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INTEGRAL NEAR-OPTIMAL SOLUTIONS TO CERTAIN
CLASSES OF LINEAR PROGRAMMING PROBLEMS

by

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ABSTRACT

This thesis examines the relation between optimal rational solutions and optimal integral solutions for several classes of linear programming problems. This relationship depends in various ways on total unimodularity of certain matrices and in Chapter II this topic is reviewed. Also in this chapter several relevant examples from the literature of combinatorial optimization, including network flows, matroid intersections, matchings, bipartite graphs and balanced matrices, are discussed from a viewpoint which emphasizes the role of total unimodularity.

In Chapter III attention is focused on two special classes of linear programming problems, a "covering" problem $\min\{l \cdot y : yA \geq w, y \geq 0\}$, which is denoted $\Gamma(A,w)$, and a "packing" problem $\max\{l \cdot y : yA \leq w, y \geq 0\}$, denoted $\Pi(A,w)$, where A is a nonnegative, integral $m \times n$ matrix, w is a nonnegative, integral n -vector and l is the m -vector of ones. For a fixed matrix A , the integer round-up property (IRU) holds for $\Gamma(A,w)$ if the best integral solution value for $\Gamma(A,w)$ is the integer round-up of the best rational solution value for $\Gamma(A,w)$, for each nonnegative, integral n -vector w . Similarly, the integer round-down property (IRD) holds for $\Pi(A,w)$ if, for every nonnegative, integral n -vector w , the best integral solution value for $\Pi(A,w)$ is the integer round-down of the best rational solution value for $\Pi(A,w)$. A polyhedron P is decomposable if, for every positive integer k , each integral

vector in $kP = \{x: x/k \in P\}$ is the sum of k integral vectors of P . It is shown that for a broad class of instances of $\Gamma(A,w)$ and $\Pi(A,w)$, the IRU and IRD properties are equivalent to decomposability of an associated polyhedron. A recursive characterization of decomposable polyhedra is given and it is also demonstrated that for a given nonnegative matrix A , the IRU and IRD properties for $\Gamma(A,w)$ and $\Pi(A,w)$, respectively, can be checked through a finite process.

In Chapter IV IRU and IRD results are given for linear programming problems arising from polymatroids, the intersection of two strongly base-orderable matroids, branchings, totally unimodular matrices and the scheduling of precedence related unit execution time jobs on independent identical machines. An example in which IRD fails for a problem associated with perfect graphs is also presented.

In Chapter V strong integral min-max and max-min theorems are derived for certain of the linear programming problems considered in Chapter IV. Such combinatorial min-max (max-min) theorems are obtained by coupling IRU (IRD) results with the anti-blocking (blocking) theory of Fulkerson. A new and elementary proof of Fulkerson's Pluperfect Graph Theorem is given using the techniques developed in Chapter III for proving finite checkability of the rounding properties.

CHAPTER I
PRELIMINARIES

I.1 Introduction

The linear programming problem is that of optimizing a linear functional subject to a set of linear constraints. This problem may be stated:

$$\begin{aligned} \max \quad & c \cdot y \\ \text{subject to} \quad & yA \leq w, \end{aligned} \tag{I.1.1}^{\dagger}$$

where A is an $m \times n$ real matrix, c an m -vector of real numbers and w is an n -vector of real numbers. This problem has been studied extensively in the literature of operations research, and is "well-solved" in two respects. The duality theory of linear programming provides a solid theoretical setting for the study of the problem; that is, it provides us with a strong max-min theorem for linear programming problems. Further, though the algorithmic aspect of linear programming remains an interesting and active research area, the simplex method of Dantzig (1962)^{††} provides an algorithm for solving linear programming problems which has proved to be of tremendous practical use.

The integer programming problem arises when one insists on having integral solutions to a linear programming problem. This problem may

[†]This method of indexing is used throughout this thesis. The Roman numeral refers to the chapter, followed by the section within that chapter and the particular item indexed.

^{††}This manner is used for citing references included in the bibliography.

be stated:

$$\begin{aligned}
 &\max \quad c \cdot y \\
 &\text{s.t.} \quad yA \leq w \\
 &\quad y \text{ integral,}
 \end{aligned}
 \tag{I.1.2}$$

where A , c and w are as in (I.1.1).

Integer programming problems have also been extensively studied, and are of great interest since in many applications, only integral solutions have any physical interpretation. Further, integer programming is central to the theory of combinatorial optimization, which is the primary concern of this thesis. However, neither a strong duality theory for generating max-min results, nor any reasonably efficient method of solution is known for the general integer programming problem. Indeed, there are many well-known integer programming problems, such as the knapsack problem, the traveling salesman problem and the triple assignment problem which are notoriously difficult to solve. A theoretical explanation for this difficulty is to be found in the theory of NP-complete problems of Cook (1971) and Karp (1972).

Despite these difficulties, however, there are many important classes of combinatorial integer programming problems for which integral max-min theorems are known. In Chapter II, a survey is made of some of these, including network flows, bipartite graphs, balanced matrices, matchings and matroid intersections; in particular we view these topics from an algebraic standpoint.

Attention is then narrowed to two types of linear programming problems:

$$\begin{aligned} \min \quad & l \cdot y \\ \text{s.t.} \quad & yA \geq w \\ & y \geq 0 \end{aligned} \tag{I.1.3}$$

and

$$\begin{aligned} \max \quad & l \cdot y \\ \text{s.t.} \quad & yA \leq w \\ & y \geq 0, \end{aligned} \tag{I.1.4}$$

where in both (I.1.3) and (I.1.4), A is an $m \times n$ matrix of nonnegative integers, w is a nonnegative integral n -vector, and l is the m -vector of ones. The main topic of this thesis is a study of programming problems of type (I.1.3) and (I.1.4) for which the optimal integer solution value, though not necessarily the same as the optimal rational solution value, is the round-up (for (I.1.3)) or the round-down (for (I.1.4)) to the nearest integer of the optimal rational solution value. In Chapter III this rounding property is studied from an algebraic and polyhedral viewpoint. In Chapter IV the results of Chapter III are applied to optimization problems for a variety of combinatorial topics, including polymatroids, matroid intersection, circulations of a totally unimodular matrix, scheduling and perfect graphs. Finally, in Chapter V the relation of the blocking and antiblocking theory of Fulkerson

(1970,1971a,1972) to our earlier development is explored, and a new proof of Fulkerson's (1972) Pluperfect Graph Theorem is given.

I.2 Notation

The following notation is used throughout this thesis.

R	the real numbers
R_+	the nonnegative real numbers
R^n	Euclidean n-space
R_+^n	$\{x \in R^n: x \geq 0\}$
Z	the integers
Z_+	the nonnegative integers
Z_+^n	$\{x \in Z^n: x \geq 0\}$
$Z_+^{m \times n}$	the nonnegative integral $m \times n$ matrices
$x \leq y, \quad x, y \in R^n$	$x_i \leq y_i, \quad i = 1, \dots, n$
$x < y, \quad x, y \in R^n$	$x_i < y_i, \quad i = 1, \dots, n$
$x \cdot y, \quad x, y \in R^n$	$\sum_{i=1}^n x_i y_i$
$\lceil x \rceil, \quad x \in R_+$	the least integer greater than or equal to x
$\lfloor x \rfloor, \quad x \in R_+$	the greatest integer less than or equal to x
$ S , \quad S \text{ a set}$	the cardinality of S
$A \subseteq B, \quad A, B \text{ sets}$	A is a subset of B
$A \subset B, \quad A, B \text{ sets}$	A is a proper subset of B
$B-A, \quad A, B \text{ sets}$	$\{x \in B: x \notin A\}$
$x(S), \quad T \text{ a finite index set,}$ $S \subseteq T, \quad x \in R^{ T }$	$\sum_{i \in S} x_i$

$$(x|S), \quad T \text{ a finite index set,} \quad (x|S)_i = \begin{cases} x_i & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

$$S \subseteq T, \quad x \in \mathbb{R}^{|T|}$$

$$\overline{A}, \quad A, B \text{ sets, } A \subseteq B \quad B-A$$

I.3 Linear Programming

In this section some basic linear programming terminology is established. For a more complete discussion of the concepts introduced, and for proofs of the results stated here, a good general reference is Dantzig (1962). For matrix computations in this section and throughout the remainder of this thesis, all vectors are assumed to be appropriately dimensioned, and no distinction is made between row and column vectors.

Let A be an $m \times n$ real matrix, $w \in \mathbb{R}^n$, $c \in \mathbb{R}^m$ and consider the linear programming problem:

$$\begin{aligned} \max \quad & c \cdot y \\ \text{s.t.} \quad & yA \leq w. \end{aligned} \tag{I.3.1}$$

The linear functional $c \cdot y$ is called the objective function of (I.3.1), the matrix A is called the constraint matrix or constraint system and w is called the right-hand side. A feasible solution for (I.3.1) is any $y \in \mathbb{R}^m$ such that $yA \leq w$. If (I.3.1) has no feasible solutions, it is an infeasible problem; otherwise it is a feasible problem. An optimal solution for (I.3.1) is any feasible $z \in \mathbb{R}^m$ which maximizes the objective function $c \cdot y$, i.e., such that $c \cdot z \geq c \cdot y$ for every feasible y for (I.3.1). Given y feasible for (I.3.1), the associated value of y for the programming problem (I.3.1) is the real

number $c'y$, and the (optimal) value of the programming problem (I.3.1) is the value of any optimal solution to it. If a linear programming problem is feasible but has no optimal solution, then the problem is said to be unbounded, and its value is $+\infty$.

A set $C \subseteq \mathbb{R}^m$ is called convex if whenever $a \in C$, $b \in C$ and $0 \leq \lambda \leq 1$, we have $\lambda a + (1-\lambda)b \in C$. Given a finite set of points $\{x^1, \dots, x^k\} \subset \mathbb{R}^m$, a convex combination of those points is any $x \in \mathbb{R}^m$ which can be written in the form $x = \sum_{i=1}^k \lambda_i x^i$, such that $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geq 0$ for $i = 1, \dots, k$. Given a set $B \subseteq \mathbb{R}^m$, the convex hull of B is the set of all convex combinations of finite subsets of B .

A polyhedron in \mathbb{R}^m is the feasible solution set to a finite system of linear inequalities $yA \leq w$, where A is any real $m \times n$ matrix and $w \in \mathbb{R}^n$. Geometrically, thus, a polyhedron is the intersection of a finite number of closed half-spaces in \mathbb{R}^m . Polyhedra are convex sets. A polyhedron is called bounded if it is a bounded subset of \mathbb{R}^m . The polyhedron associated with a linear programming problem is the convex set of all the feasible solutions to the programming problem. An extreme point (or vertex) of a polyhedron P is any $x \in P$ such that there do not exist distinct vectors $y \in P$ and $z \in P$ such that $x = \lambda y + (1-\lambda)z$ for $0 < \lambda < 1$. That is, $x \in P$ is an extreme point if it cannot be written as a convex combination of other points in P . A feasible solution to the linear programming problem (I.3.1) is called extreme (or basic) if it is an extreme point of the polyhedron associated with the linear programming problem.

The following characterization of polyhedra will prove useful. For a proof, see, e.g., Rockafellar (1970).

Theorem I.3.2. $P \subseteq \mathbb{R}^m$ is a polyhedron if and only if there exist finite sets $\{x^1, \dots, x^r\} \subset \mathbb{R}^m$ and $\{y^1, \dots, y^s\} \subset \mathbb{R}^m$ such that

$$P = \{z \in \mathbb{R}^m : z = \sum_{i=1}^r \lambda_i x^i + \sum_{j=1}^s \mu_j y^j, \lambda_i \geq 0, \mu_j \geq 0, \sum_{i=1}^r \lambda_i = 1\}.$$

If P has at least one extreme point, then the set $\{x^1, \dots, x^r\}$ may be taken to be the set of extreme points of P , and if P is bounded, all the μ_j are to be taken to be equal to zero. \square

Another useful basic theorem about polyhedra is:

Theorem I.3.3. (a) A bounded polyhedron is the convex hull of its extreme points.

(b) Every nonempty polyhedron contained in \mathbb{R}_+^m has at least one extreme point. \square

The associated integer programming problem to a linear programming problem is the linear programming problem with integrality constraints added. Thus the associated integer programming problem to (I.3.1) is

$$\max \quad c \cdot y$$

$$\text{s.t. } yA \leq w \quad (\text{I.3.4})$$

$$y \text{ integral.}$$

Feasible solutions, optimal solutions, values and optimal values are defined for integer programming problems just as for linear programming problems. The polyhedron associated with an integer programming problem is the convex hull of all its feasible solutions.

CHAPTER II

BACKGROUND

In this chapter several important classes of combinatorial integer programming problems are examined from a perspective emphasizing unimodularity of the constraint matrix. Thus we first review several well-known results on total unimodularity of matrices.

II.1 Totally Unimodular Matrices

A square matrix is called unimodular if its determinant is equal to ± 1 . A matrix is called totally unimodular if each of its square submatrices has determinant equal to ± 1 or 0. Hoffman and Kruskal (1958) have shown:

Theorem II.1.1.1. If the constraint matrix A of the linear programming problem (I.3.1) is totally unimodular and if w is an integral vector, then every basic feasible solution to (I.3.1) will be integral (i.e., the polyhedron $\{y \in \mathbb{R}^m: yA \leq w\}$ will have all integral extreme points). Thus if (I.3.1) is feasible and bounded (and A is totally unimodular, $w \in \mathbb{Z}^n$), then (I.3.1) will always have an integral optimal solution.

Conversely, if the nonnegativity constraints $y \geq 0$ are added to (I.3.1) and if this modified problem has an integral optimal solution for every $c \in \mathbb{R}^m$ and integral w such that it is a feasible problem, then the constraint matrix A is totally unimodular. \square

(An elementary proof of this theorem is provided in Veinott and Dantzig (1968).)

This result is important, since it tells us that at least some integer programming problems (namely those with totally unimodular constraint systems and integral right-hand sides) are no more difficult than their associated linear programming problems. Since the simplex method for linear programming always determines a basic optimal solution (when such exists), it will automatically generate an integral optimal solution in the case of a totally unimodular constraint matrix and integral right-hand side vector. Linear programming duality theory can also be used to generate strong integral max-min theorems for such integer programming problems.

We conclude this section by giving two well-known theorems which provide sufficient conditions for a matrix to be totally unimodular; we use these results in our later development. It is not difficult to prove the following theorem using induction on the size of submatrices of A (see, e.g., Garfinkel and Nemhauser (1972)).

Theorem II.1.2. (Heller and Tompkins (1958)) Let A be a matrix with all entries 0, 1, or -1 and say that A satisfies:

- (1) no more than two nonzero entries occur in any column;
- (2) the rows of A can be partitioned into two subsets Q_1 and Q_2 such that:
 - (i) if a column contains two nonzero elements with the same sign, one element is in Q_1 and the other in Q_2 ;
 - (ii) if a column contains two nonzero elements of opposite sign, both elements are in Q_1 or both are in Q_2 .

Then A is totally unimodular. \square

Theorem II.1.3. Let A be a $(0,1)$ matrix which satisfies:

$$a_{ij} = 1 \text{ and } a_{kj} = 1 \text{ for } k > i+1 \quad (\text{II.1.4})$$

imply that $a_{i+1,j} = \dots = a_{k-1,j} = 1$ for all i,k,j . (That is, in each column of A , the entries of value one occur in consecutive row positions.) Then A is totally unimodular.

Proof: Since any square submatrix of a matrix satisfying (II.1.4) will clearly inherit this property, it is enough to show that all square matrices A satisfying (II.1.4) have determinant 0, +1, or -1.

We proceed by induction on n , the dimension of A .

If $n = 1$, the result is clear. So assume that all $(n-1) \times (n-1)$ matrices satisfying (II.1.4) have determinant 0, +1, or -1, and let A be an $n \times n$ matrix satisfying (II.1.4). We are done if A has a zero row or column, so by permuting columns we may assume that $a_{11} = 1$, and further, that the first column of A , say A^1 has a minimum number of nonzero elements among all those columns A^j of A such that $a_{1j} = 1$. We then deduce from (II.1.4) that for any i and j , $1 \leq i, j \leq n$, $a_{1j} = 1$ and $a_{i1} = 1$ implies that $a_{ij} = 1$. Let A^* be obtained from A by subtracting the column A^1 from every column A^j such that $a_{1j} = 1$, $j \neq 1$. Then A^* will still satisfy (II.1.4), and will have precisely one 1 entry in its first row. Thus, if we let B be the $(n-1) \times (n-1)$ submatrix of A^* gotten by deleting the first row and column of A^* , we have $\det(A) = \det(A^*) = \det(B) = 0, +1$ or -1 , by the induction hypothesis. \square

II.2 Network Flows

Let $G = (N, A)$ be a directed graph with node set N and arc set A . A directed arc in G (i.e., a member of A) is denoted by a pair (x, y) , $x \in N$, $y \in N$. The arc (x, y) is directed from x to y .

