{n}E\langle 112\rangle + \frac{(5-n)}{n}E\langle 1111\rangle$$

$$+ \frac{2(3-n)}{n}E\langle 1\rangle E\langle 12\rangle - \frac{2}{n}E\langle 1\rangle E\langle 3\rangle + \frac{2(n-2)}{n}E\langle 1\rangle E\langle 111\rangle$$

$$- \frac{1}{n}E\langle 2\rangle E[2]' - \frac{(n-1)}{n}E\langle 11\rangle E[2]' + 2E[1]'E[1]'E[2]'$$

$$= \frac{1}{n^2}[[4]] + \frac{1}{n}\frac{N}{(N-1)}E\left\{\langle 2\rangle\langle 2\rangle - \frac{1}{N}\langle 4\rangle\right\}$$

$$+ \frac{2}{n}\frac{N}{(N-1)}E\left\{\langle 1\rangle\langle 3\rangle - \frac{1}{N}\langle 4\rangle\right\} + \frac{(n-8)}{n}E\left\{\frac{N^2}{(N-1)(N-2)}\langle 1\rangle\langle 1\rangle\langle 2\rangle\right.$$

$$\left. - \frac{2N}{(N-1)(N-2)}\langle 1\rangle\langle 3\rangle - \frac{N}{(N-1)(N-2)}\langle 2\rangle\langle 2\rangle + \frac{2}{(N-1)(N-2)}\langle 4\rangle\right\}$$

$$+\frac{(5-n)}{n}E\left\{\frac{N^3}{(N-1)(N-2)(N-3)}<1><1><1><1>+\right.$$

$$+\frac{8N}{(N-1)(N-2)(N-3)}<1><3>+\frac{3N}{(N-1)(N-2)(N-3)}<2><2>$$

$$\left.-\frac{6N^2}{(N-1)(N-2)(N-3)}<1><1><2>-\frac{6}{(N-1)(N-2)(N-3)}<4>\right\}$$

$$+\frac{2(3-n)}{n}\frac{N}{(N-1)}E<1>E\left\{<1><2>-\frac{1}{N}<3>\right\}-\frac{2}{n}E<1>E<3>$$

$$+\frac{2(n-2)}{n}E<1>E\left\{\frac{N^2}{(N-1)(N-2)}<1><1><1>\right.$$

$$\left.-\frac{3N}{(N-1)(N-2)}<1><2>+\frac{2}{(N-1)(N-2)}<3>\right\}-\frac{1}{n}E<2>E[2]'$$

$$-\frac{(n-1)}{n}\frac{N}{(N-1)}E[2]'E\left\{<1><1>-\frac{1}{N}<2>\right\}+2E[1]'E[1]'E[2]'$$

$$=\frac{1}{n^2}[[4]]+\frac{(2Nn-N-3N^2)}{n(N-1)(N-2)(N-3)}<<4>>+\frac{N(N^2+3N-Nn-3)}{n(N-1)(N-2)(N-3)}E<2><2>$$

$$+\frac{2N(N+1)(N-n+2)}{n(N-1)(N-2)(N-3)}E<1><3>+\frac{N^2(Nn-8N+3n-6)}{n(N-1)(N-2)(N-3)}E<1><1><2>$$

$$+\frac{N^3(5-n)}{n(N-1)(N-2)(N-3)}E<1><1><1><1>$$

$$+\frac{2N(3N-nN-n)}{n(N-1)(N-2)}E<1>E<1><2>+\frac{2N(n-N)}{n(N-1)(N-2)}E<1>E<3>$$

$$+\frac{2N^2(n-2)}{n(N-1)(N-2)}E<1>E<1><1><1>+\frac{(n-N)}{n(N-1)}E<2>E[2]'$$

$$-\frac{N(n-1)}{n(N-1)}E[2]'E<1><1>-\Big\{E<1><1><2>-E<1><1><11>$$

$$-2E<1>E<1><2>+2E<1>E<1><11>-E<1><1>E[2]'\Big\}$$

$$+\Big\{E<1><1><2>-E<1><1><11>-2E<1>E<1><2>$$

$$+2E<1>E<1><11>-E<1><1>E[2]'+2E[1]'E[1]'E[2]'\Big\}$$

$$=\frac{1}{n^2}[[4]]+\frac{(2Nn-N-3N^2)}{n(N-1)(N-2)(N-3)}<<4>>+\frac{N(N^2+3N-Nn-3)}{n(N-1)(N-2)(N-3)}E<2><2>$$

