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THE WAREHOUSE SCHEDULING PROBLEM

by

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Doctor of Philosophy

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CHAPTER I

INTRODUCTION

Consider a warehouse supplying parts to a high-volume assembly plant for consumer durable goods such as refrigerators or television sets. One of the warehouse manager's concerns is to ensure that there is enough space to accomodate all the parts upon their delivery. In this thesis we explore the problem of setting parts ordering rules to satisfy the assembly schedule as well as to make efficient use of both the limited warehouse space and the parts order and delivery system. The focus of this thesis is on optimizing the space utilization.

There are two complementary techniques available for optimizing space utilization: order sizing and delivery scheduling. Reducing the size of orders has the effect of reducing the space required to hold the cycle stock. A tradeoff arises in that smaller order sizes necessitate a greater order frequency which in turn places a greater burden on the parts order and delivery system.

Delivery scheduling is the technique of coordinating the delivery times of different orders to optimize space utilization. The greatest demand for warehouse space occurs if all parts deliveries arrive at the warehouse simultaneously. By time phasing or staggering these deliveries, the peak demand for warehouse space can be moderated.

We refer to the combined problem of order sizing and delivery scheduling as the Warehouse Scheduling Problem.

The Warehouse Scheduling Problem, WSP, bears many similarities to the Economic Lot Scheduling Problem, ELSP, in which production runs of different products must be sized and sequenced in a single time-constrained facility. As a result, the evolution of research on the WSP has closely paralleled that of the ELSP. These two areas are of considerable academic interest because they serve as paradigms for the general problem of integrating lot sizing procedures with procedures to coordinate the detailed timing of operations. In this thesis, we build upon recent scheduling advances in both areas and suggest, for further research, how the procedures developed here for the WSP can be applied to the ELSP problem as well.

The following assumptions are common in the literature of the WSP:

Parts Demand. The demand for parts occurs continuously in time at known constant rates and backorders are not allowed. This assumption is reasonable in many high-volume assembly environments because the assembly schedule is frequently expressed in terms of a production rate and a production mix. Both the rate and the mix are fixed for periods of times that are long relative to the parts ordering cycle. Furthermore, the assembly cycle is often short so that most of the end items in the product mix are assembled on a daily or even hourly basis. Consequently, parts must be supplied to the assembly plant at rates that are stable over time. A part stockout can imply the shutdown of the assembly plant. The assembly plant carries no buffer stock so backorders are not allowed.

Parts Supply. Problems associated with supply uncertainty are not considered here. In general, the warehouse must carry safety stock of each part to buffer the assembly schedule from uncertainties in both timing and quantities deliveries. To focus on the relationship between lot sizing and scheduling, we restrict attention to deterministic part supply: part deliveries occur at a fixed lead time after part orders and deliveries quantities exactly equal the matching order quantities. We further assume that the lead times are zero. (It is a trivial matter to offset a delivery schedule with known positive lead times to derive an order schedule.) Consequently, in what follows an order is synonymous with a delivery.

Schedule. An order covers a single part number. It consists of a specification of the part number ordered, the quantity ordered, and the delivery time of the order. Order quantities are positive and finite. Delivery times are non-negative with time zero representing immediate delivery. A schedule is a list of orders sorted by increasing delivery times. A schedule is said to be demand-satisfying if there exists a finite amount of initial inventory for each part such that if the schedule is followed there will never be stockout. A demand-satisfying schedule is therefore an infinite list of orders. A schedule and a vector of initial inventories are said to be space-bounded if there exists an upper bound on the warehouse space required to implement the schedule, starting with the initial inventories. Finally, a cyclic schedule is a finite list of orders, together with a cycle length, τ, such that the delivery time of each order lies in the closed interval $[0,\tau]$. A key requirement of a cyclic schedule is that the total quantity of each part ordered in the cycle equal the total demand for the part over the length of the cycle. Schedules generated from cyclic schedules are demand-satisfying and space-bounded. This schedule is repeated periodically every τ units of times with the same starting and ending inventories for each part. A relative cyclic schedule is a cyclic schedule with a cycle length of 1. A cyclic schedule can be expressed as a relative cyclic schedule together with a cycle length τ, since it is a simple matter to scale the relative delivery times and the relative order sizes by τ.

Performance Criteria. Schedules are evaluated over an infinite time horizon. The different performance measures of interest are:

1) Maximum space used. Clearly, a local maximum of the space used will occur at each of the delivery times. The relative maximum space used is the the largest of these local maxima over one cycle.

2) Long run ordering cost per unit time. We assume that the diseconomies of frequent orders for a part can be represented in terms of a fixed order cost levied against each order for the part. Such a cost may consist of an implicit component such as the economic value of time lost in changing over the order and delivery system from processing one order to processing another. A recent example applying such a distinction between explicit and implicit order costs in multi-stage production systems may be found in Jackson, Maxwell, and Muckstadt [88]. This distinction is not considered further in this thesis although a similar distinction arises in the consideration of space costs. The long run ordering cost per unit time is the sum, over all parts, of the fixed order cost per cycle divided by the cycle length.

3) Long run holding cost per unit of time. The holding cost for each part is proportional to its average inventory. The coefficient of proportionality, called the carrying charge, is the cost of holding one unit of the part for one unit of time. The long run holding cost per unit of time is the sum, over all parts, of the holding cost per cycle divided by the cycle length.

4 ~~5~~) Long run inventory costs per unit time. This is the sum of the long run ordering cost per unit time and the long run holding cost per unit

time.

5) Long run space cost per unit time. This is the product of the maximum space used over the cycle with the opportunity cost of one space unit of the warehouse for one unit time. If the long run holding cost per unit time and the long run space cost are both incorporated in the cost function to be minimized, we assume that the carrying charge (the inventory holding rate is the product of the inventory carrying charge and the unit variable cost (Hadley and Whitin[63])) does not include any component for the cost of renting or leasing the warehouse or the cost of operating the warehouse. These latter costs must be incorporated in the opportunity cost for space.

Many authors have considered the WSP with additional assumptions. Churchman et al[57], Holt[58], Buchan and Koenigsbergs[63], Hadley and Whitin[63], Parsons[66], Thompson[67], Lewis[70], Starr[70], and Johnson and Montgomery[74] have ignored the scheduling aspect of the problem and considered order sizing exclusively. Homer[66], Page and Paul[76], Zoller[77], and Hall[85] have considered scheduling but have restricted the reorder intervals to be equal for all parts.

Churchman et al[57], applied the Lagrangian Multipliers Technique, LMT, as a solution procedure to solve the problem when the scheduling aspect is ignored. This procedure minimizes the long run average inventory cost subject to the constraint that the total space required does not exceed the space available. This constraint formulation allows for the possibility of receiving simultaneously all the parts at one point in time. Consequently, the maximum space required obtained by the LMT is an upper bound on the optimal value obtained by any other technique.

Staggering the deliveries over the cycle to avoid simultaneous replenishment is an alternative method to order sizing to reduce the maximum space required. Staggering was known to authors who used LMT but it was not exploited. Hadley and Whitin[63] state:"Here we shall not attempt to account for the possibility that orders can be phased in the certainty case so that it will never be necessary to have the maximum quantity of each item on hand at the same time". Buchan and Koenigsbergs[63], Thompson[67], and Lewis[70] insert a normalizing factor which lies between 0.5 and 1 in the constraint in recognition of the fact that staggering would take place in practice so such a conservative bound is not necessary.

In 1966, Homer renewed interest in the subject by considering the scheduling aspect of the problem. He drew on the ELSP by using the Common Cycle, CC, approach (Elmaghraby[78]) where the reorder intervals are restricted to be equal for all parts. In addition, he determines the best way of spreading out the deliveries over the common cycle length to avoid simultaneous deliveries so as to minimize the maximum space required. Page and Paul[76] show that the CC yields a better utilization of the warehouse space than LMT. However, the CC does not always produce a better cost solution than LMT. The imposed common reorder interval may be very different to the optimum unrestricted reorder interval for some parts. They refine the CC by developing a grouping heuristic where the parts within each group share the same reorder interval. However, the reorder interval of different groups can be different. After computing the maximum space used for each group, they use the LMT to size the cycle of each group under the assumption of simultaneous maximum space used across all groups. Goyal[78] shows that the Page and Paul refined approach can be further improved by allowing parts to be ordered more than once. The same result of the Common Cycle approach with staggering was also obtained, independently, by Zoller[77], and by Hall[85]. Hall includes in his cost function the cost related to the storage space used. Hartley and Thomas[82] solve the WSP to optimality for the case of two products using the staggering technique without the restriction of equal reorder intervals.

As an extension of his model, Homer[66] suggests that some parts may be ordered more than once in the cycle, or once every several cycles if the peak inventory space derived by CC exceeds the space available. He says:" The problem then becomes one of deciding how many items may be subject to split deliveries, how many deliveries per item to allow and the size of each delivery, as well as the optimum spacing of the deliveries. Preliminary work indicates that these additional variables are not independent of the sequence in which items are received, thereby introducing a combinatorial problem of some consequence." No further studies were conducted to investigate the effectiveness of his suggestion. In this thesis, we will consider general schedules in which more than two products may be ordered several times in different amounts during the cycle. Only cyclic schedules will be examined. The Zero Switch Rule, ZSR,(Maxwell[64]) in which the quantity ordered for each part should be sufficient to last until its next order will be followed.

As recognized by Homer, the problem of integrating lot sizing and scheduling is a challenging problem. This integration was originally examined by Maxwell[64] for the ELSP. Maxwell[64] proposes a two stage approach to develop a cyclic schedule. First, the production sequence in which the parts may be produced more than once in the cycle is determined, and then the production timing and production quantities are computed. Delporte and Thomas[77] formulate the problem as a mixed integer quadratic programming problem to joinly determine the sequence and the production lengths. However, because of the complexity of a such formulation, they construct heuristics to determine the frequencies and the sequence. The production lengths are then derived by solving a quadratic programming problem. Dobson[87] uses a similar procedure to that of Maxwell and that of Delporte and Thomas. He first determines approximate Power-of-Two frequencies, and then he obtains the sequence by using the Power-of-Two Bin Packing Heuristic. The optimal schedule for the derived sequence can then be found by using a parametric quadratic programming algorithm (Zipkin[87]). In this thesis, a similar approach where the problem is split in different stages to determine the ordering frequencies, the sequence, the cycle length, the delivery timing and the delivery quantities will be used.

In Chapter II we show that the optimal cyclic schedules satisfy the ZSR. We present a general formulation of the WSP that includes binary variables to determine the positions of each part within the ordering sequence. For a given sequence, we demonstrate that the formulated problem can be approximated into two subproblems: a linear program that minimizes the relative maximum space used and a quadratic program that minimizes the holding cost.

Chapter III presents sequencing heuristics. We extend the Power-of–Two Bin Packing Heuristic which is restricted to the power-of-two integer order frequencies to the case of arbitrary integer frequencies.

Chapter IV deals with the time varying lot sizes models. In the derivation of these models, we assume that the order frequencies and the order sequence are given. We reformulate the linear program that minimizes the relative maximum space used and show that its optimal solution is characterized by filling the warehouse at each order. We give conditions under which this linear program and the quadratic program that minimizes the holding cost have the same optimal solution. When these conditions are not satisfied, we derive a bound on the cost penalty that results when using the optimal solution of the linear program as a solution to the quadratic program. Finally in this chapter, we determine the optimal cycle length for the model that minimizes the long run inventory costs per unit time under the storage space constraint.

In Chapter V we consider the equal lot sizes models. We conduct an empirical investigation to estimate a bound on the space penalty by imposing the equal lot sizes restriction. We develop an efficient algorithm to minimize the maximum space used that either solves the linear progam or suggests that a change in order sequence may be desirable.

Chapter VI integrates all the heuristics and algorithms developed in the previous chapters into one iterative algorithm. The algorithm computes iteratively the ordering frequencies, the cycle length, the delivery times, the maximum space used, and the long run inventory costs per unit time. We show that the algorithm compares favorably with other methods encountered in the literature.

Finally, Chapter VII summarizes the results obtained in this thesis and presents possible extensions for further research. In particular, the relation between this work and the ELSP is explored.

CHAPTER II

MIXED INTEGER NONLINEAR PROGRAMMING MODELS FOR THE WSP

The WSP is a complex problem in which the ordering frequencies, the sequence in which products are ordered, the length of the cycle and the timing of the deliveries must be determined. It can not readily be formulated as a single problem. This difficulty in formulation is also encountered in the ELSP. Most frequently, for both the ELSP and the WSP, a two stage approach has been suggested in the literature. First, the production frequencies and cycle length are determined. In this step the sequencing aspect is ignored and a relaxed problem is solved. Some authors have used iterative methods to generate near-optimal cycle length and frequencies (Doll and Whybark[73], Goyal[75], Haessler[79], Delporte and Thomas[77]). The second step of the approach uses these frequencies and finds a feasible schedule. This two stage approach is also adopted in this thesis to solve the WSP with the exception that the cycle length is not fixed by the first stage of the approach; cycle length is determined in the second stage.

This chapter details several models for the second stage of the general approach. The models can be combined with heuristics to determine the order frequencies and cycle length; Chapter VI illustrates one such combination. For the balance of the chapter, we assume that the frequencies are given. All of the models presented are based on the Zero Switch Rule since it will be shown that this rule is consistent with optimality in the WSP.The models differ with regard to the performance criteria.

Notation

Before presenting the mathematical formulation, it is useful to introduce the notation used to describe the parameters and the decision variables.

Hereafter, the indices i and r are products designators: i,r=1,...,n, where n is the number of parts. The indices j,k,and l designate orders: j,k,l=1,...,m, where m is the number of orders placed.

Parameters

The following parameters are assumed to be given:

V= warehouse space available;

m_i= number of orders for product i, i=1,...,n;

λ_i= demand rate for product i in unit space/unit time, i=1,...,n;

h_i= holding cost rate for part i in \$/unit space/unit time, i=1,...,n;

K_i= setup cost(ordering cost) in \$/order, i=1,...,n.

Note that demand rates and holding cost rates have been expressed using units of space. Conversion from units of part to units of space is a trivial matter given the space required per unit of part.

Decision Variables

The decision variables are:

Z^b_{ij}= inventory of product i immediately prior to delivery of the jth order;

Z^a_{ij}= inventory of product i immediately after delivery of the jth order;

$Z_{ij}= Z^a_{ij} - Z^b_{ij}$, lot size of product i on the jth order(equals zero if part i is not ordered on the jth order);

Z_j= total inventory immediately after delivery of the jth order (= $\sum_{i=1}^{n} Z^a_{ij}$);

W= maximum space used(since the demand rate is expressed in unit space per unit time, maximum space used and maximum inventory will be used interchangeably throughout this thesis);

U_j= time interval between jth and (j+1)st order, interpreted cyclically;

T_j= reorder interval of the part ordered on the jth order;

τ = cycle length;

$$\delta_{ij}=\begin{cases} 1 \text{ if part i is ordered on the jth order,} \\ 0 \text{ otherwise; and} \end{cases}$$

$$\delta_{ijk}=\begin{cases} 1 \text{ if the next order of part i after the jth order is on the kth order,} \\ 0 \text{ otherwise.} \end{cases}$$

In the following, a mixed integer nonlinear programming model is formulated for the problem of minimizing the long run average inventory costs per unit time. First the objective function is presented, then the consistency of the ZSR with the WSP is discussed, and finally the constraints to be satisfied are presented.

The long run average inventory costs per unit time includes the long run average setup cost per unit time and the long run holding cost per unit time. The average holding cost for product i per unit time is proportional to the sum of the area of the inventory trapezoids depicted in Figure 2-1 divided by the cycle length. The constant of proportionality is h_i for product i. The long run average setup cost per unit time is the sum, over all orders, of the setup cost of the product replenished at each order divided by the cycle length. Therefore, the long run average inventory costs per unit time is:

$$\frac{1}{\tau}\left(\sum_{i=1}^{n} K_i\, m_i + \text{Total average holding cost for all products}\right). \qquad (2\text{–}1)$$

A mathematical expression for the total average holding cost for all products will be derived below after discussing the use of the ZSR for the WSP.

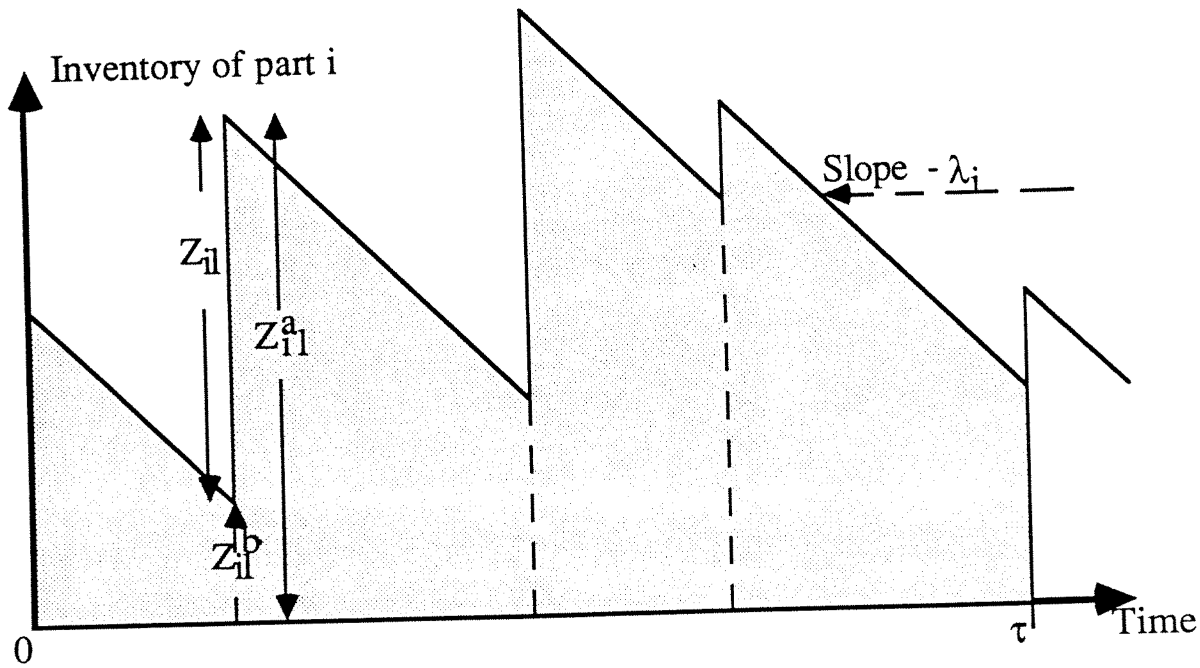

Figure 2-1 Inventory of Part i

The Zero Switch Rule, ZSR, requires that the quantity ordered for each part be just sufficient to last until its next order (Maxwell[64]). This rule is widely used for the ESLP to simplify the problem. It is clear that ZSR is consistent with the objective of minimizing the maximum space used because no extra inventory will be carried. The next theorem shows that ZSR is also consistent with the overall WSP.

Theorem II-1. The optimal cyclic schedule for WSP satisfies the ZSR.

Proof.(By contradiction)

Let Schedule 1 be an optimal schedule for WSP, that is; a schedule that does not violate the space restriction at any point in time during the cycle and that minimizes the long run inventory costs per unit time. To prove the theorem, it is sufficient to show that if the ZSR is not satisfied for schedule 1, then it is not optimal. The proof is most easily understood by an example.

For the sake of clarity, suppose that , according to schedule 1, part i is ordered on the 2nd, 5th, and 9th orders (see Figure 2-2).