G is called a two-terminal, capacitated network if it has two specified nodes, $s \in N$ called the source, and $t \in N$ called the sink, and a capacity function on the arcs, $c: A \rightarrow R_+$. A feasible flow of value v in G is a function $f: A \rightarrow R_+$ which satisfies

$$\sum_{(s,y) \in A} f(s,y) - \sum_{(y,s) \in A} f(y,s) = v \quad (\text{II.2.1})$$

$$\sum_{(x,y) \in A} f(x,y) - \sum_{(y,x) \in A} f(y,x) = 0, \quad x \in N, \quad x \neq s, t \quad (\text{II.2.2})$$

$$\sum_{(t,y) \in A} f(t,y) - \sum_{(y,t) \in A} f(y,t) = -v \quad (\text{II.2.3})$$

$$0 \leq f(x,y) \leq c(x,y), \quad (x,y) \in A. \quad (\text{II.2.4})$$

Given any $X \subseteq N$ such that $s \in X$, $t \notin X$, a cut separating s from t in G is the set of arcs $\{(x,y) \in A: x \in X, y \in N-X\}$. Denote such a cut by (X, \bar{X}) . The capacity of cut (X, \bar{X}) , denoted $c(X, \bar{X})$, is given by $\sum_{(x,y) \in (X, \bar{X})} c(x,y)$. Ford and Fulkerson (1962) have shown:

Theorem II.2.5. (Max-flow, min-cut) For any two-terminal, capacitated network $G = (N, A)$, the maximum value of a feasible flow in G is equal to the minimum cut capacity of any cut separating the source s from the sink t . \square

Thus, given an integral capacity function $c: A \rightarrow \mathbb{Z}_+$, the problem

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & \text{(II.2.1)-(II.2.4)} \end{aligned} \tag{II.2.6}$$

will certainly have an integer optimum value. That the problem (II.2.6) will also always have an integral optimal solution follows from the fact that the constraints (II.2.1)-(II.2.3) define a totally unimodular matrix. The total unimodularity of this constraint matrix follows trivially from (II.1.2), since the matrix corresponding to these constraints has precisely one +1 entry and one -1 entry in each column, and all other entries are equal to zero. If we call the matrix associated with the constraints (II.2.1)-(II.2.3) A , and let I denote the $|A| \times |A|$ identity matrix, then it is clear that (II.2.6) may be expressed:

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & \begin{bmatrix} A \\ -A \\ I \\ -I \end{bmatrix} [x] \leq \begin{bmatrix} 0 \\ 0 \\ c \\ 0 \end{bmatrix} \end{aligned} \tag{II.2.7}$$

Since the constraint matrix for (II.2.7) is plainly still totally unimodular and (II.2.7) is feasible for any $c \geq 0$, we deduce from Theorem (II.1.1) that (II.2.7) will have an integral optimal solution whenever c is integral.

Ford and Fulkerson (1962) have also provided an efficient algorithm for finding a flow of maximum value. As this algorithm starts with any feasible flow (e.g., $f(x,y) = 0$ for every $(x,y) \in A$) and generates an optimal flow through various integral additions and deletions of flow from arcs, this algorithm can also be seen as ensuring the existence of optimal integral flows.

Interestingly, as elementary as the maximum flow problem and its solution appear, many other important combinatorial theorems can be easily derived from it. Among these are (see Ford and Fulkerson (1962), Chapter 2) the capacitated supply-demand theorem of Gale (1957), Hoffman's (1960) circulation theorem, the theorem of Hall (1935) on systems of distinct representatives of a family of sets, the theorem of Fulkerson (1971b) on disjoint common partial transversals of two families of sets, Birkhoff's (1946) theorem on permutation matrices, and the theorem of Dilworth (1950) on covering the elements of a partially ordered set with a minimum number of chains in the partial order. Each of the above results may be stated as a combinatorial (i.e., integral) max-min theorem, and thus in each instance total unimodularity of the constraint system provides an algebraic explanation for an integral max-min theorem.

II.3 Bipartite Graphs

Let $G = (N,A)$ be an undirected graph with node (or vertex) set N and arc (or edge) set A . Assume further that G is loopless (i.e., $\{x,y\} \in A$ implies $x \neq y$) and that G has no multiple edges (i.e., there is at most one edge between any two nodes $x \in N$, $y \in N$). G

is bipartite if there exists a partition of the nodes N of G into two sets S and T such that $S \cup T = N$, $S \cap T = \emptyset$, and $\{x,y\} \in A$ implies either $x \in S$ and $y \in T$ or $x \in T$ and $y \in S$.

Let A be the node-arc incidence matrix of G , i.e.,

$$a_{ij} = \begin{cases} 1 & \text{if node } i \text{ is an endpoint of arc } j \\ 0 & \text{otherwise.} \end{cases}$$

It is then easy to see that S and T induce exactly the kind of partition of the rows of A which guarantees that A is totally unimodular by Theorem II.1.2. Thus any bounded, feasible linear programming problem of the form

$$\begin{aligned} \max \quad & c \cdot x \\ \text{s.t.} \quad & Ax \leq w \end{aligned} \tag{II.3.1}$$

will have an integral optimal solution whenever w is integral (Theorem II.1.1).

An example of a linear programming problem on a bipartite graph is the Hitchcock transportation problem (see for instance Ford and Fulkerson (1962)). Let $G = (S \cup T, A)$ be the complete bipartite graph on $S \cup T$; i.e., let $A = \{\{s_i, t_j\} : s_i \in S, t_j \in T\}$. Associate to every $s_i \in S$ a nonnegative, integral supply a_i , to every $t_j \in T$ associate a nonnegative, integral demand b_j , and assume $\sum_{\{i:s_i \in S\}} a_i = \sum_{\{j:t_j \in T\}} b_j$. Further, to every $\{s_i, t_j\} \in A$ associate a nonnegative cost c_{ij} . The

Hitchcock problem may then be stated:

$$\begin{aligned}
 & \min \sum_{i,j} c_{ij} x_{ij} \\
 & \text{s.t.} \quad \sum_{j=1}^{|T|} x_{ij} \leq a_i, \quad s_i \in S \\
 & \quad \sum_{i=1}^{|S|} x_{ij} \geq b_j, \quad t_j \in T \\
 & \quad x_{ij} \geq 0,
 \end{aligned} \tag{II.3.2}$$

or equivalently,

$$\begin{aligned}
 & \max(-\sum_{i,j} c_{ij} x_{ij}) \\
 & \text{s.t.} \quad \sum_{j=1}^{|T|} x_{ij} \leq a_i, \quad s_i \in S \\
 & \quad -\sum_{i=1}^{|S|} x_{ij} \leq -b_j, \quad t_j \in T \\
 & \quad -x_{ij} \leq 0,
 \end{aligned} \tag{II.3.3}$$

which is of the type (II.3.1) for a directed bipartite graph G . One can think of the a_i 's as representing supply at vertex s_i , the b_j 's as representing demand and vertex b_j , and the c_{ij} 's as unit shipping costs from s_i to t_j ; hence the term "transportation problem."

II.4 Common Independent Sets of Two Matroids

A matroid $M = (E, I)$ is a finite set $\{e_1, \dots, e_n\}$ and a family of subsets I of E which satisfy two axioms:

$$(1) \text{ if } A \subseteq B \in I, \text{ then } A \in I \quad (\text{II.4.1})$$

$$\text{and } (2) \text{ if } A \subseteq E, \text{ then all maximal } B \subseteq A \text{ such that } B \in I \text{ have the same cardinality.} \quad (\text{II.4.2})$$

The members of I are called the independent sets of M ; the maximal members of I are called the bases of M . The rank of $A \subseteq E$ is the cardinality of (any) maximal independent set contained in A , and is denoted $r(A)$.

Matroids provide an abstract setting for the study of many combinatorial problems; two examples of matroids are:

- (1) E = any finite set of vectors in R^n and I = the linearly independent subsets of E , and
- (2) E = the edges of a graph, and I = the subsets of E which contain no cycles (see Section IV.3).

For a good introduction to matroid optimization and the polyhedral study of matroids, see Edmonds (1970) and Giles (1975).

Given two matroids on a set E , $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$, with rank functions r_1 and r_2 respectively, Edmonds (1970) has shown that the extreme solutions to the system,

$$x(A) \leq r_1(A), \quad A \subseteq E \quad (\text{II.4.3})$$

$$x(B) \leq r_2(B), \quad B \subseteq E$$

$$x \geq 0,$$

are precisely the incidence vectors of the subsets of E which are independent in both M_1 and M_2 . His proof of this result uses the following interesting technique. It is not difficult to see that among the extreme solutions to (II.4.3) are all the incidence vectors of sets independent in both M_1 and M_2 , and that any integral solution to (II.4.3) will be such an incidence vector. However, it seems perfectly possible that (II.4.3) might have some fractional extreme solutions.

Although it is not generally the case that the constraint system of (II.4.3) will be totally unimodular, Edmonds (1970) has shown that, given any extreme solution to (II.4.3), the constraints satisfied at equality by that solution generate a submatrix which can be transformed via elementary row operations into a totally unimodular matrix. From this and Theorem II.1.1, it follows easily that all extreme solutions to (II.4.3) must indeed be integral, giving the desired result.

Thus, although no totally unimodular system yielding the common independent sets of two matroids as extreme solutions is known, total unimodularity is again central in establishing this combinatorial theorem. We use a similar technique in Section IV.1 to show that certain linear programming problems for polymatroid optimization satisfy the rounding property discussed in Section I.1.

II.5 Matchings

Let $G = (N, A)$ be a finite undirected graph without loops or multiple edges, and let A be its node-arc incidence matrix. The maximum matching problem for G is given by:

$$\begin{aligned}
 &\max \quad 1 \cdot x \\
 &\text{s.t.} \quad Ax \leq 1 \\
 &\quad \quad x \geq 0 \\
 &\quad \quad x \text{ integral.}
 \end{aligned}
 \tag{II.5.1}$$

Given any $S \subseteq N$, let $A(S) = \{e \in A: e = \{u, v\}, u \in S, v \in S\}$ and consider the following linear programming problem, with the integrality constraints of (II.5.1) replaced by additional linear constraints:

$$\begin{aligned}
 &\max \quad 1 \cdot x \\
 &\text{s.t.} \quad Ax \leq 1 \\
 &\quad \quad \sum_{e \in A(S)} x_e \leq \frac{|S|-1}{2}, \quad S \subseteq N, \quad |S| \geq 3 \text{ and odd} \\
 &\quad \quad x \geq 0.
 \end{aligned}
 \tag{II.5.2}$$

Edmonds (1965a) has shown that the basic feasible solutions to (II.5.2) correspond precisely to the feasible solutions to (II.5.1); thus the integer programming problem (II.5.1) may be "replaced" by the linear programming problem (II.5.2). R. Oppenheim (1973) has shown that a

further set of linear constraints may be added to (II.5.2) such that for every extreme solution to this system, the constraints satisfied at equality generate a full-rank unimodular submatrix. This provides a "unimodularity oriented" proof of Edmonds' result. Oppenheim calls this property "local unimodularity", and similarly we may call the property discussed in (II.4), "local total unimodularity". The proof of Oppenheim's result is quite complicated, and thus can hardly be seen as motivation for Edmonds' result, however it does demonstrate the role of unimodularity in another important family of combinatorial problems where integrality results are known.

II.6 Balanced Matrices

A $(0,1)$ -matrix A is called balanced if it contains no square submatrix of odd size all of whose row and column sums are equal to 2. Consider the following linear programming problems, where A is a $(0,1)$ -matrix, and w and c are nonnegative integral vectors;

$$\begin{aligned} \min \quad & l \cdot y \\ \text{s.t.} \quad & yA \geq w \end{aligned} \tag{II.6.1}$$

$$0 \leq y \leq c,$$

$$\begin{aligned} \max \quad & l \cdot y \\ \text{s.t.} \quad & yA \leq w \end{aligned} \tag{II.6.2}$$

$$0 \leq y \leq c.$$

Fulkerson, Hoffman and Oppenheim (1974) have demonstrated the following theorem, some special cases of which were proved in Berge (1972).

Theorem II.6.3. If A is balanced, then (II.6.1) and (II.6.2) always have an integral optimal solution. \square

Fulkerson, Hoffman and Oppenheim (1974) also give the following property of balanced matrices, which they use in proving Theorem II.6.3.

Theorem II.6.4. If A is balanced and $\{x: Ax = 1, x \geq 0\}$ is nonempty, then every vertex of this polyhedron is $(0,1)$ -valued. \square

To prove (II.6.4), they actually prove the equivalent property for A :

Property II.6.5. Let A be balanced and say there exists $x > 0$ satisfying $Ax = 1$. Then there exists a set of nonoverlapping columns a_{j_1}, \dots, a_{j_k} of A (i.e., $a_{j_r} \cdot a_{j_s} = 0$ for $r \neq s$) whose sum is the vector of all ones. \square

However, using the cofactor decomposition of determinants and the nonoverlapping property of the columns involved, it is clear that the submatrix of A specified in (II.6.5) must be totally unimodular. Thus Fulkerson, Hoffman and Oppenheim have actually shown that a balanced matrix has certain unimodularity properties which are responsible for the result given in Theorem II.6.4.

CHAPTER III

INTEGER ROUNDING AND POLYHEDRAL DECOMPOSITION

III.1 Integer Rounding

The problems considered in Chapter II ((II.2.6), (II.3.1), (II.3.2), (II.4.3), (II.5.2), (II.6.1) and (II.6.2)) had the property that the optimal linear programming value and that of the associated integer programming problem were the same. However, some interesting cases are also known where the optimal value in an integer maximization problem is the integer round-down of the optimal value of the associated linear programming problem, and where the optimal value in an integer minimization problem is the integer round-up of the value of the associated linear programming problem.

An example of the former is the work of Fulkerson and Weinberger (1975) on packing integral, feasible flows of a supply-demand network into an integral vector. Examples of the latter include the work of Hu (1961) on scheduling problems with assembly tree precedence structures, Edmonds' (1965b) results on partitioning a matroid into a minimum number of independent sets, and Weinberger's (1976) work on covering an integral vector with integral, feasible flows in a capacitated supply-demand network. All of these problems will be examined in some detail in Chapter IV, and it will be shown that unimodularity plays a central role in establishing the rounding property for each of them.

In the remainder of this section, we fix notation that will be useful throughout the rest of this thesis and take a brief look at certain results in Fulkerson and Weinberger (1975), to provide an introduction to the rounding problem.

Let A be an $m \times n$ matrix of nonnegative integers with no zero columns, let B be an $m \times n$ matrix of nonnegative integers with no zero rows, and let w be an n -vector of nonnegative integers. We will be primarily concerned with the following two linear programming problems and their associated integer programming problems:

$$\min \quad l \cdot y$$

$$\text{s.t.} \quad yA \geq w \quad \Gamma(A, w)$$

$$y \geq 0$$

and

$$\max \quad l \cdot y$$

$$\text{s.t.} \quad yB \leq w \quad \Pi(B, w)$$

$$y \geq 0.$$

$\Gamma(A, w)$ is commonly called a covering problem, since it is the problem of minimally covering the "weight" vector w with rows of A . As long as A is nonnegative, and A has no zero columns, $\Gamma(A, w)$ will always be a feasible problem with a finite optimal value.

$\Pi(B, w)$ is commonly called a packing problem, since it is the problem of packing a maximum number of the rows of B into the vector w . The nonnegativity of w ensures that $\Pi(B, w)$ will always be feasible (the vector of all zeros is always a feasible solution), and that B is nonnegative and has no zero rows guarantees that $\Pi(B, w)$ will also have a finite optimal value.

Given nonnegative integral matrices A and B as above, the following notation will be used:

y_w^* = an optimal solution for $\Gamma(A,w)$ or $\Pi(B,w)$,

r_w^* = the value of an optimal solution to $\Gamma(A,w)$ or $\Pi(B,w)$,

$$\text{i.e., } r_w^* = 1 \cdot y_w^*,$$

z_w^* = an optimal integral solution for $\Gamma(A,w)$ or $\Pi(B,w)$,

and s_w^* = the value of an optimal integral solution to $\Gamma(A,w)$ or

$$\Pi(B,w), \text{ i.e., } s_w^* = 1 \cdot z_w^*.$$

With A and B as above fixed, we say that the integer round-up property (IRU) holds for $\Gamma(A,w)$ if $\lceil r_w^* \rceil = s_w^*$ for every $w \in Z_+^n$. ($\lceil r_w^* \rceil$ denotes the least integer greater than or equal to r_w^* .) The integer round-down property (IRD) holds for $\Pi(B,w)$ if $\lfloor r_w^* \rfloor = s_w^*$ for every $w \in Z_+^n$. ($\lfloor r_w^* \rfloor$ denotes the greatest integer less than or equal to r_w^* .)

Fulkerson and Weinberger (1975) consider the following problem.

Let $G = (N,A)$ be a directed graph whose nodes have been partitioned into $N = S \cup R \cup T$. For every $x \in S$, associate a nonnegative integer supply $a(x)$, for every $x \in T$, associate a nonnegative integer demand $b(x)$, and for every $(x,y) \in A$, associate a nonnegative integer capacity $c(x,y)$. Such a system is called a capacitated, integral, supply-demand network.

A feasible flow for (N,A) is a function $f: A \rightarrow R_+$ which satisfies:

$$\sum_{\{y \in N: (x,y) \in A\}} f(x,y) - \sum_{\{y \in N: (y,x) \in A\}} f(y,x) \leq a(x), \quad x \in S \quad (\text{III.1.1})$$

$$\sum_{\{y \in N: (x,y) \in A\}} f(x,y) - \sum_{\{y \in N: (y,x) \in A\}} f(y,x) = 0, \quad x \in R \quad (\text{III.1.2})$$

$$\sum_{\{y \in N: (y,x) \in A\}} f(y,x) - \sum_{\{y \in N: (x,y) \in A\}} f(x,y) \geq b(x), \quad x \in T \quad (\text{III.1.3})$$

$$0 \leq f(x,y) \leq c(x,y), \quad (x,y) \in A. \quad (\text{III.1.4})$$

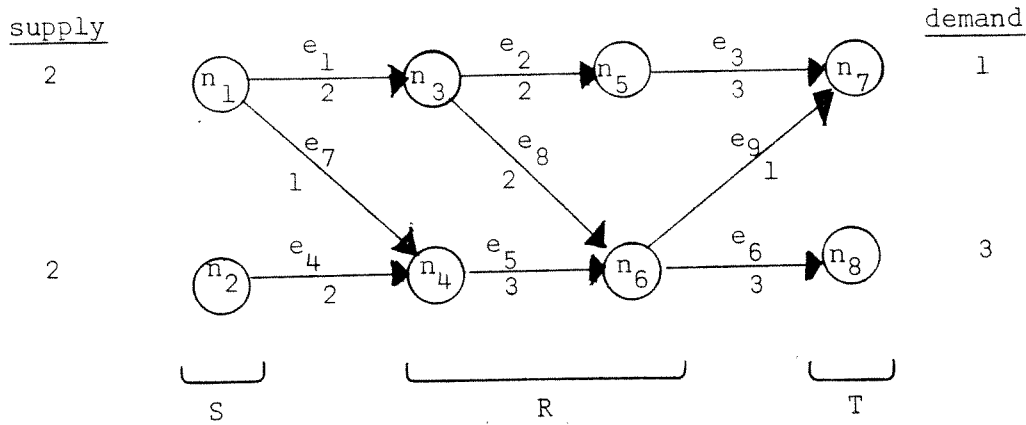
Gale (1957) has shown that all the extreme solutions to (III.1.1)-(III.1.4) are integral (and again one may deduce this from Theorem II.1.1, since the system of constraints given by (III.1.1)-(III.1.4) is totally unimodular), and has provided simple necessary and sufficient conditions for (III.1.1)-(III.1.4) to have a solution.

