

# ZERO-ONE REPRESENTATION AND BOUNDS FOR FRACTIONAL FACTORIAL DESIGNS

D. A. ANDERSON\*

*Department of Statistics, University of Wyoming, Laramie, WY 82071, U.S.A.*

W. T. FEDERER

*Biometrics Unit, Cornell University, Ithaca, NY 14853, U.S.A.*

*Abstract:* Let  $T$  denote a main effect plan for the  $s^n$  factorial with  $N$  treatments, that is  $T$  is an  $N \times n$  matrix with elements from the  $\{0, 1, \dots, s-1\}$ . Denote by  $T_0, T_1, \dots, T_{s-1}$  the  $N \times n$  incidence matrices of  $0, 1, \dots, s-1$ , respectively, so that  $T = \sum_i T_i$  and  $\sum_i T_i = J_{N \times n}$ . In the usual way we write  $E\{Y\} = X\alpha$ , where  $X$  is the  $(0, 1)$  design matrix corresponding to  $T$ . A transformation  $G$  is obtained for which  $X = X^*G'$ , where  $X^* = \begin{bmatrix} 1 & \vdots & T_1 & \vdots & \dots & \vdots & T_{s-1} \end{bmatrix}$  thus giving a representation of the design matrix directly in terms of a full rank  $(0, 1)$ -incidence matrix. The determinants of  $X'X$ ,  $X^*'X^*$ , and  $G'G$  are evaluated. The determinant of the information matrix is directly expressible in terms of the determinant of a  $(0, 1)$ -matrix. These results are extended to include the general asymmetrical factorial  $\Pi s_i$ . Upper and lower bounds are obtained for the determinant values of  $X^*$  when  $X^*$  is square and in general for  $X^*'X^*$ . One important aspect of this representation is that the construction of main effect plans and an assessment of their goodness via the determinant criteria can be studied directly in terms of  $(0, 1)$  matrices. An extension to include interaction terms for the  $s^n$  factorial where  $s$  is a prime or prime power and for the  $\Pi s_i$  factorial is given.

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## 1. Introduction

In a factorial experiment involving  $n$  factors with the  $i$ th factor at  $s_i$  levels  $i=1,2,\dots,n$ , the total number of treatment combinations is  $\prod_{i=1}^n s_i$ . A design is usually represented as an  $N \times n$  matrix  $T$  whose  $N$  rows  $i=1$  denote the particular treatment combinations and whose  $n$  columns correspond to the levels of the  $n$  factors. The elements of a column of  $T$  corresponding to a factor at  $s_i$  levels are integers from  $\{0,1,\dots,s_i-1\}$ ,  $i=1,2,\dots,n$ , to denote the  $s_i$  levels.

In the analysis, the matrix  $T$  is often replaced by a matrix  $X_{N \times n}$  which reflects the  $v$  single degree of freedom parametric contrasts in the parametric vector  $\beta_{v \times 1}$  from the usual regression equation  $E\{Y\} = X\beta$  and  $\text{Cov}\{Y\} = \sigma^2 I_N$ . The normal equations are  $X'X\hat{\beta} = X'Y$ , and solutions to the normal equations provide best linear unbiased estimates of estimable functions of the parameters in  $\beta$ . The matrix  $X'X$  is called the information matrix of the design  $T$ , and if  $X'X$  is nonsingular the variance-covariance matrix of  $\beta$  is proportional to  $(X'X)^{-1}$ . Most criteria of goodness of a design depend upon some function of  $(X'X)^{-1}$  as, for example, the determinant, trace, and maximum root criteria. If  $X'X$  is singular we consider a conditional inverse  $(X'X)^{-}$  and restrict to estimable functions of the  $\beta$ 's.

Since the matrix  $X$  is obtained directly from the matrix  $T$ , all of the information concerning the goodness of the design (in terms of some function of  $(X'X)^{-1}$ ) is contained within  $T$ . Thus for the purpose of

constructing good designs and the comparison of designs, the simplest and most direct representation of this property in terms of  $T$  itself would be useful.

Raktoe and Federer [1970] obtained such a representation directly in terms of the  $(0,1)$  matrix  $(1:T)$  for main effect plans for the  $2^n$  factorial, where  $1$  denotes a vector with every element unity. In Section 2 of this paper we present a similar representation for the  $\prod_{i=1}^n s_i$  factorial, and we represent  $|X^*X'|$  directly in terms of this representation, where  $X = X^*G'$ . In the third section an upper bound on the  $|X^*|$  is obtained for both the symmetric and asymmetric factorials, and the minimum nonzero value of this determinant is indicated.

The importance of the representation presented lies in the insight that may be gained toward the construction of fractional factorial plans and the assessment of their goodness via the determinant criteria as shown in Section 4. This is illustrated by the four methods of construction, using  $(0,1)$ -matrices, presented.

## 2. Representation of main effect plans in terms of (0,1)-matrices

To illustrate the procedure, let us consider the  $s^n$  symmetrical factorial and the corresponding main effect plan  $T$  for estimating the  $v = 1+n(s-1)$  mean and main effect contrasts under the assumption that all two-factor and higher-factor interactions are zero. Let  $T_{N \times n}$  be an  $N \times n$  matrix of elements  $j_1 j_2 \cdots j_n$ , where  $j_i = 0, 1, \dots, s-1$  for all  $i$ ,  $i = 1, 2, \dots, n$ . The  $N \times 1$  observation vector corresponding to plan  $T$  is denoted as  $Y_T$ . Let  $T_j$ ,  $j=0, 1, \dots, s-1$ , be the  $N \times n$  (0,1)-incidence matrix for element  $j$ . That is, an element of  $T_j$  is one or zero as the corresponding element of  $T_j$  is  $j$  or not. Then,

$$\sum_{j=0}^{s-1} T_j = J_{N \times n} \quad \text{and} \quad T = \sum_{j=0}^{s-1} j T_j, \quad (2.1)$$

where  $J$  is a matrix for which every element is one. Further, let  $\alpha_{1 \times ns} = (\alpha_0^{(1)}, \alpha_0^{(2)}, \dots, \alpha_0^{(n)}, \alpha_1^{(1)}, \dots, \alpha_1^{(n)}, \alpha_2^{(1)}, \dots, \alpha_{s-1}^{(n)})$  be the overparameterized vector of parameters corresponding to the  $j$ th level of factor  $i$ , i.e.,  $\alpha_j^{(i)}$ . Note that a mean parameter  $\mu$  is not included but could be added if desired. For the above overparameterized model, the usual incidence matrix is:

$$X_{N \times ns} = \begin{bmatrix} T_0 & T_1 & T_2 & \cdots & T_{s-1} \end{bmatrix}, \quad (2.2)$$

where the vector of ones  $1_{N \times 1}$  for the mean effect  $\mu$  is omitted.