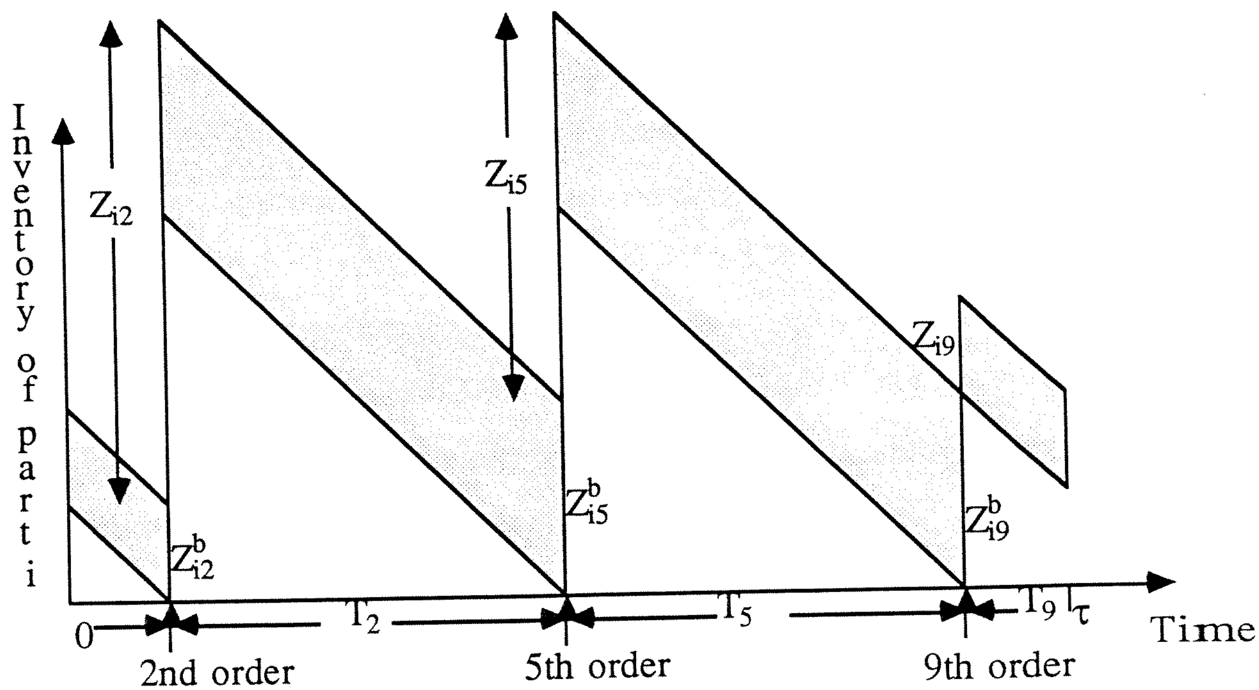

Figure 2-2 Consistency of the ZSR with the WSP

Consider another schedule that orders $(Z_{i2}+Z^b_{i2}-Z^b_{i5})$, $(Z_{i5}+Z^b_{i5}-Z^b_{i9})$,and

$(Z_{i9}+Z^b_{i9}-Z^b_{i2})$ on the 2nd, 5th, and 9th order respectively. The new schedule sets the inventory of part i immediately prior to each delivery of part i to zero. Clearly, the new schedule is also feasible since the total inventory at each order is smaller than the total inventory at each order for schedule 1. Moreover, the average holding cost of part i for this new schedule is less than the average holding cost for schedule 1 by the amount of the shaded parallelograms in Figure 2-2 ($\lambda_i(Z^b_{i5}\,T_2+Z^b_{i9}\,T_5+Z^b_{i2}\,T_9)$). This contradicts the assumption that schedule 1 was optimal. Note that by (2-1), for given frequencies and cycle length, the two schedules have the same long run setup cost per unit time. (E.O.P)

Although it is commonly used in formulating the problem, the ZSR is not necessarily optimal in the ELSP. In the ELSP, the production rate is finite, so having inventory on hand at the beginning of a production run allows the run to finish earlier, thereby freeing the scarce ressource(time on the bottleneck machine) for another use. In the WSP, inventory on hand when an order arrives creates no similar advantage since the constraint is on the maximum inventory. Throughout this thesis, all the models to be discussed assume the ZSR.

Under the ZSR rule, the average holding cost of product i becomes the sum of the area of the inventory triangles pictured in Figure 2-3 (after rotating the schedule so that an order occurs at time zero and runout occurs at time τ). Note that an order of part i corresponds to the jth order in the overall sequence if and only if $\delta_{ij}=1$. Thus, the long run average holding cost per unit time is:

$$\frac{1}{2\tau}\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\,h_i\,\delta_{ij}\,T_j^2, \tag{2–2}$$

and the long run average inventory cost per unit of time is:

$$\frac{1}{\tau}\left\{ \sum_{i=1}^{n} K_i\, m_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i\, h_i\, \delta_{ij}\, T_j^2 \right\}. \tag{2–3}$$

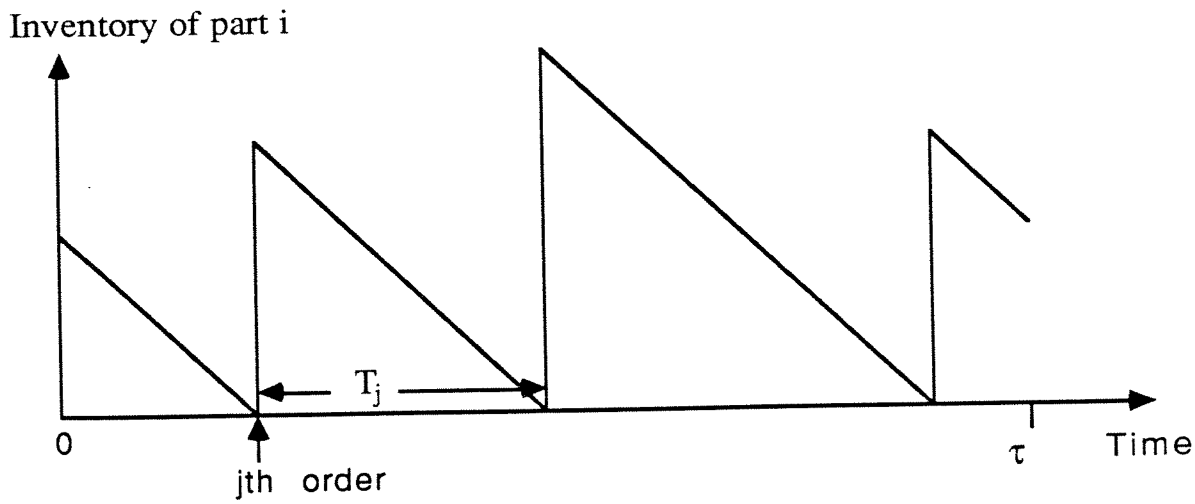

Figure 2-3 Inventory of Part i under the ZSR

Next , the constraints for the general model of WSP will be examined.

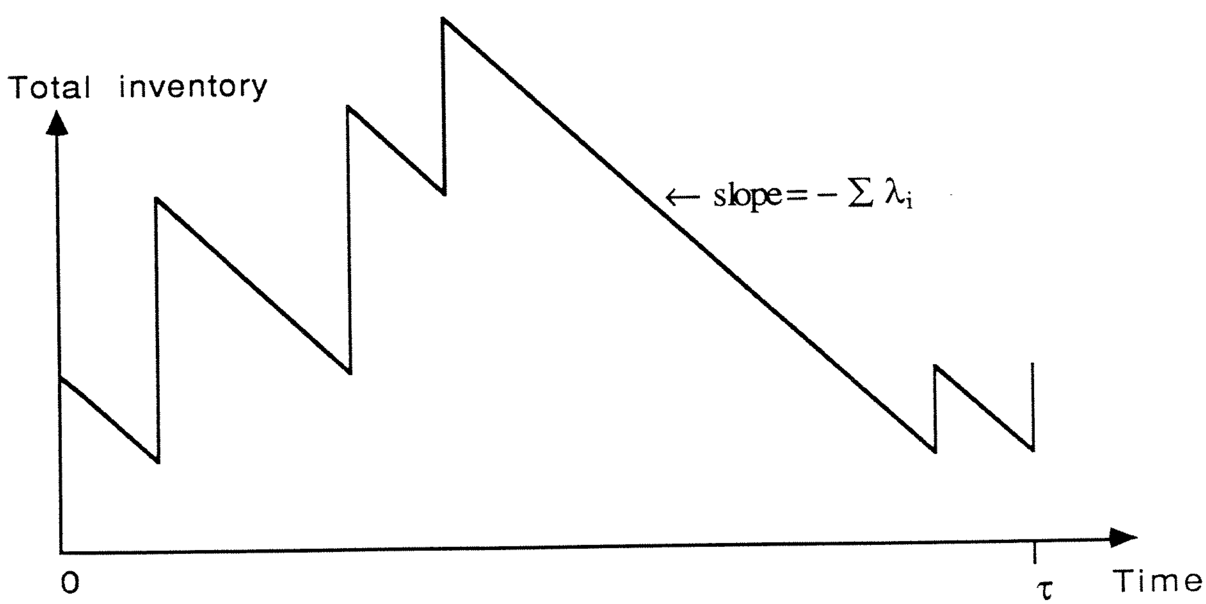

Figure 2-4 Total Inventory of all Parts

Figure 2-4 is a sample plot of total inventory for a particular schedule. As can been seen from Figure 2-4, the maximum inventory must occur on one of the replenishment points when the orders are received, or equivalently on one of the ordering points since replenishment is assumed to be instantaneous. Elsewhere in time, the inventory is depleted with rate equal to the sum, over all products, of the demand rates.

Therefore, the maximum inventory, W, must satisfy:

$$W \geq Z_j, \qquad j=1,\ldots,m, \tag{2–4}$$

and the warehouse space availability constraint is:

$$W \leq V. \tag{2–5}$$

Let S_{ij} = time interval until part i is ordered again after the jth order,

and let $\Delta_{ijk} = \sum_{l=j+1}^{k} \delta_{ijl}$.

Throughout this thesis, such summations are to be interpreted in a cyclic fashion. That is, if k≤j, then

$$\Delta_{ijk} = \sum_{l=j+1}^{m} \delta_{ijl} + \sum_{l=1}^{k} \delta_{ijl}\,.$$

Note that Δ_{ijk}=1 if and only if the next order of product i after the jth order is one of {j+1, j+2,..., k}, interpreted cyclically if k≤j. Otherwise Δ_{ijk}=0. Using this fact, S_{ij} can be expressed as:

$$S_{ij} = U_j + \sum_{k=j+1}^{j-1} (1 - \Delta_{ijk})\, U_k. \tag{2–6}$$

To see this, suppose $\delta_{ijj'}$=1; ie, the first time part i is delivered after the jth order is on the j'th order. In this case, Δ_{ijk}=0 for k=j+1,j+2,...,j'-1 and Δ_{ijk}=1 for k=j',j'+1,.....,j. Hence, $S_{ij} = U_j+U_{j+1}+\ldots+U_{j'-1}$, as desired.

By the ZSR, the amount of inventory of part i on the jth order should be equal to the amount needed to satisfy the demand until the next order of part i. Therefore,

$$Z^a_{ij} = \lambda_i S_{ij}, \qquad j=1,\dots,m \text{ and } i=1,\dots,n,$$

and

$$Z_j = \sum_{i=1}^{n} \lambda_i S_{ij}, \qquad j=1,\dots,m. \qquad (2\text{–}7)$$

Now, to determine T_j, the time interval until the order on which the part ordered on the jth order is reordered, it is necessary to know which part is ordered on the jth order. Using the definition of δ_{ij} and S_{ij}, T_j can be written as:

$$T_j = \sum_{i=1}^{n} \delta_{ij} S_{ij}, \qquad j=1,\dots,m. \qquad (2\text{–}8)$$

Moreover, the sum of the reorder intervals for each product should equal τ.

$$\sum_{j=1}^{m} \delta_{ij} T_j = \tau, \qquad i=1,\dots,n. \qquad (2\text{–}9)$$

In addition, the time intervals, U_j, should also add to the cycle length.

$$\sum_{j=1}^{m} U_j = \tau. \qquad (2\text{–}10)$$

Finally, the logical constraints involving the binary variables are:

$$\sum_{j=1}^{m} \delta_{ij} = m_i, \qquad i=1,\dots,n, \qquad (2\text{–}11)$$

$$\sum_{i=1}^{n} \delta_{ij} = 1, \qquad j=1,\dots,m, \qquad (2\text{–}12)$$

$$\sum_{k=1}^{m} \delta_{ijk} = 1, \qquad j=1,...,m \text{ and } i=1,...,n, \quad (2\text{–}13)$$

$$\delta_{ijk} - \delta_{ik} \leq 0, \qquad j,k,=1,...,m \text{ and } i=1,...,n, \quad (2\text{–}14)$$

and

$$\sum_{l=j+1}^{k} \delta_{il} \leq m_i \sum_{l=j+1}^{k} \delta_{ijl}, \qquad j=1,...,m,\ k=j+1,...,j \text{ and } i=1,...,n. \quad (2\text{–}15)$$

Constraints (2-11) require that each part i should be ordered exactly m_i times. Constraints (2-12) state that exactly one part should be replenished on each order. Constraints (2-13) indicate that there has to be precisely one order on which the next order of part i after the jth order is placed. Constraints (2-14) ensures that if $\delta_{ijk} = 1$, then necessarily $\delta_{ik} = 1$. These are called contingency constraints. Finally, constraints (2-15), together with constraints (2-13), ensure that no order of part i is placed between the jth order and the first order of part i after the jth order.

The general model, GM, that we just formulated is a very large mixed integer nonlinear programming problem. It consists of ($3m^2n+2mn+4m+2n+2$) constraints excluding the nonnegativety constraints, and ($mn+2m+2$) continuous variables excluding the slack variables and ($2m^2n+mn$) discrete variables.

To recapitulate, the GM can be written in a condense form as:

$$\text{Min } \frac{1}{\tau}\left\{ \sum_{i=1}^{n} m_i K_i + 0.5 \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i h_i \delta_{ij} T_j^2 \right\}$$

s.t.

$$
\begin{array}{lll}
 & W \le V, & \\
 & Z_j \le W, & j=1,\ldots,m, \\
 & Z_j = \sum_{i=1}^{n} \lambda_i S_{ij}, & j=1,\ldots,m, \\
 & T_j = \sum_{i=1}^{n} \delta_{ij} S_{ij}, & j=1,\ldots,m, \\
 & \sum_{j=1}^{m} \delta_{ij} T_j = \tau, & i=1,\ldots,n, \\
 & S_{ij} = U_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) U_k, & j=1,\ldots,m \text{ and } i=1,\ldots,n, \\
\text{(GM)} & \sum_{j=1}^{m} U_j = \tau, & \\
 & \sum_{j=1}^{m} \delta_{ij} = m_i, & i=1,\ldots,n, \\
 & \sum_{i=1}^{n} \delta_{ij} = 1, & j=1,\ldots,m, \\
 & \sum_{k=1}^{m} \delta_{ijk} = 1, & j=1,\ldots,m \text{ and } i=1,\ldots,n, \\
 & \delta_{ijk} - \delta_{ik} \le 0, & j,k=1,\ldots,m \text{ and } i=1,\ldots,n, \\
 & \sum_{l=j+1}^{k} \delta_{il} \le \sum_{l=j+1}^{k} \delta_{ijl}, & j=1,\ldots,m,\ k=j+1,\ldots,j \text{ and } i=1,\ldots,n, \\
 & \Delta_{ijk} = \sum_{l=j+1}^{k} \delta_{ijl}, & j,k=1,\ldots,m \text{ and } i=1,\ldots,n, \\
 & U_j \ge 0, & j=1,\ldots,m, \\
 & \delta_{ijk}, \delta_{ij} \in \{0,1\}, & j,k=1,\ldots,m \text{ and } i=1,\ldots,n.
\end{array}
$$

Hereafter, because of the complexity of the general model formulated for WSP, we assume that the sequence is given by some sequencing heuristic. Such heuristics are discussed in the next chapter. In this case, the GM model reduces to the following nonlinear program.

$$\text{Min } \frac{1}{\tau}\left\{ \sum_{i=1}^{n} m_i K_i + 0.5 \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i h_i \delta_{ij} T_j^2 \right\}$$

s.t.

$$W \le V,$$

$$Z_j \le W, \qquad j=1,\ldots,m,$$

$$Z_j = \sum_{i=1}^{n} \lambda_i \left(U_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) U_k \right), \qquad j=1,\ldots,m,$$

$$\text{(NLP)} \qquad T_j = \sum_{i=1}^{n} \delta_{ij} \left(U_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) U_k \right), \qquad j=1,\ldots,m,$$

$$\sum_{j=1}^{m} \delta_{ij} T_j = \tau, \qquad i=1,\ldots,n,$$

$$\sum_{j=1}^{m} U_j = \tau,$$

$$U_j \ge 0, \qquad j=1,\ldots,m,$$

where δ_{ij} and $\Delta_{ijk} = \sum_{l=j+1}^{k} \delta_{ijl}$ are known for all i,j, and k.

Let:
$t_j = T_j / \tau$ relative reorder interval;
$u_j = U_j / \tau$ relative time interval;
$z_j = Z_j / \tau$ relative total inventory on the jth order;
$w = W / \tau$ relative maximum inventory.

After this transformation the above model becomes:

$$\text{Min } \frac{1}{\tau}\sum_{i=1}^{n} m_i\,K_i + \frac{\tau}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\,h_i\,\delta_{ij}\,t_j^2$$

s.t.

$$w\ \tau \le V,$$

$$\sum_{i=1}^{n}\lambda_i\left(u_j+\sum_{k=j+1}^{j-1}(1-\Delta_{ijk})u_k\right) \le w, \qquad j=1,\ldots,m,$$

$$t_j-\sum_{i=1}^{n}\delta_{ij}\left(u_j+\sum_{k=j+1}^{j-1}(1-\Delta_{ijk})u_k\right) = 0, \qquad j=1,\ldots,m,$$

$$\sum_{j=1}^{m}\delta_{ij}\,t_j = 1, \qquad i=1,\ldots,n,$$

$$\sum_{j=1}^{m}u_j = 1,$$

$$u_j \ge 0, \qquad j=1,\ldots,m.$$

Now, let π be the dual variable associated with the constraint $w\ \tau \le V$. A Lagrangian relaxation of the above model can be obtained by moving the constraint $w\ \tau \le V$ to the objective function:

$$\text{Min } \frac{1}{\tau}\sum_{i=1}^{n} m_i\,K_i + \tau\left(\pi\ w + 0.5\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_i\,h_i\,\delta_{ij}\,t_j^2\right) - \pi\ V$$

s.t.

$$\sum_{i=1}^{n}\lambda_i\left(u_j+\sum_{k=j+1}^{j-1}(1-\Delta_{ijk})u_k\right) \le w, \qquad j=1,\ldots,m,$$

$$t_j-\sum_{i=1}^{n}\delta_{ij}\left(u_j+\sum_{k=j+1}^{j-1}(1-\Delta_{ijk})u_k\right) = 0, \qquad j=1,\ldots,m,$$

$$\sum_{j=1}^{m}\delta_{ij}\,t_j = 1, \qquad i=1,\ldots,n,$$

$$\sum_{j=1}^{m}u_j = 1,$$

$$u_j \ge 0, \qquad j=1,\ldots,m.$$

It is clear that this model is separable in terms of the cycle length and the remaining decision variables. First, the optimal (u_j^*, t_j^*, w^*) is determined by the following quadratic program.

$$\text{Min} \quad \pi w + 0.5 \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i h_i \delta_{ij} t_j^2$$

s.t.

$$\sum_{i=1}^{n} \lambda_i \left(u_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) u_k \right) \le w, \qquad j=1,\ldots,m,$$

$$t_j - \sum_{i=1}^{n} \delta_{ij} \left(u_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) u_k \right) = 0, \qquad j=1,\ldots,m,$$

$$\sum_{j=1}^{m} \delta_{ij} t_j = 1, \qquad i=1,\ldots,n,$$

$$\text{(GQP)} \qquad \sum_{j=1}^{m} u_j = 1,$$

$$u_j \ge 0, \qquad j=1,\ldots,m.$$

This model will be referred as the General Quadratic Program, GQP. Then, given the optimal solution (u_j^*, t_j^*, w^*) to QP, the optimal cycle length, τ^*, is given by the solution to the following problem.

$$\text{(RNLP)} \qquad \text{Min} \quad \frac{\sum_{i=1}^{n} m_i K_i}{\tau} + \tau \left(\pi w^* + 0.5 \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i h_i \delta_{ij} (t_j^*)^2 \right).$$

This model will be called the Relaxed NonLinear Program, RNLP. If all parts are ordered only once during the cycle, then RNLP reduces to the model discussed by Hall[85].

