SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
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TECHNICAL REPORT NO. 695

April 1986

A MODEL FOR THE FAILURE PROCESS OF SEMI-CRYSTALLINE POLYMER MATERIALS UNDER STATIC FATIGUE*

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AMS Classification: 73K20, 60K10, 73M10

Kew Words: Composite material, Polymers, Chain-of-bundles, Static Fatigue, Stress rupture, Reliability, Load sharing, Weibull distribution.

Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant Number AFOSR-84-0165.

ABSTRACT

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UNDER

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A semi-crystalline polymer fiber is a composite material consisting of difficult to deform crystals joined by more easily deformed and more easily broken amorphous materials. The failure process begins at the atomic level in the amorphous regions where random thermal fluctuations cause, at some time, a molecule to slip relative to other molecules or to rupture at one of its atomic bonds. The frequency of such random events in greatly enhanced by small increases in stress. As molecules slip of rupture, neighboring molecules become overloaded, thus increasing their failure rates. Such molecule failures accumulate locally and give rise to growing microcracks, although the exact kinetic mechanisms are not well understood. These growing minute cracks are the irreversible changes in the microstructure of the material that ultimately lead to macroscopic failure of the fiber

We model this material as being partitioned into a weakest link series of independent short sections, called bundles, along the axis of the tensile force. Each bundle has discrete elements arranged in a geometrical network. The tensile stress on the system causes initial element failures to begin to occur at

random throughout the bundle. When an element fails, its load is shifted to neighboring nonfailed elements, increasing their failure rates. Soon some neighbors to the initial failures begin to fail, forming microcracks of two adjacent failed elements. Elements adjacent to some of the microcracks begin to fail, and the microcracks grow stochastically. Ultimately all elements in the bundle fail.

The simple model that we propose has two phases. Cracks are initiated throughout the bundle in accordance with a Poisson process whose intensity is jointly proportional to the number of elements in the bundle and the initial failure rate of a single element. Each crack initiation begins a crack growth process, a pure birth Markov process whose birth rates are jointly proportional to the number of most highly stressed elements adjacent to a crack, and the failure rate of each of these elements. These cracks are assumed to grow independently of one another. They grow to infinite size in finite time. The bundle fails at the earliest of the crack initiation plus crack growth to infinite size times.

The model is compared numerically with a classic model that allows smaller cracks to join and form larger ones. The excellent agreement tells us something about how such materials behave: Initial failures occur throughout the material. While these initial failures may be large in absolute number, they are small relative to the material size. The initial failures become growing microcracks, but the material effectively fails before the microcracks form any significant volume of material.

Additional mathematical aspects of the model are pursued. An explicit lower tail formula for the distribution function of failure time is given. The concept of a critical crack size is explored, and the distribution function for the failure time when there are failures present in the material at time zero is derived. A variety of load sharing assumptions are examined.

A MODEL FOR THE FAILURE PROCESS OF SEMI-CRYSTALLINE POLYMER MATERIALS UNDER STATIC FATIGUE

1. Introduction and summary. If a Kevlar fiber is suspended under a constant weight, it will eventually fail, an example of what is commonly called **static fatigue** or **stress rupture.** The stress induced in the fiber by the suspended weight causes irreversible changes in the microstructure of the fiber that ultimately lead to macroscopic fracture.

A single semi-crystalline polymer fiber is itself a composite of crystal and amorphous materials [Young (1983), p. 279]. The typical polymer molecule might have a length to diameter ratio of 10,000 to 1 [ibid., p. 151]. With this length of molecule, it is impossible for the amount of organization required to form a 100 percent crystalline polymer to take place during crystallization from a solution or melt. A semi-crystalline polymer is comprised of, typically, 40%-80% of material in the crystalline state, and the remainder, in a glassy or amorphous state. Individual molecules have the probability of being incorporated into more than one crystal, giving rise to intercrystalline links. The Newel-Rosenstock model [Gotlib, et al (1973)] views the polymer as a three dimensional network of sites representing the difficult-to-deform crystalline parts of the polymer, which are joined by elastic bonds representing the

more easily deformable and more easily broken amorphous material [Dobrodumov and El'yashevich (1973)]. A tensile force is applied in one direction, and the network is viewed as a series system of planar subsystems.

The failure process begins at the atomic level in the amorphous regions, where the molecules undergo a process of random thermal vibrations over time. The random fluctuations will cause, at some time, sufficient thermal energy to exist in a molecule to overcome certain local energy barriers, and the molecule may slip relative to other molecules or rupture at one of its atomic bonds. The frequency of such random events is greatly enhanced by small increases in temperature or stress, or by the presence of certain chemical agents (such as moisture on glass fibers) [Phoenix and Tierney (1983)].

As molecules slip or rupture, neighboring molecules become overloaded, thus increasing their failure rates. Such molecule failures accumulate locally and give rise to growing microcracks, although the exact kinetic mechanisms are not well understood [Phoenix and Tierney (1983)]. These growing minute cracks are the irreversible changes in the microstructure of the material that ultimately lead to macroscopic failure of the fiber.

2. The chain-of-bundles model. We model the material as being partitioned into a series of m short sections, called **bundles**, along the axis of the tensile force, as depicted in Figure 1. The bundles are assumed to be statistically independent, and the

series arrangement implies a weakest link formulation where-in the material or system fails when the first bundle fails. Since the number of bundles in the material is on the order of Avagadro's number, we are interested in extremely small bundle failure probabilities.