**Example 2.1.** Let  $T$ ,  $T_0$ ,  $T_1$ , and  $T_2$  for a saturated main effect plan for a  $3^4$  factorial be:

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix}, \quad T_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then,  $T_{9 \times 4} = (0)T_0 + (1)T_1 + (2)T_2$ ,  $J_{9 \times 4} = T_0 + T_1 + T_2$ , and  $X_{9 \times 12} = [T_0 : T_1 : T_2]$ . The parameter vector  $\alpha' = (\alpha_0^{(1)}, \alpha_0^{(2)}, \alpha_0^{(3)}, \alpha_0^{(4)}, \alpha_1^{(1)}, \alpha_1^{(2)}, \alpha_1^{(3)}, \alpha_1^{(4)}, \alpha_2^{(1)}, \alpha_2^{(2)}, \alpha_2^{(3)}, \alpha_2^{(4)})$ . Note that  $E[Y_T] = (T_0 : T_1 : T_2)\alpha$  where  $Y_T' = (Y_{0000}, Y_{0111}, Y_{0222}, Y_{1012}, Y_{1120}, Y_{1201}, Y_{2021}, Y_{2102}, Y_{2210})$ .

To make  $X_{N \times ns}$  of full column rank  $v$  for  $N \geq v$  and  $T$  a connected main effect plan, we use the (0,1)-matrix

$$X_{N \times v}^* = \begin{bmatrix} 1 & : & T_1 & : & T_2 & : & \cdots & : & T_{s-1} \end{bmatrix} . \quad (2.3)$$

This is done to utilize the theory and properties associated with (0,1)-matrices and results obtained by Raktue and Federer (1970) and Anderson and Federer (1975). Using these results, the theory for  $s_n$  and  $\prod_{i=1}^n s_i$  factorial will be extended. It should be noted that a variety of  $X$ -matrices could be used and could be converted to  $X^*$  matrices through the transformation  $XG = X^*$ . Originally, the Helmert matrix was used as the  $X$  matrix by Anderson and Federer (1973,1981). It was noted that orthogonal polynomials could also have been used. Using the Helmert orthogonal contrast matrix, it was possible to utilize full rank square matrices which have some advantages. However, we shall use the  $X$ -matrix of (2.2) following a suggestion by an associate editor and a referee. In the usual formulation, let  $E(Y_T) = X\alpha$ . In order to obtain unique solutions, we reparameterize as follows:

$$\sum_{i=1}^n \alpha_0^{(i)} = \beta_0, \quad \beta_j^{(i)} = \alpha_j^{(i)} - \alpha_0^{(i)}, \quad \text{and} \quad \beta_{v \times 1} = G'_{v \times ns} \alpha_{ns \times 1} \quad (2.4)$$

where

$$G_{ns \times v} = \begin{bmatrix} 1_{n \times 1} & \vdots & -1'_{1 \times (s-1)} (\times) I_n \\ \cdot & \cdot & \cdot \\ 0_{n(s-1) \times 1} & \vdots & I_{s-1} (\times) I_n = I_{n(s-1)} \end{bmatrix} , \quad (2.5)$$

where  $\mathbf{1}$  is a vector of ones,  $\mathbf{0}$  is a vector of zeros,  $\mathbf{I}_n$  is the identity matrix of size  $n \times n$  which has ones on the diagonal and zeros elsewhere, and  $(\times)$  denotes the Kronecker product of matrices. Under this formulation,

$$\mathbf{X}_{N \times ns} = \mathbf{X}_{N \times v}^* \mathbf{G}'_{v \times ns} \text{ and}$$

$$(\mathbf{G}'\mathbf{G})_{v \times v} = \begin{bmatrix} n & \vdots & -\mathbf{1}'_{1 \times n(s-1)} \\ \vdots & \ddots & \vdots \\ -\mathbf{1}_{n(s-1) \times 1} & \begin{matrix} 2\mathbf{I}_n & \mathbf{I}_n & \cdots & \mathbf{I}_n \\ \mathbf{I}_n & 2\mathbf{I}_n & \cdots & \mathbf{I}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{I}_n & \mathbf{I}_n & \cdots & 2\mathbf{I}_n \end{matrix} \end{bmatrix}. \quad (2.6)$$

Rearrangement of the rows and columns of (2.6) allows us to write

$$(\mathbf{G}'\mathbf{G})_{v \times v} = \begin{bmatrix} n/s & \vdots & \mathbf{0}'_{1 \times n(s-1)} \\ \vdots & \ddots & \vdots \\ -\mathbf{1}_{n(s-1) \times 1} & \vdots & \mathbf{I}_n (\times) (\mathbf{I}_{s-1} + \mathbf{J}_{s-1}) \end{bmatrix}, \quad (2.7)$$

where  $\mathbf{J}_{s-1}$  is a square matrix of side  $s-1$  and with all elements being ones. This rearrangement would result from using  $\boldsymbol{\alpha}' = \left( \alpha_0^{(1)}, \alpha_1^{(1)}, \dots, \alpha_{s-1}^{(1)}, \alpha_0^{(2)}, \dots, \alpha_{s-1}^{(2)}, \alpha_0^{(3)}, \dots, \alpha_{s-1}^{(n)} \right)$  as the parameter vector and rearranging the columns accordingly. The minus ones in the first row of (2.4) were eliminated by adding  $1/s$  times the sum of all the other rows to the first row. In this form, it is simple to evaluate the determinant of  $\mathbf{G}'\mathbf{G}$  by noting that  $|\mathbf{I}_{s-1} + \mathbf{J}_{s-1}| = s$ . Hence,

$$|\mathbf{G}'\mathbf{G}| = \left( \frac{n}{s} \right)^n s^n = ns^{n-1}. \quad (2.8)$$

**Example 2.2.** For  $\mathbf{T}$  and  $\mathbf{X}$  of Example 2.1,  $\boldsymbol{\alpha} = \left( \alpha_0^{(1)}, \alpha_0^{(2)}, \alpha_0^{(3)}, \alpha_0^{(4)}, \alpha_1^{(1)}, \alpha_1^{(2)}, \alpha_1^{(3)}, \alpha_1^{(4)}, \alpha_2^{(1)}, \alpha_2^{(2)}, \alpha_2^{(3)}, \alpha_2^{(4)} \right)$  and  $\mathbf{G}$  and  $\mathbf{G}'\mathbf{G}$  are:

$$G_{12 \times 9} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad G'G_{9 \times 9} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Rearranging the columns of  $G'G$  we obtain:

$$|G'G|_{9 \times 9} = \begin{vmatrix} 4/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{vmatrix} = \frac{4}{3}(3^4) = 4(3^3) = 108.$$