Fulkerson and Weinberger (1975) consider a matrix A whose rows are indexed by the collection of integral, feasible solutions to (III.1.1)-(III.1.4), and whose columns are indexed by the elements of A . The entry a_{ij} of A is the value of the i th integral, feasible flow on arc j .

Example III.1.5. Let $G = (N,A)$ be a network with

$N = \{n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8\}$, and $A = \{e_1 = (n_1, n_3), e_2 = (n_3, n_5), e_3 = (n_5, n_7), e_4 = (n_2, n_4), e_5 = (n_4, n_6), e_6 = (n_6, n_8), e_7 = (n_1, n_4), e_8 = (n_3, n_6), e_9 = (n_6, n_7)\}$. Let $S = \{n_1, n_2\}$ with $a(n_1) = a(n_2) = 2$, let $T = \{n_7, n_8\}$ with $b(n_7) = 1$, $b(n_8) = 3$, and let $R = \{n_3, n_4, n_5, n_6\}$. Let c be given by: $c(e_7) = c(e_9) = 1$, $c(e_1) = c(e_2) = c(e_4) = c(e_8) = 2$, and $c(e_3) = c(e_5) = c(e_6) = 3$.

Schematically, this supply-demand network is given by the following diagram.



Then we have

$$A = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\ 1 & 1 & 1 & 2 & 3 & 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 3 & 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 2 & 3 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 & 2 & 3 & 0 & 2 & 1 \end{bmatrix} \quad \square$$

Fulkerson and Weinberger (1975) proved the following.

Theorem III.1.6. IRD holds for $\Pi(A, w)$ for every $w \in \mathbb{Z}_+^n$, where A is the matrix of integral, feasible flows in a capacitated, integral, supply-demand network. \square

A proof of this result, motivated by the work in Fulkerson and Weinberger (1975), but simpler and more generally applicable, is contained in Trotter and Weinberger (1976). The proof relies on two lemmas. Let v represent the given supply-demand system (N, A) with supplies a , demands b and capacities c . For $p \in \mathbb{R}_+$, let v_p represent the

same supply-demand system, but with supplies p_a , demands p_b and capacities p_c . Then we have:

Lemma III.1.7. Every integral, feasible flow for v_k , where $1 \leq k \in \mathbb{Z}_+$, is the sum of k integral flows, each feasible for v . \square

This lemma is proved by induction on k . Given f a feasible, integral flow for v_k , we need to extract f' an integral "subflow" of f so that f' is feasible for v and $f-f'$ is feasible for v_{k-1} . It is not difficult to see that such an f' will exist if and only if there exists a feasible flow for the supply-demand network with arc capacities given by $\min(f, c)$ and arc flow lower bounds (all zero in the original network) given by $\max(f(x, y) - (k-1)c(x, y), 0)$ (i.e., $f'(x, y)$ must be greater than or equal to this quantity.) It is clear that the flow f/k will satisfy these arc capacities and lower bounds, and the total unimodularity of (III.1.1)-(III.1.4) implies that the existence of such a feasible flow automatically guarantees the existence of a feasible, integral f' .

Lemma III.1.8. There exists a solution to $\Pi(A, w)$ of value $r \geq 0$ if and only if the system v_r with modified arc capacities $c'(x, y) = \min(w(x, y), rc(x, y))$ is feasible. \square

We use a similar lemma to establish results about integer rounding in various contexts (see Lemmas III.3.2, IV.1.30 and IV.3.7).

Using Lemmas III.1.7 and III.1.8, one can establish the IRD property for $\Pi(A, w)$. We show in Section III.3 that polyhedral

decomposition of the type described in (III.1.7) is actually equivalent to the rounding property stated in Theorem III.1.6 in certain instances.

Although the main emphasis of Fulkerson and Weinberger (1975) was not the IRD result, but actual determination of the value of $\Pi(A, w)$ through the use of the blocking theory of Fulkerson (1970, 1971a), this paper did first display techniques and a point of view which motivated much of the work in the remainder of this thesis.

III.2 Lower and Upper Comprehensive Polyhedra

Given P and Q nonempty polyhedra in R_+^m , we say that: P is lower comprehensive (LC) if P is bounded, and $x \in P$, $0 \leq y \leq x$ implies $y \in P$; Q is upper comprehensive (UC) if $x \in Q$ and $y \geq x$ implies $y \in Q$.

The following three theorems about LC and UC polyhedra are well-known. Suppose A is an $m \times n$ matrix of nonnegative real numbers with no zero rows, B is a nonnegative $m \times n$ real matrix with no zero columns and w is a nonnegative real n -vector. Then the following is clear.

Theorem III.2.1. (1) The polyhedron $P = \{y \in R^m: yA \leq w, y \geq 0\}$ is LC; (2) the polyhedron $Q = \{y \in R^m: yB \geq w, y \geq 0\}$ is UC. \square

Conversely, we also obtain

Theorem III.2.2. Let P be a LC polyhedron, Q a UC polyhedron, each contained in R_+^m . Then:

- (1) there exists a nonnegative matrix A with no zero rows and a nonnegative vector w , such that $P = \{y \in R^m: yA \leq w, y \geq 0\}$;
- (2) there exists a nonnegative matrix B with no zero columns and a nonnegative vector v , such that $Q = \{y \in R^m: yB \geq v, y \geq 0\}$.

Proof: (1) Let $P = \{y: yA \leq w, y \geq 0\}$ for some A and w . Since P is nonempty and bounded, A will have no zero rows. It is also clear that w must be nonnegative, since P LC implies $0 \in P$.

If (1) fails, we may thus assume that there are some negative entries in A^1 , the first column of A . So say $a_{i1} < 0$ for $1 \leq i \leq k$ and $a_{i1} \geq 0$ for $k+1 \leq i \leq m$. Let \tilde{A} be the same as A except that $\tilde{a}_{i1} = 0$ for $1 \leq i \leq k$, and let $\tilde{P} = \{y \in R^m: y\tilde{A} \leq w, y \geq 0\}$. To prove (1), it is then enough to show that $\tilde{P} = P$. It is clear that $\tilde{P} \subseteq P$, so suppose there is a $y = (y_1, \dots, y_m)$ such that $y \in P - \tilde{P}$. Then $y \cdot A^1 \leq w_1$, but $y \cdot \tilde{A}^1 > w_1$. Let $\tilde{y} = (0, \dots, 0, y_{k+1}, \dots, y_m)$. Then $\tilde{y} \cdot \tilde{A}^1 = y \cdot \tilde{A}^1 > w_1$. On the other hand, $\tilde{y} \cdot \tilde{A}^1 = \tilde{y} \cdot A^1$; thus $\tilde{y} \cdot A^1 > w_1$ and so $\tilde{y} \notin P$. But $0 \leq \tilde{y} \leq y \in P$, in contradiction with P being LC.

(2) Let $Q = \{y \in R^m: yB \geq v, y \geq 0\}$ for some B and v .

Suppose $b_{11} < 0$. Let $y = (y_1, \dots, y_m)$ be any vector such that $yB \geq v, y \geq 0$, and let B^1 be the first column of B . Then since $b_{11} < 0$, there exists $c > 0$ large enough so that if we let $\tilde{y} = (y_1 + c, y_2, \dots, y_m)$, then $\tilde{y} \cdot B^1 < v_1$, and so $\tilde{y} \notin Q$. But $\tilde{y} \geq y \in Q$, in contradiction with Q being UC. Thus B is nonnegative. If $v_i \leq 0$, then the constraint $y \cdot B^i \geq v_i$ of Q is implied by the non-negativity requirement $y \geq 0$, so all such constraints may be deleted, leaving matrix B' . If the resulting matrix B' is vacuous we take $v = 0$ and $B = I_{m \times m}$, the $m \times m$ identity matrix. If B' is nonvacuous we are done, since Q nonempty implies B' has no zero columns. \square

Lower and upper comprehensive polyhedra have been extensively studied, in particular in the antiblocking and blocking theories of

Fulkerson (1970,1971a,1972). Upper comprehensive polyhedra are studied in blocking theory, and Fulkerson (1970) has shown that upper comprehensive polyhedra occur in "dual" pairs, the extreme points of one determining the constraint system of the other. Similarly, in anti-blocking theory Fulkerson (1972) demonstrates that lower comprehensive polyhedra also occur in "dual" pairs, the extreme points of one determining the constraint system of the other. Antiblocking theory and blocking theory will be discussed further in Chapter V.

Given any polyhedron R , let $M(R)$ denote the set of the maximal integral points in R , and $m(R)$ denote the set of minimal integral points in R . The theorem which follows is used in Section III.3.

Theorem III.2.3. (1) If $P \subset \mathbb{R}_+^m$ is LC, then $M(P)$ is finite and nonempty.

(2) If $Q \subseteq \mathbb{R}_+^m$ is UC, then $m(Q)$ is finite and nonempty.

Proof: (1) The result is immediate, since P is bounded and $0 \in P$. \square

(2) By Theorem I.3.2, there exist points x^1, \dots, x^r and y^1, \dots, y^s in \mathbb{R}^m such that $Q = \{z \in \mathbb{R}^m: z = \sum_{i=1}^r \lambda_i x^i + \sum_{j=1}^s \mu_j y^j, \lambda_i \geq 0, \mu_j \geq 0, \sum_{i=1}^r \lambda_i = 1\}$. Since Q is UC, it is clear that all the x^i and y^j , $i = 1, \dots, r$, $j = 1, \dots, s$ are nonnegative. Thus if we let $Q^B = \{z \in Q: z = \sum_{i=1}^r \lambda_i x^i, \lambda_i \geq 0, \sum_{i=1}^r \lambda_i = 1\}$ then Q^B will contain all the minimal points of Q . Now let $Q^I = \{u+v \in \mathbb{R}_+^m: u \in Q^B, 0 \leq v \leq 1\}$. Then $Q^I \subset Q$, Q^I is bounded and $m(Q) \subseteq Q^I$. (If $w \in Q$, w integral, then there is a $z = (z_1, \dots, z_m) \in Q^B$ such that $z \leq w$. But then $z' = (\lceil z_1 \rceil, \dots, \lceil z_m \rceil)$ satisfies z' integral, $z' \in Q^I$ and $z' \leq w$.) Thus $m(Q)$ is finite, and since $Q \neq \emptyset$, clearly $m(Q) \neq \emptyset$. \square

III.3 Polyhedral Decomposition

Given any polyhedron $R \subseteq R_+^n$ and real number $r \geq 0$, let rR denote the polyhedron given by $rR = \{ry: y \in R\}$. Say that the decomposition property holds for R if for every $k \in \mathbb{Z}_+$, $k \geq 1$, we have that y integral and $y \in kR$ imply $y = \sum_{i=1}^k x^i$, $x^i \in R$, x^i integral for $i = 1, \dots, k$.

Now assume that P and Q are polyhedra in R_+^n with integral extreme points, and assume further that P is LC and Q is UC. Let A be the matrix whose rows are the elements of $M(P)$, and let B be the matrix whose rows are the elements of $m(Q)$. By Theorem III.2.3, $M(P)$ and $m(Q)$ are finite nonempty sets. The following theorem shows an equivalence between the decomposition property and rounding. In Chapter IV we make extensive use of this theorem to show that certain classes of linear programming problems possess the rounding property. Assume that A (as above) has no zero columns, and B (as above) has no zero rows. Then:

Theorem III.3.1. (a) IRU holds for $\Gamma(A, w)$ for every $w \in \mathbb{Z}_+^n$ if and only if the decomposition property holds for P .

(b) IRD holds for $\Pi(B, w)$ for every $w \in \mathbb{Z}_+^n$ if and only if the decomposition property holds for Q .

Before proving (III.3.1) we establish the following lemma.

Lemma III.3.2. Let $r \geq 0$ be rational. Then

- (a) $\Gamma(A, w)$ has a feasible solution of value $r = 1 \cdot y$ if and only if there is an $x \in rP$ such that $x \geq w$;

(b) $\Pi(B,w)$ has a feasible solution of value $r = 1 \cdot y$ if and only if there is an $x \in rQ$ such that $x \leq w$.

Proof: Necessity for (a): Let y be feasible for $\Gamma(A,w)$, $1 \cdot y = r$.

Then, $yA \in rP$ and $yA \geq w$.

Sufficiency for (a): The case $r = 0$ is clear, so assume $r > 0$. We

have $x \in rP$, $x \geq w$. Let E index the extreme points of P . Then

we have $\frac{x}{r} \in P$, and thus we can write $\frac{x}{r} = \sum_{i \in E} \lambda_i x^i$ with $\sum_{i \in E} \lambda_i = 1$, $\lambda_i \geq 0$ for each $i \in E$, and each x^i an (integral) extreme point of

P . (Note that since P is bounded, it is the convex hull of its extreme points--see (I.3.3).)

But then for each i , we may find a row a^i of A such that $x^i \leq a^i$ (the a^i need not be distinct). Thus $\frac{x}{r} \leq \sum_{i \in E} \lambda_i a^i$ and $x \leq \sum_{i \in E} (r\lambda_i) a^i$, so $w \leq \sum_{i \in E} (r\lambda_i) a^i$. Thus, putting weight $r\lambda_i$ on row a^i of A for $i \in E$ and weight 0 on other rows of A , gives the desired solution to $\Gamma(A,w)$ of value r .

Necessity for (b): Let y be feasible for $\Pi(B,w)$, $1 \cdot y = r$. Then, $yB \in rQ$ and $yB \leq w$.

Sufficiency for (b): If $r = 0$, the result is clear, so assume $r > 0$.

Let E index the extreme points of Q . We have $x \in rQ$, $x \leq w$ and

so $\frac{x}{r} \in Q$. We may then by (I.3.2) write $\frac{x}{r} = \sum_{i \in E} \lambda_i x^i + \sum_{j=1}^s \mu_j y^j$ where

the x^i are (integral) extreme points of Q , $\lambda_i \geq 0$, $\sum_{i \in E} \lambda_i = 1$,

and $\mu_j \geq 0$. Since Q is UC, we further know that $y^j \in R_+^n$ for each j ;

i.e., we may write $\frac{x}{r} = (\sum_{i \in E} \lambda_i x^i) + y$ as above with $y \in R_+^n$. Thus

$$\frac{x}{r} \geq \sum_{i \in E} \lambda_i x^i.$$

But the x^i are integral points of Q , so for each i , we may find a row b^i of B such that $x^i \geq b^i$ (not all the b^i need be distinct). Then, $\frac{x}{r} \geq \sum_{i \in E} \lambda_i b^i$ and $x \geq \sum_{i \in E} (r\lambda_i) b^i$, so $w \geq \sum_{i \in E} (r\lambda_i) b^i$. Thus, putting weight $r\lambda_i$ on row b^i of B for $i \in E$, and weight 0 on the other rows of B gives the desired solution to $\Pi(B, w)$ of value r . \square

Proof of Theorem III.3.1: Necessity for (a): Say IRU holds for $\Gamma(A, u)$ for every $u \in Z_+^n$. Let $k \in Z_+$, $k \geq 1$, $w \in kP$, $w \geq 0$ and integral. Then by Lemma III.3.2, $\Gamma(A, w)$ has a solution of value k . But $k \in Z_+$ and IRU for $\Gamma(A, w)$ imply that $\Gamma(A, w)$ has an integral solution of size k ; i.e., there are (not necessarily distinct) rows a^1, \dots, a^k of A that are integral points of P for which $\sum_{i=1}^k a^i \geq w$. Thus, since P is LC, there are $x^i \in P$, x^i integral, such that $\sum_{i=1}^k x^i = w$. This is the desired decomposition of w .

Sufficiency for (a): Say that the decomposition property holds for P , and given $w \in Z_+^n$, say an optimal solution to $\Gamma(A, w)$ has value r_w^* . If $r_w^* = 0$, then it is clear that $s_w^* = 0$ and we are done. So assume $r_w^* > 0$, and let $s = \lceil r_w^* \rceil$.

By Lemma III.3.2, there is an $x \in r_w^* P$, $x \geq w \geq 0$. But then $w \in r_w^* P$, since P LC implies $r_w^* P$ LC. Clearly then $w \in sP$, and so the decomposition property for P gives us $w = \sum_{i=1}^s x^i$, $x^i \in P$, x^i integral for each i . But then by definition of A , there are rows a^i of A such that $a^i \geq x^i$ for each i (not all the a^i need be distinct), and so $w \leq \sum_{i=1}^s a^i$ gives the desired integral solution to $\Gamma(A, w)$ of value $s = \lceil r_w^* \rceil$.

Necessity for (b): Say IRD holds for $\Pi(B,u)$ for every $u \in Z_+^n$.

Let $k \geq 1$ be an integer, $w \in kQ$, $w \geq 0$ and integral. By

Lemma III.3.2, $\Pi(B,w)$ has a solution of value k . Then $k \in Z_+$

and IRD for $\Pi(B,w)$ imply that $\Pi(B,w)$ has an integral solution of

value k . Thus there are rows b^1, \dots, b^k of B (not necessarily all

distinct) such that $\sum_{i=1}^k b^i \leq w$. But b^1, \dots, b^k are integral points

of Q , so since Q is UC, there are points $x^i \in Q$, x^i integral,

such that $\sum_{i=1}^k x^i = w$, which is the desired decomposition of w .

Sufficiency for (b): Say the decomposition property holds for Q ,

and given $w \in Z_+^n$ suppose $\Pi(B,w)$ has optimal value r_w^* . If

$0 \leq r_w^* < 1$, then it is clear that IRD holds for this w (the vector

of all zeros is always feasible for $\Pi(B,w)$), so assume $r_w^* \geq 1$. Let

$s = \lfloor r_w^* \rfloor$.