Let n be the number of elements in a bundle and let m be the number of bundles in the system. Let $H_{m,n}(t)$ be the cumulative distribution function for the random time of system failure, and let $G_n(t)$ be the cumulative distribution function for the time of bundle failure. Because the bundles are assumed to be statistically independent and identical, and in a weakest link series arrangement, the system survival function $1 - H_{m,n}(t)$ is related to the bundle survival function $1 - G_n(t)$ by

$$1 - H_{m,n}(t) = [1 - G_n(t)]^m. \tag{2.1}$$

Each bundle has discrete elements arranged in a geometrical network. The tensile force on the system causes initial element failures to begin to occur at random throughout the bundle. When an element fails, its load is shifted to neighboring nonfailed elements in the network, and these overloads on nonfailed neighbors increase their failure rates. Soon some neighbors to the initial failures begin to fail, forming microcracks consisting of two adjacent failed elements, and further increasing the stress on elements near the microcracks. Elements adjacent to

some of the microcracks fail, and the microcracks grow stochastically in time. Ultimately all elements in the bundle fail.

We call this the **chain-of-bundles** model. It is a series arrangement of bundles, where each bundle has elements in parallel. Two criteria have been studied in the past, the static strength model in which each element has a random strength and the focus is on determining the probability distribution of the tensile strength of the material, and the static fatigue model where one wants to determine the probability distribution of the time of material failure under constant load. In this paper, we consider only the time to failure problem.

3. Element failure behavior. We make assumptions about the failure behavior of the individual elements, and about how the failure of some elements affects the failure rates of others, and then try to deduce the behavior of the material as a whole. In this paper we allow an element to exist in only two states, failed or nonfailed. (In the study of certain types of creep behavior a larger state space is required. Each "element" in such a model is itself a system of more basic particles that describe the slippage of molecules and the breaking of bonds between them.) Each element begins in the nonfailed state and then fails at some random time. The hazard or failure rate of the element is determined by the time history of load that that element supported. Let ζ be the failure time of a single element that carries the stress or load $\sigma(t)$ up to the time of

its failure. In this paper we assume that the instantaneous failure rate of an element depends only on its current load and not on its previous load history in the sense that

$$P(t < \zeta < t+dt \quad t < \zeta, \sigma(u) \text{ for } 0 \leq u \leq t) = K(\sigma(t))dt, \qquad (3.1)$$

where $K(\sigma)$ is an increasing convex function of the load σ, called the **breakdown rule.** With this assumption, a fiber that is given a constant load of σ will have a failure time that is exponentially distributed with mean $1/K(\sigma)$. The survival probability for an element subject to the time varying load $\sigma(u)$ for $u > 0$ is

$$P(\zeta > t) = \exp\{-\int_0^t K(\sigma(u))du\}. \qquad (3.2)$$

Two cases are of special interest, **power law breakdown** in which

$$K(\sigma) = \alpha\sigma^{\rho}, \quad \text{for } \sigma > 0, \qquad (3.3)$$

and **exponential law breakdown** in which

$$K(\sigma) = \alpha \exp\{\beta\sigma\}, \quad \text{for } \sigma > 0.$$

Under power law breakdown, the log mean time to element failure will plot linearly with log load, while under exponential breakdown, the log mean time to element failure plots linearly

with load. The exponential breakdown law is attributed to Zhurkov (1965) in the form

$$K(\sigma) = \tau_0^{-1} \exp\{-U_0/kT\} \exp\{\gamma\sigma/kT\}, \qquad (3.4)$$

where τ_0 is a constant related to the period of bond vibration, U_0 is the activation energy of the fracture process in the absence of load, k is Boltzmann's constant, T is absolute temperature, and γ is called the activation volume. Phoenix (1983) summarizes the arguments leading to (3.4), and points out some serious drawbacks to the exponential breakdown rule. He also derives the power law breakdown rule from basic assumptions in a manner analogous to that of Zhurkov, and he lists some of its advantages. As an example, under the power law breakdown rule the time to failure of a system under the nominal load per element L has the same distribution as $L^{-\rho}$ times the time to failure of the system under unit load [Tierney (1984)]. This is readily verified from (3.2) and (3.3) for a single element under a load that is constant in time. For this reason, much of the derivation under power law breakdown can be carried out assuming $L = 1$. With properly chosen parameters, both rules give qualitatively similar results. Where power law breakdown is assumed for element behavior, the same behavior will be maintained at the bundle and system level. This is not the case with exponential breakdown. Because of its advantages, we will emphasize power law breakdown in this article.

4. Load sharing among elements. Having described the single element behavior under load, we turn to describing the sharing of bundle load among the nonfailed elements. The **load sharing rule** prescribes the load carried by each nonfailed element in the bundle for each possible configuration of failed and nonfailed elements. If there is to be any hope of analyzing the model, the load sharing rule must be history independent. That is, the load carried by an element should depend only on which elements have failed and not on the sequence in which those failures occurred. A load sharing rule should also be **monotone** in that the load carried by any nonfailed element should never decrease upon the failure of an additional element. There are real systems which do not have monotone load sharing, for instance, where one element carries load from one part of the system to another so that upon its failure the load is no longer transferred and is correspondingly reduced in part of the system. For such systems the difficulty of the analysis is greatly increased. Fortunately, monotone load sharing is a reasonable assumption for the materials that we model.