$$+\frac{2N(N+1)(N-n+2)}{n(N-1)(N-2)(N-3)}E<1><3>$$

$$+ \frac{(-8N^3+9N^2n-6N^2-11Nn+6n)}{n(N-1)(N-2)(N-3)} E<1><1><2>$$

$$+ \frac{N^3(5-n)}{n(N-1)(N-2)(N-3)} E<1><1><1><1>+E<1><1><11>$$

$$+ \frac{(6N^2-8Nn+4n)}{n(N-1)(N-2)} E<1>E<1><2> -2E<1>E<1><11>$$

$$+ \frac{2N(n-N)}{n(N-1)(N-2)} E<1>E<3>+ \frac{2N^2(n-2)}{n(N-1)(N-2)} E<1>E<1><1><1>$$

$$+ \frac{(n-N)}{n(N-1)} E<2>E[2]' + \frac{(N-n)}{n(N-1)} E<1><1>E[2]' + [[112]]$$

$$= \frac{1}{n^2}[[4]] + \frac{(2Nn-N-3N^2)}{n(N-1)(N-2)(N-3)} <<4>>$$

$$+ \frac{N(N^2+3N-Nn-3)}{n(N-1)(N-2)(N-3)} E\left\{\frac{(N-1)}{N}<22>+\frac{1}{N}<4>\right\}$$

$$+ \frac{2N(N+1)(N-n+2)}{n(N-1)(N-2)(N-3)} E\left\{\frac{(N-1)}{N}<13>+\frac{1}{N}<4>\right\}$$

$$+ \frac{(-8N^3+9N^2n-6N^2-11Nn+6n)}{n(N-1)(N-2)(N-3)} E\left\{\frac{(N-1)(N-2)}{N^2}<112>+\frac{2(N-1)}{N^2}<13>\right.$$

$$\left. + \frac{(N-1)}{N^2}<22>+\frac{1}{N^2}<4>\right\}$$

$$+ \frac{N^3(5-n)}{n(N-1)(N-2)(N-3)} E\left\{\frac{(N-1)(N-2)(N-3)}{N^3}<1111>\right.$$

$$\left. + \frac{6(N-1)(N-2)}{N^3}<112>+\frac{3(N-1)}{N^3}<22>+\frac{4(N-1)}{N^3}<13>+\frac{1}{N^3}<4>\right\}$$

$$+E\left\{\frac{(N-2)(N-3)}{N^2}<1111>+\frac{5(N-2)}{N^2}<112>+\frac{2}{N^2}<22>+\frac{2}{N^2}<13>\right\}$$

$$+ \frac{(6N^2-8Nn+4n)}{n(N-1)(N-2)} E<1>E\left\{\frac{(N-1)}{N}<12>+\frac{1}{N}<3>\right\}$$

$$-2E<1>E\left\{\frac{(N-2)}{N}<111>+\frac{2}{N}<12>\right\} + \frac{2N(n-N)}{n(N-1)(N-2)} E<1>E<3>$$

$$+ \frac{2N^2(n-2)}{n(N-1)(N-2)} E<1>E\left\{\frac{(N-1)(N-2)}{N^2}<111>+\frac{3(N-1)}{N^2}<12>+\frac{1}{N^2}<3>\right\}$$

$$+ \frac{(n-N)}{n(N-1)} E<2>E\left\{<2>-<11>\right\}$$

$$+ \frac{(N-n)}{n(N-1)} E\left\{ \langle 2\rangle - \langle 11\rangle \right\} E\left\{ \frac{(N-1)}{N}\langle 11\rangle + \frac{1}{N}\langle 2\rangle \right\} + [[112]]$$

$$= \frac{1}{n^2}[[4]] - \frac{1}{N^2}\langle\langle 4\rangle\rangle + \left\{ \frac{(N-n)}{nN} + \frac{3}{N^2} \right\} \langle\langle 22\rangle\rangle$$