As was noted above, we are using the transformation  $X_{N \times ns} = X_{N \times v}^* G'_{v \times ns}$ .  
From the fundamental matrix equation (Searle, 1982, Sections 11.2 and 11.7)

$$X'Xu = \lambda u = GX^*X^*G'u, \quad (2.9)$$

where  $\lambda$  is a vector of the eigenvalues of  $X'X$ ,  $u$  is any vector, and  $X'X$  is nonnegative definite. Then,

$$G'GX^*X^*(G'u) = \lambda(G'u), \quad \lambda \geq 0, \quad (2.10)$$

and

$$G'GX^*X^* = \lambda. \quad (2.11)$$

Therefore,

$$|G'GX^*X^*| = |G'G| \cdot |X^*X^*| = \text{product of positive eigenvalues of } X'X, \quad (2.12)$$

or

$$|X^*X^*| = |G'G|^{-1} \Pi(\text{positive eigenvalues of } X'X).$$

The above results are summarized in the following theorem:

**Theorem 2.1.** *Under the transformation  $X = X^*G'$ , we have*

- a)  $X_{N \times v}^* = [1 : T_1 : T_2 : \dots : T_{s-1}]$ ,
- b)  $|G'G| = ns^{n-1}$ ,
- c)  $ns^{n-1} |X^*X| = \text{product of positive eigenvalues of } X'X$ ,
- d)  $X_{N \times v}^* = X_{N \times ns} G_{ns \times v} (G'G)^{-1}_{v \times v}$ , and when  $N=v$  and  $T$  is a saturated plan, then
- e)  $|X^*| = \left[ (\text{product of positive eigenvalues of } X'X) / ns^{n-1} \right]^{\frac{1}{2}}$ .

The results for the  $s^n$  factorial are now extended to the general asymmetrical factorial  $s_1 \times s_2 \times \dots \times s_n = \prod_{i=1}^n s_i$  for  $s_1 \leq s_2 \leq s_3 \leq \dots \leq s_n$ . Since factors may have different numbers of levels, we may change  $X$ ,  $X^*$ ,  $G$ , and  $\alpha$  accordingly. Let  $k = \sum_{i=1}^n s_i$ ; then  $X$  is  $N \times k$ ,  $X^*$  is  $N \times v$ , and  $G$  is  $k \times v$  for  $v = 1+k-n$ . Let each  $T_j$  of (2.1) be  $N \times n$ , and let  $T_j^*$  be  $T_j$  with all columns of zeros deleted. Then, let

$$X_{N \times k} = [T_0 : T_1 : T_2^* : T_3^* : \dots : T_{s_n-1}^*] \quad (2.13)$$

and

$$X_{N \times v}^* = [1 : T_1 : T_2^* : T_3^* : \dots : T_{s_n-1}^*] \quad (2.14)$$

For  $\alpha'_{1 \times k} = (\alpha_0^{(1)}, \alpha_1^{(1)}, \dots, \alpha_{s_1-1}^{(1)}, \alpha_0^{(2)}, \alpha_1^{(2)}, \dots, \alpha_{s_2-1}^{(2)}, \dots, \alpha_0^{(n)}, \alpha_1^{(n)}, \dots, \alpha_{s_n-1}^{(n)})$ , the above  $X$  and  $X^*$  matrices will need to have the columns rearranged from  $\alpha'_{1 \times k} = (\alpha_0^{(1)}, \alpha_0^{(2)}, \dots, \alpha_0^{(n)}, \alpha_1^{(1)}, \dots, \alpha_1^{(n)}, \dots, \alpha_{s_n-1}^{(n)})$ . The following  $G$  is for the former  $\alpha$  arrangement:

$$G_{k \times v} = \begin{bmatrix} 1 & -1'_{1 \times (s_1-1)} & 0 & 0 & 0 \\ 0_{1 \times (s_1-1)} & I_{(s_1-1)} & & & \\ 1 & 0 & -1'_{1 \times (s_2-1)} & 0 & 0 \\ 0_{1 \times (s_2-1)} & & I_{(s_2-1)} & & \\ 1 & 0 & 0 & -1'_{1 \times (s_3-1)} & 0 \\ 0_{1 \times (s_3-1)} & & & I_{s_3-1)} & \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & 0 & -1'_{1 \times (s_n-1)} \\ 0_{1 \times (s_n-1)} & & & & I_{(s_n-1)} \end{bmatrix} \quad (2.15)$$

Then,

$$(G'G)_{v \times v} = \begin{bmatrix} n & \vdots & -1'_{1 \times (k-n)} & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1_{(k-n) \times 1} & (I+J)_{(s_1-1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \vdots & 0 & (I+J)_{(s_2-1)} & 0 & 0 & 0 \\ 0 & \vdots & 0 & 0 & (I+J)_{(s_3-1)} & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & \vdots & 0 & 0 & & \dots & (I+J)_{(s_n-1)} \end{bmatrix} \quad (2.16)$$

Adding suitable multiples of the remaining rows to the first row, the scalar

$n$  becomes  $n - \sum_{i=1}^n \frac{(s_i-1)}{s_i}$  and then all other elements of row one are zero.

In this form, the determinant of  $G'G$  is obtained as:

$$|G'G| = \left( n - \sum_{i=1}^n \frac{(s_i-1)}{s_i} \right) \prod_{i=1}^n s_i, \quad (2.17)$$

since the determinant of  $(I+J)_{(s_i-1)}$  is  $s_i$ . Note that this reduces to  $ns^{s-1}$

when all  $s_i = s$  as given in Theorem 1.

These results are combined in the following theorem:

**Theorem 2.2.** Under the transformation  $X_{N \times k} = X_{N \times v}^* G'_{v \times k}$ , we have

- a)  $X_{N \times v}^* = \begin{bmatrix} 1 & : & T_1 & : & T_2^* & : & T_3^* & : & \cdots & : & T_{s_n-1}^* \end{bmatrix}$ ,
- b)  $|G'G| = \left( n - \sum_{i=1}^n \frac{s_i^{-1}}{s_i} \right) \prod_{i=1}^n s_i$ ,
- c)  $\left( n - \sum_{i=1}^n \frac{s_i^{-1}}{s_i} \right) \prod_{i=1}^n s_i |X^* X^*| = \text{product of positive eigenvalues of } X'X$ ,
- d)  $X_{N \times v}^* = X_{N \times k} G_{k \times v} (G'G)^{-1}_{v \times v}$ , and when  $N=v$  and  $T$  is a connected saturated plan, then
- e)  $|X^*| = \left[ \text{product of positive eigenvalues of } X'X \left( n - \sum_{i=1}^n \frac{s_i^{-1}}{s_i} \right) \prod s_i \right]^{\frac{1}{2}}$ .