The solution of RNLP is a square root formula and the dual variable $\pi \ge 0$ can be chosen to satisfy the omitted constraint $\tau^* w^* \le V$.

$$\tau^*(u_j^*, t_j^*, w^*, \pi) = \sqrt{\frac{\sum_{i=1}^{n} m_i K_i}{\pi w^* + 0.5 \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i h_i \delta_{ij} (t_j^*)^2}}.$$

The two extreme cases where π is very large (high storage space cost) and π is very small (low storage space cost) will also be considered in this dissertation. If π is very large, that is ; if the holding cost is insignificant when compared to the storage space cost, then the following linear program model is appropriate:

Min w

s.t.

$$\sum_{i=1}^{n} \lambda_i \left(u_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) u_k \right) \le w, \qquad j=1,\dots,m,$$

$$t_j - \sum_{i=1}^{n} \delta_{ij} \left(u_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) u_k \right) = 0, \qquad j=1,\dots,m,$$

$$\sum_{j=1}^{m} \delta_{ij} t_j = 1, \qquad i=1,\dots,n,$$

(LP)

$$\sum_{j=1}^{m} u_j = 1,$$

$$u_j \ge 0, \qquad j=1,\dots,m.$$

On the other hand, for a given cycle length, if the holding cost dominates the storage space cost (π is very small), then a quadratic program is appropriate:

$$\text{Min} \quad \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i\, h_i\, \delta_{ij}\, t_j^2$$

s.t.

$$\tau \sum_{i=1}^{n} \lambda_i \left(u_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) u_k \right) \le V, \qquad j=1,\ldots,m,$$

$$t_j - \sum_{i=1}^{n} \delta_{ij} \left(u_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) u_k \right) = 0, \qquad j=1,\ldots,m,$$

$$\sum_{j=1}^{m} \delta_{ij}\, t_j = 1, \qquad i=1,\ldots,n,$$

$$\text{(QP)} \qquad \sum_{j=1}^{m} u_j = 1,$$

$$u_j \ge 0, \qquad j=1,\ldots,m.$$

Finally, another model which is worth investigating is the one with a restriction on the lot sizes. The quantity replenished for each part is restricted to be the same whenever the part is ordered. Mathematically, the new set of constraints is :

$$\left(\sum_{i=1}^{n} \lambda_i\, \delta_{ij} \right) T_j = \left(\sum_{i=1}^{n} \lambda_i\, \delta_{ij} \right) \sum_{i=1}^{n} \frac{\tau}{m_i}\, \delta_{ij}, \qquad j=1,\ldots,m,$$

or, equivalently,

$$t_j = \sum_{i=1}^{n} \frac{\delta_{ij}}{m_i}, \qquad j=1,\ldots,m.$$

Using (2-6) and (2-8) this is equivalent to:

$$\sum_{i=1}^{n} \delta_{ij} \left(u_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) u_k \right) = \sum_{i=1}^{n} \frac{\delta_{ij}}{m_i}, \qquad j=1,\ldots,m. \tag{2–16}$$

The new linear program that minimizes the maximum inventory can be expressed as:

$$
\begin{array}{llr}
\text{Min} & w & \\
\text{s.t.} & & \\
& \displaystyle\sum_{i=1}^{n} \lambda_i \left(u_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) u_k \right) \le w, & j=1,\ldots,m, \\
& \displaystyle\sum_{i=1}^{n} \delta_{ij} \left(u_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) u_k \right) = \sum_{i=1}^{n} \frac{\delta_{ij}}{m_i}, & j=1,\ldots,m, \\
\text{(RLP)} & \displaystyle\sum_{j=1}^{m} u_j \quad = 1, & \\
& u_j \ge 0, & j=1,\ldots,m.
\end{array}
$$

This model will be referred as the Restricted Linear Program, RLP.

NLP, QP, LP, and RLP will be discussed in more detail in chapters IV and V.

CHAPTER III

SEQUENCING HEURISTICS

As pointed out in the previous chapter, although the problem in which the sequence must be determined for given order frequencies can be formulated as a mixed integer nonlinear program, it is computationally impractical for large problems. The complexity of this model is mainly due to the combinatorial aspect of the problem. When solving the ELSP, the same difficulty has been circumvented by developing heuristics to generate promising sequences (Haessler and Hogue [76], Delporte and Thomas [77], Dobson [87]). Those heuristics are similar in a sense that they try to evenly space the orders of each product. Similarly, in this chapter a sequencing heuristic is constructed to develop sequences to be used for the WSP. First, Dobson's bin packing heuristic which is restricted to the power-of-two frequencies is reviewed, and then a more general sequencing heuristic is presented.

III-1 Power-of-Two Bin Packing Heuristic

The first idea behind the Power-of-Two Bin Packing Heuristic is to produce an ordering sequence for which the orders of each product can be scheduled at equally spaced points in time over the cycle. Equal intervals are desirable from a holding cost perspective. In the case of power-of-two frequencies, this is easily done. Suppose $m_i=2^{\alpha_i}$ with α_i integer for all products i. Let m^+ denote the largest frequency. Divide the time scale

[0,1) into m^+ equal sized intervals called bins. Number the bins consecutively $1,2,..,m^+$ and define the distance between bin k and bin j to be $(j-k)/m^+$ if $k \leq j$ and $(m^+ + k - j)/m^+$ if $k > j$. Let b_{ik} denote the bin to which the kth order of product i is assigned. The inter-order distance for product i is said to be equalized if the distance between b_{ik} and b_{ik+1}, interpreted cyclically, is the same for all orders $k = 1,2,.....,m_i$. For power-of-two frequencies, m^+ is an integer multiple of the frequency for every product. Consequently, setting $b_{ik}=(b_{ik-1}+m^+/m_i) \bmod m_i$ ensures that inter-order distances are equalized. If inter-order distances are equalized for all products, then it is trivial to produce a schedule for which the inter-order time intervals are equalized: simply schedule each order to occur at the beginning of the time interval, or bin, to which is assigned.

The second idea behind the Power-of-Two Bin Packing Heuristic is to spread the peak demand for space evenly over the cycle. This is approximated by assigning orders to bins so that the maximum load across all the bins is minimized. The <u>load</u> for a bin is defined to be the total space size of the orders assigned to that bin, where the space size of each order of product i is given by $\psi_i = \lambda_i/m_i$. Note that this approximation implicitly assumes that (a) all orders for the same product have the same lot size, (b) all orders within a bin occur simultaneously, and (c) orders in other bins have no effect on the peak demand for space within a given bin. Assumption (a) is consistent with the first objective of the bin packing heuristic but assumptions (b) and (c) are almost certainly violated by the resulting schedule. Nevertheless, Theorem IV-3 below, indicates that, under certain conditions, this approximation is consistent with the objective of minimizing the maximum space required per cycle.

The problem of assigning orders to bins to equalize inter-order distances and to minimize the maximum load per bin is in itself a difficult combinatorial problem. It can be formulated as a mixed integer-linear problem, but we omit the formulation. The heuristic used to solve the assignment problem is outlined in Figure 3-1. The heuristic assumes that the products have first been arranged in a lexicographically decreasing order by (m_i , ψ_i): that is, first by frequency, then in case of a tie by order space size. Bin(k) is an ordered list of part numbers assigned to bin number k. Load(k) is the load of the kth bin`.

1. Load(k) = 0   k = 1,2,....,m^+.
   Bin(k) = ∅   k = 1,2,....,m^+.
2. For i = 1 to n do:
   begin,
   3. find b* s.t. Load(b*)=min{Load(b):b=1,2,....,(m^+/m_i)},
   4. for k = b*,b*+m^+/m_i,....,b*+((m_i-1) (m^+/m_i)) do:
      begin:
      5. Load(k) ← Load(k)+ ψ_i,
      6. append i to end of list Bin(k),
      end,
   end.
7. The sequence P = (Bin(1), Bin(2), ,Bin(m^+))

Figure 3-1 Power-of-Two Bin Packing Heuristic

Numerical Example 1

The Power-of-Two Bin Packing Heuristic is illustrated using the example presented in Table 3-1. Note that the parts have been sorted in lexicographically decreasing order. The different steps of the heuristic when applied to example 1 are shown in Figure 3-2.

Product	1	2	3	4	5
Demand rate	12	6	4	3	2
Frequency	4	2	1	1	1
Order space size	3	3	4	3	2

Table 3-1 Example 1 Data

```
2. i=1,
   Load(k) = 3    k=1,2,3,4,
   Bin(k)  = (1)    k=1,2,3,4.
2. i=2.
   3. b* = 1.
   4. k  = 1.
         5. Load(1) = 6.
         6. Bin(1)  = (1,2).
   4. k  =3.
         5. Load(3) = 6.
         6. Bin(3)  = (1,2).
2. i=3.
   3. b* = 2.
   4. k  = 2.
         5. Load(2) = 7.
         6. Bin(2)  = (1,3).
2. i=4.
   3. b* = 4.
   4. k  = 4.
         5. Load(4) = 6.
         6. Bin(4)  = (1,4).
2. i=5.
   3. b* = 1.
   4. k  = 1.
         5. Load(1) = 8.
         6. Bin(1)  = (1,2,5).
7. P = ( 1,2,5,1,3,1,2,1,4 ).
```

Figure 3-2 Illustration of the Power-of-Two Bin Packing Heuristic

III-2 The Arbitrary Frequencies Bin Packing Heuristic

Consider the example of a single product which is ordered 11 times at equally spaced points in time during a cycle of length one unit of time. If its frequency is rounded to 8, the nearest power-of-two integer, the maximum inventory will increase by (11/8 - 1) = 37.5 %. Thus the round–off to a power-of-two can increase significantly the maximum warehouse space used, and it is economically worthwile to consider arbitrary integer frequencies.

The chief problem with considering arbitrary integer freqencies lies in generating sequences for which the orders for a product can be placed at equal intervals. That is, the fixed order quantity model, RLP, for a given sequence with arbitrary integer frequencies, may not be feasible. For example, consider the following sequence: P = (1, 2, 2, 1, 1). It is clear that the set of equations (2-16) of RLP is infeasible for this sequence since product 2 with the largest reorder interval is ordered twice between two orders of product 1.

The Arbitrary Frequencies Bin Packing Heuristic, presented below, is an extension of Dobson's heuristic to the case of arbitrary integer frequencies that guarantees the feasibility of the fixed order quantity model.

Let:

(i,k) index the kth order of product i : $k = 1,2,\ldots\ldots,m_i$,

$m^+ = \max \{ m_i \quad i = 1,2,\ldots\ldots,n\}$,

n^+ = number of products with frequency m^+,

y_{ik} = trial relative order time of the kth order of product i. (It is called trial because the exact order time is established using methods of Chapters IV to VI, and called relative because $y_{ik} \in [0,1)$),

b_{ik} = bin number of the kth order of product i,

$p(b)$ = number of product orders in bin b

$= |\{ (i,k) : b_{ik} = b \}|$,

$Bin(b)$ = ordered list of product orders in bin b

$= ((i_1,k_1) , (i_2,k_2),..........(i_{p(b)},k_{p(b)}))$,

$$Load(b) = \sum_{r=1}^{p(b)} \psi_{i_r} \quad \text{space load of bin b.}$$

Bin(b) is said to be a time product ordering if it is ordered lexicographically increasing in (y_{ik}, i).

If $(i,k) \notin Bin(b)$ but $b_{ik} = b$, then we define an operation INCLUDE on a time-product ordered Bin(b) as follows:

INCLUDE ((i,k),b)

1. $Load(b) \leftarrow Load(b) + \psi_i$.
2. $p(b) \leftarrow p(b)+1$.
3. INSERT ((i,k),Bin(b)), where the INSERT opertion inserts (i,k) into the ordered list Bin(b) while preserving the time product ordering property.

The computational complexity of the INCLUDE operation is O(log(n)) since, for any bin b, the maximum number of elements in Bin(b) is n, (Aho et al [74]). Figure 3-3 details the different steps of the Arbitrary Frequencies Bin Packing Heuristic.

1. Renumber the products so that they are lexicographically decreasing in (m_i,ψ_i) for i = 1,2,.....,n.
2. Initialization:

   For k = 1 to m^+ do:

   begin,

   3. for i=1 to n^+ do: $y_{ik} = (k-1)/m^+$,
   4. Bin(k) = $((1,k),(2,k),.....,(n^+,k))$,
   5. p(k) = n^+,
   6. Load(k) = $\sum_{i=1}^{n^+} \psi_i$,

   end.

7. For i = n^++1 to n do:

   begin,

   8. $b^* \leftarrow \text{argmin}\{\psi(b) : b=1,2,....,m^+\}$,
   9. $y_{i1} \leftarrow y_{i_{p(b^*)}k_{p(b^*)}}$,
   10. INCLUDE ((i,1),b*),
   11. for k = 2 to m_i do:

       begin,

       12. $y_{ik} \leftarrow (y_{ik-1}+1/m_i)$ Modulo 1,
       13. $b_{ik} \leftarrow \min\{ \text{integer } b : y_{ik}\, m^+ < b \}$,
       14. INCLUDE $((i,k),b_{ik})$,

       end,

   end.

15. Concatenate the ordered lists (Bin(b): b = 1,2,....,m^+) to obtain the sequence P.

Figure 3-3 The Arbitrary Frequencies Bin Packing Heuristic

The computational complexity of the heuristic is no greater than $O(n\, m^+ \log(n))$. Step 1 of the heuristic needs at most $n \log(n)$ computer operations. The loop beginning with step 8 is evaluated once per product, the loop beginning with step 12 is evaluated at most m^+ times for each product and the most time-consuming step, step 14, requires $O(\log(n))$ operations.

Numerical example 2

The example in Table 3-2 is used to demonstrate the differents steps of the heuristic which are illustrated in detail in Figures 3-4 to 3-8.

Product	1	2	3	4
Demand rate	10	8	3	2
Frequency	5	4	3	2
Order space size	2	2	1	1

Table 3-2 Example 2 Data

$m^+ = 5$ and $n^+=1$

step1: The products have already been arranged in a lexicographically decreasing order by (m_i,ψ_i).

Step 2: Initialization:

3. $y_{1k} = 0.2\,(k-1)$, $k = 1,2,3,4,5$.
4. $p(k) = 1$, $k = 1,2,3,4,5$.
5. $\mathrm{Bin}(k) = (1,k)$, $k = 1,2,3,4,5$.
6. $\mathrm{Load}(k) = 2$, $k = 1,2,3,4,5$.

Figure 3-4 Initialization Step

Step 7. i=2.

8. $b^* = 1$.

9. $y_{21} = 0$.

10. Load(1) = 4,

p(1) = 2,

Bin(1) = ((1,1),(2,1)).

11. k = 2.

12. $y_{22} = 0.25$.

13. $b_{22} = 2$.

14. Load(2) = 4,

p(2) = 2,

Bin(2) = ((1,2),(2,2)).

11. k = 3.

12. $y_{23} = 0.5$.

13. $b_{23} = 3$.

14. Load(3) = 4,

p(3) = 2,

Bin(3) = ((1,3),(2,3)).

11. k = 4.

12. $y_{24} = 0.75$,

13. $b_{24} = 4$.

14. Load(4) = 4,

p(4) = 2,

Bin(4) = ((1,4),(2,4)).

Figure 3-5 Assignment of the Orders of Product 2

Step 7. i=3.

8. b* = 5.

9. y_{31} = 0.8.

10. Load(5) = 3,

p(5) = 2,

Bin(5) = ((1,5),(3,1)).

11. k = 2.

12. y_{32} = 0.133.

13. b_{32} = 1.

14. Load(1) = 5,

p(1) = 3,

Bin(1) = ((1,1),(2,1),(3,2)).

11. k = 3.

12. y_{33} = 0.467.

13. b_{33} = 3.

14. Load(3) = 5,

p(3) = 3,

Bin(3) = ((1,3),(3,3),(2,3)).

Figure 3-6 Assignment of the Orders of Product 3

Step 7. i=4.

8. b* = 5.

9. $y_{41} = 0.8$.

10. Load(5) = 4,

p(5) = 3,

Bin(5) = ((1,5),(3,1),(4,1)).

11. k = 2.

12. $y_{42} = 0.3$.

13. $b_{42} = 2$.

14. Load(2) = 5,

p(2) = 3,

Bin(2) =((1,2),(2,2),(4,2)).

Figure 3-7 Assignment of the Orders of Product 4

Step 8

P = (1,2,3,1,2,4,1,3,2,1,2,1,3,4).

Figure 3-8 Concatenation Step

The sequence obtained by the Arbitrary Frequencies Bin Packing Heuristic is feasible for the fixed order quantity model since a feasible solution to RLP can be derived from the trial relative order times y_{ik}.

Most steps of the heuristic are similar to the steps of the Power of Two Bin Packing Heuristic except for those involving the relative trial order times (steps 3,9,12, and 13). These steps are needed to determine the bin numbers in which to assign the orders of each product (step 13) after assigning the first order for that product (step 8), and to position the order within the sequence for that bin (step 14).

If the frequencies are power-of-two integers, then the Arbitrary Frequencies Bin Packing Heuristic reduces to the Power-of-Two Bin Packing Heuristic. To see this, note that once b* is determined for each product i (step 8), then all its subsequent orders are placed in every m^+/m_i bins. By step 12, $y_{ik}=y_{ik-1}+1/m_i$ (without loss of generality, suppose that $y_{ik}<1$), then $y_{ik}\, m^+ = y_{ik-1}\, m^+ + m^+/m_i$ or $b_{ik} = b_{ik-1} + m^+/m_i$. Moreover, in the Power of Two Bin Packing Heuristic the operation INCLUDE is trivial: each order assigned to bin b is appended to the end of the ordered list Bin(b).

More elaborate sequencing heuristics could easily be constructed. For example the selection of the bin for which to assign the first order of product i is currently made on the basis of a simple comparison (step 8). Instead, we could try assigning the first order of product i to different bins, compare the vector of bin loads that would result and select the best one (the bin load vector with the minimum maximum load). Such a comparison is unnecessary in the Power-of-Two Bin Packing Heuristic because subsequent orders of a product are always placed in bins that have the same load as the first order of the product; but it could prove valuable in the context of arbitrary frequencies. The simple comparison in step 8 appears to work well so experimentation with alternative schemes is left for further research.

CHAPTER IV

TIME VARIANT LOT SIZES MODELS

Because of the complexity of the General Model formulated at the second stage of the approach to handle the WSP, the combinatorial and continuous aspects of the problem were separated. In Chapter III an Arbitrary Frequencies Bin Packing Heuristic was developed to generate the sequences. In this chapter, we assume that the sequence is given and we focus on lot sizing, delivery timing, cycle length, and on maximum space used. The parts will be allowed to be ordered several times in different amounts during the cycle.

This chapter is organized as follows: Section IV-1 recalls the assumptions and the notation. Section IV-2 reviews the formulation of the Linear Programming model , LP, that minimizes the maximum space used under the Zero Switch Rule, and presents a characterization of the optimal solution to the LP. Section IV-3 revises the Quadratic Program, QP, that minimizes the average holding cost and gives conditions under which a time phasing policy may be optimal for both LP and QP. Finally, Section IV-4 derives a bound on the quality of the LP for the NLP model.

IV-1 Assumptions and Notation

Before proceeding to the formulation, it is worthwhile recalling the assumptions and notation introduced in Chapter II.

For a given sequence,the Warehouse Scheduling Problem is described as follows:

1- there are n parts to be stocked in a warehouse with a limited space;

2- each part is ordered m_i times according to the sequence $P=(P_1,..., P_m)$, where m is the total number of orders;

3- the demand rate for each part is known and constant, and backorders are not allowed;

4- the production rate for each part is infinite and the replenishment is instantaneous;

5- there is a constant setup cost associated with each part;

6- an inventory cost proportional to the value of the stock held and proportional to the time for which stock is carried is charged to each part;

7- a storage space cost based on the maximum inventory held is incorporated in the cost function of RNLP;

8- the problem is to determine the ordering schedule: the timings of the m deliveries and the quantities delivered, to be repeated periodically every τ units of time.