Then, by Lemma III.3.2, there is an $x \in r_w^*Q$, $x \leq w$. But then

$w \in r_w^*Q$, since Q UC implies r_w^*Q UC. So, $w \in sQ$ (note that

$r_w^*Q \subseteq sQ$ because $s \leq r_w^*$), and the decomposition property for Q

implies that $w = \sum_{i=1}^s x^i$, $x^i \in Q$, x^i integral for each i . Thus by

definition of B , there are rows b^i of B such that $b^i \leq x^i$

for each i (the b^i need not all be distinct). Hence $w \geq \sum_{i=1}^s b^i$

gives the desired solution to $\Pi(B,w)$ of value $s = \lfloor r_w^* \rfloor$. \square

This completes the proof of Theorem III.3.1.

III.4 "Excess" Polyhedra and Decomposition

Let P be a polyhedron in R_+^n with integral extreme points,

Q a polyhedron in R_+^n with integral extreme points, and assume P

is given by the constraints,

$$\begin{aligned}
f_1(x) &\leq \alpha_1 \\
f_2(x) &\leq \alpha_2 \\
&\dots \\
f_p(x) &\leq \alpha_p \\
x &\geq 0,
\end{aligned}
\tag{III.4.1}$$

and Q is given by the constraints,

$$\begin{aligned}
g_1(x) &\geq \beta_1 \\
g_2(x) &\geq \beta_2 \\
&\dots \\
g_q(x) &\geq \beta_q \\
x &\geq 0,
\end{aligned}
\tag{III.4.2}$$

where for all i, j , $\alpha_i \in R_+$, $\beta_j \in R_+$ and $f_i: R^n \rightarrow R$ and $g_j: R^n \rightarrow R$ are linear functionals. The following theorem gives a characterization of decomposition which is recursive in nature.

Theorem III.4.3. (a) The decomposition property holds for P if and only if, for every $w \in Z_+^n$ and every integer $k \geq 1$, we have that $w \in kP$ implies that the polyhedron given by the solutions to

$$f_i(w) - (k-1)\alpha_i \leq f_i(x) \leq \alpha_i, \quad i = 1, \dots, p \tag{III.4.4}$$

$$0 \leq x \leq w$$

always contains an integral point.

(b) The decomposition property holds for Q if and only if, for every $w \in Z_+^n$ and every integer $k \geq 1$, we have that $w \in kQ$ implies that the polyhedron given by the solutions to

$$\beta_j \leq g_j(x) \leq g_j(w) - (k-1)\beta_j, \quad j = 1, \dots, q \quad (\text{III.4.5})$$

$$0 \leq x \leq w,$$

always contains an integral point.

Proof: (a) Necessity: Decomposition for $w \in kP$ implies that there is an $x \in P \cap Z_+^n$ such that $w - x \in (k-1)P$. This x must satisfy (III.4.4).

Sufficiency: Let $x \in Z_+^n$ satisfy (III.4.4). Then $f_i(x) \leq \alpha_i$, $i = 1, \dots, p$, and $x \geq 0$, so $x \in P$. On the other hand, $f_i(w) - (k-1)\alpha_i \leq f_i(x)$ implies $f_i(w-x) \leq (k-1)\alpha_i$ for all i , and we have $w-x \geq 0$, so $w-x \in (k-1)P$. Now apply the same argument to $w' = w-x \in (k-1)P \cap Z_+^n$, and inductively obtain the desired decomposition of w .

(b) Similar to that for (a). \square

For $w \in Z_+^n \cap kP$ ($k \geq 1$ an integer), define the excess polyhedron of P with respect to w and k , denoted $E(P, w, k)$, to be the polyhedron given by (III.4.4). Similarly, for $w \in Z_+^n \cap kQ$ ($k \geq 1$ an integer), define the excess polyhedron of Q with respect to w and k , denoted $E(Q, w, k)$, to be the polyhedron given by (III.4.5).

In many of the examples discussed in Chapter IV for which the rounding property holds, the excess polyhedron not only always contains an integer point, but has all integral extreme points. For instance, the excess polyhedron has all integer extreme points for every $w \in \mathbb{Z}_+^n$ and integer $k \geq 1$ in the cases of integral, feasible flows in capacitated supply-demand networks discussed in (III.1), polymatroid problems discussed in (IV.1), circulations of a totally unimodular matrix discussed in (IV.4), and assembly tree scheduling problems discussed in (IV.5). In each of these cases, unimodularity properties of the constraint systems (III.4.4) and (III.4.5) will be seen to explain this integrality property for the excess polyhedron.

However, it is not always the case that the rounding property implies all integral vertices for the excess polyhedron. This stems from the fact that, given $w \in \mathbb{Z}_+^n$ and an integer $k \geq 1$, $\frac{w}{k}$ may be expressible as a convex combination of extreme points of P in more than one way, with some of the expressions resulting in integral decompositions, while others do not. This is illustrated in the following example.

Example III.4.6. Consider the problem $\Gamma(A, w)$ where A is given by:

$$A = \begin{matrix} & \begin{matrix} a^1 \\ a^2 \\ a^3 \\ a^4 \\ a^5 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

IRU holds for $\Gamma(A, w)$ for every $w \in \mathbb{Z}_+^n$. In Section III.5 it will be shown that it is enough to check the rounding property for this A for integral vectors w such that $0 \leq w \leq (2, 2, 2, 1, 1, 1)$. Thus it is tedious, but straightforward, to check that IRU holds for all w .

Let $P = \{x \in \mathbb{R}_+^6 : x \leq \sum_{i=1}^5 \lambda_i a^i, \sum_{i=1}^5 \lambda_i = 1, \lambda_i \geq 0\}$.

Then by Theorem III.3.1(a), P satisfies the decomposition property.

Now consider the problem $\Gamma(A, w)$ for $w = (1, 1, 1, 1, 1, 1)$. Then

$w \in 2P$, and we have $w = a^4 + a^5$, giving the decomposition of w and integral optimal solution to $\Gamma(A, w)$. Thus $a^4, a^5 \in E(P, w, 2)$, and

it is further clear that these are the only two integral points in $E(P, w, 2)$. (If $x \in E(P, w, 2)$, then $(w-x) \in P$.) However, if we let $b^1 = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ and $b^2 = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$, then we have $b^1 = \frac{1}{2}(a^1 + a^2)$ and $b^2 = \frac{1}{2}(a^3 + a^4)$; thus $b^1 \in P$ and $b^2 \in P$. Further, $w = b^1 + b^2$, thus $w - b^1 = b^2 \in P$. Thus $b^1 \in E(P, w, 2)$. However, it is not possible to write b^1 as a convex combination of a^4 and a^5 , which are the only two integral points in $E(P, w, 2)$. Since $E(P, w, 2)$ is a bounded polyhedron, Theorem I.3.3 shows that $E(P, w, 2)$ is the convex hull of its extreme points, and thus must have some non-integral extreme points.

Note that the difficulty arises out of the non-uniqueness of $\frac{w}{k} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ as a convex combination of the rows of A ; i.e., we have $\frac{w}{k} = \frac{1}{2}(a^4 + a^5)$, which leads to an integral decomposition, and $\frac{w}{k} = \frac{1}{4}(a^1 + a^2 + a^3 + a^4)$, which does not. \square

Examples such as (III.4.6) make it appear less likely that there is a simple general relation between unimodularity properties of the

constraints (III.4.4) and (III.4.5) and the rounding property. However, the fact that the excess polyhedra must always contain an integral point if rounding is valid does indicate a possible relation between some "local unimodularity" properties of (III.4.4) and (III.4.5) (see Section II.6) and rounding, though no such result is yet known.

III.5 A Reduction Theorem

In this section we prove a reduction theorem for the problems $\Gamma(A,w)$, $\Pi(B,w)$, $w \in Z_+^n$. As a consequence of this theorem, it is then shown that IRU and IRD are finitely checkable properties for $\Gamma(A,w)$ and $\Pi(B,w)$. In Section V.3 we use a similar proof technique to obtain a new and elementary proof of Fulkerson's Pluperfect Graph Theorem.

Let A be a nonnegative integral $m \times n$ matrix without zero columns, B a nonnegative integral $m \times n$ matrix with no zero rows, and let $w \in Z_+^n$. Say $w \in I$ if y_w^* optimal for $\Gamma(A,w)$ (or $\Pi(A,w)$) implies $y_w^* < 1$ (i.e., given any optimal vector, all components are < 1).

Theorem III.5.1. (a) IRU holds for $\Gamma(A,w)$ for every $w \in Z_+^n$ if and only if it holds for all $w \in I$.

(b) IRD holds for $\Pi(B,w)$ for every $w \in Z_+^n$ if and only if it holds for all $w \in I$.

Proof: Necessity for both (a) and (b) is obvious.

Sufficiency for (a): We proceed by induction on $1 \cdot w$. If $1 \cdot w = 0$, then clearly $w \in I$. So assume $1 \cdot w = k \geq 1$, and inductively suppose that IRU holds for all $w' \in Z_+^n$ such that $1 \cdot w' < k$. Further assume

$w \notin I$. Thus without loss of generality there is an optimal vector $y_w^* = ((y_w^*)_1, \dots, (y_w^*)_m)$ for $\Gamma(A, w)$ such that $(y_w^*)_1 \geq 1$. Let (a_{11}, \dots, a_{1n}) be the first row of A and let $w' = (\max(w_1 - a_{11}, 0), \dots, \max(w_n - a_{1n}, 0)) \in Z_+^n$. Then the optimality of y_w^* and $(y_w^*)_1 \geq 1$ imply that $1 \cdot w' \leq (1 \cdot w) - 1$. Let $y' = ((y_w^*)_1 - 1, (y_w^*)_2, \dots, (y_w^*)_m)$. Then $y'A \geq w'$, $y' \geq 0$ and $1 \cdot y' = (1 \cdot y_w^*) - 1$. Further, optimality of y_w^* for $\Gamma(A, w)$ implies y' is optimal for $\Gamma(A, w')$, since given any feasible vector $y = (y_1, \dots, y_m)$ for $\Gamma(A, w')$, the vector $(y_1 + 1, y_2, \dots, y_m)$ must be feasible for $\Gamma(A, w)$. Thus by the induction hypothesis there is a $z' \in Z_+^m$ such that $1 \cdot z' = \lceil 1 \cdot y' \rceil$ and $z'A \geq w'$. But then $z = (z'_1 + 1, z'_2, \dots, z'_m)$ is an integral, feasible vector for $\Gamma(A, w)$, and $1 \cdot z = 1 \cdot z' + 1$, so that $1 \cdot z' = \lceil 1 \cdot y' \rceil$ implies $1 \cdot z = \lceil 1 \cdot y_w^* \rceil$. \square

Sufficiency for (b): Again we proceed by induction on $1 \cdot w$. If $1 \cdot w = 0$, clearly $w \in I$. So assume $1 \cdot w = k \geq 1$ and inductively suppose that IRD holds for all $w' \in Z_+^n$ such that $1 \cdot w' < k$. Further assume $w \notin I$.

Thus without loss of generality there is an optimal vector

$y_w^* = ((y_w^*)_1, \dots, (y_w^*)_m)$ for $\Pi(B, w)$ such that $(y_w^*)_1 \geq 1$. Subtract the first row of B from w to obtain the vector w' . Note that $w' \geq 0$, since $y_w^* B \leq w$ and $(y_w^*)_1 \geq 1$. Further since B has no zero rows, $1 \cdot w' \leq 1 \cdot w - 1$. Let $y' = ((y_w^*)_1 - 1, (y_w^*)_2, \dots, (y_w^*)_m)$. Then $y'B \leq w'$, $y' \geq 0$ and $1 \cdot y' = (1 \cdot y_w^*) - 1$. Further, optimality of y_w^* for $\Pi(B, w)$ implies y' must be optimal for $\Pi(B, w')$, since given any feasible vector $y = (y_1, \dots, y_m)$ for $\Pi(B, w')$, the vector $(y_1 + 1, y_2, \dots, y_m)$ must be feasible for $\Pi(B, w)$. Thus by the induction hypothesis there is a $z' \in Z_+^m$ such that $1 \cdot z' = \lfloor 1 \cdot y' \rfloor$ and $z'B \leq w'$.

But then $z = (z'_1+1, z'_2, \dots, z'_m)$ is an integral, feasible vector for $\Pi(B, w)$, and $1 \cdot z = 1 \cdot z' + 1$, so that $1 \cdot y_w^* = 1 \cdot y' + 1$ and $1 \cdot z' = \lfloor 1 \cdot y' \rfloor$ imply $1 \cdot z = \lfloor 1 \cdot y_w^* \rfloor$. \square

We now show that IRU and IRD are finitely checkable properties for $\Gamma(A, w)$ and $\Pi(B, w)$.

Corollary III.5.2. Let c_i be the i th column sum of A , $i = 1, \dots, n$, d_i be the i th column sum of B , $i = 1, \dots, n$, $c' = (c_1-1, c_2-1, \dots, c_n-1)$ and $d = (d_1, d_2, \dots, d_n)$. Then:

- (a) IRU holds for $\Gamma(A, w)$ for every $w \in Z_+^n$ if and only if it holds for every integral w such that $0 \leq w \leq c'$;
- (b) IRD holds for $\Pi(B, w)$ for every $w \in Z_+^n$ if and only if it holds for every integral w such that $0 \leq w \leq d$.

Proof of (a): If $w \in I$ for $\Gamma(A, w)$, then clearly $w \leq c'$, so that the result follows directly from (III.5.1).

Proof of (b): By (III.5.1) it is enough to show that if IRD holds for $\Pi(B, w)$ for every integral w , $0 \leq w \leq d$, then it holds for $\Pi(B, w)$ for every $w \in I$.

So we suppose $w \in I$ and $w \not\leq d$. Since $w \in I$, all optimal vectors for $\Pi(B, w)$ have all coordinates < 1 . Thus, for every optimal vector y_w^* for $\Pi(B, w)$, we have $y_w^* B \leq d$. So if we let $w' = (\min(w_1, d_1), \dots, \min(w_n, d_n))$, then $\Pi(B, w')$ has exactly the same optimal solutions as does $\Pi(B, w)$. But $w' \leq d$, so by assumption there exists an integral, feasible solution z' to $\Pi(B, w')$ such that $1 \cdot z' = \lfloor 1 \cdot y_w^* \rfloor$. Now since $w' \leq w$, z' is certainly also feasible for $\Pi(B, w)$; thus IRD holds for $\Pi(B, w)$, as desired. \square

CHAPTER IV

EXAMPLES

In this chapter we examine integer rounding and decomposition for several combinatorial problems.

IV.1 Polymatroids

Polymatroids, a polyhedral generalization of matroids, were introduced by Edmonds (1970). The following definitions and theorems are from Edmonds (1970) and Giles (1975).

Let $E = \{1, \dots, n\}$. A polymatroid P in R^n is a compact nonempty subset of R_+^n such that:

- (1) $0 \leq x \leq y \in P$ implies $x \in P$;
- (2) For each $a \in R_+^n$, every maximal $x \in P$ such that $x \leq a$ has the same component sum $x(E) = \sum_{j \in E} x_j$, called the rank, $r(a)$, of a (with respect to P). Such a maximal x is called a basis of a in P .

A polymatroid is called integral if (2) also holds when x and a are restricted to be integer-valued.

Let $L_E = \{A: A \subseteq E\}$. A real-valued function f on L_E is called a β -function if

- (1) $f(A) \geq 0$, $A \subseteq E$;
- (2) $A \subseteq B$ implies $f(A) \leq f(B)$, $A \subseteq E$, $B \subseteq E$;
- (3) f is submodular, i.e., $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$,
 $A \subseteq E$, $B \subseteq E$.

f is called an integral β -function if f is integer-valued and a β -function.

We may use β -functions to define certain polyhedra which are polymatroids as follows:

Theorem IV.1.1. Let f be a β -function on L_E , and let $P(E,f)$ be the polyhedron in R_+^n given by $P(E,f) = \{x \in R_+^n: 0 \leq x(A) \leq f(A), A \in L_E\}$. Then $P(E,f)$ is a polymatroid. Further, if f is an integral β -function, then $P(E,f)$ is an integral polymatroid. \square

On the other hand, all polymatroids correspond to certain β -functions in the following sense.

Theorem IV.1.2. Given any polymatroid $P \subset R_+^n$, let $a \in R_+^n$ be an integral vector such that $x < a$ for every $x \in P$. Where r is the rank function of P , let $f_p(A) = r(a|A)$ (i.e., the rank of the vector a restricted to the subset A) for $A \subseteq E$. Then f_p is a β -function, and $P = P(E, f_p)$. Further, if P is an integral polymatroid and $P = P(E, f)$, then f is an integral β -function. \square

Note that Theorem IV.1.2 implies that all polymatroids are polyhedra.

Polymatroids are related to matroids (see Section II.4) by the following theorem.

Theorem IV.1.3. A function f on L_E is the rank function of a matroid $M = (E, I)$ if and only if it is an integral β -function such that $f(\{j\}) = 1$ or 0 for every $j \in E$. Such an f determines M by: $J \in I$ if and only if $J \subseteq E$ and $|J| = f(J)$, where I is the family of independent sets of M . \square

Thus the rank function f of a matroid M may be used to define the integral polymatroid $P(E, f)$. It is well-known (see Edmonds (1971))

that the extreme points of the polymatroid $P(E, f)$ are precisely the incidence vectors of the independent sets of M . Thus $P(E, f)$ has all integral vertices in this case. This is always true when f is an integral β -function, as stated in the following theorem. This theorem follows easily from Lemma IV.1.22 below, but was originally proved by Edmonds (1970) in a different manner.

Theorem IV.1.4. If $P(E, f)$ is an integral polymatroid, then every extreme point of $P(E, f)$ is integral. \square

We now establish unimodularity properties for certain classes of matrices which will be useful in proving a decomposition theorem for integral polymatroids (IV.1.24). Let $V = \{W_1, \dots, W_k\}$ be a family of subsets of E which satisfies:

for any $R \in V$ and $S \in V$, either $R \cap S = \emptyset$ or $R \cap S \in V$. (IV.1.5)

Let A be the incidence matrix of such a family V , i.e.,

$$a_{ij} = \begin{cases} 1 & \text{if } j \in W_i \\ 0 & \text{otherwise.} \end{cases}$$

Then A is not necessarily totally unimodular. However, we do have the following result of Edmonds (1970).

Lemma IV.1.6. One can obtain from A , by subtracting certain rows from others, the incidence matrix of a family of disjoint subsets of E .