$$+ \frac{2(N^2-Nn+2n)}{nN^2}\langle\langle 13\rangle\rangle + \frac{4(-2N^2+2Nn-3n)}{nN^2}\langle\langle 112\rangle\rangle$$

$$+ \frac{(5N^2-5nN+6n)}{nN^2}\langle\langle 1111\rangle\rangle + \frac{6(N-n)}{nN}\langle\langle 1\rangle\rangle\langle\langle 12\rangle\rangle$$

$$- \frac{2(N-n)}{nN}\langle\langle 1\rangle\rangle\langle\langle 3\rangle\rangle - \frac{4(N-n)}{nN}\langle\langle 1\rangle\rangle\langle\langle 111\rangle\rangle$$

$$- \frac{(N-n)}{nN}\langle\langle 11\rangle\rangle\langle\langle 11\rangle\rangle + \frac{2(N-n)}{nN}\langle\langle 2\rangle\rangle\langle\langle 11\rangle\rangle$$

$$- \frac{(N-n)}{nN}\langle\langle 2\rangle\rangle\langle\langle 2\rangle\rangle + [[112]]$$

$$= \frac{1}{n^2}[[4]] - \frac{1}{N^2}\langle\langle 4\rangle\rangle + \frac{3}{N^2}\langle\langle 22\rangle\rangle + \frac{4}{N^2}\langle\langle 13\rangle\rangle - \frac{12}{N^2}\langle\langle 112\rangle\rangle$$

$$+ \frac{6}{N^2}\langle\langle 1111\rangle\rangle + \left\{ \frac{(N-n)}{nN}\langle\langle 22\rangle\rangle - \frac{(N-n)}{nN}\langle\langle 2\rangle\rangle\langle\langle 2\rangle\rangle \right.$$

$$- \frac{2(N-n)}{nN}\langle\langle 112\rangle\rangle + \frac{2(N-n)}{nN}\langle\langle 11\rangle\rangle\langle\langle 2\rangle\rangle + \frac{(N-n)}{nN}\langle\langle 1111\rangle\rangle$$

$$\left. - \frac{(N-n)}{nN}\langle\langle 11\rangle\rangle\langle\langle 11\rangle\rangle \right\} + \left\{ \frac{2(N-n)}{nN}\langle\langle 13\rangle\rangle - \frac{2(N-n)}{Nn}\langle\langle 1\rangle\rangle\langle\langle 3\rangle\rangle \right.$$

$$+ \frac{6(N-n)}{nN}\langle\langle 1\rangle\rangle\langle\langle 12\rangle\rangle - \frac{6(N-n)}{nN}\langle\langle 112\rangle\rangle - \frac{4(N-n)}{nN}\langle\langle 1\rangle\rangle\langle\langle 111\rangle\rangle$$

$$\left. + \frac{4(N-n)}{nN}\langle\langle 1111\rangle\rangle \right\} + [[112]]$$

$$= \frac{1}{n^2}[[4]] - \frac{1}{N^2}[[4]] + \frac{(N-n)}{Nn}[[22]] + \frac{2(N-n)}{nN}[[13]] + [[112]]$$

$$= \left(\frac{1}{n^2} - \frac{1}{N^2}\right)[[4]] + \left(\frac{1}{n} - \frac{1}{N}\right)[[22]] + 2\left(\frac{1}{n} - \frac{1}{N}\right)[[13]] + [[112]]$$

$$[[1]'[1]'[1]'[1]'] = E[1]'[1]'[1]'[1]' - 3E[1]'[1]'E[1]'[1]' - 4E[1]'E[1]'[1]'[1]'$$

$$+ 12E[1]'E[1]'E[1]'[1]' - 6E[1]'E[1]'E[1]'E[1]$$

$$= E\langle 1\rangle\langle 1\rangle\langle 1\rangle\langle 1\rangle - 3E\langle 1\rangle\langle 1\rangle E\langle 1\rangle\langle 1\rangle - 4E\langle 1\rangle E\langle 1\rangle\langle 1\rangle\langle 1\rangle$$

$$+12E\langle 1\rangle E\langle 1\rangle E\langle 1\rangle\langle 1\rangle - 6E[1]'E[1]'E[1]'E[1]'$$