For the sake of clarity in the formulation, the following matrices will be used in addition to the notation introduced in Chapter II.

$F^{(j)}$ an n by m 0–1 matrix defined by:

$$F_{il}^{(j)} = \begin{cases} 1 & \text{for } l=j,j+1,...,k-1 \text{ where } \delta_{ijk}=1, \text{ interpreted cyclically,} \\ 0 & \text{otherwise;} \end{cases}$$

E an m by m 0–1 matrix defined by:

$$E_{jl} = \begin{cases} 1 & \text{if } l \text{ is the position of the next order of the part ordered on the jth order,} \\ 0 & \text{otherwise;} \end{cases}$$

L an m by m 0–1 matrix whose entries are given by:

$$L_{jl} = \begin{cases} 1 & \text{for } l=j,j+1,\ldots,k-1 \text{ where } \delta_{ij}=1 \text{ and } \delta_{ijk}=1, \text{ interpreted cyclically,} \\ 0 & \text{otherwise;} \end{cases}$$

R an n by m 0–1 matrix defined by:

$$R_{ij} = \begin{cases} 1 & \text{if } \delta_{ij} = 1, \\ 0 & \text{otherwise;} \end{cases}$$

H an m by m diagonal matrix with:

$H_{jj} = \lambda_i h_i$ where $\delta_{ij} = 1$;

T,t m column vectors, $T=(T_j)_{j=1}^{m}$ and $t=(t_j)_{j=1}^{m}$;

U,u m column vectors, $U=(U_j)_{j=1}^{m}$ and $u=(u_j)_{j=1}^{m}$;

λ an n row vector, $\lambda=(\lambda_i)_{i=1}^{n}$;

K an n row vector, $K=(K_i)_{i=1}^{n}$;

A an m by m matrix whose each row A_l is defined by:

$A_l = \lambda F^{(l)}$;

e_m an m column vector of one's;

e_n an n column vector of one's.

Note that each entry i of the row vector $(F^{(j)} u)$ is the time interval until part i is ordered after the jth order; that is, $S_{ij} = (F^{(j)} u)$.

The notation introduced above is best understood through the following example. Consider the simple example of two parts where part one is ordered twice and part two is ordered only once according to the following sequence: P = (1,2,1).

The parameters of the example are: $\lambda_1 = 2$, $\lambda_2 = 1$, $h_1 = h_2 = 1$.

Using the definition of the binary variables introduced in Chapter II, it is easily seen that:

$\delta_{11} = 1, \delta_{22} = 1, \delta_{13} = 1,$
$\delta_{113} = 1, \delta_{123} = 1, \delta_{131} = 1,$
$\delta_{212} = 1, \delta_{222} = 1, \delta_{232} = 1.$

Thus

$$F^{(1)} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad F^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad F^{(3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 0 & 3 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Next, the formulations of the Linear Program, LP, that minimizes the maximum inventory, and the Quadratic Program, QP, that minimizes the average holding cost will be derived.

IV-2 Formulation of the Linear Programming Model

The objective function to be minimized for the LP model is the maximum inventory, ie;

$$W = \mathrm{Max}\left\{ \sum_{i=1}^{n} Z^{a}_{ij} : j=1,\dots,m \right\}, \tag{4-1}$$

where Z^{a}_{ij} is the amount of inventory of part i after receiving the jth delivery. The initial inventory for each part must be set so that backorders will not be incurred. Since the ZSR is assumed, the stock of any part on the jth order should be enough to last until its next order. Thus, Z^{a}_{ij} can be written as:

$$Z^a_{ij} = \lambda_i S_{ij}, \tag{4-2}$$

and the total inventory on the jth order is:

$$Z_j = \sum_{i=1}^{n} \lambda_i S_{ij}. \tag{4-3}$$

By definition, T_j is the time interval between the jth order and the next order of the part delivered on the jth order; ie,

$$T_j = \sum_{i=1}^{n} \delta_{ij} S_{ij}, \qquad j=1,\ldots,m. \tag{4-4}$$

Moreover, the reorder intervals for each part should sum to the cycle length.

$$\sum_{j=1}^{m} \delta_{ij} T_j = \tau, \qquad i=1,\ldots,n.$$

Therefore, the problem that minimizes the maximum inventory can be written as:

$$\text{Min} \left(\text{Max} \{ Z_j : j=1,\ldots,m \} \right)$$

s.t.

$$Z_j = \sum_{i=1}^{n} \lambda_i S_{ij}, \qquad j=1,\ldots,m,$$

$$T_j = \sum_{i=1}^{n} \delta_{ij} S_{ij}, \qquad j=1,\ldots,m,$$

$$\sum_{j=1}^{m} \delta_{ij} T_j = \tau, \qquad i=1,\ldots,n,$$

$$S_{ij} = U_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) U_k, \qquad j=1,\ldots,m \text{ and } i=1,\ldots,n,$$

$$U_j \geq 0, \qquad j=1,\ldots,m.$$

As it stands, the above model is not a linear program, but it can be transformed into one by observing that it is equivalent to minimizing W subject to $W \geq \text{Max}\ \{ Z_j : j=1,\ldots,m \}$ and the remaining equations. Furthermore, $W \geq \text{Max}\ \{ Z_j : j=1,\ldots,m \}$ is equivalent to $W \geq Z_j$ for $j=1,\ldots,m$. Thus, the problem becomes:

$$
\begin{array}{llll}
\text{Min} & W & & \\
\text{s.t.} & & & \\
& \displaystyle\sum_{i=1}^{n} \lambda_i \left(U_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) U_k \right) & \leq W, & j=1,\ldots,m, \\
& \displaystyle T_j - \sum_{i=1}^{n} \delta_{ij} \left(U_j + \sum_{k=j+1}^{j-1} (1-\Delta_{ijk}) U_k \right) & = 0, & j=1,\ldots,m, \\
& \displaystyle\sum_{j=1}^{m} \delta_{ij}\, T_j & = \tau, & i=1,\ldots,n, \\
& U_j & \geq 0, & j=1,\ldots,m.
\end{array}
$$

Or using the matrix notation introduced in the previous section, LP can be formulated as:

$$
\begin{array}{lllll}
\text{Min} & W & & & (4\text{–}5) \\
\text{s.t.} & & & & \\
& A\,U & -W\, e_m & \leq 0, & (4\text{–}6) \\
& -L\,U + T & & = 0, & (4\text{–}7) \\
& R\,T & -\tau\, e_n & = 0, & (4\text{–}8) \\
& U & & \geq 0. &
\end{array}
$$

Lemma IV-1. The above linear program can be simplified to:

$$
\begin{array}{lllll}
\text{Min} & W & & & (4\text{–}5) \\
\text{s.t.} & & & & \\
& A\,U & -W\, e_m & \leq 0, & (4\text{–}6) \\
& e_m'\, U & -\tau & = 0, & (4\text{–}9) \\
& U & & \geq 0. &
\end{array}
$$

Proof:

Substitution of (4-7) into (4-8) yields $R\,L\,U = \tau\, e_n$

The elements of the matrix (R L) are given by:

$$(R\,L)_{ij} = \sum_{l=1}^{m} R_{il}\, L_{lj} = \sum_{\{l:\, \delta_{il}=1\}} L_{lj} = 1,$$

since, by definition, $R_{il} = 1$ if part i is ordered on the lth order, and the rows of L that correspond to the orders on which part i is delivered partition the columns of L (see the first and third rows of the matrix L in the example presented above). To see this, consider the example presented above.

$$R\,L = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore, the system of equations $R\,L\,U = \tau\, e_n$ is composed of n identical equations $e_m' \; U = \tau$. (E.O.P)

The new linear program can be further simplified when written in terms of the relative maximum inventory, w, and the vector of relative time intervals, u.

$$\text{Min } w \tag{4–10}$$

s.t.

$$A\,u \quad -w\, e_m \le 0, \tag{4–11}$$

$$e_m'\, u \qquad = 1, \tag{4–12}$$

$$u \qquad \ge 0.$$

Finally, it is clear that the minimum value of w is positive, since the matrix A is a nonnegative matrix with positive diagonal elements (see Appendix I for the properties of the matrix A). Then, if we let $x = u / w$, (4-12) becomes

$$e_m'\, x = w^{-1}.$$

Moreover, note that minimizing w is equivalent to maximizing w^{-1}.

Thus, the above linear program can be transformed to:

$$\text{Max } \; e_m' \, x \tag{4–13}$$

s.t.

$$A \; x \le e_m, \qquad x \ge 0. \tag{4–14}$$

The next theorem gives a characterization of the optimal solution that minimizes the maximum inventory.

Theorem IV-1. The schedule that minimizes the maximum inventory fills the warehouse at each order; ie,

$$Z_1 = Z_2 = \ldots\ldots = Z_m. \tag{4-15}$$

Proof (By marginal analysis)

In the proof by marginal analysis many cases have to be examined. We illustrate the proof for only one case to convey the essential idea. A complete proof using the theory of M-matrices is presented in Appendix II.

Suppose that the total inventories, that correspond to the optimal schedule do not satisfy (4-15).

Let:

k be the order number such that $Z_k = \text{Min} \{ Z_j : j=1,\ldots,m \}$;

$i = P_k$, the number of the part ordered on the kth order;

$y^{(k)}$ be the ordering time of the kth order derived from the optimal schedule.

Now, consider another schedule where the kth order is placed at time $y^{(k)}+\varepsilon$, where ε is a very small real number.

Suppose that part i is ordered only once, then

$Z_j^{new} = Z_j - \varepsilon \, \lambda_i$ for all j such that $j \neq k$, and $Z_k^{new} = Z_k + \varepsilon \sum_{r=1,\, r\neq i}^{n} \lambda_r$.

So, for a small ε, $\underset{j}{\mathrm{Max}}\,(Z_j^{new}) < \underset{j}{\mathrm{Max}}\,(Z_j)$,

which contradicts that $\{Z_j: j=1,...,m\}$ is optimal (More cases must be considered if part i is ordered more than once and the maximum inventory does not occur between order k and the next order of part i).

(E.O.P)

By the above theorem, the optimal schedule that minimizes the maximum inventory can be obtained without using the simplex method to solve LP. The following Corollary gives the explicit optimal solution of LP.

Corollary IV-1. The optimal vector of relative time intervals, u, is given by:

$$u^* = \frac{A^{-1}\, e_m}{e_m'\, A^{-1}\, e_m}, \tag{4–16}$$

and the optimal relative maximum inventory is:

$$w^* = \frac{1}{e_m'\, A^{-1}\, e_m}. \tag{4–17}$$

Proof:

By Theorem IV-1, the optimal policy is to fill the warehouse at each order. The nonsingularity of A is established in Appendix II. Thus, the optimal solution to LP corresponds to binding constraints; ie,

$A\, x^* = e_m$ or, equivalently, $x^* = A^{-1}\, e_m$,

and $(w^*)^{-1} = e_m'\, x^* = e_m' A^{-1} e_m$,

or, equivalently, $w^* = (e_m' A^{-1} e_m)^{-1}$,

and $u^* = w^*\, x^* = (A^{-1}\, e_m) \,/\, (e_m' A^{-1} e_m)$. (E.O.P)

The following Lemmas and Propositions are also a consequence of Theorem IV-1.

Lemma IV-2. No more than one order can be placed at any point in time during the cycle when minimizing the maximum inventory.

Proof:

The proof is presented in Appendix II.

Proposition IV-1. The average warehouse utilization rate, ρ, is given by:

$$\rho = 1 - \frac{1}{2} \frac{\sum_{j=1}^{m} \sum_{i=1}^{n} \delta_{ij} (\lambda_i t_j^*)^2}{w^* \lambda_s}, \tag{4-18}$$

where $\lambda_s = \sum_{i=1}^{n} \lambda_i$.

Proof:

Using the result of Theorem IV-1, the reduction in space after the jth order should be equal to the space needed for the part ordered on the (j+1)st order(see Figure 4-1).

Thus, $$u_j^* \lambda_s = t_{j+1}^* \sum_{i=1}^{n} \lambda_i \delta_{ij+1},$$

or $$u_j^* = \frac{t_{j+1}^* \sum_{i=1}^{n} \lambda_i \delta_{ij+1}}{\lambda_s}, \quad j=1,\ldots.,m, \tag{4-19}$$

where (4-19) is to be interpreted cyclically. Therefore, the shaded area of unused capacity (Figure 4-1) between the jth order and the (j+1)st order is:

$$\frac{1}{2}\,(u_j^*)\left(t_{j+1}^*\sum_{i=1}^{n}\lambda_i\,\delta_{ij+1}\right)=\frac{1}{2}\,\frac{\left(t_{j+1}^*\displaystyle\sum_{i=1}^{n}\lambda_i\,\delta_{ij+1}\right)^2}{\lambda_s}. \qquad (4\text{–}20)$$

Hence, the total space used during the cycle of length 1 is:

$$w^* - \frac{1}{2\lambda_s}\sum_{j=1}^{m}\sum_{i=1}^{n}\delta_{ij}\,(\lambda_i\,t_j^*)^2.$$

Finally, the average space utilization is given by:

$$\rho = 1-\frac{1}{2\,\lambda_s\,w^*}\sum_{j=1}^{m}\sum_{i=1}^{n}\delta_{ij}\,(\lambda_i\,t_j^*)^2. \qquad (\text{E.O.P})$$

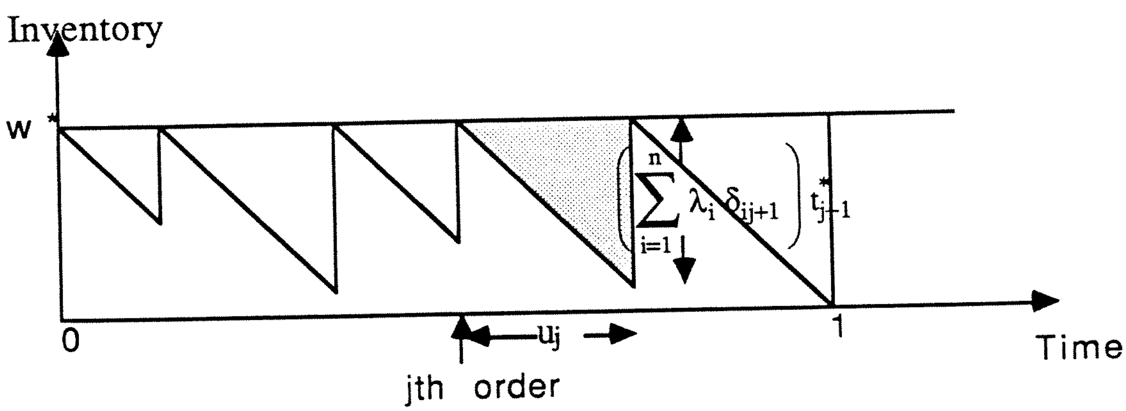

Figure 4-1 Filling the Warehouse

In the next proposition, an explicit formula for the optimal relative maximum inventory will be derived using the above proposition.

Proposition IV-2. The optimal relative maximum inventory is given by:

$$w^* = \frac{1}{2}\sum_{j=1}^{m}\left(\frac{\displaystyle\sum_{i=1}^{n}\lambda_i^2\,\delta_{ij}}{\lambda_s}+\sum_{i=1}^{n}\lambda_i\,\delta_{ij}\right)(t_j^*)^2. \qquad (4\text{–}21)$$

Proof:

Using the definition of the warehouse space utilization as the ratio of the average inventory to the maximum inventory, the warehouse space utilization can be rewritten as:

$$\rho = \frac{\frac{1}{2}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\lambda_i\,\delta_{ij}\right)(t_j^*)^2}{w^*}, \tag{4–22}$$

where the numerator is the average inventory.

Equating (4-22) to (4-18) yields:

$$\frac{\frac{1}{2}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\lambda_i\,\delta_{ij}\right)(t_j^*)^2}{w^*} = 1 - \frac{\sum_{j=1}^{m}\sum_{i=1}^{n}\lambda_i^2\,\delta_{ij}\,(t_j^*)^2}{2\,w^*\,\lambda_s},$$

or,

$$w^* = \frac{1}{2}\sum_{j=1}^{m}\left(\frac{\sum_{i=1}^{n}\lambda_i^2\,\delta_{ij}}{\lambda_s} + \sum_{i=1}^{n}\lambda_i\,\delta_{ij}\right)(t_j^*)^2. \tag{E.O.P}$$

In the following Lemma, the results of Homer[66], Page and Paul[76], Zoller[77], and Hall[85] will be derived using the above formula.

Lemma IV-3. If each part is ordered only once during the cycle, then

$$u_j^* = \frac{\sum_{i=1}^{n}\lambda_i\,\delta_{ij+1}}{\lambda_s}, \qquad j=1,\ldots,m, \tag{4–23}$$

and

$$w^* = \frac{1}{2}\left(\lambda_s + \frac{\sum_{i=1}^{n}\lambda_i^2}{\lambda_s}\right). \tag{4–24}$$

Proof:

If n=m, then

$$t_j = 1 \qquad j=1,\dots,m, \tag{4-25}$$

and (4-23) and (4-24) are obtained by substituting (4-25) into (4-19) and (4–21) respectively. (E.O.P)

Clearly, by (4-24) the optimal relative maximum inventory is sequence independent when m=n.

IV-3 Quadratic Programming Model

Since it assumed that the sequence is given, and therefore the frequencies are fixed, the long run setup cost per unit time is constant for a given cycle length. In this case, the objective function to be minimized is:

$$\frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_i\, h_i\, \delta_{ij}\, T_j^2. \tag{4–28}$$

Using the matrix notation introduced in first section, the Quadratic Program, QP, can be formulated as:

$$\text{Min } \frac{1}{2} T' H T$$

s.t.

$$\text{(QP)} \qquad \begin{array}{lll} AU & & \leq V e_m, \qquad (4\text{–}29) \\ & RT & = \tau e_n, \qquad (4\text{–}30) \\ LU - & T & = 0, \qquad (4\text{–}31) \\ U & & \geq 0, \qquad (4\text{–}32) \end{array}$$

where τ is assumed to be given.

Clearly, the optimal T is unique since H is a positive diagonal matrix.

However, U need not be unique since L is of rank (m-n+1).

Proposition IV-3. A lower bound for QP is given by $\frac{1}{2}\hat{T}'H\hat{T}$, where

$$\hat{T}_j = \sum_{i=1}^{m} \delta_{ij}\frac{\tau}{m_i} \quad \text{for } j=1,\ldots,m.$$

Proof:

It is clear that the following quadratic program

$$\text{Min } \frac{1}{2} T' H T$$

s.t.

$$R T = \tau e_n,$$

whose optimal solution is

$$\hat{T}_j = \sum_{i=1}^{m} \delta_{ij}\frac{\tau}{m_i} \quad \text{for } j=1,\ldots,m\ ,$$

constitutes a lower bound for QP. This optimal solution states that the quantity to be replenished for each part i whenever it is ordered is

$\tau \lambda_i / m_i$. (E.O.P)

The next two Lemmas and Theorem give conditions under which the same schedule minimizes both the maximum inventory and the average holding cost.

Lemma IV-4. Suppose that the optimal maximum inventory of LP is equal to the warehouse space available, V, then both LP and QP have the same optimal solutions.

Proof:

Suppose that one of the constraints, say the jth constraint, of (4-29) has a no zero slack. Then, the total inventory on the jth order is smaller than $V=\tau w^*$, where w^* is the optimal relative maximum inventory of LP, which contradicts that w^* is optimal. Moreover, by the uniqueness of the optimal solution of LP, (u^*,w^*), then the two problems have the same optimal solutions. (E.O.P)

Lemma IV-5. Suppose that the solution to LP yields a schedule with equal lot sizes and that the optimal maximum inventory does not exceed the warehouse space available, that is;

$$U^* = \tau \frac{A^{-1} e_m}{e_m' A^{-1} e_m}, \quad T_j^* = \sum_{i=1}^{n} \delta_{ij} \frac{\tau}{m_i} \quad \text{and } w^* \tau \leq V, \tag{4–33}$$

then (U^*,T^*) is also optimal for QP.