Statements c) and e) follow from the results given in equations (2.9) to (2.12). The parameters are given by  $\beta = G'\alpha$ , where  $\beta_0 = \sum_{i=1}^n \alpha_0^{(i)}/n$  and  $\beta_j^{(i)} = \alpha_j^{(i)} - \alpha_0^{(i)}$ . The following example illustrates the results.

**Example 2.3.** Consider the following saturated main effect plan for a  $2 \times 3 \times 4 \times 5$  factorial. Let

$$T_{11 \times 4} = \begin{bmatrix} 0000 \\ 0111 \\ 0222 \\ 0033 \\ 0104 \\ 1210 \\ 1021 \\ 1132 \\ 1203 \\ 1014 \\ 1120 \end{bmatrix}, T_0 = \begin{bmatrix} 1111 \\ 1000 \\ 1000 \\ 1100 \\ 1010 \\ 0001 \\ 0100 \\ 0000 \\ 0010 \\ 0100 \\ 0001 \end{bmatrix}, T_1 = \begin{bmatrix} 0000 \\ 0111 \\ 0000 \\ 0000 \\ 0100 \\ 1010 \\ 1001 \\ 1100 \\ 1000 \\ 1010 \\ 1100 \end{bmatrix}, T_2 = \begin{bmatrix} 0000 \\ 0000 \\ 0111 \\ 0000 \\ 0000 \\ 0100 \\ 0010 \\ 0001 \\ 0100 \\ 0000 \\ 0010 \end{bmatrix}, T_3 = \begin{bmatrix} 0000 \\ 0000 \\ 0000 \\ 0011 \\ 0000 \\ 0000 \\ 0000 \\ 0010 \\ 0001 \\ 0000 \\ 0000 \end{bmatrix}, \text{ and } T_4 = \begin{bmatrix} 0000 \\ 0000 \\ 0000 \\ 0000 \\ 0001 \\ 0000 \\ 0000 \\ 0000 \\ 0000 \\ 0001 \\ 0000 \end{bmatrix}.$$

Form  $T_2^*$ ,  $T_3^*$ , and  $T_4^*$  by deleting the first, the first two, and the first three columns, respectively. Then  $X_{11 \times 11}^* = \begin{bmatrix} 1 : T_1 : T_2^* : T_3^* : T_4^* \end{bmatrix}$ , which with columns rearranged to correspond to the  $\alpha$  and  $\beta$  used, is:

$$X_{11 \times 11}^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad |X^*| = -11 \quad .$$

$$X^{*'}X^* = \begin{bmatrix} 11 & 6 & 4 & 3 & 3 & 3 & 2 & 2 & 2 & 2 \\ 6 & 6 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 4 & 2 & 4 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 3 & 2 & 0 & 3 & 1 & 1 & 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 3 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad |X^{*'}X^*| = 121 \quad .$$

The above development has been for main effect fractions. Interaction terms may be included as well. For example, consider that there are  $s_1$  levels of  $F_1$  and  $s_2$  levels of  $F_2$ , and it is desired to retain all of the interaction parameters in the parameter vector  $\beta$ . Further, suppose that there are  $s_3$  levels of  $F_3$  and  $s_4$  levels of  $F_4$  for the four-factor fractional replicate. A saturated main effect plan would consider  $n=4$  factors with  $(s_1-1) + (s_2-1) + (s_3-1) + (s_4-1) + 1 = v$  parameters and combinations. To include the interaction parameters of  $F_1$  and  $F_2$  in a saturated fraction, use a three-factor fraction with  $s_1s_2$  levels of the first factor,  $s_3$  levels of the second factor, and  $s_4$  levels of the third factor with  $(s_1s_2-1) + (s_3-1) + (s_4-1) + 1 = v$  parameters and combinations.

**Example 2.4.** Let  $n=5$ ,  $s_1=2$ ,  $s_2=s_3=s_4=s_5=3$ , and suppose that the interactions of  $F_2$  with  $F_3$  and of  $F_4$  with  $F_5$  are desired. The comparable three-factor factorial is  $2(9^2)$  and for  $v=N=18$  runs, a saturated fraction  $T$  would be: 000, 101, 011, 112, 022, 123, 033, 134, 044, 145, 055, 156, 066, 167, 077, 178, 088, 180.

To include geometrical components of an interaction, simply consider each component as another main effect. If there are  $m$  geometrical components and  $n$  main effects, use an  $n+m$  factor design. For example, for factors  $F_1$ ,  $F_2$ , and  $F_3$  plus a geometrical component of  $F_1$  and  $F_3$ , one could use the design  $T$  in Example 2.1, where the last column refers to the three levels of the geometrical component of the interaction (see Pesotan and Raktoe, 1975).

### 3. Bounds on the determinants of nonsingular design matrices

The transformation from  $X$  to  $X^*$  provides a simple proof that the determinant of  $X'X$  is invariant to any change of level designation for any factor. Any permutation of the nonzero levels results only in a corresponding permutation of columns in  $X^*$ , which of course does not change the value of the determinant. Likewise, any nonzero level may be interchanged with the zero level for any specified factor. The corresponding change in  $X^*$  is a linear combination of the first column of all ones and the columns of  $T_1, T_2, \dots, T_{s-1}$  corresponding to that factor. Again, this does not change the determinant. This invariance property is a well known result (see, for example, Paik and Federer [1970], Joiner [1973], and Srivastava, Raktoe, and Pesotan [1976]), but the representation in terms of  $(0,1)$  matrices makes it more apparent.

Raktoe and Federer [1970] obtained the following bound on  $\|X^*\|$  using Hadamard's theorem:

$$\|X^*\| \leq (n+1)^{(n+1)/2} 2^{-n} \quad (3.1)$$

Since  $|X^*|$  must be an integer, we take the integer part of the right-hand side of (3.1) as the upper bound. We now obtain a generalization of their result for  $X^*$  matrices, and consequently  $X$  matrices, for main effect plans from the symmetrical  $s^n$  factorial.

**Theorem 3.1.** *Let  $T$  be a saturated main effect plan for the  $s^n$  factorial with  $N = n(s-1) + 1$ . If  $X^* = [1 : T_1 : T_2 : \cdots : T_{s-1}]$ ,*

$$\|X^*\| \leq \text{integer part of } N^{N/2} s^{-sn/2} \quad (3.2)$$

and when  $s=2$ , (3.2) reduces to equation (3.1).