A number of approaches have been taken to determine the load supported by a nonfailed element in studies of similar network configurations of failed and operational elements. Early work on the chain-of-bundles model assumed equal load sharing among the nonfailed elements in each bundle [Daniels (1945), Borges (1983), Gucer and Gurland (1962), Rosen (1964), and Smith (1982)]. Coleman (1957) introduced unequal load sharing in what he called

a **tight bundle**, but only rigorously developed results under equal load sharing. A more realistic assumption is that the stress concentrations are confined in the near vicinity of failed elements. In the study of fiber-matrix composites, a number of authors tried to characterize such stress concentrations [Argon (1972, 1974), Rosen (1964), Scop and Argon (1969), Zweben (1968) and Zweben and Rosen (1970)], but lacked rigor because their models were not completely or precisely defined. As a result, it is impossible to assess the correctness of their approximations. It should be emphasized that if the model is to be complete, then the model must specify the load carried by every nonfailed element in every possible configuration of failed and nonfailed elements. In a simulation study of a chain-of-bundles type model for polymer materials, Dobrodumov and El'yashevich (1973) attempted to do this by recalculating, after each simulated node failure, the stresses at every node of a network "spring" model attributed to Newel and Rosenstock (1953) . A similar approach was taken by Termonia, Meakin and Smith (1985).

To properly describe the failure process, the load sharing rule must focus stress concentrations in the near vicinity of failed elements. On the other hand, if one is to hope for success in an analytical study of the model, it appears necessary to consider only load sharing rules in which the microcrack is constrained to grow linearly. This simplification was made by Gotlib, Dobrodumov, El'yashevich and Swetlov (1973) in their analytical study of the simulation model of Dobrodumov and

El'yashevich (1973). In any local configuration of failed and nonfailed elements, they allowed only the most stressed element to fail. A different approach was taken by Harlow (1977) and subsequently developed by Harlow and Phoenix (1978a, 1978b, 1979, 1981a, 1981b, 1982). Instead of trying to calculate the stress concentrations exactly, they worked with an idealized model, called **local load sharing**, which captures the essential features of failure, but is simple enough to allow exact calculations. Harlow postulated that the elements are arranged in a circle so that each element has precisely two neighbors. The load on any nonfailed element depends on how many failed elements are adjacent to it. Specifically, if the nominal load per element is σ, then the load on a nonfailed element that is adjacent to j failed elements is $\varphi_j\sigma$ where φ_j, the **stress concentration factor**, is given by

$$\varphi_j = \begin{cases} 1 + j/2, & \text{for } 0 \le j \le n-1, \\ n, & \text{for } j = n-1, \end{cases} \tag{4.1}$$

where n is the number of elements in the bundle. Local load sharing concentrates all of a failed element's load onto the two nearest nonfailed neighbors in a circular configuration. This stress concentration is more intense than that yielded by more realistic approaches. Therefore, one would conjecture that local load sharing would be a conservative assumption in that it would

predict times that are earlier than would actually occur. Unfortunately, a proof of this conjecture has not yet been found.

It may be possible to more closely simulate more diffuse and realistic load sharing rules and, at the same time, retain the analytical tractibility of local load sharing, by introducing a constant δ, with $0 < \delta \leq 1$, and setting $K_r = 1 + \delta r/2$ for $r = 0, 1, \ldots, n-1$. Observe that this modification does not conserve load in that the sum of the loads on all nonfailed system elements does not equal the original bundle load. Tierney (1984) discusses this modification and Pitt and Phoenix (1982) have done some calculations in a static strength model that support this approximation.

Gotlib, Dobrodumov, El'yashevich and Svetlov (1973) study three distinct load sharing setups. Their model is incomplete in the sense that they do not prescribe the load on every nonfailed element in every configuration of failed and nonfailed elements in the bundle. Instead, they assume that the stress concentration factors in the vicinity of a microcrack depend only on the size of the microcrack and not on its individual configuration. They consider only single microcracks, and do not compute stress concentration factors for two microcracks near each other. Considering a microcrack of arbitrary shape in a general planar bundle, they assume that the most highly stressed nonfailed element adjacent to the microcrack is the one to fail next, and thus they need to determine the stress concentration factor for this element alone. Their rate for the growth of a

microcrack of size j into one of size $j+1$ depends on both the number n_j of adjacent elements carrying the maximal stress on the boundary of the microcrack, and on the corresponding stress concentration factor φ_j. The first case, based on Gotlib, El'yashevich and Svetlov (1973), is determined from a linear microcrack growing in a planar lattice (and then assumed to hold for microcracks of all shapes). The maximum stress occurs at each end of the microcrack, so that $n_j = 2$, and the stress concentration factors are

$$\varphi_j = c(1 + j)^{1/2}, \quad \text{for } j = 1,2,... \tag{4.2}$$

The constant c depends on the elastic moduli assumed for the system, and is an approximation to a nearly constant function of j.

The second case is derived assuming a symmetric microcrack growing in a planar lattice. The symmetry implies that all nonfailed elements bounding the microcrack carry equal stress, and they estimate the number of elements adjacent to a microcrack of area j by $n_j = 4(1 + j)^{1/2}$. They use the stress concentration factor

$$\varphi_j = c(1 + j)^{1/4}, \text{ for } j = 1,2,... . \tag{4.3}$$

Finally they introduce a third and weaker form of load sharing in which

$$n_j = 4(1 + j)^{1/2}, \quad \text{and} \quad \varphi_j = c(1 + j)^{1/8}, \quad j = 1,2,\ldots . \quad (4.4)$$

5. The simple model. Here we set forth a simple model which assumes that microcracks grow independently of one another, two smaller microcracks never coalescing to form a single larger one. We assume that isolated initial failures appear throughout the bundle according to a Poisson process whose rate $\nu = nK(L)$ is jointly proportional to the bundle size and the breakdown rate of a single element under the nominal load per element L. From each such initial failure, a microcrack grows according to a pure birth process in which the rate at which a microcrack of size j extends to size $j+1$ is the birth parameter

$$\lambda_j = n_j K(\varphi_j L), \text{ for } j = 1,2,\ldots . \quad (5.1)$$

Figure 2 illustrates the crack initiation-crack growth process. Let $X(t)$ be the random process describing the size of a single microcrack as a function of time t as shown in Figure 3. The pure birth parameters given in (5.1) are interpreted as the infinitesimal transition rates of the process $X(t)$ according to

$$P(X(t + \delta t) = j + 1 \quad X(t) = j) = \lambda_j(\delta t) + o(\delta t) \quad (5.2)$$

where $o(\delta t)$ is a remainder term of order less than δt as $\delta t \to 0$ [See Taylor and Karlin (1984), p. 211].