<u>Proof</u>:

Suppose that $(A\,U^* = w^* \tau) < V$. The case of equality is considered in Lemma IV-4. The Karush-Kuhn-Tucker conditions for QP are:

$$\begin{array}{llllll}
H\,T & & +\ \varphi - R'\zeta & & = 0 & (4–34)\\
 & A'\phi & -\ L'\varphi & -\xi & = 0 & (4–35)\\
 & A\,U & & & \leq V e_m & (4–36)\\
-\ T + L\,U & & & & = 0 & (4–37)\\
R\,T & & & & = \tau e_n & (4–38)\\
 & (AU - V)\ \phi & & & = 0 & (4–39)\\
 & U\xi & & & = 0 & (4–40)\\
 & \phi & & & \geq 0 & (4–41)\\
 & & & \xi & \geq 0. & (4–42)
\end{array}$$

Where ϕ, ζ, φ, ξ, are the Lagrangian multipliers of (4-29,30,31,32) respectively. Since $U^*>0$ (see Appendix II) and $A\,U^*<V$, then (4-39) and (4-40) are satisfied with $\xi = \phi = 0$. It is also clear that (4-36,37,38) are satisfied and (4–35) holds with $\varphi = 0$. Now, consider the system of equations $H\,T^* = R'\,\zeta$. This system is infeasible for unequal lot sizes. To see this, consider the example presented in Section IV-1. For this example, the system can be written as:

$$2\,T_1^* = \zeta_1$$
$$T_2^* = \zeta_2$$
$$2\,T_3^* = \zeta_1.$$

Clearly, if $T_1^* \neq T_3^*$, the system is infeasible. (E.O.P)

In the above Lemma, it has been supposed that the solution to LP yielded a schedule with equal lot sizes. This is not always the case unless some conditions on the parameters and on the sequence are met. In the next theorem, it will be shown that under some conditions the maximum inventory will be minimized with a schedule having equal reorder intervals for each part. Furthermore, under these conditions the theorem reveals that an explicit solution to LP is available without using the simplex method or inverting the matrix A.

We now recall some notation introduced in Chapter III.

Let:

$$m^+ = \operatorname*{Max}_i \{ m_i \};$$

$$\psi_i = \frac{\lambda_i}{m_i};$$

$$\psi_s = \sum_{j=1}^{m} \sum_{i=1}^{n} \delta_{ij}\, \psi_i.$$

Note that $\psi_s = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \delta_{ij} \right) \psi_i = \sum_{i=1}^{n} m_i\, \psi_i = \sum_{i=1}^{n} \lambda_i = \lambda_s.$

Theorem IV-3. Given that:

(i) $m_i = 2^{\alpha_i}$ $i=1,\dots,n$ and α_i integer;

(ii) the sequence $P = (P_1, P_2, \dots, P_m)$ is obtained by the Power-of-Two Bin Packing Heuristic;

(iii) $\sum_{i \in Bin(k)} \psi_i = \eta$ for k=1,...,m^+ and $\eta \geq 0$,

where $Bin(k) \subset \{1,2,...,n\}$ (see Section III–1 for the definition of Bin(k)).

Then

$$t_j^* = \sum_{i=1}^{n} \frac{\delta_{ij}}{m_i}, \qquad j=1,...,m, \tag{4–43}$$

$$u_j^* = \sum_{i=1}^{n} \frac{\psi_i}{\psi_s} \delta_{ij+1}, \qquad j=1,...,m, \tag{4–44}$$

$$w^* = \frac{1}{2}\left(\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{\psi_i}{m_i} \delta_{ij} + \frac{\sum_{j=1}^{m} \sum_{i=1}^{n} (\psi_i \, \delta_{ij})^2}{\psi_s} \right). \tag{4–45}$$

<u>Proof</u>:

To prove this theorem, we first show that under the above three conditions, the relative time intervals are given by (4-44). Then, we argue that the t_j's that correspond to the u_j's given by (4-44) satisfy (4-43). Finally, we show that these u_j's are optimal.

An important property of the sequences generated by the Power-of-Two Bin Packing Heuristic that is useful for the proof of this theorem is that for a given part i, all parts ordered before part i are the same in each bin where part i is assigned.

Now, if each bin is considered separately, the problem of minimizing the maximum inventory will be transformed to the case where the number of orders is equal to number of parts with ψ_i as the new demand rate for the ith part and $(m^+)^{-1}$ as the new cycle length. Then, for the kth bin, by Lemma IV-3, the time interval between the jth and (j+1)st orders is:

$$\frac{1}{m^+}\frac{\sum_{i=1}^{n}\psi_i\,\delta_{ij+1}^{(k)}}{\sum_{i\in Bin(k)}\psi_i} = \frac{1}{m^+\eta}\sum_{i=1}^{n}\psi_i\,\delta_{ij+1}^{(k)} \quad \text{for} \qquad 1\leq j\leq p(k), \tag{4–46}$$

where $\delta_{ij}^{(k)}$ is defined as before according to the sequence of the kth bin. Moreover, by condition (iii) and after summing over all bins, we have:

$$\sum_{k=1}^{m^+}\sum_{i\in Bin(k)}\psi_i = \eta\, m^+,$$

or
$$\sum_{i=1}^{n} m_i\,\psi_i = \eta\, m^+,$$

or
$$\psi_s = \eta\, m^+.$$

Hence, the time interval between between the jth and (j+1)st orders for the kth bin is proportional to the space size of the part ordered on the (j+1)st order:

$$\sum_{i=1}^{n}\frac{\psi_i}{\psi_s}\,\delta_{ij+1}^{(k)}.$$

By (4–46), it can be seen that if the part ordered on (j+1)st order is ordered more than once, then for each bin where it is assigned, the relative time interval, u_j, is the same. In addition, since the sequence is generated by the Power-of-Two Bin Packing Heuristic, the parts ordered before the jth order of the kth bin are the same in each bin where part P_j is placed. Consequently, the ordering time

$$y_j = \sum_{i\in\sigma_{bk}}\frac{\psi_i}{\psi_s}, \qquad \text{where } \sigma_{bk} = \{\, i : i = P_l \ \text{ for } 1<l<j \,\},$$

is constant in all these bins. Moreover, since the length of each bin is $(m^{+})^{-1}$, then the time interval ($(m^{+})^{-1} - y_j$) is also constant in each bin where P_j is assigned. Hence, the relative reorder interval of the part ordered on the jth order of the kth bin (see Figure 4-2) is:

$$t_j = \sum_{i \in \sigma_{bk}} \frac{\psi_i}{\psi_s} + \left(\frac{m^+}{m_r} - 1 \right) \frac{1}{m^+} + \left(\frac{1}{m^+} - \sum_{i \in \sigma_{bk}} \frac{\psi_i}{\psi_s} \right)$$

$$= \frac{1}{m_r}, \qquad \text{where } P_j = r.$$

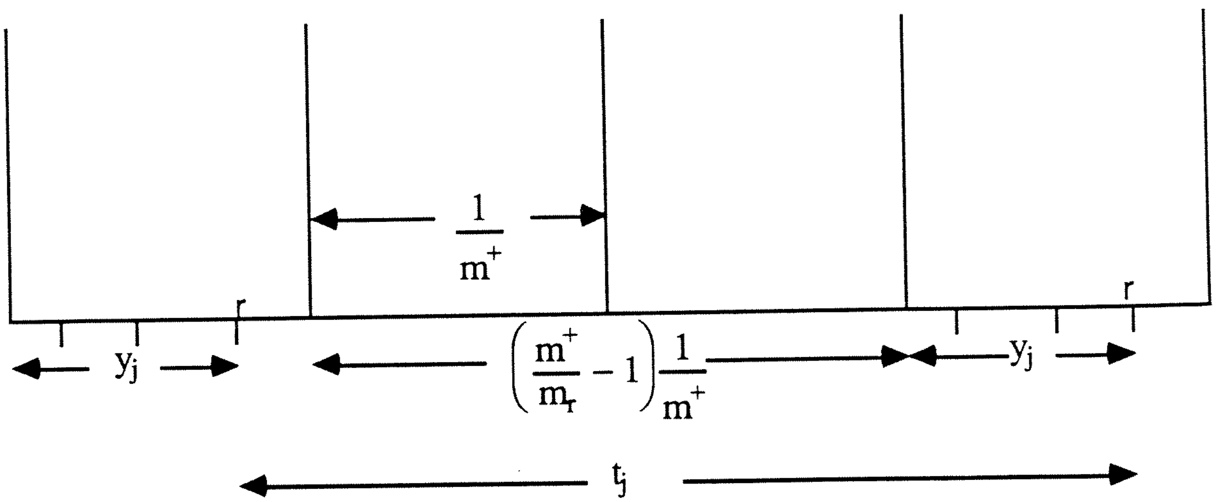

Figure 4-2 Relative Time Interval

Now, it remains to show that the relative time intervals given by (4–46) are optimal, that is; they correspond to the policy of ordering up to the maximum at each order. It suffices to show that :

$$\sum_{j=1}^{m} A_{lj}\, u_j = \sum_{j=1}^{m} A_{kj}\, u_j \qquad \text{for } l \neq k.$$

Without loss of generality, we take $l=k-1$ and we suppose that $m_i>1$, where $P_k=i$. By means of the properties of the matrix A presented in Appendix I, it can be shown that:

$$\sum_{j=1}^{m} (A_{kj}\, u_j - A_{lj}\, u_j) = -\lambda_s\, u_{k-1} + \lambda_i\, S_{ik}$$

$$= -\lambda_s\, u_{k-1} + \lambda_i\, t_k$$

$$= -\lambda_s \frac{\psi_i}{\psi_s} + \frac{\lambda_i}{m_i} \quad \text{by (4–46) and (4–43)}$$

$$= \psi_i \left(-\frac{\lambda_s}{\psi_s} + 1 \right)$$

$$= 0 \quad \text{since } \lambda_s = \psi_s. \qquad \text{(E.O.P)}$$

IV-4 Solution to NLP

NLP was defined in Chapter II as the problem minimizing the long run inventory costs per unit time subject to the space availability constraints. Using matrix notation, NLP can be formulated as:

$$\text{Min } \frac{1}{\tau}\left(K\, R\, e_m + \frac{1}{2} T'\, H\, T \right)$$

s.t.

$$\text{(NLP)} \quad \begin{array}{llll} A\,U & & \leq V\, e_m, \\ L\,U & - & T & = 0, \\ e_m'U & & & = \tau, \\ & & R\,T & = \tau\, e_n, \\ U & & & \geq 0. \end{array}$$

It is clear that a lower bound for NLP is given by the following problem:

$$\text{Min } \frac{1}{\tau}\left(K\, R\, e_m + \frac{1}{2} T'\, H\, T \right)$$

s.t.

$$R\,T \quad = \tau\, e_n.$$

Lemma IV-6. The optimal solution to the lower bound problem is given by:

$$T_j^* = \frac{\tau^*}{m_i} \qquad \text{for all j such that } P_j = i, \tag{4–47}$$

and

$$\tau^* = \sqrt{\frac{2\sum_{i=1}^{n} m_i K_i}{\sum_{i=1}^{n} \frac{\lambda_i h_i}{m_i}}}. \tag{4–48}$$

Proof:

The Lagrangian of the relaxed problem of NLP is

$\tau^{-1}(K R e_m + 0.5 T' H T) - \zeta(R T - \tau e_n)$.

Then, the Karush-Kuhn-Tucker conditions are:

$$H T \tau^{-1} - R'\zeta = 0 \quad \text{or} \quad T = \tau H^{-1} R' \zeta \tag{4–49}$$

$$-\tau^{-2}(K R e_m + 0.5 T' H T) + \zeta' e_n = 0 \tag{4–50}$$

$$R T = \tau e_n \quad \text{or using (4–49)} \quad \zeta = (R H^{-1} R')^{-1} e_n . \tag{4–51}$$

Using the definition of the R and H matrices and after some algebraic manipulations, it can be shown that:

$$\zeta_i^* = \frac{\lambda_i h_i}{m_i}, \qquad i=1,\ldots,n, \tag{4–52}$$

and thus $T_j^* = \dfrac{\tau^*}{m_i}$ $\qquad \forall$ j st $P_j = i$.

Substituting (4-49) and (4-51) in (4-50) yields:

$$\frac{KRe_m}{\tau^2} + \frac{1}{2} e_n'(RH^{-1}R')^{-1}RH^{-1}HH^{-1}R'(RH^{-1}R')^{-1}e_n - e_n'(RH^{-1}R')^{-1}e_n = 0,$$

or $\dfrac{KRe_m}{\tau^2} - \dfrac{1}{2} e_n'(RH^{-1}R')^{-1}e_n = 0,$

$$\text{or } \tau^* = \left(\frac{2\,KRe_m}{e_n'(RH^{-1}R')^{-1}e_n} \right)^{\frac{1}{2}} = \left(\frac{2\sum_{i=1}^{n} m_i\,K_i}{\sum_{i=1}^{n} \frac{\lambda_i\,h_i}{m_i}} \right)^{\frac{1}{2}}. \qquad \text{(E.O.P)}$$

A feasible solution to NLP can be obtained by the following algorithm:

1– Solve LP to get $\bar{w}$, $\bar{u}$, and $\bar{t} = L\,\bar{u}$.

2– If $\tau^*\,\bar{w} > V$, then set $\tau^* = V / \bar{w}$,
thus τ^* is given by:

$$\tau^* = \text{Min}\left\{ \left(\frac{2\sum_{i=1}^{n} m_i\,K_i}{\sum_{i=1}^{n} \frac{\lambda_i\,h_i}{m_i}} \right)^{\frac{1}{2}}, \frac{V}{\bar{w}} \right\}. \qquad (4\text{–}53)$$

If conditions of Theorem IV-3 are not satisfied, then $\bar{T} \neq T^*$. However, $\bar{T}$ can be written as:

$$\bar{T}_j = T_j^* + \theta_{ij} \qquad \forall\, j \text{ such that } P_j = i, \qquad i=1,\ldots,n, \qquad (4\text{–}54)$$

$$\text{where } \sum_{i=1}^{n} \delta_{ij}\,\theta_{ij} = 0, \qquad i=1,\ldots,n. \qquad (4\text{–}55)$$

Lemma IV-7. If $|\theta_{ij}| \leq \varepsilon\,T_j^*$, that is; if the reorder intervals obtained by solving LP differ by at most 100ε% of the optimal reorder intervals of NLP, then the error incurred for NLP by using the solution to LP is at most $100\varepsilon^2\%$.

Proof:

Let $C^{(lb)}$ be the lower bound on the long run inventory costs of NLP.

$$C^{(lb)} = \frac{\sum_{i=1}^{n} m_i K_i}{\tau^*} + \frac{\tau^*}{2} \sum_{i=1}^{n} \frac{\lambda_i h_i}{m_i}.$$

The long run inventory cost per unit time of NLP using the solution to LP is:

$$\frac{1}{\tau^*}\left\{ \sum_{i=1}^{n} m_i K_i + \frac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i h_i \delta_{ij} (\bar{T}_j)^2 \right\}$$

$$= \frac{1}{\tau^*}\left\{ \sum_{i=1}^{n} m_i K_i + \frac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i h_i \delta_{ij} (T_j^* + \theta_{ij})^2 \right\}$$

$$= \frac{1}{\tau^*}\left\{ \sum_{i=1}^{n} m_i K_i + \frac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i h_i \delta_{ij} ((T_j^*)^2 + 2\,\theta_{ij} T_j^* + \theta_{ij}^2) \right\}$$

$$= C^{(lb)} + \left\{ \sum_{i=1}^{n} \frac{\lambda_i h_i}{m_i} \sum_{j=1}^{m} \delta_{ij} \theta_{ij} \right\} + \frac{1}{2\tau^*} \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i h_i \delta_{ij} \theta_{ij}^2$$

$$= C^{(lb)} + 0 \qquad\qquad + \frac{1}{2\tau^*} \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i h_i \delta_{ij} \theta_{ij}^2 ,$$

by (4-55). Dividing the right hand side by $C^{(lb)}$ yields:

$$1 + \frac{\frac{1}{2\tau^*} \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i h_i \delta_{ij} \theta_{ij}^2}{C^{(lb)}} < 1 + \frac{\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i h_i \delta_{ij} \theta_{ij}^2}{(\tau^*)^2 \sum_{i=1}^{n} \frac{\lambda_i h_i}{m_i}}$$

$$< 1 + \frac{\sum_{i=1}^{n} \lambda_i h_i \sum_{j=1}^{m} \delta_{ij} \; \varepsilon^2 (\tau / m_i)^2}{(\tau^*)^2 \sum_{i=1}^{n} \frac{\lambda_i h_i}{m_i}}$$

$$< 1 + \varepsilon^2. \qquad \text{(E.O.P)}$$

Thus, the error cost incurred by using the optimal solution of the time variant lot sizes may not be important. A 20% maximum error of adjustment of the reorder intervals would cause an error of at most 4% for the long run average inventory costs per unit time.

Reviewing the results of this chapter we highlight the following observations. Given the order frequency and the order sequence, the problem of minimizing the maximum space used can be formulated as a linear program, LP. The optimal solution to LP is characterized by the property that every order fills the warehouse to the same level of space used. Furthermore, no two orders occur at the same time. A single matrix inversion step is all that is required to solve LP. Explicit formulas are possible for all variables of interest, including the average warehouse utilization rate. These results generalize the results obtained by Homer, Page and Paul, Zoller, and Hall under the Common Cycle assumption.

The solution to LP can be used to generate a solution to NLP formulated in Chapter II. Under certain circumstances, the solution to LP has the property of equal lot sizes even though the ELS restriction is not imposed in LP. In these circumstances, it follows that the solution to LP yields an optimal solution to NLP. These circumstances, correspond to a lower bound of the power-of-two bin packing problem considered in Chapter III. Hence, this result motivates the sequencing heuristics of

Chapter III.

In other cases, the solution to LP may not have the property of equal lot sizes or equal reorder intervals but the cost penalty of using LP solution in NLP is less than $100\varepsilon^2\%$ of the optimal NLP solution, where $100\varepsilon\%$ is the maximum percentage deviation of the LP solution from the equal reorder intervals.

In the next chapter we consider imposing the requirement that the lot sizes be equal for multiple orders of the same product.

CHAPTER V

EQUAL LOT SIZES MODELS

The previous chapter focused on the time variant lot sizes solutions which were obtained by solving LP. However, it was shown in section IV–5 that the equal lot sizes solution constitutes a lower bound for NLP. Moreover, the equal lot sizes solution presents some managerial advantages for the processing and the receipt of the orders.

This chapter deals entirely with models employing the equal lot size (ELS) restriction. In Section V-1 a formulation of the problem to minimize maximum inventory, RLP, will be developed. In Section V-2 we compare the solution of RLP with that of LP and determine an exact bound on the space penalty by imposing the equal lot sizes for a special case. We present an empirical comparison for the case of large problems. Section V-3 presents an efficient algorithm for minimizing the maximum space used. The procedure either solves RLP or suggests an improvement in the order sequence. Finally, we finish this chapter by deriving the optimal cycle length for the NLP model with the ELS restriction.

V-1 Formulation Based on the LP of Chapter IV

A formulation of the ELS problem to minimize the maximum inventory can be obtained from the linear program developed in Chapter IV by introducing the following set of constraints:

$$L\,u = \hat{t}, \tag{5–1}$$

where

$$\hat{t}_j = \sum_{i=1}^{n} \frac{\delta_{ij}}{m_i}, \qquad j=1,\ldots,m. \qquad (5-2)$$

Since $e_m' L = n\, e_m$ and $e_m' \hat{t} = n$, the constraint $e_m' u = 1$ is redundant.

Thus ELS can be formulated as:

$$
\begin{array}{lll}
\text{Min} & w & \\
\text{s.t.} & & \\
 & A\,u & \le w\, e_m, \\
(\text{RLP}) & L\,u & = \hat{t}, \\
 & u & \ge 0.
\end{array}
$$

This new linear program will be referred as the restricted linear program, RLP. Unlike the linear program of Chapter IV, the feasibility of RLP is not assured. The following example illustrates that RLP may be infeasible for certain sequence:

$P = (1,2,3,2,1,1,2,1,3)$

$\lambda_1 = 3 \quad \lambda_2 = 2 \quad \lambda_3 = 1$

$m_1 = 4 \quad m_2 = 3 \quad m_3 = 2$.