**Corollary 3.1.** *Let  $T$  be a main effect plan for an  $s^n$  factorial experiment with  $N \geq n(s-1) + 1$ . Then*

$$|X^*X^*| \leq \text{integer part of } N^{n(s-1)+1} s^{-ns} . \quad (3.3)$$

**Proof.** Various paths to proving the above theorem and corollary are possible. One method is that of Anderson and Federer (1973). A second and shorter proof (suggested by a referee) follows. Consider the eigenvalues of  $X'X$ . The eigenvalue zero occurs  $n-1$  times. Since  $X_{N \times ns} \mathbf{1}_{ns \times 1} = n \mathbf{1}_{ns \times 1}$ , the maximum eigenvalue  $\lambda_{\max}$  of  $X'X$  is greater than or equal to

$$n^2 |\mathbf{1}'_{1 \times N} \mathbf{1}_{N \times 1}| / |\mathbf{1}'_{1 \times ns} \mathbf{1}_{ns \times 1}| = \frac{n^2 N}{ns} = nN/s . \quad (3.4)$$

The trace of  $X'X$  is  $nN$ . The average of the remaining  $n(s-1)$  positive eigenvalues is

$$(nN - nN/s)/n(s-1) = N/s . \quad (3.5)$$

Since  $N/s \leq nN/s$ , the product of the eigenvalues is maximized when  $\lambda_{\max} = nN/s$  and all the other eigenvalues are  $N/s$ . Therefore,

$$|X^*X^*| \leq \left(\frac{nN}{s}\right) \left(\frac{N}{s}\right)^{n(s-1)} / |G'G| = N^v s^{-ns} \quad , \quad (3.6)$$

and since  $X_{v \times v}^*$  is a square matrix

$$\|X^*\| \leq N^{v/2} s^{-ns/2} \quad . \quad (3.7)$$

This completes the proof.  $\square$

**Note:** For  $\sum_{i=1}^r \lambda_i = P$ , the  $\prod_{i=1}^r \lambda_i$  is maximum when  $\lambda_i = P/r$  for all  $i$ . The proof of this statement is as follows. Let  $\theta = \prod_{i=1}^r \lambda_i - \rho(\sum_{i=1}^r \lambda_i - P)$ , where  $\rho$  is a Lagrange multiplier. Then,

$$\left. \begin{aligned} \frac{\partial \theta}{\partial \lambda_i} &= \frac{\prod_{i=1}^r \lambda_i}{\lambda_i} - \rho = 0 \\ \text{and} \\ \frac{\partial \theta}{\partial \rho} &= \sum_{i=1}^r \lambda_i - P = 0 \end{aligned} \right\} \quad (3.8)$$

From the above,

$$\prod_{i=1}^r \lambda_i = \lambda_i \rho \quad \text{and} \quad r \prod_{i=1}^r \lambda_i = \rho \sum_{i=1}^r \lambda_i = \rho P \quad , \quad (3.9)$$

or

$$\rho = r \prod_{i=1}^r \lambda_i / P \quad . \quad (3.10)$$

Then,

$$\lambda_i = \prod_{i=1}^r \lambda_i / \rho = P/r \quad . \quad (3.11)$$

Since there is a largest eigenvalue, the product is maximized when all eigenvalues  $\lambda_i = N/s$  except for the largest one.<sup>1</sup>

**Example 3.1.** Consider a set of  $t$  orthogonal latin square of order  $s$ . This set may be regarded as an orthogonal main effect plan for the  $s^{t+2}$  factorial with  $N=s^2$ . If  $t=s-1$ , which is possible whenever  $s$  is a prime or prime power, the set forms a saturated main effect plan. The  $s^2 \times [1+(t+2)(s-1)]$

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<sup>1</sup>This note was kindly provided by S.R. Searle, Cornell University.

matrix  $X^*$  is given by  $X^* = \begin{bmatrix} 1 & T_1 & T_2 & \cdots & T_{s-1} \end{bmatrix}$  where  $T_i' T_i = (s-1)I + J$  and  $T_i' T_j = J - I$ ,  $i \neq j$ . Thus

$$X^* X^* = \begin{bmatrix} s^2 & s1' & s1' & \cdots & s1' \\ s1 & sI+(J-I) & J-I & \cdots & J-I \\ s1 & J-I & sI+(J-I) & \cdots & J-I \\ \vdots & & & & \vdots \\ s1 & J-I & J-I & \cdots & sI+(J-I) \end{bmatrix}. \quad (3.12)$$

The determinant of  $X^* X^*$  is  $|X^* X^*| = s^{(t+2)(s-2)+2}$ , which equals the bound given in Theorem 3.1. When  $t=s-1$ , the design is saturated. Then  $|X^* X^*| = s^{s(s-1)}$  and  $|X^*| = s^{s(s-1)/2}$ , which equals the bound given in Theorem 3.1.

**Example 3.2.** Suppose  $T$  is an orthogonal array of size  $N$ ,  $n$  constraints,  $s$  levels, of strength 2, and index  $\lambda$  denoted by  $(N, n, s, 2)$ . That is,  $T$  is a fraction for an  $s^n$  factorial in  $N$  runs such that for any pair of factors each of the  $s^2$  possible combinations of levels occurs exactly  $\lambda$  times. Clearly  $N = \lambda s^2$  and the matrix  $X^* X^*$  for this fraction is exactly  $\lambda$  times the matrix (3.12) in Example 3.1. The determinant of  $X^* X^*$  also attains the upper bound given in Corollary 3.1.

By suitable modifications, Theorem 3.1 may be extended to cover the asymmetrical case, as indicated in the following theorem and corollary:

**Theorem 3.2.** Let  $T$  be a saturated main effect plan for a general asymmetrical  $\prod_{i=1}^n s_i$  factorial with  $N = 1 + \sum_{i=1}^n (s_i - 1)$  runs and let  $X^* = \begin{bmatrix} 1 & T_1 & T_2 & \cdots & T_{s_n-1} \end{bmatrix}$ . Then

$$\|X^*\| \leq \text{integer part of } N^{N/2} \prod_{i=1}^n s_i^{-s_i/2}. \quad (3.13)$$

**Corollary 3.2.** Let  $T$  be a main effect plan for a general asymmetrical  $\prod_{i=1}^n s_i$  factorial with  $N \geq 1 + \sum_{i=1}^n (s_i - 1)$  runs and let  $X^*$  be the corresponding  $(0,1)$ -matrix representation. Then

$$|X^* X^*| \leq \text{integer part of } N^v \prod_{i=1}^n s_i^{-s_i}, \quad v = 1 + \sum_{i=1}^n (s_i - 1). \quad (3.14)$$