The pure birth process $X(t)$ begins in state 1 and sojourns there for a random time S_1 which is exponentially distributed with parameter λ_1, and then moves to state 2, where it sojourns for a random time S_2 which is exponentially distributed with parameter λ_2, and so on [ibid, p. 213]. The sum $Y = S_1 + S_2 + \ldots$ is the time it takes the initial failure to grow to a crack of infinite size, called the **explosion time** of the birth process. Let $T_1, T_2, \ldots$ denote the successive times of appearance of the initial failures in the bundle, and let $Y_1, Y_2, \ldots$ denote the respective explosion times of the birth processes that these failures initiate. The bundle failure time Z is the smallest of $T_1 + Y_1, T_2 + Y_2, \ldots$. It is not necessarily the case that the first crack to begin will cause bundle failure since a later crack may grow faster. If we plot the successive (T_k, Y_k) pairs in the (t,y) plane, as in Figure 2, the points form a spatial Poisson point process whose intensity is $g(t,y) = \nu f(y)$, where $f(y)$ is the common probability density function of the explosion times [ibid, pp. 205-207]. The event $\{Z > z\}$ is the event $\{\min\{T_k + Y_k\} > z\}$ and it corresponds in the (t,y) plane to the event that no points fall in the triangle $\Lambda = \{(t,y): t + y \leq z, t \geq 0, y \geq 0\}$. The number of points $N(\Lambda)$ falling in the triangle Λ has a Poisson distribution with mean $\mu(\Lambda)$ given by

$$\mu(\Lambda) = \int\int_{\Lambda} v \, dt \, f(y)dy = nK(L) \int_0^z F(y)dy, \tag{5.3}$$

where $F(y)$ is the cumulative distribution function of the explosion time. From this we easily obtain the the system survival time distribution:

$$1 - G_n(z) = 1 - P(Z > z) = P(N(\Lambda) = 0) \tag{5.4}$$

$$= \exp\{-nK(L)\int_0^z F(y)dy\}.$$

Define the **characteristic distribution function** $W(z)$ by

$$1 - W(z) = \exp\{-K(L)\int_0^z F(y)dy\}. \tag{5.5}$$

Note that in this model we have the exact reverse weakest link relationship

$$1 - W(z) = [1 - G_n(z)]^{1/n} .$$

The characteristic function $W(z)$, as given in (5.5), is readily computed. As mentioned earlier, the explosion time Y is the sum of the sojourn times S_1, S_2, ... which are indepen-

dent random variables, and S_j has an exponential distribution with parameter λ_j. Let

$$\zeta_j = S_1+S_2+\ldots+S_j = \zeta_{j-1} + S_j ,$$

and

$$F_j(t) = P(\zeta_j \leq t) = P(\zeta_{j-1}+ S_j \leq t)$$

$$= \int_0^t F_{j-1}(t-u)\, \lambda_j e^{-\lambda_j u} du, \qquad (5.6)$$

where

$$F_1(t) = 1 - e^{-\lambda_1 t}, \quad \text{for} \quad t \geq 0.$$

Defining

$$W^{[k]}(z) = 1 - \exp\{-K(L)\int_0^z F_{k-1}(y)dy\}, \qquad (5.7)$$

then the time τ_k of the first appearance in the bundle of a microcrack of k adjacent failed elements has the cumulative distribution function $G^{[k]}(z) = P(\tau_k \leq z)$ given by

$$G_n^{[k]}(z) = 1 - \{1 - W^{[k]}(z)\}^n$$

$$= 1 - \exp\{-nK(L)\int_0^z F_{k-1}(y)dy\}. \qquad (5.8)$$

Letting $k\to\infty$, since

$$F_k(y) = P(S_1+\ldots+S_k \le y) \to P(S_1+S_2+\ldots \le y) = F(y),$$

it is clear that $W^{[k]}(z)$ as given in (5.7) converges to $W(z)$. Figure 4 plots $W^{[k]}(t)$ versus time t under the power law breakdown of (3.3) with $K(L) = 1$, $\rho = 10$ and for $k = 1,\ldots,9$. The coordinates are such that a Weibull distribution plots as a straight line (log log $1/[1 - W^{[k]}(t)]$ vs. log t). Note the evident convergence of $W^{[k]}(t)$ to the characteristic distribution function $W(t)$. The manner of convergence is such that in any portion of the graph, there is an integer k such that the first appearance of a microcrack of size k is tantamount to bundle failure. From the graph we can infer that

$$1 - G_n(t) = [1 - W(t)]^n \simeq [1 - W^{[9]}(t)]^n \qquad (5.9)$$

for values of $k \ge 9$ and for values of n up to about 10^{16}, and thus we can begin to predict the behavior of systems of realistic size. Using (2.1) we deduce the survival function for the system of m bundles in series to be

$$1 - H_{m,n}(t) = [1 - G_n(t)]^n \tilde{=} [1 - W^{[k]}(t)]^{mn}. \qquad (5.10)$$