Clearly, P is infeasible since product 2 is ordered twice between two orders of product 1 whose reorder interval is smaller than the reorder interval of product 2. However, it was shown in Chapter III that the Arbitrary Frequencies Bin Packing Heuristic guarantees the feasibility of RLP. Hereafter, all the sequences are supposed to be generated by the Arbitrary Frequencies Bin Packing Heuristic.

It was shown in Chapter IV that joint replenishment does not occur for the time variant lot sizes model, LP. This property may not hold for ELS model. However, if joint replenishment does occur, it is possible to modify the sequence and perhaps further minimize the maximum space.

Lemma IV-1. If, in an optimal solution to RLP, the jth optimal relative interval, u_j^*, is zero, then the maximum space used can not be increased by swapping the jth and (j+1)st orders.

Proof:

Let (u^*, w^*) be the optimal solution of RLP, and A^S, L^S be the matrices obtained from A and L respectively by swapping the jth and (j+1)st orders. Thus the new RLP can be written as:

$$\text{Min}\{\ w \ \text{ st } \ A^S u \le w\, e_m\,;\, L^S u = \hat{t}^{\,S}\,;\, u \ge 0\ \}\,. \tag{5–3}$$

Let $((u^S)^*, (w^S)^*)$ denote the new optimal solution .

For any feasible solution, u, of (5-3), we have:

$(w^S)^* = \underset{k}{\text{Max}}\ A_k^S\,(u^S)^* \le \underset{k}{\text{Max}}\ A_k^S\, u,$

where A_k^S is the kth row of A^S.

Since $u_j^* = 0$, it is clear that u^* is feasible for (5-3),

hence $(w^S)^* \le \underset{k}{\text{Max}}\ A_k^S\, u^*$.

Moreover, since $u_j^*=0$, $\underset{k}{\text{Max}}\ A_k^S\, u^* = \underset{k}{\text{Max}}\ A_k\, u^*$,

thus $(w^S)^* \le \underset{k}{\text{Max}}\ A_k\, u^* = w^*$. (E.O.P)

In the next section, an exact bound on the space penalty for restricting the reorder intervals to be equal for each product will be determined for a special case, and empirical results for the case of large problems will be presented.

V-2 Bounds on the Space Penalty for Using ELS

V-2-1 Exact bound for special case

Consider the problem of two products where product 1 is ordered ν times and product 2 is ordered only once (on the second position of the sequence). The sequence is generated by the Arbitrary Frequencies Bin Packing Heuristic. Then, after long algebraic manipulations, it can be shown that for the time variant lot sizes model, LP, the optimal solution is:

$$u_1^* = \frac{\lambda_2}{\lambda_s}, \qquad \text{where } \lambda_s = (\lambda_1 + \lambda_2), \tag{5-4}$$

$$u_2^* = \left(\frac{\lambda_1}{\lambda_s}\right)^{m-j} \frac{\lambda_1 \lambda_2 \lambda_s^{\nu-2}}{\lambda_s^{\nu} - \lambda_1^{\nu}}, \qquad j=2,\ldots,\nu+1, \tag{5-5}$$

$$w^* = \frac{\lambda_2}{\lambda_s} \frac{\lambda_s^{\nu+1} - \lambda_1^{\nu} \lambda_2}{\lambda_s^{\nu} - \lambda_1^{\nu}}, \tag{5-6}$$

and for the equal lot sizes model, RLP, the optimal solution is:

$$\hat{u}_1 = \frac{\lambda_2}{\lambda_s}, \tag{5-7}$$

$$\hat{u}_2 = \frac{\lambda_1}{\lambda_s}, \tag{5-8}$$

$$\hat{u}_j = \frac{1}{\nu}, \qquad j=3,\ldots,\nu+1, \tag{5-9}$$

$$\hat{w} = \frac{1}{\lambda_s}\left(\frac{\lambda_1^2}{\nu} + \lambda_s \lambda_2\right). \tag{5-10}$$

After letting $\beta = \dfrac{\lambda_2}{\lambda_1}$, the ratio of the optimal solution of RLP to the optimal solution of LP can be written as:

$$\frac{\hat{w}}{w^*} = 1 + \frac{1}{\nu} \frac{(\beta+1)^{\nu} - \nu\beta - 1}{\beta\,((\beta+1)^{\nu+1} - \beta)}, \tag{5-11}$$

or, equivalently, $\frac{\hat{w}}{w^*} = 1 + f(\beta,\nu)$,

where

$$f(\beta,\nu) = \frac{1}{\nu} \frac{(\beta+1)^{\nu} - \nu\,\beta - 1}{\beta\,((\beta+1)^{\nu+1} - \beta)}. \tag{5-12}$$

The bound on the space penalty by using the ELS is 100 $f(\beta,\nu)$%. Using a search technique to determine the maximum of 100 $f(\beta,\nu)$, it is found that:

$$100\, f(\beta,\nu) \leq 32 \qquad \forall\ \beta > 0 \text{ and } \nu = 1,2........$$

Thus the space penalty on the optimal maximum space used incurred by imposing the ELS restriction is no more than 32% for this special case.

V-2-2 Empirical results

An analytical bound on the increase of the maximum space used by restricting the reorder intervals to be equal for each product for general frequencies is very hard to obtain. Consequently, an empirical study was conducted to develop some understanding on the space penalty of the restriction. A set of 145 problems was randomly generated and solved. The data were generated from uniform distribution on the following intervals:

Number of products	(discrete)	[2,10],
Frequencies	(discrete)	{1,2,4,8} and [1,8],
demand rates	(continuous)	[10,200].

The sequences were generated using the Arbitrary Frequencies Bin Packing Heuristic. The 145 problems were separated into two sets. The first set has an imposed power-of-two integer frequencies whereas the second set has arbitrary integer frequencies. Table 5-1 reports the result of this simulation. The "largest increase" in the second row of the table is defined to be the maximum space penalty incurred over all the problems tested. An examination of this table indicates that the ELS restriction has less of an impact if power-of-two frequencies are used. The average increase of the maximum space used is 8.77% and the largest increase is 21.82% when using power-of-two frequencies compared to a 25.05% and 54.57% respectively when using arbitrary integer frequencies. This conclusion is also suggested by the conditions of Theorem IV-3. Note also that the bound obtained for the special case is between the largest increase and the average increase.

	Set 1	Set 2	Total
Number of problems	72	73	145
Largest increase	21.82	54.57	54.57
Average increase	8.77	25.05	16.97

Table 5-1 Increase of the Maximum Space Used by Imposing ELS

V-3 Efficient Formulation of the ELS Model

RLP is somewhat large in size with 2 m constraints and (m+1) decision variables. However, by using the ordering time of the first order of each part as a decision variable, a new linear program with fewer constraints and decision variables can be derived. The equal lot sizes restriction implies that the relative reorder interval of product i is $(1/m_i)$ Thus, once the the ordering time of the first order of each product is fixed, the ordering times of all its subsequent orders are also determined.

Let

y_i = the ordering time of the first order of part i.

The ordering time of the kth order of product i, k = 1,2,...,m_i is given by:

$$y_{ik} = y_i + (k-1)\,\frac{1}{m_i}\ , \qquad (5\text{–}13)$$

and the ordering time of the jth order j=1,2,...,m is

$$y^{(j)} = \sum_{i=1}^{n}\left(y_i + (\Delta_{ij} - 1)\,\frac{1}{m_i}\right)\delta_{ij}\ , \qquad (5\text{–}14)$$

where $\Delta_{ij} = \sum_{l=1}^{j} \delta_{il}$ is the number of times product i is ordered before the jth order, inclusive.

Now, the amount of inventory of product i at the time of the jth order should be equal to the amount needed to last until its next order. Mathematically,

$$Z^a_{ij} = \lambda_i\left[\left(y_i + \frac{\Delta_{ij}}{m_i}\right) - y^{(j)}\right]. \qquad (5\text{–}15)$$

The first term between the brackets is the ordering time of the next order of i after the jth order. Then summing (5-15) over all parts and using (5–14), the total inventory on the jth order can be written as:

$$Z_j = \sum_{i=1}^{n}\lambda_i\, y_i - \lambda_s\sum_{i=1}^{n} y_i\,\delta_{ij} + \sum_{i=1}^{n}\lambda_i\,\frac{\Delta_{ij}}{m_i} - \lambda_s\sum_{i=1}^{n}(\,\Delta_{ij} - 1)\frac{\delta_{ij}}{m_i}\ . \qquad (5\text{–}16)$$

Note that for all j such that δ_{rj}=1, r= 1,2,...,n, (5-16) becomes

$$Z_j = \sum_{i\neq r}\lambda_i\, y_i - y_r\sum_{i\neq r}\lambda_i + \sum_{i=1}^{n}\lambda_i\,\frac{\Delta_{ij}}{m_i} - \frac{\lambda_s}{m_r}(\Delta_{rj}-1) \quad \forall j \text{ st } \delta_{rj}=1,\ r=1,..,n. \qquad (5\text{–}17)$$

The first set of constraints to be considered:

$Z_j \leq w$ j=1,2,..,m,

or, equivalently, using (5-17)

$$-\sum_{i\neq r}\lambda_i y_i + y_r\sum_{i\neq r}\lambda_i + w \geq \sum_{i=1}^{n}\lambda_i\frac{\Delta_{ij}}{m_i} - \frac{\lambda_s}{m_r}(\Delta_{rj}-1) \quad \forall\ j \text{ st } \delta_{rj}=1,\ r=1,..,n. \quad (5\text{–}18)$$

It is clear that the left hand side of (5-18) is the same for all j st $\delta_{rj}=1$, $r=1,...,n$. Letting:

$$\mu_r = \text{Max}\left\{\sum_{i=1}^{n}\lambda_i\frac{\Delta_{ij}}{m_i} - \frac{\lambda_s}{m_r}(\Delta_{rj}-1) \text{ for all j st } \delta_{rj}=1\right\} \quad r=1,2,...,n, \quad (5\text{–}19)$$

the number of constraints in (5-18) can be reduced to the following n constraints:

$$-\sum_{i\neq r}\lambda_i y_i + y_r\sum_{i\neq r}\lambda_i + w \geq \mu_r, \qquad r=1,...,n. \qquad (5\text{–}20)$$

Without loss of generality, let $y_1=0$, and introducing the surplus variables to (5-20) yields:

$$-\sum_{i=2}^{n}\lambda_i y_i + \qquad w - s_1 = \mu_1, \qquad (5\text{–}21.a)$$

$$-\sum_{i\neq r\ i\neq 1}\lambda_i y_i + y_r\sum_{i\neq r}\lambda_i + w - s_r = \mu_r, \qquad r=2,...,n. \qquad (5\text{–}21.b)$$

Now, after subtracting the first equation from the rth equation for $r=2,3,...,n$, (5-21.b) becomes:

$$\lambda_s y_r + s_1 - s_r = \mu_r - \mu_1, \qquad r=2,...,n. \qquad (5\text{–}22)$$

From equation (5-21 a), the maximum inventory can be written as:

$$w = \mu_1 + s_1 + \sum_{i=2}^{n}\lambda_i y_i\,,$$

or, after substituting $y_i = [(\mu_i - \mu_1) + (s_i - s_1)]/\lambda_s$ for $i=2,...,n$,

$$w = \sum_{i=1}^{n}\frac{\lambda_i}{\lambda_s}\mu_i + \sum_{i=1}^{n}\frac{\lambda_i}{\lambda_s}s_i\,. \qquad (5\text{–}23)$$

Proposition V-1

$$\mu_r > 0, \qquad\qquad r=1,\ldots,n.$$

Proof.

Recall that $\mu_r = \text{Max}\left\{ \sum_{i=1}^{n} \lambda_i \frac{\Delta_{ij}}{m_i} - \frac{\lambda_s}{m_r}(\Delta_{rj}-1) \text{ for all j st } \delta_{rj}=1 \right\}.$

Let k be the number of the order on which product r was ordered for the first time, then

$$\mu_r = \text{Max}\left\{ \sum_{i=1}^{n} \lambda_i \frac{\Delta_{ik}}{m_i} ; \left(\sum_{i=1}^{n} \lambda_i \frac{\Delta_{ij}}{m_i} - \frac{\lambda_s}{m_r}(\Delta_{rj}-1) \text{ for all j st } \delta_{rj}=1 \text{ and } j>k \right) \right\},$$

hence $\mu_r > 0$ since the first term is positive. (E.O.P)

The second set of constraints should ensures the feasibility of the schedule $(y^{(j)})_{j=1}^{m}$, that is;

$$y^{(j+1)} \geq y^{(j)}, \qquad j=1,2,\ldots,m-1, \text{ and}$$
$$y^{(m)} \leq \tau.$$

or, equivalently, using (5-14),

$$\sum_{i=1}^{n}\left(y_i+(\Delta_{ij+1}-1)\frac{1}{m_i} \right)\delta_{ij+1} \geq \sum_{i=1}^{n}\left(y_i+(\Delta_{ij}-1)\frac{1}{m_i} \right)\delta_{ij}, \quad j=1,\ldots,m-1, \quad (5-24)$$

and

$$\sum_{i=1}^{n}\left(y_i+(\Delta_{im}-1)\frac{1}{m_i} \right)\delta_{im} \leq \tau.$$

Finally, the linear program that minimizes the maximum inventory under the assumption of equal lot sizes is:

$$\text{Min} \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_s} \mu_i + \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_s} s_i$$

s.t.

(ELSLP)

$$\lambda_s y_i + s_i - s_1 = \mu_i - \mu_1, \qquad i=2,\ldots,n,$$

$$\sum_{i=1}^{n} (\delta_{ij+1} - \delta_{ij})\, y_i = \sum_{i=1}^{n} \left[(\Delta_{ij}-1)\frac{\delta_{ij}}{m_i} - (\Delta_{ij+1}-1)\frac{\delta_{ij+1}}{m_i} \right], \quad j=1,\ldots,m-1,$$

$$\sum_{i=1}^{n} y_i\, \delta_{im} \le \tau - \sum_{i=1}^{n} (\Delta_{im}-1)\frac{\delta_{im}}{m_i},$$

$$y_i \ge 0, \qquad i=2,\ldots n,$$

$$s_i \ge 0, \qquad i=1,\ldots,n.$$

This will be referred as the Equal Lot Sizes Linear Program, ELSLP.

Now consider the following relaxed version of ELSLP:

$$\text{Min} \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_s} \mu_i + \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_s} s_i$$

s.t.

(RELSLP)

$$\lambda_s y_i + s_i - s_1 = \mu_i - \mu_1, \qquad i=2,\ldots,n,$$

$$y_i \ge 0, \qquad i=2,\ldots n,$$

$$s_i \ge 0, \qquad i=1,\ldots,n.$$

RELSLP does not take into account the feasibility of the schedule. However, it can be solved by inspection. If $\mu_i - \mu_1 \ge 0 \;\; \forall\, i = 2,3,\ldots,n$, then $y_i = (\mu_i - \mu_1) / \lambda_s$ and the solution is optimal for RELSLP since all the surplus variables are zero. If $(\mu_i - \mu_1) < 0$ for some i, then $y_i = 0$ and $s_i = -(\mu_i - \mu_1)$ and the solution is optimal since all the surplus variables have been assigned the minimum value necessary to satisfy the equations(recall that the objective function of RELSLP is a weighted sum of the surplus variables).

Lemma V-2.

(1) RELSLP is feasible and has a unique solution, ($\bar{y}_i$, i=2,3,...,n).

(2) If (y_i^*, i=2,3,...,n) is an optimal solution of ELSLP for which constraints (5-24) are not binding, then (y_i^*, i=2,3,...,n) optimally solves RELSLP.

(3) If ($\bar{y}_i$, i=2,3,...,n) satisfies (5-24), then it optimally solves ELSLP.

(4) If($\bar{y}_i$, i=2,3,...,n) does not satisfies constraints (5-24), then the optimal solution to ELSLP satisfies (5-24) with at least one constraint binding.

Proof

Note that ELSLP is feasible since the sequence is generated by the Arbitrary Frequencies Bin Packing Heuristic.

(1), (2), and (3) are obvious.

(4). By (2), if no such constraint are binding, then (y_i^*, i=2,3,...,n) optimally solves RELSLP and by the uniqueness of the optimal solution to RELSLP, we have ($y_i^*=\bar{y}_i$, i=2,3,...,n). But ($\bar{y}_i$, i=2,3,...,n) does not sastisfy constraints (5-24), a contradiction. (E.O.P)

Next, an algorithm based on Lemmas V-1 and V-2 is developed to solve ELSLP.

Algorithm.

1- Compute μ_i i=1,2,...,n using (5-19).

2- If $\mu_i - \mu_1 \geq 0$ then $y_i^* = (\mu_i - \mu_1) / \lambda_s$ and $s_i^* = 0$,

otherwise $y_i^* = 0$ and $s_i^* = -(\mu_i - \mu_1)$.

3- $w^* = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_s} \mu_i + \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_s} s_i$.

4- If $y^{(j+1)} \geq y^{(j)}$ for j=1,...,m-1, and $y^{(m)} \leq \tau$, where

$$y^{(j)} = \sum_{i=1}^{n} \left(y_i^* + (\Delta_{ij} - 1) \frac{1}{m_i} \right) \delta_{ij} \quad j=2,3,\ldots,m,$$

then (y_i^*: i= 2,3,...,n) is optimal, stop.

Otherwise, swap the consecutive orders for which the infeasibility occurs for the same products to get a new sequence and go to 1.

Note that by Lemma V-2, the infeasibility of (y_i^*, i=2,3,...,n) is due to at least one binding constraint of (5-24). Thus, at least one relative time interval, u_j^*, is zero and by Lemma V-1 an improvement of the maximum inventory may be achieved by swapping the orders for which the infeasibility occurs. Finally, an upper bound on the number of iterations to be performed should to be imposed because of a lack of the proof of convergence. However, it has been observed that that only a small number of iterations is needed in all the examples tested. Two example are presented next to illustrate the algorithm. For the first example, the optimal solution to RELSLP is feasible for ELSLP , whereas for the second one it is not feasible. Table 5-2 presents the data for the first example.

Product	1	2	3	4	5
Demand rate	5	4	3	2	2
Frequency	4	2	2	2	2

Table 5-2 Data for Example 1

The sequence is P=(1,2,5,1,3,4,1,2,5,1,3,4).

Step 1. The vector μ =(1.5, 3.25, 7, 8, 4.25).

Step 2. The vector y*=(0, 0.109, 0.344, 0.406, 0.172).

Step 3. w* = 4.125.

Step 4. The corresponding schedule is:

(0, 0.109, 0.172, 0.25, 0.348, 0.406, 0.5, 0.609, 0.672, 0.75, 0.844, 0.906),

which is feasible; hence it is optimal for ELSLP.

The data given in Table 5-3 is used to illustrate the algorithm when the optimal solution to RELSLP is not feasible for ELSLP.

Product	1	2	3	4	5
Demand rate	4	2	2	2	2
Frequency	5	4	3	2	2

Table 5-3 Data for Example 2

The sequence is P=(1,2,3,1,2,4,5,1,3,2,1,2,4,1,3,5).

Step 1. The vector μ =(1.267, 1.3, 3, 4.267, 6).

Step 2. The vector y*=(0, 0.0027,0.144, 0.25, 0.3944).

Step 3. w*= 2.85.

Step 4. The corresponding schedule is:

(0, 0.0027, 0.144, 0.2, <u>0.2527</u>, <u>0.25</u>, 0.3944, 0.4, 0.477, 0.5027, 0.6, <u>0.7527</u>, <u>0.75</u>, 0.8, 0.811, 0.8944).

Infeasibility occurs on the 5th and 12th orders for which:

P_5=2, P_6=4 and P_{12}=2, P_{13}=4.

Swap the 5th and the 6th orders and also the 12th and 13th orders.

The new sequence is P=(1,2,3,1,4,2,5,1,3,2,1,4,2,1,3,5).