$$= E\left\{\frac{(n-1)(n-2)(n-3)}{n^3}\langle 1111\rangle + \frac{6(n-1)(n-2)}{n^3}\langle 112\rangle + \frac{3(n-1)}{n^3}\langle 22\rangle\right.$$

$$\left.+\frac{4(n-1)}{n^3}\langle 13\rangle + \frac{1}{n^3}\langle 4\rangle\right\} - 3E\left\{\frac{(n-1)}{n}\langle 11\rangle + \frac{1}{n}\langle 2\rangle\right\}E\left\{\frac{(n-1)}{n}\langle 11\rangle\right.$$

$$\left.+\frac{1}{n}\langle 2\rangle\right\} - 4E\langle 1\rangle E\left\{\frac{(n-1)(n-2)}{n^2}\langle 111\rangle + \frac{3(n-1)}{n^2}\langle 12\rangle + \frac{1}{n^2}\langle 3\rangle\right\}$$

$$+12E\langle 1\rangle E\langle 1\rangle E\left\{\frac{(n-1)}{n}\langle 11\rangle + \frac{1}{n}\langle 2\rangle\right\} - 6E[1]'E[1]'E[1]'E[1]'$$

$$= \frac{1}{n^3}\langle\langle 4\rangle\rangle - \frac{3}{n^3}\langle\langle 22\rangle\rangle - \frac{4}{n^3}\langle\langle 13\rangle\rangle + \frac{12}{n^3}\langle\langle 112\rangle\rangle - \frac{6}{n^3}\langle\langle 1111\rangle\rangle$$

$$+\frac{3}{n^2}E\langle 22\rangle + \frac{4}{n^2}E\langle 13\rangle + \frac{6(n-3)}{n^2}E\langle 112\rangle + \frac{(n^2-6n+11)}{n^2}E\langle 1111\rangle$$

$$-\frac{3(n-1)^2}{n^2}E\langle 11\rangle E\langle 11\rangle - \frac{6(n-1)}{n^2}E\langle 11\rangle E\langle 2\rangle - \frac{3}{n^2}E\langle 2\rangle E\langle 2\rangle$$

$$-\frac{4(n-1)(n-2)}{n^2}E\langle 1\rangle E\langle 111\rangle - \frac{12(n-1)}{n^2}E\langle 1\rangle E\langle 12\rangle$$

$$-\frac{4}{n^2}E\langle 1\rangle E\langle 3\rangle + \frac{12(n-1)}{n}E\langle 1\rangle E\langle 1\rangle E\langle 11\rangle$$

$$+\frac{12}{n}E\langle 1\rangle E\langle 1\rangle E\langle 2\rangle - 6E[1]'E[1]'E[1]'E[1]'$$

$$= \frac{1}{n^3}[[4]] + \frac{3}{n^2}\frac{N}{(N-1)}E\left\{\langle 2\rangle\langle 2\rangle - \frac{1}{N}\langle 4\rangle\right\} + \frac{4}{n2}\frac{N}{(N-1)}E\left\{\langle 1\rangle\langle 3\rangle\right.$$

$$\left.-\frac{1}{N}\langle 4\rangle\right\} + \frac{6(n-3)}{n^2}E\left\{\frac{N^2}{(N-1)(N-2)}\langle 1\rangle\langle 1\rangle\langle 2\rangle - \frac{2N}{(N-1)(N-2)}\langle 1\rangle\langle 3\rangle\right.$$

$$\left.-\frac{N}{(N-1)(N-2)}\langle 2\rangle\langle 2\rangle + \frac{2}{(N-1)(N-2)}\langle 4\rangle\right\}$$

$$+\frac{(n^2-6n+11)}{n^2}E\left\{\frac{N^3}{(N-1)(N-2)(N-3)}\langle 1\rangle\langle 1\rangle\langle 1\rangle\langle 1\rangle\right.$$

$$+\frac{8N}{(N-1)(N-2)(N-3)}\langle 1\rangle\langle 3\rangle + \frac{3N}{(N-1)(N-2)(N-3)}\langle 2\rangle\langle 2\rangle$$