Let us use (5.10) and Figure 4 to point out one of the difficulties faced in trying to understand these materials via simulation. The maximum number of elements that have been in any simulation study to date is on the order of thousands. Dobrodumov and El'yashevich (1973) used $nm = 4000$, while Termonia, Meakin and Smith (1985) used $nm = 36 \times 10^3$. On the other hand, an actual fiber would have a number of elements that is many orders of magnitude times this. For the sake of discussion, we will compare $nm = 4000$ with $nm = 10^{10}$, and look at the median time to failure under unit load $L = 1$ for both systems using Figure 2. For the median failure probability of 1/2, we have

$$\frac{1}{2} = 1 - H_{m,n}(t),$$

or

$$\log \log 2 = \log \log[\frac{1}{1 - H_{m,n}(t)}]$$

$$\tilde{=} \log mn + \log \log[\frac{1}{1 - W^{[9]}(t)}].$$

Using Figure 2, $\log \log 2 - \log 4000 = -8.66$ which corresponds to a median log time to failure of $\log t \tilde{=} -6.1$, where-as $\log \log 2 - \log 10^{10} = -23.39$ which gives a log median time to failure of $\log t \tilde{=} -10.25$. The point is that under the same

load, the simulation gives times to failure that are many orders of magnitude greater than would be observed with a larger, more realistic, number of elements. In order that a simulation study be useful, there must be some means of extrapolating the simulation results to systems of realistic size, taking into account the size effect that always exists whenever element failures exhibit random variation.

We now derive some auxiliary formulas that are useful in both computation and analysis. Firstly,

$$\begin{aligned} F_{k-1}(y) &= 1 - [1 - F_{k-1}(y)] \\ &= 1 - P(y < \zeta_{k-1}) \\ &= 1 - \Sigma_{j=1}^{k-1} P(\zeta_{j-1} \le y < \zeta_j), \text{ where } \zeta_0 = 0. \end{aligned} \tag{5.11}$$

Secondly,

$$\begin{aligned} \int_0^t P(\zeta_{j-1} \le y < \zeta_j)dy &= \int_0^t P(\zeta_{j-1} \le y < \zeta_{j-1} + S_j)dy \\ &= E[\int_0^t \mathbf{1}\{\zeta_{j-1} \le y < \zeta_{j-1} + S_j\}dy] \\ &= E[\min\{S_j, (t-\zeta_{j-1})^+\}] \\ &= \frac{1}{\lambda_j} E[1 - e^{-\lambda_j (t-\zeta_{j-1})^+}] \end{aligned}$$

$$= \frac{1}{\lambda_j} \int_0^t (1 - e^{-\lambda_j(t-z)}) f_{\zeta_{j-1}}(z)dz$$

$$= \frac{1}{\lambda_j} F_j(t). \qquad (5.12)$$

Thus we can write (5.7) in the form

$$W^{[k]}(z) = 1 - \exp\{-K(L)[z - \Sigma_{j=1}^{k-1} \frac{1}{\lambda_j} F_j(z)]\}. \qquad (5.13)$$

Then, in computing (5.7) as reformulated in (5.13), we use the explicit formula, derived inductively from (5.6),

$$F_k(y)dy = \Sigma_{j=1}^{k} A_{k,j}[1 - \exp\{-\lambda_j z\}] \qquad (5.14)$$

where

$$A_{k,j} = (\frac{\lambda_1}{\lambda_1 - \lambda_j}) \cdots (\frac{\lambda_{j-1}}{\lambda_{j-1} - \lambda_j}) (\frac{\lambda_{j+1}}{\lambda_{j+1} - \lambda_j}) \cdots (\frac{\lambda_k}{\lambda_k - \lambda_j}). \qquad (5.15)$$

6. Power law breakdown. A number of simplifications take place under power law breakdown that more clearly show the effects of

the model parameters on the system behavior. Observe that under power law breakdown the birth parameters λ_j factor in the form

$$\lambda_j = n_j K(\varphi_j L) = K(L) n_j (\varphi_j)^{\rho} = K(L)\theta_j \tag{6.1}$$

where

$$\theta_j = n_j (\varphi_j)^{\rho} \quad \text{for} \quad j = 1,2,\ldots . \tag{6.2}$$

As a consequence of this factorization, each sojourn time in a birth process at an arbitrary load L has the same probability distribution as $1/K(L)$ times the same sojourn time in a process for which $K(L) = 1$. Similarly, the explosion time $Y(L)$ at an arbitrary load L has the same probability distribution as $1/K(L)$ times the the explosion time Y when $K(L) = 1$. To exploit this factorization, let us now reserve $W(t)$ for the characteristic distribution under $K(L) = 1$, let Z_n be the bundle failure time under $K(L) = 1$, and let $Z_n(L)$ be the bundle failure time under an arbitrary load. Then

$$P(Z_n(L) \leq t) = 1 - \exp\{-nK(L)\int_0^t P(Y(L) \leq y)dy\}$$

$$= 1 - \exp\{-nK(L)\int_0^t P(Y \leq K(L)y)dy\}$$

$$= 1 - \exp\{-n\int_0^{K(L)t} P(Y \leq u)du\}$$

$$= 1 - [1 - W(K(L)t)]^n. \qquad (6.3)$$

Under power law breakdown, then, the load sharing rule affects the characteristic distribution function through the parameters $\theta_1,\theta_2,\ldots$, the system load per element L appears as a scale parameter $K(L)$, and the bundle size appears as a weakest link exponent.