Proof: If W_1, \dots, W_k are already disjoint, we are finished. Otherwise find any minimal W_i in V such that there exists W_j in V with $W_i \subset W_j$. Subtract the row of A that corresponds to W_i from every row in A corresponding to a W_ℓ in V such that $W_i \subset W_\ell$, and call the altered matrix A' . A' is then again the incidence matrix of a family of subsets satisfying (IV.1.5), so we may repeat the procedure until we have obtained the incidence matrix of a family of disjoint subsets of E . \square

Now let V_1 and V_2 be two families of subsets of E such that both satisfy (IV.1.5), and let B be the incidence matrix of $V_1 \cup V_2$. Then we have:

Lemma IV.1.7. By subtracting certain rows of B from others, we may obtain a totally unimodular matrix.

Proof: Say $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, B_i corresponding to V_i , $i = 1, 2$. By Lemma IV.1.6, obtain B'_1 from B_1 and B'_2 from B_2 , both the incidence matrices of families of disjoint subsets of E . Let $B' = \begin{bmatrix} B'_1 \\ B'_2 \end{bmatrix}$. B' is then a $(0,1)$ matrix which satisfies the hypotheses of Theorem II.1.2, and is thus totally unimodular. \square

Edmonds (1970) has shown:

Lemma IV.1.8. Suppose f is a submodular function on L_E , $x \in R^n$ is a vector satisfying $x(A) \leq f(A)$ for every $A \subseteq E$, and let $V = \{A \subseteq E: x(A) = f(A)\}$. Then V satisfies (IV.1.5).

Proof: Suppose $B \subseteq E$, $C \subseteq E$ are such that $x(B) = f(B)$, $x(C) = f(C)$ and $B \cap C \neq \emptyset$. Then,

$$\begin{aligned}
f(B \cap C) &\leq f(B) + f(C) - f(B \cup C) \quad (\text{by submodularity of } f) \\
&\leq x(B) + x(C) - x(B \cup C) \\
&= x(B \cap C) \\
&\leq f(B \cap C).
\end{aligned}$$

Thus, equality holds throughout, and in particular, $x(B \cap C) = f(B \cap C)$. \square

Edmonds (1970) uses (IV.1.7) and (IV.1.8) to prove that the polyhedron generated by the constraints (II.4.3) has all integral extreme points; we use it to establish a decomposition theorem (IV.1.24) for integral polymatroids.

Lemma IV.1.9. Suppose f is a submodular function on L_E , k is a positive integer and y is a fixed vector. Consider the polyhedron defined by the constraints

$$y(A) - (k-1)f(A) \leq x(A), \quad A \subseteq E. \quad (\text{IV.1.10})$$

Then, if we have $B, C \subseteq E$ and x satisfying (IV.1.10) such that

$$y(B) - (k-1)f(B) = x(B) \quad (\text{IV.1.11})$$

and

$$y(C) - (k-1)f(C) = x(C), \quad (\text{IV.1.12})$$

then either $B \cap C = \emptyset$ or x also satisfies

$$y(B \cap C) - (k-1)f(B \cap C) = x(B \cap C). \quad (\text{IV.1.13})$$

(I.e., given x satisfying (IV.1.10), the family of subsets of E corresponding to constraints (IV.1.10) which x satisfies at equality have the property (IV.1.5).)

Proof: The case $k = 1$ is straightforward, so assume $k \geq 2$.

Note that from (IV.1.11) and (IV.1.12) we get

$$f(B) = \frac{y(B) - x(B)}{k-1} \quad (\text{IV.1.14})$$

$$f(C) = \frac{y(C) - x(C)}{k-1}, \quad (\text{IV.1.15})$$

and since x satisfies (IV.1.10) for every subset of E , we also have

$$y(B \cup C) - (k-1)f(B \cup C) \leq x(B \cup C),$$

or equivalently,

$$(k-1)f(B \cup C) \geq y(B \cup C) - x(B \cup C). \quad (\text{IV.1.16})$$

Then we have,

$$\begin{aligned}
x(B \cap C) &\geq y(B \cap C) - (k-1)f(B \cap C) \quad (\text{since } x \text{ satisfies (IV.1.10)}) \\
&\geq y(B \cap C) - (k-1)[f(B) + f(C) - f(B \cup C)] \quad (\text{by the} \\
&\quad \text{submodularity of } f) \\
&= y(B \cap C) - (k-1)\left[\frac{y(B) - x(B)}{k-1} + \frac{y(C) - x(C)}{k-1} - f(B \cup C)\right] \\
&\quad (\text{by (IV.1.14) and (IV.1.15)}) \\
&= y(B \cap C) + (k-1)f(B \cup C) + x(B) + x(C) - y(B) - y(C) \\
&= y(B \cap C) + (k-1)f(B \cup C) + x(B \cup C) + x(B \cap C) - y(B \cup C) \\
&\quad - y(B \cap C) \\
&= (k-1)f(B \cup C) + x(B \cup C) - y(B \cup C) + x(B \cap C) \\
&\geq y(B \cup C) - x(B \cup C) + x(B \cup C) - y(B \cup C) + x(B \cap C) \\
&\quad (\text{by (IV.1.16)}) \\
&= x(B \cap C).
\end{aligned}$$

Thus equality holds throughout and the lemma is proved. \square

Lemma IV.1.17. Given a submodular function f on L_E and a fixed vector y , consider the constraints:

$$0 \leq x(A) \leq \min\{f(A), y(A)\}, \quad A \subseteq E. \quad (\text{IV.1.18})$$

Then, if x satisfies (IV.1.18) and x also satisfies, for some $B \subseteq E$ and $C \subseteq E$ with $B \cap C \neq \emptyset$,

$$x(B) = \min\{f(B), y(B)\} \quad (\text{IV.1.19})$$

$$x(C) = \min\{f(C), y(C)\}, \quad (\text{IV.1.20})$$

then we also have,

$$x(B \cap C) = \min\{f(B \cap C), y(B \cap C)\}. \quad (\text{IV.1.21})$$

(I.e., given any vector x satisfying (IV.1.18), the family of subsets $D \subseteq E$ such that $x(D) = \min\{f(D), y(D)\}$ have the property (IV.1.5).)

Proof: Case (i): Say $x(B) = f(B)$ and $x(C) = f(C)$. In this case, the result follows immediately from Lemma IV.1.8.

Case (ii): Say $x(B) = y(B)$. We know that $x(B') \leq y(B')$ for every $B' \subseteq B$, and so we must have $x(B') = y(B')$ for every $B' \subseteq B$. In particular, since $B \cap C \subseteq B$, $x(B \cap C) = y(B \cap C) \geq \min\{f(B \cap C), y(B \cap C)\}$ and thus, since x also satisfies (IV.1.18), we have $x(B \cap C) = \min\{f(B \cap C), y(B \cap C)\}$, as desired.

Case (iii): If $x(C) = y(C)$, we argue exactly as in Case (ii). \square

Combining the previous two results and using Lemma IV.1.7 we obtain the result that excess polyhedra for $P(E, f)$ have all integral extreme points.

Lemma IV.1.22. Let $P = P(E, f)$ be an integral polymatroid, k a positive integer, and y a fixed nonnegative integral vector. Then the polyhedron generated by

$$y(A) - (k-1)f(A) \leq x(A) \leq \min\{f(A), y(A)\}, \quad A \subseteq E \quad (\text{IV.1.23})$$

$$x \geq 0$$

has all integral extreme points.

Proof: Let x be extreme for (IV.1.23). Then by Lemma IV.1.9, the submatrix T_1 of the constraint matrix for (IV.1.23) which corresponds to constraints of form (IV.1.10) which x satisfies at equality will satisfy the hypothesis of Lemma IV.1.6. Similarly by Lemma IV.1.17, the submatrix T_2 which corresponds to the constraints of form (IV.1.18) which x satisfies at equality will also satisfy the hypothesis of Lemma IV.1.6. Finally, let I' denote the submatrix of the constraint matrix for the constraints $x \geq 0$ which x satisfies

at equality. Thus x is the unique solution to $\begin{bmatrix} T_1 \\ T_2 \\ I' \end{bmatrix} [x] = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$ where

b_1 and b_2 are integral (since P is an integral polymatroid--see Theorem IV.1.2). But then x is also the unique solution to

$\begin{bmatrix} T'_1 \\ T'_2 \\ I' \end{bmatrix} [x] = \begin{bmatrix} b'_1 \\ b'_2 \\ 0 \end{bmatrix}$ where T'_i and b'_i are obtained from T_i and b_i by

subtracting certain rows from others, $i = 1, 2$, and $\begin{bmatrix} T'_1 \\ T'_2 \end{bmatrix}$ is a totally

unimodular matrix (Lemma IV.1.7). The matrix $\begin{bmatrix} T'_1 \\ T'_2 \\ I' \end{bmatrix}$ is then also

totally unimodular (I' is a submatrix of the identity matrix). Thus,

since $\begin{bmatrix} b'_1 \\ b'_2 \\ 0 \end{bmatrix}$ is an integral vector, we have by Theorem II.1.1 that x

must be integral. \square

Note that if we choose y large enough so that $y(A) \geq f(A)$ for $A \subseteq E$, and then choose k sufficiently large that $y(A) - (k-1)f(A) \leq 0$ for $A \subseteq E$, then the constraints (IV.1.23) reduce to $0 \leq x(A) \leq f(A)$, $A \subseteq E$, and thus we may deduce Theorem IV.1.4 from Lemma IV.1.22.

We are now ready to prove:

Theorem IV.1.24. (Decomposition for integral polymatroids). Let $P = P(E, f)$ be an integral polymatroid, and let $y \in kP$ be an integral vector, where k is a positive integer. Then $y = \sum_{j=1}^k x^j$, where $x^j \in P$ and x^j is integral for each j .

Proof: We proceed by induction on k . The case $k = 1$ is trivial, so suppose the theorem is established for $(k-1)P$ and let $y \in kP$, y integral. Then it is sufficient to find $x \in P$, x integral, such that $(y-x) \in (k-1)P$. But any integral x satisfying (IV.1.23) will satisfy this. Note that $\frac{y}{k}$ satisfies (IV.1.23), so the polyhedron defined by (IV.1.23) is nonempty. Since it is also contained in R_+^n , it will have extreme points (see Theorem I.3.3), and thus by Lemma IV.1.22 it will contain an integral point. \square

Let A be the matrix whose rows are the maximal integral points (i.e., the integral bases) of the integral polymatroid $P(E, f)$. In order to ensure that A has no zero columns (or rows), we will also assume that $P(E, f)$ is loopless, i.e., $f(\{i\}) > 0$ for every $i \in E$. (Analogously, a matroid defined on the set E is called loopless if $\{i\}$ is an independent set for every $i \in E$.) Then we have:

Theorem IV.1.25. The IRU property holds for $\Gamma(A, w)$ for every $w \in Z_+^n$.

Proof: The polyhedron $P(E,f)$ is clearly lower comprehensive, and by (IV.1.4) it has integral extreme points. Thus (IV.1.25) follows from Theorem III.3.1(a). \square

Restricting our attention to matroids, we immediately get:

Corollary IV.1.26. Let A be the matrix the rows of which are the incidence vectors of the bases of a loopless matroid. Then the IRU property holds for $\Gamma(A,w)$ for every $w \in Z_+^n$. \square

Corollary IV.1.26 may also be deduced from the work of Edmonds (1965b,1971) and Fulkerson (1972)--see Section V.2.

If we let A again be the matrix whose rows are the integral bases of the loopless, integral polymatroid $P = P(E,f)$, it is natural to inquire about integrality properties of the programming problem $\Pi(A,w)$ for $w \in Z_+^n$. Not surprisingly, we obtain:

Theorem IV.1.27. The IRD property holds for $\Pi(A,w)$ for every $w \in Z_+^n$.

We will prove (IV.1.27) via a decomposition result for the polyhedron P_B , which we define to be the convex hull of the integral bases of the polymatroid P . This decomposition result follows easily from (IV.1.24):

Theorem IV.1.28. (Decomposition for P_B). Let $x \in kP_B \cap Z_+^n$, where k is a positive integer. Then $x = \sum_{i=1}^k x^i$, where each $x^i \in P_B \cap Z_+^n$.

Proof: If $x \in kP_B$, then $x \in kP$, and so $x \in Z_+^n$ implies $x = \sum_{i=1}^k x^i$, $x^i \in P$, $x^i \in Z_+^n$ by Theorem IV.1.24. But $x \in kP_B$ implies $x(E) = kf(E)$, and so $x^i(E) = f(E)$ for $i = 1, \dots, k$. Thus $x^i \in P_B$ for $i = 1, \dots, k$, which gives the desired result. \square

Note that we may not conclude (IV.1.27) directly from (IV.1.28) and (III.3.1), since P_B is not upper comprehensive. Indeed, in Section IV.6, we provide an example of a bounded polyhedron in R_+^n with integral extreme points such that the decomposition property holds for the polyhedron, but IRD does not hold for $\Pi(B, w)$, where B has as rows the integral extreme points of the polyhedron. In order to prove IV.1.27, we will need the two following lemmas, the first of which is due to C. McDiarmid (1976a), and the second of which is very similar to Lemma III.3.2.

Lemma IV.1.29. Let P_1 and P_2 be integral polymatroids, k and ℓ integers, $v \in Z_+^n$, $w \in Z_+^n$. Then the set of vectors x such that $x \in P_1 \cap P_2$, $k \leq x(E) \leq \ell$ and $v \leq x \leq w$ is the convex hull of its integral elements. \square

Recall that A is the matrix whose rows are the integral bases of the loopless, integral polymatroid P . Let $w \in R_+^n$. We then have:

Lemma IV.1.30. $\Pi(A, w)$ has a feasible solution of value $r = l \cdot y$ ($r \geq 0$ arbitrary) if and only if there exists $x \in rP_B$ such that $x \leq w$.

Proof: Necessity: Let y be feasible for $\Pi(A, w)$ and $l \cdot y = r$.

Then $yA \in rP_B$ and $yA \leq w$.

Sufficiency: Let $x \in rP_B$. If $r = 0$, the result is clear, so assume $r > 0$. Then $\frac{x}{r} \in P_B$, and so if we let R index the integral bases of P , we have $\frac{x}{r} = \sum_{i \in R} \lambda_i x^i$, $x^i \in P_B$, $x^i \in Z_+^n$, $\sum_{i \in R} \lambda_i = 1$ and $\lambda_i \geq 0$ for every $i \in R$. Thus $x = \sum_{i \in R} (r\lambda_i) x^i$ yields the desired

packing vector by putting weight $r\lambda_i$ on the row of A which corresponds to the integral basis x^i . \square

Proof of IV.1.27: Given $w \in Z_+^n$, let r_w^* be the optimal value for $\Pi(A, w)$ and let $s = \lfloor r_w^* \rfloor$. If $s = 0$, the result is clear, so assume $s \geq 1$. Then by Lemma IV.1.30, there exists $x \in r_w^* P_B$ such that $x \leq w$, and thus the vector $u = sx/r_w^*$ satisfies $u \in sP_B$ and $u \leq w$. But applying Lemma IV.1.29 with $k = \ell = sf(E)$, $v = 0$ and $P_1 = P_2 = P$, we have that there exists $u' \in sP_B$, $0 \leq u' \leq w$, u' integral. Thus by Theorem IV.1.28, $u' = \sum_{j=1}^s x^j$, $x^j \in P_B$, x^j integral, which gives the desired integral solution of value $s = \lfloor r_w^* \rfloor$ to $\Pi(A, w)$. \square

We immediately conclude,

Corollary IV.1.31. Let A be the matrix whose rows are the incidence vectors of the bases of a loopless matroid. Then the IRD property holds for $\Pi(A, w)$ for every $w \in Z_+^n$. \square

Corollary IV.1.31 can also be derived from work of Edmonds and Fulkerson (1965), Edmonds (1971) and Fulkerson (1970).

Given an integral polymatroid $P = P(E, f)$, let $P_k = \{x \in P : x(E) \leq k\}$ for $k = 1, 2, \dots, f(E)$. Then it is easy to show that P_k is also an integral polymatroid, called a truncation of P . Let P_{kB} be the polyhedron which is the convex hull of the integral bases of P_k and let A_k be the matrix whose rows are the integral points in P_{kB} . Then we conclude:

Theorem IV.1.32. Let P be an integral, loopless polymatroid. Then with P_{kB} and A_k as above we have (1) the IRD property holds for

$\Pi(A_k, w)$ for every $w \in Z_+^n$, (2) P_{kB} is decomposable in the sense of Theorem IV.1.29, (3) the IRU property holds for $\Gamma(A_k, w)$ for every $w \in Z_+^n$ and (4) P_k satisfies the decomposition property. \square

IV.2 Matroid Intersection: Strongly Base-Orderable Matroids

In the next two sections we investigate rounding properties for packing and covering problems arising from the common independent sets of two matroids. Let M_1 and M_2 be two matroids on $E = \{1, \dots, n\}$ with rank functions r_1 and r_2 , respectively, and let $t = \max\{|C| : C \in \mathcal{C}\}$, where we use \mathcal{C} to denote the family of subsets of E which are independent in both M_1 and M_2 . For $j = 1, \dots, t$ let \mathcal{C}_j denote the family of subsets of E which are independent in both M_1 and M_2 and which have cardinality equal to j . Let A be the matrix whose rows are the incidence vectors of the maximal members of \mathcal{C} , and let B_j , for $j = 1, \dots, t$, be the matrix whose rows are the incidence vectors of the members of \mathcal{C}_j .

The problems $\Gamma(A, w)$, $\Gamma(B_t, w)$ and $\Pi(B_t, w)$ ($w \in R_+^n$) have been studied in terms of blocking and antiblocking theory (we elaborate on this in (V.2)); however, little is known in general about the problem $\Pi(A, w)$. Integer rounding does not necessarily hold for any of these problems, as the following example of McDiarmid (1976b) shows.