**Proof.** Let  $S$  be a diagonal matrix with diagonals  $(s_1, s_1, \dots, s_2, \dots, s_n, \dots, s_n)$  where  $s_i$  is repeated  $s_i$  times. Then, transform  $X$ ,  $G$ , and  $\alpha$  as:

$$\left. \begin{aligned} \tilde{X} &= X S^{\frac{1}{2}} & \text{and} & & \tilde{\alpha} &= S^{-\frac{1}{2}} \alpha \\ \tilde{G} &= S^{\frac{1}{2}} G & \text{and} & & \tilde{\beta} &= \tilde{G} \tilde{\alpha} \end{aligned} \right\} \quad (3.15)$$

Now, we follow the proof given previously for Theorem 3.1. Then,

$$|X^* X^*| = (\text{product of positive eigenvalues of } (\tilde{X}' \tilde{X}) \cdot |\tilde{G}' \tilde{G}|^{-1}). \quad (3.16)$$

After orthogonalization, the first row of  $\tilde{G}$  is  $S^{-\frac{1}{2}} \mathbf{1}_{k \times 1}$  with  $\mathbf{1}' S^{-\frac{1}{2}} S^{-\frac{1}{2}} \mathbf{1}_{k \times 1} = n$ .

Therefore,

$$|\tilde{G}' \tilde{G}| = n \prod_{i=1}^n s_i^{s_i - 1} \left| I_{s_i - 1} + J_{s_i - 1} \right| = n \prod_{i=1}^n s_i^{s_i}. \quad (3.16)$$

$$\tilde{X} \left( S^{-\frac{1}{2}} \mathbf{1}_{k \times 1} \right) = X \mathbf{1}_{k \times 1} = n \mathbf{1}_{N \times 1}. \quad (3.17)$$

The maximum eigenvalue  $\lambda_{\max}$  of  $X'X$  is

$$\lambda_{\max}(X'X) \geq n^2 \mathbf{1}'_{1 \times N} \mathbf{1}_{N \times 1} / \left( \mathbf{1}'_{1 \times k} S^{-\frac{1}{2}} S^{-\frac{1}{2}} \mathbf{1}_{k \times 1} \right) = nN. \quad (3.18)$$

Now,

$$\text{trace}(\tilde{X}' \tilde{X}) = N s_1 + N s_2 + \dots + N s_n = kN. \quad (3.19)$$

The average of the remaining  $k-n$  positive eigenvalues is less than or equal to  $(kN - nN)/(k-n) = N$ . Following the same steps as before, we obtain

$$|X^*X^*| \leq nN^{k-n} \left( \prod_{i=1}^n s_i \right)^{-1} = N^v \prod_{i=1}^n s_i^{-s_i}.$$

and when T is a saturated main effect plan,

$$|X^*| \leq N^{v/2} \prod_{i=1}^n s_i^{-s_i/2}.$$

This completes the proof.  $\square$

The above relates to an upper bound on designs T. Designs T which achieve this bound are denoted as D-optimal. The question arises as to the conditions required to achieve this upper bound. There is no assurance that it can be obtained for all T. The T producing the largest value of  $|X^*X^*|$  will be denoted as D-optimal. The following theorem indicates the conditions under which the upper bound can be achieved.

**Theorem 3.3.** *Necessary and sufficient conditions for achieving the upper bound of Theorem 3.2 are that*

- i) *all levels of a factor occur with equal frequency and*
- ii) *all possible pairs of levels of two factors appear with the same frequency.*

**Proof.<sup>2</sup>** First let us suppose that the upper bound is attained for the matrix  $X^*$  corresponding to a design T. Then, the matrix  $\tilde{X}'\tilde{X}$  has one and only one eigenvalue  $nN$  and  $k-n$  eigenvalues equal to  $N$ . Let  $n(i_j, i'_j)$  be the number of treatments of T having factor  $i$  at level  $j$  and factor  $i'$  at level  $j'$  for  $j, j' = 1, 2, \dots, s_i$  (for convenience of notation). When  $i_j = i'_j$ , then the number of treatments having factor  $i$  at level  $j$  is denoted as  $n(i_j)$ . Note that these frequencies are the elements of the matrix  $X'X$ .

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<sup>2</sup>These results follow from the comments of a referee, the necessary part, and from Anderson and Federer [1973,1981], the sufficient part.

The eigenvector corresponding to  $\lambda_{\max} = nN$  has to be  $S^{-\frac{1}{2}} \mathbf{1}_k$  since the Raleigh quotient for this vector equals  $nN$ . Then,

$$\begin{aligned} \tilde{X}' \tilde{X} S^{-\frac{1}{2}} \mathbf{1}_{k \times 1} &= n S^{\frac{1}{2}} X' \mathbf{1}_{N \times 1} = n S^{\frac{1}{2}} \left[ n(1_1), \dots, n(1_{s_1}), n(2_1), \dots, n(n_{s_n}) \right] \\ &= n N S^{-\frac{1}{2}} \mathbf{1}_{k \times 1} \quad . \end{aligned} \quad (3.20)$$

This implies that  $n(i_j) = N/s_i$  for all  $i_j$ . Let  $e(i_j, i_{j'})$  be a  $k \times 1$  vector having a plus one at the coordinate corresponding to level  $j$  of factor  $i$ , a minus one at level  $j'$  of factor  $i$ ,  $j \neq j'$ , and zeros elsewhere. Then,  $S^{-\frac{1}{2}} e(i_j, i_{j'})$  is orthogonal to  $S^{-\frac{1}{2}} \mathbf{1}_{k \times 1}$  and to the kernel of  $\tilde{X}' \tilde{X}$ . This vector is an eigenvector corresponding to  $\lambda = N$ . From these results,

$$\begin{aligned} \tilde{X}' \tilde{X} S^{-\frac{1}{2}} e(i_j, i_{j'}) &= S^{\frac{1}{2}} \begin{bmatrix} n(1_1, i_j) & -n(1_1, i_{j'}) \\ \vdots & \vdots \\ n(1_{s_1}, i_j) & -n(1_{s_1}, i_{j'}) \\ n(2_1, i_j) & -n(2_1, i_{j'}) \\ \vdots & \vdots \\ n(n_{s_n}, i_j) & -n(n_{s_n}, i_{j'}) \end{bmatrix} \\ &= N S^{-\frac{1}{2}} [0 \dots 0 : 10 \dots -10 \dots \dots 0]' \quad , \end{aligned} \quad (3.21)$$

where the +1 is for factor  $i$  at level  $j$  and the -1 is for factor  $i$  at level  $j'$ . Since

$$\begin{aligned} n(i_j, i_{j'}) &= \text{zero for } j \neq j', \quad n(i_j, i_j) - n(i_j, i_{j'}) \\ &= n(i_j) = N/s_i, \quad \text{and} \quad n(i_j, i_{j'}) - n(i_{j'}, i_j) \\ &= -n(i_{j'}, i_j) = -N/s_i \quad , \end{aligned}$$

the equalities in the rows of factor  $i$  are satisfied. The remaining equalities imply  $n(i'_j, i_j) = n(i'_j, i_{j*})$ . Then,

$$\begin{aligned} n(i'_j, i_j) &= \frac{1}{s_i} \sum_{j^*} n(i'_j, i_{j^*}) = \frac{1}{s_i} n(i'_j, ) \\ &= N/s_i s_i, i \neq i' \text{ and all } jj' . \end{aligned} \quad (3.22)$$