Let us write out explicitly how the load sharing assumptions determine the parameters $\theta_1,\theta_2,\ldots$ of the characteristic distribution function W. Under local load sharing, we have

$$\theta_j = 2(1 + \tfrac{j}{2})^\rho$$

$$= 2^{1-\rho}(2 + j)^\rho, \text{ for } j = 1,2,\ldots . \qquad (6.4)$$

Under the load sharing of (4.2), we have

$$\theta_j = 2c^\rho(1 + j)^{\rho/2}, \text{ for } j = 1,2,\ldots . \qquad (6.5)$$

Finally, under (4.3) and (4.4), we have, respectively,

$$\theta_j = 4c^\rho(1 + j)^{[(1/2)+(\rho/4)]}, \text{ for } j = 1,2,\ldots . \qquad (6.6)$$

and,

$$\theta_j = 4c^{\rho}(1 + j)^{[(1/2)+(\rho/8)]}, \text{ for } j = 1,2,\ldots . \quad (6.7)$$

Using (5.6), it is easily worked out by induction that

$$\int_0^t F_{j-1}(y)dy = \frac{\theta_1\theta_2\cdots\theta_{j-1}}{j!}\, t^j + o(t^j). \quad (6.8)$$

Also, for any $k = 1,2,\ldots,$

$$\begin{aligned} F(y) = P(Y \le y) &= F_{k-1}(y) - [F_{k-1}(y) - F(y)] \\ &= F_{k-1}(y) - [P(\zeta_{k-1} \le y) - P(Y \le y)] \quad (6.9) \\ &= F_{k-1}(y) - P(\zeta_{k-1} \le y < Y) \end{aligned}$$

and

$$P(\zeta_{k-1} \le y < Y) = \Sigma_{j=k}^{\infty} P(\zeta_{j-1} \le y < \zeta_j). \quad (6.10)$$

Putting (5.12), (6.9) and (6.10) together, for any value $k = 1,2,\ldots,$ we obtain

$$\int_0^z F(y)dy = \int_0^z F_{k-1}(y)dy - \Sigma_{j=k}^{\infty} \frac{1}{\theta_j} F_j(z). \quad (6.11)$$

We will use Equations (6.8)-(6.11) to derive an asymptotic result that describes the characteristic distribution $W(t)$ and tells us something about the failure process. This result is motivated by an equivalent result in a far more complex model that is due to Tierney (1980). Define $r(1) = 0$, and for $j = 1,2,\ldots,$ let

$$r(j) = \frac{1}{\rho} \log \frac{(\theta_{j-1})^j}{\theta_1 \cdots \theta_{j-1}} , \tag{6.12}$$

where θ_j is any of (6.4) to (6.7). Observe that $r(j)$ increases with j, and that while $r(j)$ varies with ρ, it is asymptotically constant as $\rho \to \infty$ when θ_j is given by one of (6.4) to (6.7). Let us consider letting $n \to \infty$ and $\rho \to \infty$, with ρ being proportional to $\log n$, say $C = (\log n)/\rho$, and pick out the unique value k for which

$$\lim_{\rho \to \infty} r(k) \leq \frac{\log n}{\rho} = C < \lim_{\rho \to \infty} r(k+1). \tag{6.13}$$

Fixing this value for k, define

$$a_n = \left[\frac{k!}{n\theta_1\theta_2 \cdots \theta_{k-1}}\right]^{1/k}. \tag{6.14}$$

Theorem 1. With $K(L) = 1$, let $n \to \infty$ and $\rho \to \infty$, with $C = \rho/\log n$ satisfying (6.13). Under these circumstances

$$\lim_{n\to\infty} P(Z_n \le a_n t) = 1 - \exp\{-t^k\}. \tag{6.15}$$

Proof. In view of (5.4) with $K(L) = 1$, it suffices to show that

$$n \int_0^{a_n t} F(y)dy \to t^k, \tag{6.16}$$

and using (6.11), we need to show that both

$$n \int_0^{a_n t} F_{k-1}(y)dy \to t^k, \tag{6.17}$$

and

$$\frac{n}{\theta_j} F_j(a_n t) \to 0 \quad \text{for} \quad j \ge k. \tag{6.18}$$

To establish (6.17), observe that $a_n t \to 0$, and use (6.8) and (6.14) to get

$$n \int_0^{a_n t} F_{k-1}(y)dy = n \frac{\theta_1 \cdots \theta_{k-1}}{k!} (a_n t)^k + o((a_n t)^k)$$

$$= t^k + o((a_n t)^k) \to t^k.$$

To obtain (6.18), because $F_j(t) \ge F_{j+1}(t)$ and $\theta_j \le \theta_{j+1}$, we need only consider the case $j = k$. Then

$$-\frac{n}{\theta_k} F_k(a_n t) = n \int_0^{a_n t} F_{k-1}(y)\, e^{-\theta_k(a_n t - y)}\, dy$$

$$= n \frac{\theta_1 \cdots \theta_{k-1}}{(k-1)!} \int_0^{a_n t} y^{k-1} e^{-\theta_k(a_n t - y)}\, dy + o(a_n t)$$

$$= \frac{k}{(a_n)^k} \int_0^1 (a_n t z)^{k-1} e^{-\theta_k a_n t(1-z)}\, a_n t\, dz + o(a_n t)$$

$$= k t^k \int_0^1 z^{k-1} e^{-\theta_k a_n t(1-z)}\, dz.$$

We see now that (6.18) will hold provided that $\theta_k a_n \to \infty$ as $n \to \infty$. But $\log n$ is proportional to ρ, with proportionality constant C, and by (6.13), $C < r(k+1)$, which implies that