$$\left.-\frac{6N^2}{(N-1)(N-2)(N-3)}\langle 1\rangle\langle 1\rangle\langle 2\rangle - \frac{6}{(N-1)(N-2)(N-3)}\langle 4\rangle\right\}$$

$$- \frac{3(n-1)^2}{n^2} E \Big\{ \frac{N}{(N-1)} \langle 1 \rangle \langle 1 \rangle - \frac{1}{(N-1)} \langle 2 \rangle \Big\} E \Big\{ \frac{N}{(N-1)} \langle 1 \rangle \langle 1 \rangle$$

$$- \frac{1}{(N-1)} \langle 2 \rangle \Big\} - \frac{6(n-1)}{n^2} E \langle 2 \rangle E \Big\{ \frac{N}{(N-1)} \langle 1 \rangle \langle 1 \rangle - \frac{1}{(N-1)} \langle 2 \rangle \Big\}$$

$$- \frac{3}{n^2} E \langle 2 \rangle E \langle 2 \rangle - \frac{4(n-1)(n-2)}{n^2} E \langle 1 \rangle E \Big\{ \frac{N^2}{(N-1)(N-2)} \langle 1 \rangle \langle 1 \rangle \langle 1 \rangle$$

$$- \frac{3N}{(N-1)(N-2)} \langle 1 \rangle \langle 2 \rangle + \frac{2}{(N-1)(N-2)} \langle 3 \rangle \Big\}$$

$$- \frac{12(n-1)}{n^2} E \langle 1 \rangle E \Big\{ \frac{N}{(N-1)} \langle 1 \rangle \langle 2 \rangle - \frac{1}{(N-1)} \langle 3 \rangle \Big\} - \frac{4}{n^2} E \langle 1 \rangle E \langle 3 \rangle$$

$$+ \frac{12(n-1)}{n} E \langle 1 \rangle E \langle 1 \rangle E \Big\{ \frac{N}{(N-1)} \langle 1 \rangle \langle 1 \rangle - \frac{1}{(N-1)} \langle 2 \rangle \Big\}$$

$$+ \frac{12}{n} E \langle 1 \rangle E \langle 1 \rangle E \langle 2 \rangle - 6E[1]' E[1]' E[1]' E[1]'$$

$$= \frac{1}{n^3} [[4]] + \frac{(-7N^2-N+12Nn-6n^2)}{n^2(N-1)(N-2)(N-3)} \langle\langle 4 \rangle\rangle + \frac{3N(N^2+N-2Nn+n^2-1)}{n^2(N-1)(N-2)(N-3)} E \langle 2 \rangle \langle 2 \rangle$$

$$+ \frac{4N(N^2+4N-3Nn+2n^2-3n+1)}{n^2(N-1)(N-2)(N-3)} E \langle 1 \rangle \langle 3 \rangle$$

$$+ \frac{6N^2(Nn-3N+3n-n^2-2)}{n^2(N-1)(N-2)(N-3)} E \langle 1 \rangle \langle 1 \rangle \langle 2 \rangle$$

$$+ \frac{N^3(n^2-6n+11)}{n^2(N-1)(N-2)(N-3)} E \langle 1 \rangle \langle 1 \rangle \langle 1 \rangle \langle 1 \rangle$$

$$-3\left(\frac{n-1}{n}\right)^2 \left(\frac{N}{N-1}\right)^2 E \langle 1 \rangle \langle 1 \rangle E \langle 1 \rangle \langle 1 \rangle$$

$$+ \frac{6N(n-1)(n-N)}{n^2(N-1)^2} E \langle 1 \rangle \langle 1 \rangle E \langle 2 \rangle - \frac{3(N-n)^2}{n^2(N-1)^2} E \langle 2 \rangle E \langle 2 \rangle$$

$$- \frac{4(N-n)(N-2n)}{n^2(N-1)(N-2)} E \langle 1 \rangle E \langle 3 \rangle + \frac{12(N-n)}{n(N-1)} E \langle 1 \rangle E \langle 1 \rangle E \langle 2 \rangle$$

$$- \frac{4(n-1)(n-2)N^2}{n^2(N-1)(N-2)} E \langle 1 \rangle E \langle 1 \rangle \langle 1 \rangle \langle 1 \rangle$$