Example IV.2.1. Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Let M_1 be the matroid whose bases are all subsets of cardinality three of E except $\{e_1, e_3, e_5\}$, $\{e_1, e_2, e_4\}$, $\{e_2, e_3, e_6\}$ and $\{e_4, e_5, e_6\}$. Let M_2 be the matroid which has bases $\{e_1, e_3, e_5\}$, $\{e_1, e_4, e_5\}$, $\{e_1, e_2, e_3\}$, $\{e_1, e_2, e_4\}$, $\{e_2, e_4, e_6\}$, $\{e_2, e_3, e_6\}$, $\{e_3, e_5, e_6\}$ and $\{e_4, e_5, e_6\}$. We indicate that

M_1 and M_2 are indeed matroids in the following section. Then we have

$$A = B_t = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{matrix}.$$

Let $y = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $w = (1, 1, 1, 1, 1, 1)$. Then y optimizes $\Gamma(A, w)$, $\Pi(A, w)$, $\Gamma(B_t, w)$ and $\Pi(B_t, w)$ with value $1 \cdot y = 2$. However, the optimal integral solution for $\Gamma(A, w)$ and $\Gamma(B_t, w)$ has value 3, and the optimal integral solution for $\Pi(A, w)$ and $\Pi(B_t, w)$ has value 1. \square

However, McDiarmid (1976b) has identified a class of matroids for which we do get rounding results for the problems $\Gamma(A, w)$, $\Gamma(B_j, w)$ and $\Pi(B_j, w)$. Again nothing is known for the problem $\Pi(A, w)$. A matroid M is called strongly base-orderable if for each pair of bases B and B' of M there is a bijection $f: B \rightarrow B'$ such that $(B' - f(A)) \cup A$ is a basis of M for each subset A of B . McDiarmid and Davis (1975) and McDiarmid (1976b) have shown:

Theorem IV.2.2. Let M_1 and M_2 be loopless, strongly base-orderable matroids. Then

- (a) IRU holds for $\Gamma(A, w)$ for every $w \in \mathbb{Z}_+^n$;
- (b) IRD holds for $\Pi(B_j, w)$ for every $w \in \mathbb{Z}_+^n$ ($j = 1, \dots, t$);
- (c) if every element of E is in some member of \mathcal{C}_j , then IRU holds for $\Gamma(B_j, w)$ ($j = 1, \dots, t$).

IV.3 Matroid Intersection: Branchings

Another matroid intersection problem for which rounding results hold is that of branchings in a graph. Branchings have been studied extensively; see Edmonds (1968, 1970, 1972), Fulkerson (1974), Giles (1975), Tarjan (1977), Lovász (1976) and Harding (1977).

Let $G = (N, A)$ be a loopless directed graph. A cycle of G is a sequence $(n_0, a_1, n_1, \dots, n_{k-1}, a_k, n_k = n_0)$ such that:

- (i) the n_i are distinct for $i \in \{0, 1, \dots, k-1\}$;
- (ii) $a_i \in A$, either $a_i = (n_{i-1}, n_i)$ or $a_i = (n_i, n_{i-1})$ and the a_i are distinct for $i \in \{1, \dots, k\}$.

A subgraph of $G = (N, A)$ is any graph $G' = (N', A')$ such that $N' \subseteq N$, $A' \subseteq A$, and $(x, y) \in A'$ implies $x \in N'$ and $y \in N'$.

A forest in G is any subgraph of G which contains no cycles. Given

$n \in N$, let $\delta_G^-(n)$ denote the number of arcs in G directed to n

from any other node in N . A branching in G is any forest $B = (N', A')$ such that $\delta_B^-(n) \leq 1$ for every $n \in N'$.

The branchings of a graph can be viewed as the common independent sets of two matroids defined on the arc set of the graph as follows.

Given a finite set E and a partition of E , $E = E_1 \cup E_2 \cup \dots \cup E_k$,

where $E_i \cap E_j = \emptyset$ for $1 \leq i < j \leq k$, and nonnegative integers

a_1, \dots, a_k , the partition matroid on E generated by E_1, \dots, E_k and

a_1, \dots, a_k is the matroid in which $F \subseteq E$ is independent if

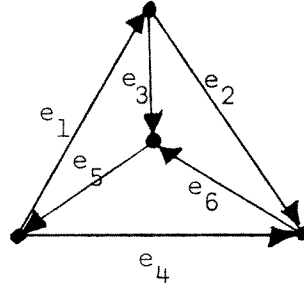
$|F \cap E_i| \leq a_i$ for $i = 1, \dots, k$. Given a graph $G = (N, A)$, the

forest matroid of G is the matroid on A such that $\{e_1, \dots, e_k\} \subseteq A$

is independent if it is the edge set of a forest in G . It is easy

to check that partition matroids and forest matroids are indeed matroids.

Example IV.2.1 (revisited). The matroid M_1 of (IV.2.1) is actually the forest matroid of the graph



The matroid M_2 of (IV.2.1) is a partition matroid on $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ given by $E_1 = \{e_1, e_6\}$, $E_2 = \{e_2, e_5\}$, $E_3 = \{e_3, e_4\}$ and $a_1 = a_2 = a_3 = 1$. \square

For $G = (N, A)$, let $M_1(G)$ be the forest matroid of G and let $M_2(G)$ be the partition matroid on A generated by $A_n = \delta_G^-(x)$ and $a_n = 1$ for $n \in N$. (Note that this is not the partition matroid induced in Example IV.2.1. This is vital, since we will get rounding result for programming problems associated with the common independent sets of $M_1(G)$ and $M_2(G)$, whereas we saw that these rounding results failed for the matroids considered in Example IV.2.1.) Let $M_1(G)$ and $M_2(G)$ have rank functions r_1 and r_2 , respectively, and let $P(G)$ be the polyhedron in $R_+^{|A|}$ whose extreme points are the incidence vectors of sets independent in both $M_1(G)$ and $M_2(G)$. Then Edmonds (1970) has shown:

Lemma IV.3.1. (a) $P(G) = \{x \in R_+^{|A|} : x(A) \leq \min(r_1(A), r_2(A)), A \subseteq A\}$
 (b) $x = (x_1, \dots, x_{|A|})$ is an extreme point of $P(G)$ if and only if $\{e_i \in A : x_i = 1\}$ is the edge set of a branching in G . \square

In order to prove rounding results for branchings, we first prove a decomposition theorem for $P(G)$. This theorem will follow from a theorem of Edmonds (1968). For $r \in N$, a branching rooted at r is a branching $B = (N, A')$ of $G = (N, A)$ such that $\delta_B^-(r) = 0$ and $\delta_B^-(n) = 1$ for $n \in N - \{r\}$. Edmonds' theorem states:

Theorem IV.3.2. The maximum number of edge-disjoint branchings of G rooted at $r \in N$ equals the minimum of $|X, \bar{X}|$ over all X such that $r \in X \subset N$. \square

We will also need the following well-known lemma (see, e.g., Harary (1969)).

Lemma IV.3.3. Let $F = (N', A')$ be a forest of $G = (N, A)$. Then F has at most $|N| - 1$ arcs. \square

We are now ready to prove a decomposition theorem for $P(G)$, from which several rounding results for programming problems related to branchings follow.

Theorem IV.3.4. The decomposition property holds for $P(G)$.

Proof[†]: Given a positive integer k and $x = (x_1, \dots, x_{|A|}) \geq 0$ such that $x \in kP(G)$, it is by IV.3.1(b) sufficient to show that x is the sum of k vectors, each of which is the incidence vector of a branching in $G = (N, A)$. Let $G_x = (N, A_x)$ have arc set consisting of x_i copies of arc e_i , $i = 1, \dots, |A|$, and for $r \notin N$ consider the graph $G' = (N \cup \{r\}, A')$, where $A' = A_x \cup A_r$ and A_r is the arc set

[†]The author is indebted to F. R. Giles for first giving a proof of Theorem IV.3.4.

consisting of $(k - \delta_{G_x}^-(n))$ copies of the arc (r, n) for every $n \in N$. Note that $k - \delta_{G_x}^-(n) \geq 0$ for every $n \in N$, since by IV.3.1 and IV.1.24 x is the sum of the incidence vectors of k independent sets in $M_2(G)$.

Let $r \in X \subset N \cup \{r\}$. By Lemma IV.3.3 the number of arcs of G' which have both their endpoints in \bar{X} is at most $k(|\bar{X}| - 1)$, since by IV.3.1 and IV.1.24 x is also the sum of k incidence vectors of independent sets of $M_1(G)$ (forests of G). But clearly $\delta_{G'}^-(n) = k$ for every $n \in \bar{X}$, and thus $|(X, \bar{X})| \geq k$. So by Theorem IV.3.2 there exist k edge-disjoint branchings of G' , each rooted at r . Say the edge sets of these branchings are A'_1, \dots, A'_k and let $A_i = A'_i \cap A_x$ for $i = 1, \dots, k$. Then the A_i are precisely the edge sets of branchings in G , the sum of whose incidence vectors gives the vector x . \square

Let A be the matrix whose rows are the maximal integral points of $P(G)$, i.e., the incidence vectors of the maximal branchings in G . Then we get:

Theorem IV.3.5. IRU holds for $\Gamma(A, w)$ for every $w \in \mathbb{Z}_+^{|A|}$.

Proof: $P(G)$ is lower comprehensive by (IV.3.1(a)) and (III.2.1). Thus, (III.3.1) and (IV.3.4) imply the rounding property for $\Gamma(A, w)$. \square

Using (IV.3.4) we can derive two further rounding results for programming problems associated with branchings. Let B be the matrix whose rows are the incidence vectors of the edge sets of the maximum cardinality branchings of G , and let P_B be the polyhedron which is the convex hull of the rows of B . We first show that P_B is decomposable.

Theorem IV.3.6. Suppose k is a positive integer and $x \in kP_B \cap Z_+^{|A|}$.
Then $x = \sum_{i=1}^k x^i$, where $x^i \in P_B \cap Z_+^{|A|}$ for $i = 1, \dots, k$.

Proof: If $x \in kP_B$, then $x \in kP(G)$, so $x \in Z_+^{|A|}$ implies $x = \sum_{i=1}^k x^i$, where $x^i \in P(G) \cap Z_+^{|A|}$ for $i = 1, \dots, k$, by (IV.3.4). But $x \in kP_B$ implies $x(A) = kt$, where t is the size of a maximum cardinality branching in G . Thus $x^i(A) = t$ for each i , and so $x^i \in P_B$ for each i , which gives the desired result. \square

We use the following lemma to obtain rounding results for $\Gamma(B, w)$ and $\Pi(B, w)$.

Lemma IV.3.7. (a) $\Pi(B, w)$ has a solution of value $r = 1 \cdot y$ if and only if there exists $x \in rP_B$ such that $x \leq w$.

(b) $\Gamma(B, w)$ has a solution of value $r = 1 \cdot y$ if and only if there exists $x \in rP_B$ such that $x \geq w$.

Proof: Exactly as in (IV.1.30). \square

(IV.3.7) and (IV.3.6) together with Lemma IV.1.29 gives us the desired rounding results for $\Pi(B, w)$ and $\Gamma(B, w)$.

Theorem IV.3.8. (a) IRD holds for $\Pi(B, w)$ for all $w \in Z_+^{|A|}$.

(b) IRU holds for $\Gamma(B, w)$ for all $w \in Z_+^{|A|}$, provided B has no zero columns, i.e., every arc of G is in some maximum cardinality branching.

Proof: Since the polyhedron P_B is neither UC nor LC, we may not directly apply Theorem III.3.1. However, similar techniques do give a proof (see Theorem IV.1.27).

Proof of (a): Given $w \in Z_+^{|A|}$ let r_w^* be the optimal value of $\Pi(B, w)$, and let $s = \lfloor r_w^* \rfloor$. If $0 \leq r_w^* < 1$, the result is clear so assume $r_w^* \geq 1$. From (IV.3.7) we see that there exists a vector $y \in r_w^* P_B$ such that $y \leq w$, and so there exists $u \in sP_B$ such that $u \leq w$. But then by Lemma IV.1.29 and Lemma IV.3.1, there exists $u' \in sP_B$, $u' \leq w$, u' integral. Thus by Theorem IV.3.6, $u' = \sum_{j=1}^s x^j$, $x^j \in P_B \cap Z_+^{|A|}$, for $j = 1, \dots, s$, which gives the desired integral solution of value $s = \lfloor r_w^* \rfloor$ to $\Pi(B, w)$.

Proof of (b): Given $w \in Z_+^{|A|}$, let r_w^* be the optimal value of $\Gamma(B, w)$ and let $s = \lceil r_w^* \rceil$. If $r_w^* = 0$, the result is clear, so assume $r_w^* > 0$. By IV.3.7 there exists $y \in r_w^* P_B$ such that $y \geq w$, and so there exists $u \in sP_B$ such that $u \geq w$. But then by Lemma IV.1.29 and Lemma IV.3.1, there exists $u' \in sP_B$, $u' \geq w$, u' integral. Thus, by Theorem IV.3.6, $u' = \sum_{j=1}^s x^j$, $x^j \in P_B \cap Z_+^{|A|}$, for $j = 1, \dots, s$, which gives the desired integral solution of value $s = \lceil r_w^* \rceil$ to $\Gamma(B, w)$. \square

IV.4 Rounding and Polyhedra with Totally Unimodular Constraint Systems

In this section we investigate rounding properties for certain programming problems arising from polyhedra defined by totally unimodular constraint systems. First we consider lower and upper comprehensive polyhedra, and then "circulations" of a totally unimodular matrix (see Trotter and Weinberger (1976)).

Let M be a $(0,1)$ totally unimodular matrix with no zero rows or columns, let w be a nonnegative integral vector and let

$P = \{x \in \mathbb{R}^n: Mx \leq w, x \geq 0\}$, $Q = \{x \in \mathbb{R}^n: Mx \geq w, x \geq 0\}$. Let A be the matrix the rows of which are the maximal integral points of P , and B be the matrix whose rows are the minimal integral points of Q . Then we have:

Theorem IV.4.1. (a) The decomposition property holds for P .

(b) The decomposition property holds for Q .

Proof of (a): Suppose $x \in kP \cap \mathbb{Z}_+^n$ for some positive integer k .

Inductively, it is enough to show that there exists an integral vector x^k such that $x^k \in P$ and $x - x^k \in (k-1)P$. Such an x^k will exist if and only if there is an integral solution y to the system of constraints

$$Mx - (k-1)w \leq My \leq w; y \geq 0; x - y \geq 0. \quad (\text{IV.4.2})$$

Note that the point $y = \frac{x}{k}$ (so that $x - y = \frac{k-1}{k}x$) obviously satisfies (IV.4.2), although it may not be integral. But (IV.4.2) generates a bounded, nonempty polyhedron and M is totally unimodular. Thus by Theorem II.1.1 we see that (IV.4.2) has an integral (extreme) solution.

Proof of (b): For k a positive integer, let $x \in kQ \cap \mathbb{Z}_+^n$. It is then enough to show that there is an integral solution y to the system

$$w \leq My \leq Mx - (k-1)w; y \geq 0; x - y \geq 0. \quad (\text{IV.4.3})$$

Again $y = \frac{x}{k}$ satisfies (IV.4.3), so by the total unimodularity of M , we get the desired integral solution to (IV.4.3). \square

Theorem IV.4.4. (a) IRU holds for $\Gamma(A, w)$ for every $w \in Z_+^n$, provided that A has no zero columns.

(b) IRD holds for $\Pi(B, w)$ for every $w \in Z_+^n$, provided that B has no zero rows.

Proof: By (III.2.1), P is LC and Q is UC. Thus (IV.4.4) follows directly from (IV.4.1) and (III.3.1). \square

Trotter and Weinberger (1976) have given rounding results for another class of programming problems associated with totally unimodular matrices. Let N be any totally unimodular matrix. The elements of the vector subspace of R^n given by $\{x \in R^n: Nx = 0\}$ are called the circulations of the matrix N . Let $a \in Z_+^n$, $b \in Z_+^n$, $a \leq b$, and consider the polyhedron $R = \{x \in R^n: Nx = 0, a \leq x \leq b\}$. Assume $R \neq \emptyset$, and let C be the matrix whose rows are the (finitely many) integral points of R . The following two theorems are from Trotter and Weinberger (1976); for completeness, proofs are also given here.

Theorem IV.4.5. The decomposition property holds for R .

Proof: Suppose $x \in kR \cap Z_+^n$ for a positive integer k . As in (IV.4.1), it is enough to show that there is always an integral solution to

$$Ny = 0; \quad a \leq y \leq b; \quad (k-1)a \leq x-y \leq (k-1)b. \quad (\text{IV.4.6})$$

Since N is totally unimodular, the constraint system (IV.4.6) will plainly also be totally unimodular, and so it is enough to show that (IV.4.6) has any solution. But $y = \frac{x}{k}$ obviously satisfies (IV.4.6). \square

Theorem IV.4.6. (a) IRU holds for $\Gamma(C, w)$ for all $w \in Z_+^n$, provided that C has no zero columns.

(b) IRD holds for $\Pi(C, w)$ for all $w \in Z_+^n$, provided that C has no zero rows.

Proof of (a): Given $w \in Z_+^n$, let r_w^* be the value of an optimal solution to $\Gamma(C, w)$. If $r_w^* = 0$, the result is clear, so assume $r_w^* > 0$. Let $s = \lceil r_w^* \rceil$ and suppose y_w^* is an optimal solution to $\Gamma(C, w)$. Then $y_w^* C \in r_w^* R$. Thus $N(y_w^* C) = 0$ and $r_w^* a \leq y_w^* C \leq r_w^* b$. Now let $x' = \frac{s}{r_w^*} y_w^* C$. Then x' satisfies

$$Nx' = 0; \quad sa \leq x' \leq sb; \quad x' \geq w. \quad (\text{IV.4.7})$$

But since N is totally unimodular, the constraint system (IV.4.7) is also totally unimodular, and thus there is an integral vector x which satisfies (IV.4.7); i.e., $x \in sR$, $x \geq w$.

Now by (IV.4.5), $x = \sum_{i=1}^s x^i$, where each x^i is an integral vector in R . By definition of C , each x^i is a row of C , and thus we get the desired integral solution to $\Gamma(C, w)$ of value s .