$$n = e^{C\rho} < e^{r(k+1)\rho}$$

and

$$1 = \frac{1}{n} e^{C\rho} < \frac{1}{n} e^{r(k+1)\rho} = e^{[r(k+1) - C]\rho} \to \infty,$$

so that by (6.12) and (6.14),

$$\frac{1}{n} e^{r(k+1)\rho} = \frac{(\theta_k)^k}{n\, \theta_1 \cdots \theta_{k-1}} = \frac{(\theta_k a_n)^k}{k!}$$

whence $\theta_k a_n \to \infty$ as it was to be shown. ■

In practical terms, the result asserts that, when n is large, and ρ is moderate, and k is chosen so that (6.13) is satisfied, then we have the approximation

$$G_n(z) \tilde{=} 1 - \exp\{-(z/a_n)^k\}. \tag{6.19}$$

The integer k is termed the **critical crack size.** Under the asymptotic conditions of the theorem, the appearance of a microcrack comprised of k adjacent failed elements is tantamount to bundle failure. This leads to (6.19) which asserts that the system failure time has a distribution which is approximately Weibull with shape parameter k.

From (6.19) we see that when k is the critical crack size, then

$$a_n = a_{n,k} = \left[\frac{k!}{n\theta_1 \cdots \theta_{k-1}}\right]^{1/k} \tag{6.20}$$

is approximately the median in that $G_n(z)$ is close to one for $z > a_{n,k}$, and $G_n(z)$ is close to zero for $z < a_{n,k}$.

Figure 5 plots the approximation of (6.19) on the same graph with a plot of $W^{[9]}$ as computed earlier. We see that the straight lines given by the right side of the approximation (6.19) form an envelope about the $W^{[9]}(z) \tilde{=} W(z)$ curve. Equating $(z/a_{n,k})^k$ to $(z/a_{n,k+1})^{k+1}$ shows that the change in critical crack size occurs at $z = (k+1)/\theta_k$. That is, (6.19)

may be rewritten

$$W(z) \tilde{-} 1 - \exp\{- \frac{\theta_1 \cdots \theta_{k-1}}{k!} z^k\} \quad \text{for} \quad \frac{k+1}{\theta_k} < z < \frac{k}{\theta_{k-1}}.$$

Finally, if we let $Z_n(L)$ be the failure time of a system under the nominal load per element L, and let $Z_n = Z_n(1)$, then according to our earlier remarks,

$$\begin{aligned} P(Z_n(L) \leq t) &= P(L^{-\rho} Z_n \leq t) = P(Z_n \leq L^{\rho} t) \\ &= 1 - [1 - W(L^{\rho} t)]^n \qquad (6.21) \\ &\tilde{-} 1 - \exp\{-n \frac{\theta_1 \cdots \theta_{k-1}}{k!} L^{k\rho} t^k\} \end{aligned}$$

$$\text{for} \quad (k+1)/\theta_k < L^{\rho} t < k/\theta_{k-1}.$$

From (6.21) we deduce that the median time to failure t_m and the critical crack size k are determined by jointly solving

$$\frac{1}{2} = \exp\{-n \frac{\theta_1 \cdots \theta_{k-1}}{k!} L^{k\rho} (t_m)^k\}, \qquad (6.22)$$

or, using the same reasoning that justifies a_n as given in (6.20) as an approximate median,

$$t_m \tilde{-} L^{-\rho} [\frac{k!}{n\theta_1 \cdots \theta_{k-1}}]^{1/k}, \qquad (6.23)$$

and

$$\frac{k+1}{\theta_k} < L^{\rho} t_m < \frac{k}{\theta_{k-1}}. \tag{6.24}$$

7. A deterministic analysis. Gotlib, Dobrodumov, El'yashevich and Svetlov (1973) give a deterministic argument that leads to a critical crack size that is close to that derived in the previous section. Again we assume that $L = 1$ and let $n_j(t)$ be the number of microcracks of size j at time t. They argue, essentially, that for small values of j, the populations of j-size microcracks should be large, and thus $n_j(t)$ can be well approximated by its mean value. This leads to the differential equations

$$\frac{dn_1(t)}{dt} = n - \theta_1 n_1(t), \tag{7.1}$$

and

$$\frac{dn_{j+1}(t)}{dt} = \theta_j n_j(t) - \theta_{j+1} n_{j+1}(t), \text{ for } j = 1,2,\ldots \tag{7.2}$$

Next, they claim that $n_{j+1}(t)$ is small relative to $n_j(t)$ so that (7.1) and (7.2) are approximately

$$\frac{dn_1(t)}{dt} = n \tag{7.3}$$

and

$$\frac{dn_{j+1}(t)}{dt} = \theta_j n_j(t) \quad \text{for } j = 1,2,\ldots . \tag{7.4}$$

The solution is

$$n_1(t) = nt \tag{7.5}$$

and,

$$n_j(t) = n \frac{\theta_1\theta_2\ldots\theta_{j-1}}{j!} t^j, \quad \text{for } j = 2,3,\ldots . \tag{7.6}$$