Step 1. The vector μ =(1.267, 1.3, 3, 3.767, 6).

Step 2. The vector y*=(0, 0.0027, 0.144, 0.2083, 0.3944).

Step 3. w*= 2.767.

Step 4. The corresponding schedule is:

(0, 0.0027, 0.144, 0.2, 0.2083, 0.2527, 0.3944, 0.4, 0.477, 0.5027, 0.6, 0.7083, 0.7527, 0.8, 0.811, 0.8944),

which is feasible; hence it is optimal for ELSLP (with the modified sequence).

V-4 Optimal Cycle Length for the NLP Model

In this last section, we will recall the optimal cycle length for the NLP model which was introduced in Chapter II.

If NLP is to be solved, then the optimal cycle length would be:

$$\tau^* = \mathrm{Min}\left\{ \left(\frac{2\sum_{i=1}^{n} m_i K_i}{\sum_{i=1}^{n} \frac{\lambda_i h_i}{m_i}} \right)^{\frac{1}{2}} ; \frac{V}{w^*} \right\}.$$

CHAPTER VI

THE CONVERGENT FREQUENCIES ALGORITHM

In this chapter we integrate the techniques of the previous chapters into an optimization based heuristic algorithm that determines a complete solution to the WSP: ordering frequencies, ordering sequence, a relative cyclic delivery schedule, and a cycle length. Section VI-1 introduces the algorithm , and Section VI-2 shows that the algorithm compares favorably with other methods presented in the literature.

VI-1 The Convergent Frequencies Algorithm

Many heuristics have been developed in the ELSP area to find near optimal frequencies and cycle length(see Elmaghraby[78] for an excellent review of these heuristics). The heuristics alternate between an optimization of the long run inventory costs per unit time with respect to the cycle length for given frequencies and an optimization of the same objective function with respect to the frequencies for a given cycle length. As pointed out by Schweitzer and Silver[83], the long run average inventory costs per unit time need not have a minimum, but have generally an infimum when minimized with respect to the frequencies and the cycle length. The algorithm that we are about to present is based on a similar approach. Computational experience with the algorithm shows that it yields near-optimal frequencies and cycle length after a small number of iterations.

The Convergent Frequencies Algorithm, CFA, is an iterative algorithm that integrates all the heuristics and methods developed in the previous chapters. The starting frequencies and cycle length are based on the solution of the problem in which each part is ordered independently of the others, the so-called "Independent Solution". The Arbitrary Frequencies Bin Packing Heuristic is then used to generate the sequence. Given the sequence and the cycle length, the optimal maximum inventory is determined using the algorithm developed in Chapter V. In subsequent iterations, candidate frequencies are generated for longer and longer cycle lengths. The corresponding reorder intervals converge to the reorder intervals of the independent solution. In this way the inventory cost of constraining all parts to share the same cycle length approaches an infimum. In practice, convergence is very rapid. The algorithm is based on the equal lot sizes restriction because of the managerial advantages of equal lot sizes and to facilitate comparison with other algorithms in the literature. A version of the algorithm using time variant lot sizes is easily developed.

The optimal reorder intervals for the independent solution are given by:

$$T_i^I = \sqrt{\frac{2\,K_i}{\lambda_i\,h_i}}, \qquad i=1,\ldots,n, \tag{6–1}$$

and the corresponding minimum cost is:

$$C^I = \sum_{i=1}^{n} \sqrt{2\,\lambda_i\,h_i\,K_i}. \tag{6–2}$$

Let T^+ denote any base cycle length such that

$$T^{+} \geq \operatorname{Max} \{ T_i^I : i=1,...,n\}, \tag{6–3}$$

and

$$m_i(k)=\left[\frac{k\,T^{+}}{T_i^I}\right] \qquad i=1,\ldots,n \;\text{ and } k=1,2,\ldots, \tag{6–4}$$

where the symbol [ ] gives the nearest integer, and k is an arbitrary constant, called the cycle factor.

Let $T_i(k)$ denote the reorder interval of part i given a cycle length of $k\,T^{+}$, a frequency of $m_i(k)$, and the assumption of equal lot sizes:

$$T_i(k)= \frac{k\,T^{+}}{m_i(k)} \qquad i=1,\ldots,n \;\text{ and } k=1,2\ldots\ldots \tag{6–5}$$

Lemma VI-1. $T_i(k)$ converges to T_i^I as k goes to infinity.

<u>Proof</u>:

By (6-4) and (6-5),

$$T_i(k) = \frac{k\,T^{+}}{\left[\frac{k\,T^{+}}{T_i^I}\right]},$$

$$\text{then } T_i(k) - T_i^I = \frac{k\,T^{+}}{\left[\frac{k\,T^{+}}{T_i^I}\right]} - T_i^I,$$

$$\text{but } \frac{k\,T^{+}}{T_i^I} - 0.5 \leq \left[\frac{k\,T^{+}}{T_i^I}\right] \leq \frac{k\,T^{+}}{T_i^I} + 0.5,$$

$$\text{thus } \frac{k\,T^{+}}{\frac{k\,T^{+}}{T_i^I}+0.5} - T_i^I \leq T_i(k) - T_i^I \leq \frac{k\,T^{+}}{\frac{k\,T^{+}}{T_i^I}-0.5} - T_i^I,$$

hence as $k \rightarrow \infty$, $T_i(k) \rightarrow T_i^I$, $\forall$ i. (E.O.P)

Note that the same result holds if the frequencies $m_i(k)$ are obtained by rounding up or down the quotient $(k\, T^{+} / T_i^I)$.

VI-1-1 CFA for solving NLP

In this section we present the CFA algorithm, discuss the steps of the algorithm, and illustrate it with an example from the WSP literature. When solving the NLP with the restriction of equal lot sizes, the algorithm is composed of the following steps:

1) Compute $T_i^I = \sqrt{(2\, K_i)/(\lambda_i\, h_i)}$ and $C^I = \sum_{i=1}^{n} \sqrt{2\, K_i\, \lambda_i\, h_i}$,

let $T^{+} = \max_i T_i^I$, and $k=1$.

2) Let

$$m_i^1(k) = [\, k\, T^{+} / T_i^I \,] \qquad i=1,\dots,n,$$

$$m_i^2(k) = \lceil k\, T^{+} / T_i^I \rceil \qquad i=1,\dots,n,$$

$$m_i^3(k) = \lfloor k\, T^{+} / T_i^I \rfloor \qquad i=1,\dots,n,$$

$$m_i^4(k) = \begin{cases} m_i^2 & \text{if } C_i^2 < C_i^3, \\ m_i^3 & \text{otherwise,} \end{cases} \qquad i=1,\dots,n,$$

where $C_i^j = \dfrac{K_i\, m_i^j(k)}{k\, T^{+}} + \dfrac{1}{2}\lambda_i\, h_i\, \dfrac{k\, T^{+}}{m_i^j(k)}$.

Find the set of frequencies that gives the minimum cost for fixed frequencies to the unconstrained problem; ie, find

$C^{j^*}(k) = \text{Min}\,\{\, C^j(k) : j=1,2,3,4 \}$,

where $C^j(k) = \sqrt{2\left(\sum_{i=1}^{n} K_i\, m_i^j(k)\right)\left(\sum_{i=1}^{n} \dfrac{\lambda_i\, h_i}{m_i^j(k)}\right)}$,

and let $m_i(k) = m_i^{j^*}(k)$ for $i=1,\dots,n$, and $C(k) = C^{j^*}(k)$.

3) Compute the cycle length $\tau(k)=\sqrt{2\left(\sum_{i=1}^{n} K_i\, m_i(k)\right) / \left(\sum_{i=1}^{n} \frac{\lambda_i\, h_i}{m_i(k)}\right)}$

4) Generate the sequence using the Arbitrary Frequencies Bin Packing Heuristic.

5) Compute the relative equal lot sizes maximum space used, w(k), using the algorithm developed in Chapter V.

6) If $\tau(k)\, w(k) < V$, then $C^*(k) = C(k)$,

otherwise, $\tau(k)=V/w(k)$, and $C^*(k)=\left(\sum_{i=1}^{n} K_i m_i(k)\right)/\tau(k)+\tau(k) \sum_{i=1}^{n} (\lambda_i h_i)/(2m_i(k))$.

7) If $(C(k) - C^I) / C^I \leq \varepsilon$ go to 8, otherwise $k \leftarrow k+1$ and go to 2.

8) If $C^*(k) = C(k)$, then $C^* = C(k)$, otherwise, $C^* = \text{Min}\{\, C(l): l=1,\ldots,k\}$.

In Step 2, the set of frequencies is selected among four candidates and it corresponds to the one that yields the smallest cost for a fixed frequencies.

For given frequencies and the equal lot sizes assumption, the long run average inventory cost per unit time is:

$$\sum_{i=1}^{n}\left(\frac{K_i\, m_i}{\tau} + \frac{\tau}{2} \frac{\lambda_i\, h_i}{m_i} \right),$$

thus the optimal cycle length for these given frequencies is:

$$\sqrt{2\left(\sum_{i=1}^{n} K_i\, m_i\right) / \left(\sum_{i=1}^{n} \frac{\lambda_i\, h_i}{m_i}\right)},$$

and the optimal cost is:

$$\sqrt{2\left(\sum_{i=1}^{n} K_i\, m_i\right) \left(\sum_{i=1}^{n} \frac{\lambda_i\, h_i}{m_i}\right)}.$$

In Steps 1 through 5 and Step 7, the algorithm ignores the space constraint to compute the frequencies and the cycle length, and in Step 6 the cycle length is adjusted, if necessary, to ensure feasibility. Step 8 is needed because the cost is not monotonically decreasing with k. Finally, a cycle factor of the form $k^{\left(\frac{1}{\alpha}\right)}$,where α is a given integer, would give a better solution, but convergence would be slower.

The algorithm is illustrated with the two product example of Thomas and Hartley[83]. The data is reported in Table 6-1

Product	Demand Rate	Setup Cost	Holding Cost
1	1000	25	0.5
2	1000	20	2

Table 6-1 Thomas and Hartley Example

The space available is 450 sqft and the optimal cost of the independent solution is 440.95659 \$. In the first iteration, we have:

$k=1$: $\tau(1)=0.2944$ $m_1(1)=1$ $m_2(1)=2$ $C^*(1)=441.59$ $W(1)=367.99$,

where $W(k)= \tau(k)\ w(k)$. In the fourth iteration, we have:

$k=4$: $\tau(4)=1.267$ $m_1(4)=4$ $m_2(4)=9$ $C^*(4)=440.96$ $W(4)=440.96$.

At the fourth iteration $\varepsilon=0.00044\%$, thus $C^*=440.96$.

The result of Thomas and Hartley are given in the table below and compared with the above solution.

Technique	Max Space Used	Cost	m1	m2	Cycle length	Num Iter
Thomas & Hartley	449	440.96	9	20	2.835	20
CFA	440.96	440.96	4	9	1.26996	4

Table 6-2 Comparison of CFA Solution with Thomas and Hartley Solution

The CFA algorithm needed only four iterations to converge whereas the Thomas and Hartley algorithm required twenty iterations to complete. (Comparison of iteration counts can only suggest computational differences, because the Thomas and Hartley algorithm is a completely different approach) Moreover, the solution obtained by CFA is optimal in this case since Thomas and Hartley showed that their procedure yields an optimal solution for the two product problem.

Next, we will show that the CFA converges after a finite number of iterations for a given cost error ε.

Lemma VI-2. The Convergent Frequencies Algorithm converges after a finite number of iterations.

<u>Proof</u>.

Recall that the stopping criterion for CFA is $(C(k) - C^I) / C^I \leq \varepsilon$, where ε is a small number. Recall also that

$$C(k)=\sum_{i=1}^{n}\left(\frac{K_i\, m_i(k)}{\tau(k)}+\frac{1}{2}\,\frac{\lambda_i\, h_i\, \tau(k)}{m_i(k)}\right),$$

$$\tau(k)=\sqrt{2\left(\sum_{i=1}^{n} K_i\, m_i(k)\right)\Big/\left(\sum_{i=1}^{n}\frac{\lambda_i\, h_i}{m_i(k)}\right)},$$

$$C^I=\sum_{i=1}^{n}\sqrt{2\lambda_i\, h_i\, K_i}.$$

Since $\tau(k)$ is optimal for the given $m_i(k)$'s, no other cycle length can yield a lower cost. In particular,

$$C(k) \le \sum_{i=1}^{n} \left(\frac{K_i\, m_i(k)}{k\, T^+} + \frac{1}{2} \frac{\lambda_i\, h_i\, k\, T^+}{m_i(k)} \right).$$

Moreover, since

$$\frac{k\, T^+}{T_i^I} - 0.5 \le \left(m_i(k) = \left[\frac{k\, T^+}{T_i^I} \right] \right) \le \frac{k\, T^+}{T_i^I} + 0.5,$$

(we suppose, without loss of generality, that the frequencies are given by rounding to the nearest integer the quotient ($k\, T^+ / T_i^I$)) then,

$$C(k) \le \sum_{i=1}^{n} \left(\frac{K_i}{k\, T^+} \left(\frac{k\, T^+}{T_i^I} + 0.5 \right) + \frac{1}{2} \lambda_i\, h_i \frac{k\, T^+}{\dfrac{k\, T^+}{T_i^I} - 0.5} \right)$$

$$\le \sum_{i=1}^{n} \frac{K_i}{T_i^I} + \frac{0.5 \sum_{i=1}^{n} K_i}{k\, T^+} + 0.5\, k\, T^+ \sum_{i=1}^{n} \frac{\lambda_i\, h_i\, T_i^I}{k\, T^+ - 0.5\, T_i^I}$$

$$\le \sum_{i=1}^{n} \frac{K_i}{T_i^I} + \frac{0.5 \sum_{i=1}^{n} K_i}{k\, T^+} + 0.5 \frac{k}{k-0.5} \sum_{i=1}^{n} \lambda_i\, h_i\, T_i^I$$

$$\le \frac{2\, k - 0.5}{k - 0.5} \sum_{i=1}^{n} \sqrt{0.5\, \lambda_i\, h_i\, K_i} + \frac{a'}{k},$$

where $a' = \dfrac{0.5 \sum_{i=1}^{n} K_i}{T^+}$.

After dividing by C^I, we get:

$$\frac{C(k)}{C^I} = \frac{k - 0.25}{k - 0.5} + \frac{a}{k}, \quad \text{where} \quad a = \frac{a'}{C^I}.$$

Next, the values for k such that $\frac{k-0.25}{k-0.5} + \frac{a}{k} \leq 1+\varepsilon$ are to be determined.

$\frac{k-0.25}{k-0.5} + \frac{a}{k} \leq 1+\varepsilon$ is equivalent to:

$$-\varepsilon k^2 + (0.25 + a + 0.5\,\varepsilon)\,k - 0.5\,a \leq 0, \tag{6–6}$$

which holds for:

$$k \geq \frac{(0.25 + a + 0.5\,\varepsilon) + \sqrt{(0.25 + a + 0.5\,\varepsilon)^2 - 2\,a\,\varepsilon}}{2\,\varepsilon} \equiv k_1,$$

and for

$$k \leq \frac{(0.25 + a + 0.5\,\varepsilon) - \sqrt{(0.25 + a + 0.5\,\varepsilon)^2 - 2\,a\,\varepsilon}}{2\,\varepsilon} \equiv k_2,$$

Note that $\left[\frac{k\,T^+}{T_i^I}\right] > 0$ if $k > 0.5$ for i=1,...,n.

It can easily be proven that $k_2 < 0.5$, thus (6-6) holds for $k \geq k_1$.

So for any $k \geq k_1$,

$$\frac{C(k) - C^I}{C^I} \leq \varepsilon\,. \qquad \text{(E.O.P)}$$

VI-2 Computational Comparisons

The algorithm is first compared with the Doll and Whybark[73] algorithm and the Brown[67] algorithm using a simple example provided in the Doll and Whybark paper. The algorithm is also compared with Goyal[78] results using the Hadley and Whitin[63] example and the Johnson and Montgomory[74] example. Then, we show that CFA yields an optimal solution in each of the six problems of two-product problems considered in Thomas and Hartley[83]. Finally, we present the results of two problems whose data are randomly generated. The data for all the problems are reported in Tables 6-3 to 6-8.

Product	Demand Rate	Setup Cost	Holding Cost
1	7200	9	1
2	20000	4	1

$V=\infty$

$C^I=760\$$

Table 6-3 Doll and Whybark Example

Product	Demand Rate	Setup Cost	Holding Cost
1	50000	50	0.2
2	20000	50	0.2
3	160000	50	0.2

V=15000 sqft

$C^I=3421.31\$$

Table 6-4 Johnson and Montgomory Example

Product	Demand Rate	Setup Cost	Holding Cost
1	20000	50	0.2
2	50000	75	0.2
3	100000	100	0.2

V=14000 sqft

$C^I=3857.2$ $

Table 6-5 Hadley and Whitin Example

Problem	1	2	3	4	5	6
Setup Cost(1)	10	10	20	25	25	25
Setup Cost(2)	45	10	10	20	20	20
Holding Cost(1)	1	1	1	0.5	0.5	0.5
Holding Cost(2)	1	1	1	2	2	2
Demand Rate(1)	2000	2000	2000	1000	1000	1000
Demand Rate(2)	1000	500	5000	1000	1000	1000
V	440	84	598	450	300	400
Cost of "IS"	500	300	5999.1	440.96	440.96	440.96

Table 6-6 Thomas and Hartley Examples

Product	Demand Rate	Setup Cost	Holding Cost
1	400	10	0.5
2	850	3	0.1
3	400	4	3
4	300	7	0.067
5	160	5	0.1

V=700 sqft

C^I=213.1906$

Table 6-7 Test Problem 1

Product	Demand Rate	Setup Cost	Holding Cost
1	1000	100	20
2	3000	120	7.5
3	4000	70	20
4	9200	160	2.5

V=1550 sqft

C^I=10383.36$

Table 6-8 Test Problem 2

Comparisons of the results are shown in the next Tables 6-9 to 6-12.

Technique	m1	m2	Cycle Length	Cost	Num Iter
D&W	1	3	0.5532	763	2
Brown	1	2	0.0444	765	x
CFA	2	5	0.1	760	2

x not reported.

Table 6-9 Comparison Using Doll and Whybark Example

Problem	Johnson and Montgomory			Hadley and Whitin			
Technique	J&M	Goyal	CFA	H&W	P&P	Goyal	CFA
Max Space	15000	14200	14903	14000	14000	13751.2	14000
Excess Cost %	0.9	0.1	0	5.3	1.2	1	0.8
m1	x	3	5	x	1	2	2
m2	x	2	3	x	1	3	3
m3	x	6	9	x	1	3	3
Cycle Length	x	0.2919	0.4967	x	x	0.291	0.296

x not reported.

Table 6-10 Comparison Using J &M and H&W Examples

Problem	1	1	2	2	3	3	4	4	5	5	6	6
Technique	T&H	CFA	T&H	CFA	T&H	CFA	T&H	CFA	T&H	CFA	T&H	CFA
Max space	433	433	84	84	595	577	449	441	300	300	368	368
Excess Cost %	0	0	70.9	70.9	0	0	0	0	2.2	2.2	0.1	0.1
m1	3	3	2	2	72	4	9	4	1	1	1	1
m2	1	1	1	1	161	9	20	9	2	2	2	2
Cycle Length	0.3	0.3	0.06	0.06	10.2	0.57	2.84	1.27	0.24	0.24	0.29	0.29
Num Iter	3	1	7	1	169	4	20	4	1	4	3	4

Table 6-11 Comparison to Thomas and Hartley results

Problem	1	2
Number of Products	5	4
Max Space Used	582.117	1458.771
Cost	213.226	10383.96
Frequencies	5-6-20-2-2	7-7-17-6
Cycle length	1.61332	0.71071
Num Iterations	2	6

Table 6-12 Result of Test Problems 1&2

In the above tables, the excess cost is defined as the cost of the solution in excess of the independent solution cost.