$$+ \frac{12N(n-1)(n-N)}{n^2(N-1)(N-2)} E \langle 1 \rangle E \langle 1 \rangle \langle 2 \rangle + \frac{12N(n-1)}{n(N-1)} E \langle 1 \rangle E \langle 1 \rangle E \langle 1 \rangle \langle 1 \rangle$$

$$-6E[1]' E[1]' E[1]' E[1]' + E \langle 1 \rangle \langle 1 \rangle \langle 1 \rangle \langle 1 \rangle$$

$$-3E\langle 1\rangle\langle 1\rangle E\langle 1\rangle\langle 1\rangle - 4E\langle 1\rangle E\langle 1\rangle\langle 1\rangle\langle 1\rangle$$

$$+12E\langle 1\rangle E\langle 1\rangle E\langle 1\rangle\langle 1\rangle - E\langle 1\rangle\langle 1\rangle\langle 1\rangle\langle 1\rangle$$

$$+3E\langle 1\rangle\langle 1\rangle E\langle 1\rangle\langle 1\rangle + 4E\langle 1\rangle E\langle 1\rangle\langle 1\rangle\langle 1\rangle$$

$$-12E\langle 1\rangle E\langle 1\rangle E\langle 1\rangle\langle 1\rangle$$

$$= \frac{1}{n^3}[[4]] + \frac{(-7N^2-N+12Nn-6n^2)}{n^2(N-1)(N-2)(N-3)}\langle\langle 4\rangle\rangle$$

$$+ \frac{3N(N^2+N-2Nn+n^2-1)}{n^2(N-1)(N-2)(N-3)} E\langle 2\rangle\langle 2\rangle$$

$$+ \frac{4N(N^2+4N-3Nn+2n^2-3n+1)}{n^2(N-1)(N-2)(N-3)} E\langle 1\rangle\langle 3\rangle$$

$$+ \frac{6N^2(Nn-3N+3n-n^2-2)}{n^2(N-1)(N-2)(N-3)} E\langle 1\rangle\langle 1\rangle\langle 2\rangle$$

$$+ \frac{(6N^2n^2-6N^3n-11n^2N+11N^3+6n^2)}{n^2(N-1)(N-2)(N-3)} E\langle 1\rangle\langle 1\rangle\langle 1\rangle\langle 1\rangle$$

$$+ \frac{3(n-N)(n-2nN+N)}{n^2(N-1)^2} E\langle 1\rangle\langle 1\rangle E\langle 1\rangle\langle 1\rangle$$

$$+ \frac{6N(n-1)(n-N)}{n^2(N-1)^2} E\langle 1\rangle\langle 1\rangle E\langle 2\rangle - \frac{3(N-n)^2}{n^2(N-1)^2} E\langle 2\rangle E\langle 2\rangle$$

$$- \frac{4(N-n)(N-2n)}{n^2(N-1)(N-2)} E\langle 1\rangle E\langle 3\rangle + \frac{12(N-n)}{n(N-1)} E\langle 1\rangle E\langle 1\rangle E\langle 2\rangle$$

$$+ \frac{4(N-n)(3nN-2N-2n)}{n^2(N-1)(N-2)} E\langle 1\rangle E\langle 1\rangle\langle 1\rangle\langle 1\rangle$$

$$+ \frac{12N(n-1)(n-N)}{n^2(N-1)(N-2)} E\langle 1\rangle E\langle 1\rangle\langle 2\rangle$$

$$+ \frac{12(n-N)}{n(N-1)} E\langle 1\rangle E\langle 1\rangle E\langle 1\rangle\langle 1\rangle + [[1111]]$$

$$= \frac{1}{n^3}[[4]] + \frac{(-7N^2-N+12Nn-6n^2)}{n^2(N-1)(N-2)(N-3)}\langle\langle 4\rangle\rangle$$

$$+ \frac{3N(N^2+N-2Nn+n^2-1)}{n^2(N-1)(N-2)(N-3)} E\left\{\frac{(N-1)}{N}\langle 22\rangle + \frac{1}{N}\langle 4\rangle\right\}$$

$$+ \frac{4N(N^2+4N-3Nn+2n^2-3n+1)}{n^2(N-1)(N-2)(N-3)} E\left\{\frac{(N-1)}{N}\langle 13\rangle + \frac{1}{N}\langle 4\rangle\right\}$$