This completes the proof of the necessary condition.  $\square$

To prove sufficiency, if the conditions for equations (3.20) to (3.22) are satisfied, then  $S^{-\frac{1}{2}} \mathbf{1}_{k \times 1}$  is an eigenvector of  $\tilde{X}'\tilde{X}$  corresponding to  $\lambda = nN$ ; the  $S^{-\frac{1}{2}} e(i_j, i_j)$  span an eigenspace corresponding to  $\lambda = N$  with dimension  $k-n$ . This proves the sufficiency of the above equations.

**Example 3.3.** Proportional frequency designs and sets of orthogonal  $F$  squares as discussed by Hedayat and Seiden (1970) provide examples of orthogonal main effect plans for the asymmetric factorial. It can be shown that the structure of  $X^*X^*$  for these designs is similar to (3.12).

**Theorem 3.4.** *The class of saturated main effect designs for the  $\prod_{i=1}^n s_i$  factorial contains designs for which  $|X^*| = 1$ . That is the minimum possible nonzero value is always attainable.*

**Proof.** The familiar "one at a time design" (see Anderson and Federer, 1975) has a  $(0,1)$  representation as

$$X^* = \begin{bmatrix} 1 & 00 \cdots 0 \\ 1 & I_{v-1} \end{bmatrix}, \quad v-1 = \sum_{i=1}^n (s_i - 1),$$

whose determinant is clearly one. The proof is complete since one design is exhibited for every case.  $\square$

**Corollary 3.3.** *If  $T$  is a saturated main effect plan for the  $\prod_{i=1}^n s_i$  factorial, the minimum possible value of  $|X'X|$  is  $\prod_{i=1}^n (s_i!)^2$  and this value is always attainable. Thus for any saturated design the  $|X'X|$  is a multiple of this minimum value.*

**Proof.** This follows directly from Theorems 2.2 and 3.4.  $\square$

Thus for saturated main effect plans, the smallest value of the determinant of  $X$ , or  $X^*$ , can always be attained. Plans  $T$  achieving this value are denoted as  $D$ -minimal designs. The upper bound on the determinant of  $X$ , or  $X^*$ , will be attained whenever an orthogonal saturated main effect design with equal numbers of repetitions on the levels of each factor is obtained. In the  $3^n$  series, for example, this will occur with  $n=4$  and  $N=9$  yielding  $|X^*| = 3^3$ ; the next orthogonal saturated main effect plan occurs for  $n=13$  and  $N=27$  yielding  $|X^*| = 3^{21}$ . In cases where an orthogonal design does not exist, the upper bound will not be attained.

It is noted that the spectrum of values for  $\|X_{(n+1) \times (n+1)}^*\|$  contains all values attainable for  $\|X_{n \times n}^*\|$ . This is easily demonstrated by constructing an  $X_{(n+1) \times (n+1)}^* = \begin{pmatrix} \vdots & 0'_{1 \times n} \\ 1_{(n+1) \times 1} & X_{n \times n}^* \end{pmatrix}$ , the determinant (absolute value) of which has the same spectrum of values as  $\|X_{n \times n}^*\|$ . This means that a lower bound on the upper bound of  $\|X_{(n+1) \times (n+1)}^*\|$  can be obtained from the maximum value of  $\|X_{n \times n}^*\|$ . In most cases, this latter value is unknown. For such situations, use the maximum value of  $\|X_{p \times p}^*\|$ ,  $p \leq n$ , for the  $p$  for which it is known, e.g., for an orthogonal array. Then, any design  $T$  which does not have an  $\|X_{(n+1) \times (n+1)}^*\|$  between the lower bound on the upper bound and the upper bound is not a good design with respect to  $D$ -optimality. A design  $T$  with an  $X_{(n+1) \times (n+1)}^* = \begin{pmatrix} \vdots & I_{n-p} & \vdots & 0 \\ 1_{(n+1) \times 1} & \vdots & \vdots & \vdots \\ \vdots & 0 & X_{p \times p}^* & \vdots \end{pmatrix}$  will achieve the lower bound on the upper bound.

#### 4. On the construction of main effect plans

The construction of main effect plans for the symmetric and asymmetric factorial is now directly related to constructions of (0,1) matrices with certain constraints on the columns. Thus the body of knowledge and developed theory of (0,1) matrices can be directly brought to the construction of main effect plans. In this section we illustrate this with a few types of constructions. Recall from (2.1) that for a factor at  $s$  levels there must be a corresponding set of  $(s-1)$  columns in  $X^*$  with all pairwise inner products zero and among these columns at least one row must be all zeros.

**Example 4.1.** Circulant Matrix Construct. Let  $C_{2n}$  be a  $2n \times 2n$  circulant matrix whose first row contains ones and zeros such that the  $i$ th and  $(n+i)$ th coordinates are not both one. The remaining rows of  $C$  are of course just cyclic permutations of the first row. Let  $X^*$  be

$$X^* = \begin{bmatrix} 1 & \mathbf{0}_{1 \times 2n} \\ \mathbf{1}_{2n \times 1} & C_{2n \times 2n} \end{bmatrix}.$$

This  $X^*$  matrix is appropriate for a  $3^n$  saturated main effect plan, and since the theory of circulants is well known the determinant is easy to evaluate. To illustrate, we list the first row of a suitable  $C$  matrix for the  $3^n$  factorial with  $n=3,4,5,6$ , and 7 and give the corresponding determinant of  $X^*$ .

| $n$ | First row of $C$        | $N$ | Det. of $X^*$ | Upper bound |
|-----|-------------------------|-----|---------------|-------------|
| 3   | (1 0 1 0 0 0)           | 7   | 4             | 6           |
| 4   | (1 1 0 1 0 0 0 0)       | 9   | 27            | 27          |
| 5   | (1 1 1 0 1 0 ... 0)     | 11  | 88            | 141         |
| 6   | (1 1 1 0 1 0 ... 0)     | 13  | 208           | 884         |
| 7   | (1 1 0 1 1 0 1 0 ... 0) | 15  | 420           | 6470        |

A similar construction for the  $3^n$  factorial would require a  $(s-1)n \times (s-1)n$  circulant matrix with at most one 1 in the  $i, n+i, 2n+i, \dots, (s-1)n+i$  columns  $i=1, 2, \dots, n$ .