Proof of (b): Given $w \in Z_+^n$, let r_w^* be the value of an optimal solution to $\Pi(C, w)$. If $0 \leq r_w^* < 1$, the result is clear, so assume $r_w^* \geq 1$, and let $s = \lfloor r_w^* \rfloor$. Suppose y_w^* is an optimal solution to $\Pi(C, w)$. Then $y_w^* C \in r_w^* R$, and thus $N(y_w^* C) = 0$ and $r_w^* a \leq y_w^* C \leq r_w^* b$. Now let $x' = \frac{s}{r_w^*} y_w^* C$. Then x' satisfies

$$Nx' = 0; \quad sa \leq x' \leq sb; \quad x' \leq w. \quad (\text{IV.4.8})$$

Then, since N is totally unimodular, there must be an integral vector x which satisfies (IV.4.8); i.e., $x \in sR$, $x \leq w$. Thus (IV.4.5) implies that $x = \sum_{i=1}^s x^i$, where each $x^i \in R \cap Z_+^n$. By definition of C , each x^i is a row of C , and thus we get the desired integral solution to $\Pi(C, w)$ of value s . \square

In Trotter and Weinberger (1976), it is shown that the rounding results of Fulkerson and Weinberger (1975) on packing integral, feasible flows of an integral, capacitated, supply-demand network into an integral, nonnegative vector (see (III.1)), and those of Weinberger (1976) on covering an integral, nonnegative vector with integral, feasible flows of an integral, capacitated, supply-demand network can be easily derived from Theorem IV.4.6. Other applications of (IV.4.6), including a proof that the max-flow, min-cut theorem (see II.2.5) of Ford and Fulkerson (1962) is a special case of a more general result on rounding, are also given in Trotter and Weinberger (1976).

IV.5 Job Scheduling

In this section, we consider a rounding result for the following scheduling problem. We are given a set $\{j_1, \dots, j_n\}$ of jobs which must be processed by some fixed time t , and we wish to find the minimum number of machines necessary to do so. Each job is identical, and so is each machine, in that any job requires exactly one unit of processing time, and may be processed by any machine. At any given time a single machine is allowed to be processing only one job.

We are also given a precedence relation among the jobs, and we denote the fact that job j_i precedes job j_k by $j_i < j_k$

(i.e., j_i must be fully processed before j_k is allowed to begin processing). The precedence relation is irreflexive ($j_i \not\prec j_i$) and transitive ($j_i \prec j_k \prec j_\ell$ implies $j_i \prec j_\ell$), and thus the relation may be represented by a directed graph (see Example IV.5.6) by directing an arc from j_i to j_k if $j_i \prec j_k$ but there is no ℓ such that $j_i \prec j_\ell \prec j_k$, i.e., j_k is an immediate successor of j_i . The resulting directed graph is called a (rooted) assembly tree if

(i) for each $i \in \{1, \dots, n-1\}$, there exists a unique $k \in \{1, \dots, n\}$ such that (j_i, j_k) is an arc of the directed graph (i.e., for $i \in \{1, \dots, n-1\}$, j_i has a unique immediate successor); (ii) there is no job j_k such that $j_n \prec j_k$. Job j_n is then called the root of the assembly tree.

The assembly tree scheduling problem (AS) may then be formulated: find the minimum number of processors required to complete all the jobs j_1, \dots, j_n of the assembly tree T by a given time t , subject to the constraints that (i) job j_i cannot be processed until all jobs j_k such that $j_k \prec j_i$ have been processed, and (ii) once a particular job begins processing on a particular machine, that job must be processed continuously by that machine until completion of the job.

A related scheduling problem on assembly trees arises when we no longer assume that all machines are identical, but rather assign a processing speed λ_i , $0 < \lambda_i \leq 1$, to each machine m_i . A machine with processing speed λ_i requires $\frac{1}{\lambda_i}$ time units to process any job (all jobs are still assumed to be identical), and we allow any number of machines to process a job concurrently, as long as the sum of their processing speeds is ≤ 1 . The assembly tree scheduling problem

with machine splitting (ASM) may then be formulated: given an assembly tree T and a fixed time t , find machines m_1, \dots, m_p with processing speeds $\lambda_1, \dots, \lambda_p$ such that all jobs may be processed by m_1, \dots, m_p by time t , and $\sum_{i=1}^p \lambda_i$ is minimized, subject to the same two constraints as in AS.

The problem AS has been studied by Hu (1961) and by Coffman (1976). Muntz and Coffman (1970) studied a problem very similar to ASM, and gave an algorithm for solving this problem from which an algorithm which solves ASM may be trivially derived. From the work of Hu (1961) and Muntz and Coffman (1970), it is clear that a rounding relation holds between the values of optimal solutions to AS and ASM; we now explore this rounding relation.

Given an assembly tree T with jobs (nodes) $\{j_1, \dots, j_n\}$, a path in T is any sequence of the form

$j_{i_1}, (j_{i_1}, j_{i_2}), j_{i_2}, (j_{i_2}, j_{i_3}), \dots, j_{i_{k-1}}, (j_{i_{k-1}}, j_{i_k}), j_{i_k}$, where each $(j_{i_l}, j_{i_{l+1}})$ is an arc of T . The length of a path is the number of nodes of T in it. It is easy to see that for any job j_i in the assembly tree T , there will be a unique path of the form

j_i, \dots, j_n in T , where j_n is the root of T . The length of this path is called the label or level of job j_i (j_n has level one).

If the largest level in T is $\bar{\alpha}$, we say that T has height $\bar{\alpha}$.

Given $i \in \{1, \dots, \bar{\alpha}\}$, let $p(i)$ be the number of jobs in T of level i .

Hu (1961) (see also Coffman (1976)) has shown that the minimum number of machines m necessary to process all jobs by time $t = \bar{\alpha} + c$ (c a nonnegative integer) for the problem AS is given by

$$m = \max_{1 \leq \gamma \leq \bar{\alpha}} \left\lceil \frac{1}{\gamma+c} \sum_{j=1}^{\gamma} p(\bar{\alpha}+1-j) \right\rceil. \quad (\text{IV.5.1})$$

Muntz and Coffman (1970) have shown that the minimum sum of all machine processing speeds necessary to process all jobs by time $t = \bar{\alpha} + c$ ($c \in \mathbb{Z}_+$) for the problem ASM is given by

$$\max_{1 \leq \gamma \leq \bar{\alpha}} \frac{1}{\gamma+c} \sum_{j=1}^{\gamma} p(\bar{\alpha}+1-j). \quad (\text{IV.5.2})$$

It is not difficult to see that (IV.5.1) and (IV.5.2) provide lower bounds for m for AS and ASM, and Hu and Muntz and Coffman provide simple algorithms which show that these bounds can be achieved.

It is natural to ask whether the rounding relation between AS and ASM expressed by (IV.5.1) and (IV.5.2) can be given a polyhedral explanation, as has been the case with the rounding results thus far considered. Let T be an assembly tree of height $\bar{\alpha}$ with n jobs. Say the jobs with label $\bar{\alpha}$ are j_{11}, \dots, j_{1,k_1} ; those with label $\bar{\alpha}-1$ are $j_{21}, \dots, j_{2,k_2}, \dots$ those with label 2 are $j_{\bar{\alpha}-1,1}, \dots, j_{\bar{\alpha}-1,k_{\bar{\alpha}-1}}$, and the root is $j_{\bar{\alpha}1}$. Then (IV.5.1) leads us to consider the system

[illegible]

where $I_{n \times n}$ is the $n \times n$ identity matrix.

Note that the constraint system (IV.5.3) has the "consecutive ones" property of Theorem II.1.3, and thus it is totally unimodular. So if we let B be the matrix whose rows are the maximal extreme solutions to (IV.5.3), B will have all integral (and thus $(0,1)$) entries.

Consider the programming problems:

$$\begin{aligned} \min \quad & 1 \cdot y \\ \text{s.t.} \quad & yB \geq 1 \\ & y \geq 0, \end{aligned} \tag{IV.5.4}$$

and

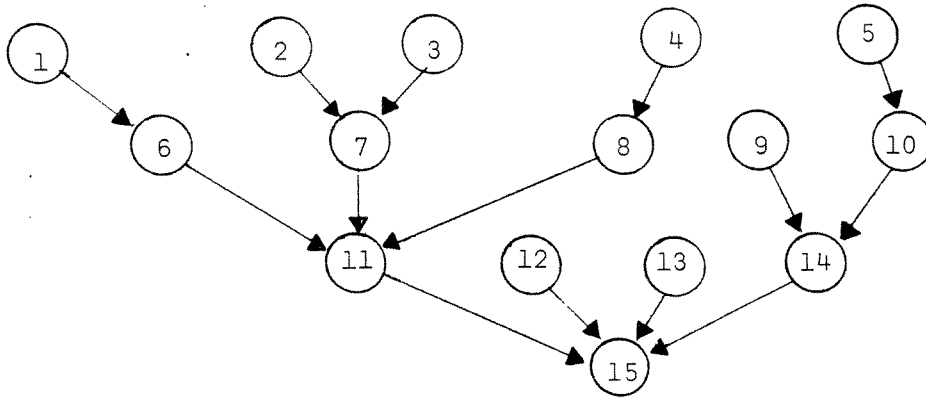
$$\begin{array}{ll} \min & 1 \cdot y \\ \text{s.t.} & y_B \geq 1 \\ & y \geq 0 \\ & y \text{ integral.} \end{array} \quad (\text{IV.5.5})$$

Let r^* be the optimal value of (IV.5.4), and s^* the optimal value of (IV.5.5). It follows easily from the antiblocking theory of Fulkerson (1972) (see Chapter V) that r^* is given by (IV.5.2), provided that $r^* \geq 1$. Since any integral solution to (IV.5.3) will clearly be an extreme solution, it follows from Theorem IV.4.4 that $s^* = \lceil r^* \rceil$; i.e., s^* is given by (IV.5.1).

We may associate with every row of B a "machine history" for AS by letting it represent the jobs done by one machine; i.e., associate with row B^i of B the set of jobs which correspond to the positive entries of B^i . It would then be desirable, in order to have a complete polyhedral explanation for the rounding relation between (IV.5.1) and (IV.5.2), to have a one-to-one correspondence between optimal solutions to (IV.5.5) and optimal solutions to AS. That is, given $y^* = (y_1^*, \dots, y_m^*)$ an optimal solution to (IV.5.5), it would be desirable to have a solution to AS in which we have a machine m_i corresponding to every y_i^* such that $y_i^* = 1$ (note that y^* will be a $(0,1)$ -vector), with machine m_i processing precisely those jobs which correspond to positive entries in the row B^i of B .

However, for the matrix B of (IV.5.5), although every optimal solution to AS will generate an optimal solution to (IV.5.5) (by setting $y_i^* = 1$ if there exists a machine in the solution to AS which processes exactly the jobs corresponding to the positive entries of B^i , and setting $y_i^* = 0$ otherwise), the converse is not true. The following example illustrates this.

Example IV.5.6. Let T be an assembly tree with 15 precedence related jobs as diagrammed below.



Then we have $\bar{\alpha} = 4$, and if we take $c = 2$, so that $t = \bar{\alpha} + c = 6$, then (IV.5.1) implies that $m = 3$ for AS. Let $S_1 = \{1, 2, 3, 13, 14, 15\}$, $S_2 = \{6, 7, 8, 10, 11\}$ and $S_3 = \{4, 5, 9, 12\}$. Then S_1 , S_2 and S_3 are all "machine histories" but taken together, they do not generate a solution to AS which finishes by time $\bar{\alpha} + c = 6$. However, if we let B^i , $i = 1, 2, 3$ be the vector $B_j^i = \begin{cases} 1 & \text{if } j \in S_i \\ 0 & \text{otherwise} \end{cases}$, then B^i , $i = 1, 2, 3$, are rows of B , and so generate an optimal solution to (IV.5.5). \square

IV.6 Perfect Graphs

In (III.3) we saw that for lower and upper comprehensive polyhedra, the decomposition property was equivalent to integer rounding holding in certain associated programming problems. Even for polyhedra which are not comprehensive, the decomposition property often implies rounding results in associated programming problems, as was the case in Theorems IV.1.27, IV.2.2(b) and (c), IV.3.8 and IV.4.6. However, for polyhedra

which are not comprehensive, decomposition does not always lead to a rounding result. In this section we give a polyhedron associated with perfect graphs for which the decomposition property holds, but for which IRD does not hold for $\Pi(B,w)$, where B is the matrix whose rows are the integral points of the polyhedron. For a more complete discussion of perfect graphs, see Section V.3.

Let $G = (N,A)$ be an undirected, loopless graph without multiple edges. A clique in G is a set $X \subseteq N$ such that $x_1 \in X$, $x_2 \in X$ implies $\{x_1, x_2\} \in A$. An anticlique in G is a set $X \subseteq N$ such that $x_1 \in X$, $x_2 \in X$ implies $\{x_1, x_2\} \notin A$. Given $X \subseteq N$, the node-induced subgraph of G induced by the node set X is the graph $G' = (X, A')$ where $A' = \{\{x, y\} \in A : x \in X, y \in X\}$. Let A_G be the matrix whose rows are the incidence vectors of the maximal cliques of G , and let B_G be the matrix whose rows are the incidence vectors of the maximal anticliques of G . G is called perfect if, for every node-induced subgraph $G' = (X, A')$ of G , the minimum number of cliques of G' whose union is X is equal to the maximum cardinality of an anticlique of G' . That is, G is perfect if $s_1^*(G')$, the value of an optimal integral solution to $\Gamma(A_G, 1)$, is given by

$$s_1^*(G') = \max\{1 \cdot B_{G'}^j : B_{G'}^j \text{ is a row of } B_{G'}\}, \text{ for every node-induced subgraph } G' \text{ of } G.$$

Perfect graphs have been studied extensively, see Berge (1970), Chvátal (1975), Fulkerson (1971a, 1972, 1973), Lovász (1972a, 1972b), Padberg (1974), Trotter (1977) and Bland, Huang and Trotter (1976).

As will be seen in (V.3), if G is perfect, it is actually the case that s_w^* , the optimal value for $\Gamma(A_G, w)$, is given by

$s_w^* = \max\{w \cdot B_G^j : B_G^j \text{ is a row of } B_G\}$, for every nonnegative integral vector w . Let $P_B(G)$ be the convex hull of the rows of A_G , and $P(G) = \{x \in \mathbb{R}^{|N|} : 0 \leq x \leq y, y \in P_B(G)\}$. Fulkerson (1972, 1973) has shown that $P(G)$ will have all integral extreme points whenever G is perfect, and so we have by (III.3.1),

Theorem IV.6.1. If G is perfect, then the decomposition property holds for $P(G)$. \square

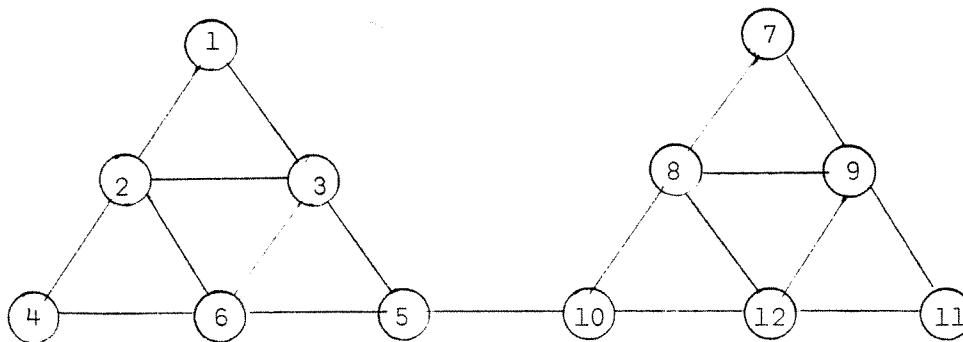
From (IV.6.1) we may deduce another decomposition result. Let G be a perfect graph, and say the largest clique in G has cardinality t . Let D be the matrix whose rows are the incidence vectors of the cliques of G of cardinality t , and let Q be the polyhedron in $\mathbb{R}_+^{|N|}$ which is the convex hull of the rows of D . Then we have:

Theorem IV.6.2. The decomposition property holds for Q .

Proof: Suppose k is a positive integer, and x is an integral point such that $x \in kQ$. Then clearly $x \in kP(G)$, so by (IV.6.1), $x = \sum_{i=1}^k x^i$ for some vectors $x^i \in P(G) \cap \mathbb{Z}_+^{|N|}$, $i = 1, \dots, k$. But $x \in kQ$ implies $x(N) = kt$, and so $x^i(N) = t$ for $i = 1, \dots, k$. Thus $x^i \in Q$ for $i = 1, \dots, k$. \square

However, IRD does not hold for the problem $\Pi(D, w)$, as the following example demonstrates.

Example IV.6.3. Let G be the graph;



Then G is perfect, and

$$D = \begin{array}{c} \begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{l} D^1 \\ D^2 \\ D^3 \\ D^4 \\ D^5 \\ D^6 \\ D^7 \\ D^8 \end{array} \left[\begin{array}{cccccc|cccccc} 1 & 1 & 1 & 0 & 0 & 0 & & & & & & \\ 0 & 1 & 0 & 1 & 0 & 1 & & & & & & \\ 0 & 0 & 1 & 0 & 1 & 1 & & & & & & \\ 0 & 1 & 1 & 0 & 0 & 1 & & & & & & \\ \hline & & & & & & 1 & 1 & 1 & 0 & 0 & 0 \\ & & & & & & 0 & 1 & 0 & 1 & 0 & 1 \\ & & & & & & 0 & 0 & 1 & 0 & 1 & 1 \\ & & & & & & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right].$$

Let w be the vector of ones in \mathbb{R}^{12} . Then $\frac{1}{2}(D^1 + D^2 + D^3 + D^5 + D^6 + D^7)$ provides a rational solution of value 3 to $\Pi(D, w)$, but the best integral solution has value 2. \square

Theorem IV.6.2 and Example IV.6.3 are not in contradiction to Theorem III.3.1, since the polyhedron Q is not upper comprehensive.

CHAPTER V

BLOCKING, ANTIBLOCKING AND ROUNDING

In Chapter IV we generally considered only the relation between the optimal rational value and the optimal integral value of a programming problem. Little attention was paid to determining the actual optimal value of the problem. However, much research has been devoted to determining the values of programming problems such as those considered in (IV.1), (IV.2), (IV.3) and (IV.4). In this chapter some results in this direction are considered, and the relation of rounding to them is shown. A good general framework for studying the problems $\Gamma(A,w)$ and $\Pi(A,w)$ (for nonnegative A and w) is the blocking and antiblocking theory of Fulkerson (1970, 1971a, 1972). We begin by reviewing certain basic facets of this theory.