$$+ \frac{6N^2(Nn-3N+3n-n^2-2)}{n^2(N-1)(N-2)(N-3)} E\left\{\frac{(N-1)(N-2)}{N^2}\langle 112\rangle + \frac{(N-1)}{N^2}\langle 22\rangle + \frac{2(N-1)}{N^2}\langle 13\rangle\right.$$

$$\left. + \frac{1}{N^2}\langle 4\rangle\right\} + \frac{(6N^2n^2-6N^3n-11n^2N+11N^3+6n^2)}{n^2(N-1)(N-2)(N-3)} E\left\{\frac{(N-1)(N-2)(N-3)}{N^3}\langle 1111\rangle\right.$$

$$\left. + \frac{6(N-1)(N-2)}{N^3}\langle 112\rangle + \frac{3(N-1)}{N^3}\langle 22\rangle + \frac{4(N-1)}{N^3}\langle 13\rangle + \frac{1}{N^3}\langle 4\rangle\right\}$$

$$+ \frac{3(n-N)(n-2nN+N)}{n^2(N-1)^2} E\left\{\frac{(N-1)}{N}\langle 11\rangle + \frac{1}{N}\langle 2\rangle\right\} E\left\{\frac{(N-1)}{N}\langle 11\rangle + \frac{1}{N}\langle 2\rangle\right\}$$

$$+ \frac{6N(n-1)(n-N)}{n^2(N-1)^2} E\langle 2\rangle E\left\{\frac{(N-1)}{N}\langle 11\rangle + \frac{1}{N}\langle 2\rangle\right\} - \frac{3(N-n)^2}{n^2(N-1)^2} E\langle 2\rangle E\langle 2\rangle$$

$$- \frac{4(N-n)(N-2n)}{n^2(N-1)(N-2)} E\langle 1\rangle E\langle 3\rangle + \frac{12(N-n)}{n(N-1)} E\langle 1\rangle E\langle 1\rangle E\langle 2\rangle$$

$$+ \frac{4(N-n)(3nN-2N-2n)}{n^2(N-1)(N-2)} E\langle 1\rangle E\left\{\frac{(N-1)(N-2)}{N^2}\langle 111\rangle + \frac{3(N-1)}{N^2}\langle 12\rangle\right.$$

$$\left. + \frac{1}{N^2}\langle 3\rangle\right\} + \frac{12N(n-1)(n-N)}{n^2(N-1)(N-2)} E\langle 1\rangle E\left\{\frac{(N-1)}{N}\langle 12\rangle + \frac{1}{N}\langle 3\rangle\right\}$$

$$+ \frac{12(n-N)}{n(N-1)} E\langle 1\rangle E\langle 1\rangle E\left\{\frac{(N-1)}{N}\langle 11\rangle + \frac{1}{N}\langle 2\rangle\right\} + [[1111]]$$

$$= \frac{1}{n^3}[[4]] - \frac{1}{N^3}\langle\langle 4\rangle\rangle + \frac{3(N^3-Nn^2+n^2)}{n^2N^3}\langle\langle 22\rangle\rangle$$

$$+ \frac{4(N^3-Nn^2+n^2)}{n^2N^3}\langle\langle 13\rangle\rangle + \frac{6(-3N^3+N^3n-N^2n^2+3Nn^2-2n^2)}{n^2N^3}\langle\langle 112\rangle\rangle$$

$$+ \frac{(6N^2n^2-6N^3n-11n^2N+11N^3+6n^2)}{n^2N^3}\langle\langle 1111\rangle\rangle$$

$$+ \frac{3(n-N)(n-2nN+N)}{n^2N^2}\langle\langle 11\rangle\rangle\langle\langle 11\rangle\rangle$$

$$+ \frac{6(n-N)(-N+nN-n)}{n^2N^2}\langle\langle 11\rangle\rangle\langle\langle 2\rangle\rangle + \frac{3(n^2-N^2)}{n^2N^2}\langle\langle 2\rangle\rangle\langle\langle 2\rangle\rangle$$