**Example 4.2.** Sum Composition - Let  $T_1, T_2, \dots, T_{s-1}$  be  $n \times n$  matrices of ones and zeros, and let

$$X^* = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ 1 & T_1 & 0 & \dots & 0 \\ 1 & 0 & T_2 & \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & T_{s-1} \end{bmatrix}.$$

Clearly  $|X^*| = \prod_{i=1}^{s-1} |T_i|$ , and the corresponding design is

$$T = \begin{bmatrix} \mathbf{0} & 1 \times n \\ T_1 \\ 2T_2 \\ 3T_3 \\ \vdots \\ (s-1)T_{s-1} \end{bmatrix}.$$

Anderson and Federer (1974) considered possible values for the determinant of  $(0,1)$ -matrices and used ten methods of construction to obtain many of the possible values. Here we present all possible determinant values attainable by the above method of construction for saturated main effect plans from the  $3^n$  series for  $n=3, 4, 5, 6$ , and  $7$ .

$$n = 3: 6^3 [0,1,2,4]$$

$$n = 4: 6^4 [0,1,2,3,6,9]$$

$$n = 5: 6^5 [0,1,2,3,4,5,6,8,9,10,12,15,20,25] = 6^5 [0,1,2,3,4,5]^2$$

$$n = 6: 6^6 [0,1,2,\dots,9]^2 = 6^6 [\text{all possible products of integers } 0,1,\dots,9]$$

$$n = 7: 6^7 [\text{all integers } \leq 18,20,24,32]^2$$

where the integers within a square bracket represent possible values for the determinant of  $X^*$ .

It should be noted that this construction is restrictive and does not provide all possible values of  $|X|$ . For example, for  $n=3$ , and for another construction, it is possible to obtain a design for which  $X=6^3(3)$  and which is not obtained via the above construction. Even though this method of construction gave the largest value obtained for  $n=3$ , it is expected that this will hold for larger  $n$ . When  $n=4$ , the orthogonal saturated design in Example 2.1 yields a design for which  $|X|=6^4(27)$ , which is three times larger than the largest value obtained from this sum composition. The spectrum of possible values or even the largest possible value of  $|X|$  is unknown at present. The transformation of  $X$  to  $X^*$ , i.e., a  $(0,1)$ -matrix, is considered to be one step toward the resolution of these problems.

**Example 4.3.** The construction of Example 4.2 can be extended to the general main effect plans. Let  $T_1, T_2, \dots, T_{s-1}$  be  $(0,1)$ -matrices of order  $N_1 \times n, N_2 \times n, \dots, N_{s-1} \times n$ , respectively, for  $N_i \geq n$ , such that  $[0 \ T_1']$  could be regarded as a main effect plan for the  $2^n$  factorial with  $N_i+1$  runs. Now, consider the following design for the  $s_n$  factorial with  $N = 1 + \sum_{i=1}^{s-1} N_i$  runs:

$$T = [0 \ T_1' \ 2T_2' \ \dots \ (s-1)T_{s-1}']' .$$

Then,

$$X^* = \begin{bmatrix} 1 & 0' & 0' & \cdots & 0' \\ 1 & T_1 & 0 & & 0 \\ 1 & 0 & T_2 & & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \cdots & T_{s-1} \end{bmatrix},$$

and

$$X^{*'}X^* = \begin{bmatrix} N & 1'T_1 & 1'T_2 & \cdots & 1'T_{s-1} \\ T_1'1 & T_1'T_1 & 0 & & 0 \\ T_2'1 & 0 & T_2'T_2 & & 0 \\ \vdots & & & & \vdots \\ T_{s-1}'1 & 0 & 0 & \cdots & T_{s-1}'T_{s-1} \end{bmatrix}.$$

Given the  $T_i$ ,  $i=1,2,\dots,s-1$ , it is a relatively simple matter to compute  $|X^{*'}X^*|$ .

To conclude, we suggest one additional construction for main effect plans from the  $s^n$  factorial. This method makes use of a  $(0,1)$ -matrix  $T$  and its complement  $(J-T)$  and by arranging these matrices to satisfy constraints (2.1) and (2.2). We illustrate the procedure for the  $3^n$ , the  $4^n$  series, and then for the  $s^n$  series.

**Example 4.4.** Let  $T$  be an  $N \times n$   $(0,1)$ -matrix of full rank with  $N \geq n$ . For the  $3^n$  series, consider the plan defined by

$$T_1 = \begin{bmatrix} T \\ J-T \\ 0 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 \\ T \\ J-T \end{bmatrix}$$

with  $3N$  runs. Each of the three levels of each factor occurs  $N$  times. For this design,

$$X^*X^* = \begin{bmatrix} 3N & N1' & N1' \\ N1 & T'T+(J-T)'(J-T) & (J-T)'T \\ N1 & T'(J-T) & T'T+(J-T)'(J-T) \end{bmatrix}.$$

If T itself is a structured matrix, then  $X^*X^*$  has a simple structure. For example, if  $T=I_n$ , then

$$X^*X^* = \begin{bmatrix} 3n & n1' & n1' \\ n1 & nI+(n-2)(J-I) & (J-I) \\ n1 & (J-I) & nI+(n-s)(J-I) \end{bmatrix}$$

and

$$|X^*X^*| = 3^{n-1}n(n^2-3n+3)^2.$$

For the  $4^n$  series, the construction is given by

$$T_1 = \begin{bmatrix} T \\ J-T \\ 0 \\ 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 \\ T \\ J-T \\ 0 \end{bmatrix}, \quad \text{and} \quad T_3 = \begin{bmatrix} 0 \\ 0 \\ T \\ J-T \end{bmatrix}.$$

In general, for the  $s^n$  factorial we let  $T_i$ ,  $i=1,2,\dots,s-1$ , be  $sN \times n$  matrices whose  $i$ th and  $(i+1)$ st blocks are  $T$  and  $J-T$ , respectively, with the remaining blocks composed of zero matrices. For this construction, we have:

$$X^*X^* = \begin{bmatrix} N1 & N1' & N1' & N'1 & \dots & N1' \\ N1 & A & B & 0 & & 0 \\ N1 & B & A & B & & 0 \\ N1 & 0 & B & A & & 0 \\ \vdots & & & & & \vdots \\ N1 & 0 & 0 & 0 & \dots & A \end{bmatrix},$$

where  $A = T'T + (J-T)'(J-T)$  and  $B = (J-T)'T$ .

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