As can be seen from the above tables, the algorithm behaves favorably in comparison to other methods in terms of computation (Thomas and Hartley example) and cost. Finally, if more iterations are carried out for the two test problems, then their cost solutions come very close to the independent solution costs:

Test problem 1:
$k=11$, $\tau=2.24$, $m_1=29$, $m_2=35$, $m_3=113$, $m_4=11$, $m_5=12$,
$C^*=213.1974$, $W=675.59$.

Test problem 2:
$k=28$, $\tau=3.303$, $m_1=33$, $m_2=32$, $m_3=79$, $m_4=28$,
$C^*=10383.36$, $W=1530.56$.

However, the "penalty" to be paid is a large cycle length and a long sequence. This was also observed by Dobson[87] for his relaxation problem to determine production frequencies, and by Muckstadt and Singer[78] and Schweitzer and Silver[83] for the independent solution.

CHAPTER VII

CONCLUSIONS AND EXTENSIONS

The goal of this thesis has been to develop a method to produce a cyclic schedule that minimizes the long run average inventory costs per unit of time without violating a warehouse space capacity constraint. We have shown that such a schedule must satisfy the Zero Switch Rule. For given frequencies, this schedule can be obtained by either solving a complex mixed integer nonlinear program or by first generating a sequence using a heuristic developed for this purpose and then solving a linear program or a quadratic program. We have developed an efficient algorithm that computes iteratively the ordering frequencies, the cycle length, the maximum space used, the timing of deliveries, and the long run average inventory costs per unit time. Compared with the existing methods in the literature, the algorithm has produced the same or better cost solutions at very little computational expense.

When solving the WSP, we have assumed that the production rates are infinite. One obvious extension is to relax this assumption. Vemuganti, Dianich, and Oblak[87] have considered the fixed cycle length version of this problem. Another extension involves the application of the Convergent Frequencies Algorithm to the ELSP. The idea of converging to the independent solution combined with the Arbitrary Frequencies Bin Packing Heuristic and Zipkin's parametric quadratic algorithm may yield good cost solutions. The results of this thesis can also be extended to the one warehouse, multi-retailer problem in which the warehouse and the retailers have a severe space limitation. Finally, the case of finite backorder costs should also be considered. In an environment different to the high-volume assembly plant, allowing the warehouse to run out of stock for some period of time will reduce the maximum space used during the cycle. Vemuganti[88] has examined the fixed cycle length, zero setup time version of this problem. Gallego and Roundy[88] have extended the ELSP to allow backorders by using an approach similar to that of Dobson[87] and Zipkin[87]. The author has already started examining the first two extensions, whereas the last important extension will be the subject of a joint research with Guillermo M. Gallego, Columbia University.

APPENDIX I

Properties of the A Matrix

In this appendix some of the properties of the A matrix will be outlined.

Lemma

(1) A is a nonnegative matrix.

(2) $A_{jj} = \lambda_s > 0$, where $\lambda_s = \sum_{i=1}^{n} \lambda_i$, and $A_{jj} > A_{hj}$ $\forall$ $h \neq j$.

(3) For arbitrary $h \in \{1,...,m\}$ let $i=P_h$.

(3-1) If $m_i=1$, then:

$$A_{hh-1} = \lambda_i, \tag{1}$$

$$A_{hj} = A_{h-1j} + \lambda_i, \qquad \text{for all } j \neq h-1. \tag{2}$$

(1) and (2) are interpreted cyclically; ie, if h=1, then h-1=m.

(3-2) If $m_i>1$, then:

$$A_{hh-1} = 0, \tag{3}$$

$$A_{hj} = A_{h-1j} + \lambda_i, \qquad j=h,...,k-1, \text{ where } \delta_{ijk}=1, \tag{4}$$

$$A_{hj} = A_{h-1j}, \qquad \text{otherwise.} \tag{5}$$

(4) For arbitrary $j \in \{1,...,m\}$, let $i=P_j$.

(4-1) If $m_i=1$, then:

$$A_{jj-1} = \lambda_i, \tag{1}$$

$$A_{hj} = A_{hj-1} - \lambda_i, \qquad \text{for all } h \neq j, \tag{6}$$

(4-2) If $m_i > 1$, then:

$$A_{jj-1} = 0, \tag{3}$$

$$A_{hj} = A_{hj-1} - \lambda_i, \qquad h=k,\dots,j-1, \text{ where } \delta_{ikj}=1 \text{ and } \delta_{ik}=1, \tag{6'}$$
$$A_{hj} = A_{hj-1}, \qquad \text{otherwise.}$$

Proof:

(1) and (2) are trivial from definitions of the A and the $F^{(h)}$'s matrices.

(3) Recall that:

$$F^{(h)}_{rj} = \begin{cases} 1 & \text{for } j=h,h+1,\dots,k-1, \quad \text{where } \delta_{rhk}=1, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Consider the two matrices $F^{(h)}$ and $F^{(h-1)}$.

Let $i=P_{h,}$.clearly,

$$F^{(h-1)}_{ih-1} = 1 \text{ and } F^{(h-1)}_{ij} = 0, \text{ for all } j\neq h-1, \tag{8}$$

and for $r\neq i$, we have:

$$F^{(h)}_{rj} = F^{(h-1)}_{rj}, \quad \text{for } j\neq h-1, \tag{9}$$

and

$$F^{(h)}_{rh-1} = 0. \tag{10}$$

(3-1) $m_i=1$ and $P_h=i$, then

$$\delta_{ihh} = 1 \text{ and } F^{(h)}_{ij}=1 \text{ for } j=1,\dots,m. \tag{11}$$

For $j\neq h-1$, we have:

$$A_{hj}-A_{h-1j}=\lambda_i(F^{(h)}_{ij}-F^{(h-1)}_{ij})+\sum_{r\neq i}\lambda_r(F^{(h)}_{rj}-F^{(h-1)}_{rj}) \text{ by the row definition of A}$$

$$=\lambda_i(1 \quad -0 \quad)+ \ 0, \qquad \text{by (11),(8), and (9).}$$

For j=h-1,

$$A_{hh-1} = \lambda_i F_{ih-1}^{(h)} + \sum_{r \neq i} \lambda_r F_{rh-1}^{(h)}$$

$$= \lambda_i \qquad + 0, \qquad \text{by (11) and (10).}$$

(3–2) $m_i>1$, $\delta_{ihk}=1$, $P_h=i$, $P_k=i$, and $k \neq h$.

By (7), $F_{ih-1}^{(h)}=0$. (12)

Then, for j=h,h+1,...,k-1, we have:

$$A_{hj}-A_{h-1j}=\lambda_i(F_{ij}^{(h)}-F_{ij}^{(h-1)})+\sum_{r \neq i}\lambda_r(F_{rj}^{(h)}-F_{rj}^{(h-1)})$$

$$=\lambda_i(1 \quad - \quad 0 \quad)+ \qquad 0, \qquad \text{by (7), (8), and (9).}$$

For j=h-1,

$$A_{hh-1} = \lambda_i F_{ih-1}^{(h)} + \sum_{r \neq i} \lambda_r F_{rh-1}^{(h)}$$

$$= \quad 0 \quad + \quad 0, \qquad \text{by (12) and (10).}$$

For j=h,...,h-2, and j=k,...,m,

$$A_{hj}-A_{h-1j}=\lambda_i(F_{ij}^{(h)}-F_{ij}^{(h-1)})+\sum_{r \neq i}\lambda_r(F_{rj}^{(h)}-F_{rj}^{(h-1)})$$

$$=\lambda_i(\ 0 \ - \ 0 \ )+ \qquad 0, \qquad \text{by (7), (8), and (9).}$$

(4-1) $m_i=1$ and $P_j=i$, then

$F_{ij}^{(h)} \quad = 0, \qquad$ for $h \neq j$,

$F_{ij-1}^{(h)} \quad =1, \qquad$ for $h \neq j$.

$F_{rj}^{(h)} = F_{rj-1}^{(h)}$ for $r \neq i$ and h=1,...,m, since only one part can be ordered on any order.

Thus, for $h \neq j$ we have:

$$A_{hj}-A_{hj-1}=\lambda_i(F_{ij}^{(h)}-F_{ij-1}^{(h)})+\sum_{r\neq i}\lambda_r(F_{rj}^{(h)}-F_{rj-1}^{(h)})$$

$$=\lambda_i(\ 0-\ 1\ )+\ 0.$$

(4–2) $m_i>1$, $\delta_{ij}=1$, $\delta_{ik}=1$, $\delta_{ikj}=1$, and $j\neq k$.

Using the definition of $F^{(h)}$, we have:

$F_{ij}^{(h)}$ $=0$, for $h=1,\dots,m$,

$F_{ij-1}^{(h)}$ $=1$, for $h=k,\dots,j-1$,

and

$F_{ij-1}^{(h)}$ $=0$, otherwise.

Moreover, since $\delta_{ij}=1$ and only one part can be ordered on any order, then

$F_{rj}^{(h)} = F_{rj-1}^{(h)}$ for $r\neq i$ and $h=1,\dots,m$.

Thus for h=k,..., j-1, we have:

$$A_{hj}-A_{hj-1}=\lambda_i(F_{ij}^{(h)}-F_{ij-1}^{(h)})+\sum_{r\neq i}\lambda_r(F_{rj}^{(h)}-F_{rj-1}^{(h)})$$

$$=\lambda_i(\ 0\ -\ 1\ \ )+\ 0,$$

and otherwise

$$A_{hj}-A_{hj-1}=\lambda_i(F_{ij}^{(h)}-F_{ij-1}^{(h)})+\sum_{r\neq i}\lambda_r(F_{rj}^{(h)}-F_{rj-1}^{(h)})$$

$$=\lambda_i(\ 0\ -\ 0\ \ )+\ 0. \qquad \text{(E.O.P)}$$

APPENDIX II

Proof of Theorem IV-1

Before presenting the algebriac proof of theorem IV-1 some definitions and theorems which will be used in the proof need to be introduced.

Definition 1

An m by m real matrix $Q = (Q_{ij})$is diagonally dominant if :

$$\sum_{h=1\ h\neq j}^{m} | Q_{hj} | \leq | Q_{jj} |, \qquad j=1,...,m \qquad (1)$$

and strictly diagonally dominant if strict inequality holds in (1) for all j; the matrix is irreducibly diagonally dominant if Q is irreducible, diagonally dominant, and strict inequality holds in (1) for at least one j.

Theorem 1 (Ortega, Rheinboldt)

The real matrix Q is irreducible if and only if , for any two indices 1 and j , there exists a sequence of nonzero elements of Q of the form :

$$\{ Q_{hh_1}, Q_{h_1h_2}, Q_{h_2h_3},...,Q_{h_kj} \}.$$

Definition 2

An m by m real matrix Q is an M-matrix if it is invertible, $Q^{-1} \geq 0$ and $Q_{hj} \leq 0$ for all $h \neq j$ and $j=1,...,m$.

Theorem 2 (Ortega , Rheinboldt)

Let Q be strictly or irreducibly diagonally dominant and assume $Q_{hj} \leq 0$ for $h \neq j$ and that $Q_{hh} > 0$ for $h=1,2,.....,m$, then Q is an M-matrix.

We now are ready to prove that the optimal schedule that minimizes the maximum inventory corresponds to the warehouse full at each order.

Proof of theorem IV-1

Recall that (LP) could be simplified to :

$$\text{Max } e_m' x$$

s.t.

$$A\ x \leq e_m,$$
$$x \geq 0.$$

We will prove that $x^* = A^{-1} e_m$ ($d^* = A'^{-1} e_m$) is a primal (dual) optimal solution to (LP). The proof is composed of the following three steps repeated for both the primal and the dual:

<u>Step 1</u>

Premultiply both sides of the system of equations $A\ x = e_m$ by the matrix defined as follows:

$G_{jj} = 1, \qquad j=1,...,m,$

$G_{jj+1} = -1, \qquad j=1,...,m-1,$

$G_{hj} = 0, \qquad \text{otherwise.}$

Let $C = G\,A$ and $\tilde{e} = G\,e_m$ where

$\tilde{e}_j \quad = 0, \qquad j=1,\ldots,m-1,$

$\tilde{e}_m \quad =1.$

and C is of the following form:

$$C = \begin{bmatrix} & & Q & & \beta \\ & & & & \\ A_{m\,1} & A_{m\,2} & \cdots & A_{m\,m-1} & A_{m\,m} \end{bmatrix},$$

where Q is an (m-1) by (m-1) matrix and β is an (m-1) column vector. Using the properties of the A matrix, the Q matrix can be defined as:

For arbitrary $h \in \{1,\ldots,m-1\}$, let $i=P_h$.

1. If $m_i=1$, then

$$Q_{h-1h-1} = A_{h-1h-1} - A_{hh-1} = \sum_{r=1}^{n} \lambda_r - \lambda_i,$$

$$Q_{h-1j} = A_{h-1j} - A_{hj} = -\lambda_i, \qquad \text{for } j\neq h-1.$$

2. If $m_i>1$, then

$$Q_{h-1h-1} = A_{h-1h-1} - A_{hh-1} = \sum_{r=1}^{n} \lambda_r,$$

$$Q_{h-1j} = A_{h-1j} - A_{hj} = -\lambda_i, \qquad \text{for } j=h,..,k-1, \text{ where } \delta_{ihk}=1.$$

$Q_{h-1j} \quad = A_{h-1j} - A_{hj} = 0, \qquad\qquad$ otherwise.

The vector β is defined as follows:

$\beta_{h-1} \quad = A_{h-1m} - A_{hm} = -\lambda_i,$ if $m \in \{h,...,k-1\}$ where $\delta_{ihk}=1$,

$\beta_{h-1} \quad = 0, \qquad\qquad$ otherwise.

Hence:

$Q_{hj} \leq 0$ for $h \neq j$ with $Q_{hh+1} < 0$,

$Q_{hh} > 0$ for $h = 1,...,m-1$,

and

$\beta_h \leq 0$ for $h=1,...,m-1$.

Now, let $k_1, k_2,...,k_{m_i}$ be the order numbers on which the m_i orders of part i were placed. Then:

$Q_{k_1-1j} \; = -\lambda_i, \qquad$ for $j=k_1,k_1+1,...,k_2-1$,

$Q_{k_2-1j} \; = -\lambda_i, \qquad$ for $j=k_2,k_2+1,...,k_3-1$,

.

.

$Q_{k_{m_i}-1j} = -\lambda_i, \qquad$ for $j=k_{m_i},k_{m_i}+1,...,k_1-1$.

Thus, the rows of Q that corresponds to part i partition the columns of Q. Hence λ_i appears at most once in each column of Q. So, for each column of Q , there are at most n+1 nonzero entries, a positive element on the diagonal, and at most n negative entries corresponding to the demand rates of each part. Therefore, Q is diagonally dominant and strict inequality holds in (1) for at least one column. The strict inequality holds for the columns $j=1,2,...,k\text{-}1$ where $\delta_{P_1 1k} = 1$.

Let Q_{hj} and $Q_{h'j'}$ be nonzero entries of Q. Since $Q_{hh}>0$ and $Q_{hh+1}<0$, then there exists a chain from Q_{hj} to $Q_{h'j'}$, ie, it exists a sequence of nonzero elements of Q which forms a chain from Q_{hj} to $Q_{h'j'}$. Starting from Q_{hj} , we follow the following chain:

$$Q_{hj} \; Q_{jj} \; Q_{jj+1} \; Q_{j+1j+1} \; Q_{j+1j+2} \cdots\cdots\cdots Q_{h'h'} \; Q_{h'j'}.$$

Therefore, Q is irreducibly diagonally dominant and by theorem 2 it is an M-matrix and $Q^{-1}\geq 0$.

Step 2

Premultiply both sides of $C\,x = \tilde{e}$ by the following matrix:

$$\begin{bmatrix} & & & & 0 \\ & & & & 0 \\ & & & & \cdot \\ & & Q^{-1} & & \\ & & & & 0 \\ & & & & 0 \\ 0 & 0 & \cdot & \cdot\; 0 & 1 \end{bmatrix}$$

to get $\tilde{C}\,x = \tilde{e}$ where

$$\tilde{C} = \begin{bmatrix} & & I & & Q^{-1}\beta \\ & & & & \\ A_{m\,1} & A_{m\,2} & \cdot & \cdot\; A_{m\,m-1} & A_{m\,m} \end{bmatrix}$$

and I is an (m-1) by (m-1) identity matrix. Since $Q^{-1} \geq 0$ and $\beta \leq 0$ we have $Q^{-1}\,\beta \leq 0$.

Step 3

Premultiply both sides of $\tilde{C}\,x = \tilde{e}$ by the following matrix:

$$\begin{bmatrix} & & & & 0 \\ & & & & 0 \\ & & I & & \\ & & & & 0 \\ & & & & 0 \\ -A_{m\,1} & -A_{m\,2} & \cdots & -A_{m\,m-1} & 1 \end{bmatrix}$$

to get the final system of equations:

$$\begin{bmatrix} & & & & \\ & & & & \\ & & I & Q^{-1}\,\beta & \\ & & & & \\ 0 & 0 & \cdots & 0 & \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \\ \\ \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ 1 \end{bmatrix} \qquad (2)$$

where $\gamma = A_{mm} - \sum_{j=1}^{m-1} \tilde{\beta}\, A_{mj}$, and $\tilde{\beta} = Q^{-1}\,\beta$.

Thus $\gamma > 0$.

Therefore, by solving (2), we get:

$x_m = \gamma^{-1} > 0,$

$x_j = -(Q^{-1}\beta)_j\,\gamma^{-1} \geq 0.$

which shows that $x^* = A^{-1} e_m \geq 0$.

In the first step of the proof for the dual the matrix G is as follows:

$G_{ij} = 1,$ $\quad j=1,\dots,m,$

$G_{jj-1} = -1,$ $\quad j=2,\dots,m-1,$

$G_{1m} = -1.$

The matrix Q is defined as:

$Q_{jh} = A'_{jh} - A'_{j-1h} = A_{hj} - A_{hj-1},$ $\quad$ for j,h=1,...,m−1.

For arbitrary $j \in \{1,\dots,m-1\}$, let $i=P_j$.

1. If $m_i=1$, then

$$Q_{jj} = A_{jj} - A_{jj-1} = \sum_{r=1}^{n} \lambda_r - \lambda_i,$$

$Q_{jh} = A_{hj} - A_{hj-1} = -\lambda_i,$ $\quad$ for all $h \neq j$.

2. If $m_i>1$, then

$$Q_{jj} = A_{jj} - A_{jj-1} = \sum_{r=1}^{n} \lambda_r,$$

$Q_{jh} = A_{hj} - A_{hj-1} = -\lambda_i,$ $\quad$ $h=k,\dots,j-1$, where $\delta_{ikj}=1$ and $\delta_{ik}=1$,

$Q_{jh} = 0,$ $\quad$ otherwise.

Finally, the vector β is defined as follows:

$\beta_j = A_{mj} - A_{mj-1} = -\lambda_i,$ $\quad$ if $m \in \{k,\dots,j-1\}$ where $\delta_{ikj}=1$,

$\beta_j = 0,$ $\quad$ otherwise.

Hence:

$Q_{jh} \leq 0$ for $h \neq j$ with $Q_{h-1h} < 0$,

$Q_{jj} > 0$ for $j = 1, \ldots, m-1$,

and

$\beta_j \leq 0$ for $j = 1, \ldots, m-1$.

The remaining part of the proof to show $d^* = (A')^{-1} e_m \geq 0$ is similar to the proof that shows $x^* = A^{-1} e_m \geq 0$.

Finally, by the Strong Duality Theorem, since $e_m' x^* = e_m' d^*$, then x^* is optimal for the primal and d^* is optimal for the dual. . (E.O.P)

Finally, we will show how the above proof can be used to prove that joint replenishment can never occur for the optimal variable lot sizes schedule that minimizes the maximum inventory.

Proposition IV-1. Joint replenishment does not occur for the optimal solution to (LP); ie, $u_j > 0$.

Proof

It suffices to show that $x_j > 0$, since $x_j = \frac{u_j}{w}$.

Step 3 of the proof of Theorem IV-1 revealed that $x_m = \gamma^{-1} > 0$. Since the order of the rows is arbitrary it must be the case that $x_j > 0$ for $j=1, \ldots, m$.

(E.O.P)

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