$$+ \frac{4(n^2-N^2)}{n^2N^2}\langle\langle 1\rangle\rangle\langle\langle 3\rangle\rangle + \frac{12(N-n)}{nN}\langle\langle 1\rangle\rangle\langle\langle 1\rangle\rangle\langle\langle 2\rangle\rangle$$

$$+ \frac{4(N-n)(3nN-2N-2n)}{n^2N^2}\langle\langle 1\rangle\rangle\langle\langle 111\rangle\rangle + \frac{12(N-n)(N-nN+n)}{n^2N^2}\langle\langle 1\rangle\rangle\langle\langle 12\rangle\rangle$$

$$+ \frac{12(n-N)}{nN}\langle\langle 1\rangle\rangle\langle\langle 1\rangle\rangle\langle\langle 11\rangle\rangle + [[1111]]$$

$$
\begin{aligned}
&= \frac{1}{n^3}[[4]] - \frac{1}{N^3}\langle\langle 4\rangle\rangle + \frac{3}{N^3}\langle\langle 22\rangle\rangle + \frac{4}{N^3}\langle\langle 13\rangle\rangle - \frac{12}{N^3}\langle\langle 112\rangle\rangle \\
&\quad + \frac{6}{N^3}\langle\langle 1111\rangle\rangle \\
&\quad + \frac{3(N^2-n^2)}{n^2N^2}\langle\langle 22\rangle\rangle - \frac{3(N^2-n^2)}{n^2N^2}\langle\langle 2\rangle\rangle\langle\langle 2\rangle\rangle - \frac{6(N^2-n^2)}{n^2N^2}\langle\langle 112\rangle\rangle \\
&\quad + \frac{6(N^2-n^2)}{n^2N^2}\langle\langle 11\rangle\rangle\langle\langle 2\rangle\rangle + \frac{3(N^2-n^2)}{n^2N^2}\langle\langle 1111\rangle\rangle \\
&\quad - \frac{3(N^2-n^2)}{n^2N^2}\langle\langle 11\rangle\rangle\langle\langle 11\rangle\rangle + \frac{4(N^2-n^2)}{n^2N^2}\langle\langle 13\rangle\rangle \\
&\quad - \frac{4(N^2-n^2)}{n^2N^2}\langle\langle 1\rangle\rangle\langle\langle 3\rangle\rangle + \frac{12(N^2-n^2)}{n^2N^2}\langle\langle 1\rangle\rangle\langle\langle 12\rangle\rangle \\
&\quad - \frac{12(N^2-n^2)}{n^2N^2}\langle\langle 112\rangle\rangle - \frac{8(N^2-n^2)}{n^2N^2}\langle\langle 1\rangle\rangle\langle\langle 111\rangle\rangle + \frac{8(N^2-n^2)}{n^2N^2}\langle\langle 1111\rangle\rangle \\
&\quad + \frac{6(N-n)}{nN}\langle\langle 112\rangle\rangle - \frac{12(N-n)}{nN}\langle\langle 1\rangle\rangle\langle\langle 12\rangle\rangle - \frac{6(N-n)}{nN}\langle\langle 11\rangle\rangle\langle\langle 2\rangle\rangle \\
&\quad + \frac{12(N-n)}{nN}\langle\langle 1\rangle\rangle\langle\langle 1\rangle\rangle\langle\langle 2\rangle\rangle - \frac{6(N-n)}{nN}\langle\langle 1111\rangle\rangle \\
&\quad + \frac{12(N-n)}{nN}\langle\langle 1\rangle\rangle\langle\langle 111\rangle\rangle + \frac{6(N-n)}{nN}\langle\langle 11\rangle\rangle\langle\langle 11\rangle\rangle \\
&\quad - \frac{12(N-n)}{nN}\langle\langle 1\rangle\rangle\langle\langle 1\rangle\rangle\langle\langle 11\rangle\rangle + [[1111]] \\
&= \left(\frac{1}{n^3} - \frac{1}{N^3}\right)[[4]] + 3\left(\frac{1}{n^2} - \frac{1}{N^2}\right)[[22]] + 4\left(\frac{1}{n^2} - \frac{1}{N^2}\right)[[13]] \\
&\quad + 6\left(\frac{1}{n} - \frac{1}{N}\right)[[112]] + [[1111]] \quad .
\end{aligned}
$$

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