V.1 Blocking and Antiblocking

Let A and B be $m \times n$ nonnegative real matrices, A with no zero columns and B with no zero rows. Let C be the polyhedron $C = \{c \in R_+^n: Ac \leq 1\}$ and D be the polyhedron $D = \{d \in R_+^n: Bd \geq 1\}$. Then the polyhedron $\bar{C} = \{a \in R_+^m: a \cdot c \leq 1 \text{ for every } c \in C\}$ is called the antiblocker of C , and the polyhedron $\hat{D} = \{b \in R_+^m: b \cdot d \geq 1 \text{ for every } d \in D\}$ is called the blocker of D . The following two theorems of Fulkerson (1970, 1972) indicate that antiblockers and blockers each occur in "dual" pairs.

Theorem V.1.1. Let A have rows a^1, \dots, a^m and let $C = \{c \in R_+^n: Ac \leq 1\}$ have extreme points c^1, \dots, c^r . Let \bar{C} be the

matrix having rows c^1, \dots, c^r , and let $A = \{a \in R_+^n: Ca \leq 1\}$. Then, $\overline{C} = A$ and $\overline{A} = C$. Thus, $\overline{\overline{C}} = C$. \square

Theorem V.1.2. Let B have rows b^1, \dots, b^m and let $\mathcal{D} = \{d \in R_+^n: Bd \geq 1\}$ have extreme points d^1, \dots, d^r . Let D be the matrix having rows d^1, \dots, d^r , and let $\mathcal{B} = \{b \in R_+^n: Db \geq 1\}$. Then $\hat{\mathcal{D}} = \mathcal{B}$ and $\hat{\mathcal{B}} = \mathcal{D}$. Thus, $\hat{\hat{\mathcal{B}}} = \mathcal{B}$. \square

We call any nonnegative matrix C such that C and A as in (V.1.1) form an antiblocking pair of polyhedra an antiblocking matrix of A . Any nonnegative matrix D such that \mathcal{D} and \mathcal{B} as in (V.1.2) form a blocking pair of polyhedra is called a blocking matrix of B . (Fulkerson (1970) restricts B and D to having only rows necessary in defining the polyhedra \mathcal{D} and \mathcal{B} (proper rows), and thus obtains unique blocking pairs of matrices.)

The next two theorems of Fulkerson (1970, 1972) demonstrate the relation of antiblocking theory and blocking theory to the optimal values of $\Gamma(A, w)$ and $\Pi(B, w)$, respectively, where $w \in R_+^n$. Given nonnegative matrices A and C , we say that the min-max equality holds for the ordered pair A, C if for every $w \in R_+^n$ we have that r_w^* , the optimal value of $\Gamma(A, w)$, is given by $r_w^* = \max\{w \cdot c^j: c^j \text{ is a row of } C\}$. Given nonnegative matrices B and D , we say that the max-min equality holds for the ordered pair B, D if for every $w \in R_+^n$ we have that r_w^* , the optimal value of $\Pi(B, w)$, is given by $r_w^* = \min\{w \cdot d^j: d^j \text{ is a row of } D\}$.

Theorem V.1.3. Let A and C be nonnegative matrices without zero columns. Then the min-max equality holds for the pair A, C if and only

if A and C are an antiblocking pair of matrices. Thus if the min-max equality holds for A, C , it also holds for C, A . \square

Theorem V.1.4. Let D and B be nonnegative matrices without zero rows. Then the max-min equality holds for the pair B, D if and only if B and D are a blocking pair of matrices. Thus if the max-min equality holds for B, D , it also holds for D, B . \square

In the following section, we investigate combinatorial min-max and max-min theorems that arise from (V.1.3) and (V.1.4) for those problems which we have studied in Chapter 4.

V.2 Relation to Rounding

We may combine our earlier results on rounding with the preceding material to immediately conclude the following two theorems from (V.1.3) and (V.1.4).

Theorem V.2.1. Let A and C be an antiblocking pair of matrices and suppose IRU holds for $\Gamma(A, w)$ for every $w \in Z_+^n$. Then the value of an optimal integral solution to $\Gamma(A, w)$ is given by

$$s_w^* = \max\{[w \cdot c^j]: c^j \text{ is a row of } C\}, \text{ for every } w \in Z_+^n. \quad \square$$

Theorem V.2.2. Let B and D be a blocking pair of matrices and suppose IRD holds for $\Pi(B, w)$ for every $w \in Z_+^n$. Then the value of an optimal integral solution to $\Pi(B, w)$ is given by $s_w^* = \min\{[w \cdot d^j]: d^j \text{ is a row of } D\}$, for every $w \in Z_+^n$. \square

Thus, if we know that IRU holds for $\Gamma(A, w)$ (IRD holds for $\Pi(B, w)$), and we know an antiblocking matrix of A (a blocking matrix of B), then

we can, via (V.2.1)((V.2.2)) generate a combinatorial (i.e., integral) min-max (max-min) theorem.

We now give the constraint systems of the blocking and antiblocking polyhedra, and the resulting combinatorial theorems, for certain of the problems studied in (IV.1)-(IV.3). Such results are also known for circulations of a totally unimodular matrix (IV.4)--see Trotter and Weinberger (1976).

Polymatroids: Let $P = P(E, f)$ be a loopless, integral polymatroid. Let A be the matrix whose rows are the integral bases of P , and let $|E| = n$, $C = \{c \in R_+^n : Ac \leq 1\}$, $D = \{d \in R_+^n : Ad \geq 1\}$. Edmonds (1965b, 1970, 1971) has shown that the antiblocker of C is given by the constraints

$$x(F) \leq f(F), \quad F \subseteq E \quad (V.2.3)$$

$$x_j \geq 0, \quad j \in E.$$

Edmonds and Fulkerson (1965) have shown that the blocker of D is given by the constraints

$$x(E-F) \geq f(E) - f(F), \quad F \subseteq E. \quad (V.2.4)$$

From (V.2.3) and (V.2.1), we conclude the following theorem, first proved algorithmically by Edmonds (1965b, 1970).

Theorem V.2.5. Let A be the matrix whose rows are the bases of an integral, loopless polymatroid $P(E, f)$. Then the value of an optimal

integral solution to $\Gamma(A, w)$ is given by $s_w^* = \max\{\lceil \frac{1}{f(X)} (w \cdot x) \rceil : x \text{ is the incidence vector of } X \subseteq E, X \neq \emptyset\}$, for every $w \in Z_+^n$. \square

From (V.2.4) and (V.2.2) we conclude (V.2.6), which was first proved by Edmonds and Fulkerson (1965) and Edmonds (1970).

Theorem V.2.6. With A as in (V.2.5), let s_w^* be the value of an optimal integral solution to $\Pi(A, w)$, with $w \in Z_+^n$. Then s_w^* is given by $s_w^* = \min\{\lfloor \frac{1}{f(E)-f(X)} (w \cdot \bar{x}) \rfloor : \bar{x} \text{ is the incidence vector of } E-X, X \subseteq E \text{ such that } f(E) - f(X) \geq 1\}$. \square

Intersection of Two Matroids: Let M_1 and M_2 be two loopless matroids defined on the set E , with rank functions r_1 and r_2 , respectively. For $X \subseteq E$, let $r_0(X) = \min\{r_1(X') + r_2(X-X') : X' \subseteq X\}$. Let A be the matrix whose rows are the incidence vectors of the maximal subsets of E which are independent in both M_1 and M_2 . Edmonds (1970) has proved that the antiblocker of $\{c \in R_+^n : Ac \leq 1\}$, for $|E| = n$, is given by

$$x(X) \leq r_0(X), \quad X \subseteq E \quad (V.2.7)$$

$$x_j \geq 0, \quad j \in E.$$

Thus, for strongly base-orderable matroids and branchings (see (IV.2) and (IV.3)), we get the result:

Theorem V.2.8. Let matrix A have as rows the incidence vectors of the maximal common independent sets of two loopless matroids. Then the value of an optimal integral solution to $\Gamma(A, w)$, is given by

$s_w^* = \max\{\lceil \frac{1}{r_0(X)} (w \cdot x) \rceil : x \text{ is the incidence vector of } X \subseteq E, X \neq \emptyset\}$, for every $w \in Z_+^n$. \square

The blocker of $\{d \in R^n: Ad \geq 1\}$ is not known.

For $1 \leq k \leq r_0(E)$, let B_k be the matrix whose rows are the incidence vectors of the subsets of E of cardinality k which are independent in both M_1 and M_2 . McDiarmid (1976a) has determined that the antiblocker of $\{c \in R_+^n: B_k c \leq 1\}$ is given by

$$x(E) \leq k$$

$$x(X) \leq r_0(X), \quad X \subseteq E \quad (V.2.9)$$

$$x(X_1 \cap X_2) \leq r_1(X_1) + r_2(X_2) - k, \quad X_1 \subseteq E, X_2 \subseteq E \text{ such that } X_1 \cup X_2 = E$$

$$x_j \geq 0, \quad j \in E.$$

McDiarmid (1976a), Cunningham (1975) and Edmonds and Giles (1976) have all independently determined that the blocker of $B_k = \{d \in R_+^n: B_k d \geq 1\}$ is given by

$$x(E-X) \geq k - r_0(X), \quad X \subseteq E. \quad (V.2.10)$$

Huang (1976) has given an additional interesting proof of this result.

In the case of strongly base-orderable matroids (IV.2) and branchings (IV.3), we deduce (V.2.11) from (V.2.9) and (V.2.1), and (V.2.12) from (V.2.10) and (V.2.2).

Theorem V.2.11. For $1 \leq k \leq r_0(E)$, let B_k be the matrix whose rows are the incidence vectors of the subsets of E of cardinality k which are independent in both M_1 and M_2 , and assume that every element of E occurs in some cardinality k independent set (i.e., B_k has no

zero columns). Then the value of an optimal integral solution to $\Gamma(B_k, w)$ ($w \in \mathbb{Z}_+^n$) is given by $s_w^* = \max(s_1, s_2, s_3)$ where $s_1 = \lceil \frac{1 \cdot w}{k} \rceil$, $s_2 = \max\{\lceil \frac{1}{r_1(X_1) + r_2(X_2) - k} (w \cdot x') \rceil : x' \text{ is the incidence vector of } X_1 \cap X_2; X_1 \subseteq E, X_2 \subseteq E \text{ such that } X_1 \cup X_2 = E \text{ and } r_1(X_1) + r_2(X_2) - k \geq 1\}$ and $s_3 = \max\{\lceil \frac{1}{r_0(X)} (w \cdot x) \rceil : x \text{ is the incidence vector of } X \subseteq E, X \neq \emptyset\}$. \square

Theorem V.2.12. Where B_k is as in (V.2.11), the value of an optimal integral solution to $\Pi(B_k, w)$ is given by $s_w^* = \min\{\lceil \frac{1}{k - r_0(X)} (w \cdot \bar{x}) \rceil : \bar{x} \text{ is the incidence vector of } E - X, X \subseteq E \text{ such that } k - r_0(X) \geq 1\}$. \square

In general, many of the constraints (V.2.3), (V.2.4), (V.2.7), (V.2.9) and (V.2.10) will be redundant. A minimal set of necessary constraints for (V.2.3) is determined in Edmonds (1970), and for (V.2.7) in Giles (1975).

V.3 The Pluperfect Graph Theorem

In this section, we examine the relation of antiblocking theory to perfect graphs (see IV.6). If A and B are nonnegative matrices, we say that the min-max equality holds strongly for the ordered pair A, B if, for all $w \in \mathbb{Z}_+^n$, $\Gamma(A, w)$ has an integral optimal solution of value $s_w^* = \max\{w \cdot b^j : b^j \text{ is a row of } B\}$. Given any nonnegative $m \times n$ matrix A with no zero columns, we say that a row a^i of A is inessential if there are nonnegative real numbers λ_j , $j = 1, \dots, m$, such that $\lambda_i = 0$, $\sum_{j=1}^m \lambda_j = 1$, and $a^i \leq \sum_{j=1}^m \lambda_j a^j$. A row of A is essential if it is not inessential.

Theorem V.3.1. (Pluperfect Graph Theorem, Fulkerson (1972)). Let A be a $(0,1)$ matrix and let B be an antiblocking matrix of A . Then the min-max equality holds strongly for A, B if and only if every essential row of B is a $(0,1)$ -vector. Hence, if the min-max equality holds strongly for A, B , it also holds strongly for B^*, A , where B^* consists of the essential rows of B . \square

We now give a new and elementary proof for (V.3.1) which is based on the following lemma, whose proof is similar to that of Theorem III.5.1.

Lemma V.3.2. Let A be a $(0,1)$ -matrix with no zero columns. If the optimal value r_w^* of $\Gamma(A, w)$ is an integer for every $w \in Z_+^n$, then $\Gamma(A, w)$ has an integral optimal solution for every $w \in Z_+^n$.

Proof: We proceed on induction on $1 \cdot w$. For $1 \cdot w = 0$, the result is clear, so assume the lemma is true for $u \in Z_+^n$ such that $1 \cdot u \leq k-1$, and let $w \in Z_+^n$, $1 \cdot w = k$. Let $y_w^* = ((y_w^*)_1, \dots, (y_w^*)_m)$ be any optimal solution for $\Gamma(A, w)$ and without loss of generality assume $(y_w^*)_1 > 0$. Let a^1 be the first row of A , and consider $w' \in Z_+^n$ given by $w'_i = \max(w_i - a_i^1, 0)$, $1 \leq i \leq n$. By the optimality of y_w^* , we have that $(y_w^*)_1 > 0$ implies $1 \cdot w' < 1 \cdot w$. So, by the induction hypothesis, $\Gamma(A, w')$ has an integral optimal solution, say $z_{w'}^*$. But note that $y' = (\max((y_w^*)_1 - 1, 0), (y_w^*)_2, \dots, (y_w^*)_m)$ is feasible for $\Gamma(A, w')$, and thus $r_{w'}^* = 1 \cdot z_{w'}^* \leq 1 \cdot y' < 1 \cdot y_w^* = r_w^*$. So $r_{w'}^* \leq r_w^* - 1$. So $1 \cdot z_{w'}^* \leq r_w^* - 1$. But then $z = z_{w'}^* + (1, 0, \dots, 0)$ is an integral feasible solution for $\Gamma(A, w)$, and $1 \cdot z = 1 \cdot z_{w'}^* + 1 \leq r_w^*$. \square

Proof of Theorem V.3.1: Sufficiency: Since A, B are an antiblocking pair of matrices, we know from (V.1.3) that the optimal value of $\Gamma(A, w)$ is given by $r_w^* = \max\{w \cdot b^j : b^j \text{ is a row of } B\}$ for every $w \in R_+^n$. But if $w \in Z_+^n$, then clearly $r_w^* \in Z_+$, since all essential rows of B are $(0,1)$ -vectors. Thus (V.3.2) implies that the min-max equality holds strongly in the direction A, B for every $w \in Z_+^n$.

Necessity: Suppose B has an essential row b^k which is not a $(0,1)$ -vector. Note that Theorem V.1.1 implies that all entries of B must be less than or equal to one, so we may assume that $0 < b_1^k < 1$. Since b^k is essential, there exists a vector $w \in Z_+^n$ such that $w \cdot b^i$ is uniquely maximized over rows b^i of B by b^k , and $w_1 \geq 1$ (see Fulkerson (1972)). Let $w' = (w_1 - 1, w_2, \dots, w_n) \in Z_+^n$. Since the min-max equality holds strongly for A, B , we know that $w \cdot b^k \in Z$. But $w' \cdot b^k = w \cdot b^k - b_1^k$, and so $w \cdot b^{k-1} < w' \cdot b^k < w \cdot b^k$. But by assumption, $\max\{w' \cdot b^i : b^i \text{ is a row of } B\}$ is an integer. So there exists a row b^j of B such that $w' \cdot b^j > w' \cdot b^k > w \cdot b^{k-1}$, and $w' \cdot b^j \in Z$. Then clearly $w' \cdot b^j \geq w \cdot b^k$, and so $w \cdot b^j \geq w \cdot b^k$, a contradiction. \square

The Pluperfect Graph Theorem is related to graphs in the following way. Let G be a loopless, undirected graph without multiple edges. Let A be the matrix whose rows are the incidence vectors of the maximal cliques of G , and let B be the matrix whose rows are the incidence vectors of the maximal anticliques of G . Recall that G is termed perfect if $\Gamma(A, w)$ has an optimal integral solution of value $\max\{w \cdot b^i : b^i \text{ is a row of } B\}$ for every $(0,1)$ -vector w . Call G pluperfect if the strong min-max equality holds for the pair A, B . Then as an immediate consequence of Theorem V.3.1, we have:

Corollary V.3.3. Suppose G is a loopless, undirected graph without multiple edges and A and B are the incidence matrices of maximal anticliques and cliques, respectively, of G . Then the following are equivalent:

- (1) G is pluperfect, i.e., the strong min-max equality holds for the pair A, B ;
- (2) the matrices A and B are an antiblocking pair. \square

It is clear that pluperfection for G implies perfection for G . Lovász (1972a, 1972b) established that the converse is also true, i.e.:

Theorem V.3.4. G is perfect if and only if G is pluperfect. \square

Theorem V.3.4 is known as the Perfect Graph Theorem, and was first conjectured (in somewhat different form) by Berge. An immediate consequence of (V.3.1) and (V.3.4) (see Chvátal (1975)) is:

Corollary V.3.5. For the loopless, undirected graph G without multiple edges, let A be its clique incidence matrix, and B its anticlique incidence matrix. Then G is perfect if and only if A and B are an antiblocking pair. \square

Fulkerson (1973) has shown the following stronger result which only requires the assumption that the matrices A and B be $(0,1)$ -valued.

Theorem V.3.6. Let A and B be $(0,1)$ -matrices. Then A, B are an antiblocking pair of matrices if and only if there is a (perfect) graph G such that the essential rows of A are precisely the incidence vectors of the maximal cliques of G , and the essential rows of B are precisely the incidence vectors of the maximal anticliques of G . \square

Theorem V.3.6 thus characterizes all pairs of matrices for which the min-max equality holds strongly for every $w \in Z_+^n$, or equivalently, all integral antiblocking pairs of matrices.

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