The large population argument breaks down at a certain crack size. Consider the crack size k such that, at the time t_m that the first crack of size k appears, so that $n_k(t_m) = 1$, the rate θ_k at which this single crack grows to size $k+1$, exceeds the rate $\theta_{k-1}n_{k-1}(t_m)$ at which a second crack of size k will appear. When this happens, the single crack of size k

will race ahead and cause bundle failure. Putting these two conditions together gives

$$1 = n_k(t_m) = n \frac{\theta_1\theta_2\cdots\theta_{k-1}}{k!} (t_m)^k, \tag{7.7}$$

whose solution agrees exactly with the approximate median given in (6.23), and

$$\theta_k \geq \theta_{k-1} n_{k-1}(t_m) = n \frac{\theta_1\theta_2\cdots\theta_{k-1}}{(k-1)!} (t_m)^{k-1} \tag{7.8}$$

$$= \frac{k}{t_m}.$$

From (7.8), the critical crack size k must satisfy

$$\frac{k}{\theta_k} < t_m < \frac{k-1}{\theta_{k-1}}. \tag{7.9}$$

The criterion for the critical crack size of the deterministic analysis given in (7.9) differs by the factor $(k+1)/k$ from that derived in the stochastic analysis and given in (6.24). In the stochastic analysis, k is such that the random time τ_k at which a microcrack of size k first appears has virtually the same probability distribution as that of the first appearance of an infinite crack. That is, τ_k is virtually identical with the

time of bundle failure. In the deterministic analysis, k is such that the first microcrack to reach size k is the microcrack that grows to ultimately cause bundle failure.

8. Validation of the simple model. In the model of Section 3, it was assumed that two smaller microcracks never coalesce to form a single larger one. We will now compare the results of that model with those of a more complete model that does allow microcracks to join. If we are to allow such interactions between microcracks, then we must specify the stress concentration factors for nonfailed elements in the vicinity of two or more microcracks. To date, the only analytically tractable load sharing rule that completely specifies the stress concentration factors for all nonfailed elements, in every possible configuration of failed and nonfailed elements, is the local load sharing of Harlow (1977) and Harlow and Phoenix (1978a,1978b,1979,1981a,1981b,1982). Recall that under local load sharing, the elements are assumed to be arranged in a circle, and the stress concentration factor for any nonfailed element is $\varphi_j = 1 + j/2$, where j is the number of failed elements adjacent to it. The failure process begins with all elements in the circle intact, and then one-by-one, elements fail until all have failed. Obviously the individual failed elements and microcracks are joining in the bundle failure process. If fact, intuitively it seems that there would be more joining in the essentially one dimensional circular arrangement than there would be in a two dimensional rectangular or hexagonal

arrangement of elements.

Because of the relatively simple geometry of the circle, Kelley (1978) was able to define a Markov process whose state at any time is the exact geometrical configuration of all failed and nonfailed elements, and to give an algorithm that computed the probability distribution of the bundle failure time exactly. [See also Taylor and Karlin (1984), pp. 264-273.] Using this algorithm, we have computed the probability distribution of bundle failure time for bundles of size $n = 1,2,...,9$. We make the **reverse weakest link transformation**, defined by

$$1 - W_n(t) = [1 - G_n(t)]^{1/n} \qquad (8.1)$$

Figure 6 plots $W_n(t)$ versus time t under the power law breakdown of (2.3) with $K(L) = 1$ and $\rho = 10$. The coordinates are such that a Weibull distribution plots as a straight line (log log $1/[1 - W_n(t)]$ vs. log t). Note the evident convergence of $W_n(t)$ to a limiting characteristic distribution function $W(t)$. We see that $W_9(t)$ coincides with $W(t)$ over the range of the graph. Table 1 compares $W_9(t)$ in this model, where microcracks are allowed to join, with $W^{[9]}$ from the simple model,where microcracks grow independently. For all practical purposes, the results are the same in both models, verifying that the interaction and joining of growing microcracks is a negligible phenonemon in the failure process.

9. An asymptotic formula for the lower tail of the distribution. In a study of the breakdown phenomenon in thin film insulators, Yashchin (1981),[See also Feigin and Yashchin (1982).], derives an asymptotic expansion for the lower tail of the distribution function of an infinite sum of independent exponentially distributed random variables. The technique used was to invert the Laplace transform of the distribution function with the "saddle point method". [See, e.g. Marsden (1973), Sec. 7.2.] In one problem, the exponential random variables had parameters $a(j+1)^{\rho}$ for $j = 0,1,\ldots,$ for a positive constant a and $\rho > 1$. They obtained the following approximation for the lower tail of the distribution function $D(t)$ of the infinite sum:

$$D(t) \sim At^{-1/2} \exp\{-(\rho-1)z_0 t^{-1/(\rho-1)}\}, \tag{9.1}$$

where,

$$\eta = \rho/(\rho-1), \tag{9.2}$$

$$A = [\eta(2\pi)^{\rho-1}/a]^{1/2}, \tag{9.3}$$

and,

$$z_0 = a^{-1/(\rho-1)} \left[\frac{\pi/\rho}{\sin(\pi/\rho)}\right]^{\eta}. \tag{9.4}$$

(We have corrected a minor error in their initial constant A.) Under local load sharing and power law breakdown, as given in (6.4), we have $a = 2^{1-\rho}$, and we have $j = 2,3,\ldots$, since the parameters then are

$$\theta_j = 2(1 + \tfrac{j}{2})^{\rho} = 2^{1-\rho}(2 + j)^{\rho} \text{ for } j = 1,2,\ldots .$$

In addition, we are interested in the lower tail of the integral

$$I(t) = \int_0^t F(y)dy, \tag{9.5}$$

and not in the lower tail of the distribution itself. These changes are so minor that we can follow their method very closely. The Laplace transform for our problem differs from theirs by having the additional factor

$$\tfrac{1}{s}(1 + \tfrac{s}{a})(1 + \frac{s}{a2^{\rho}}). \tag{9.6}$$

The effect of this is to add the corresponding factor to the approximation, with

$$s = z_0 t^{-\eta}, \text{ where } z_0 = 2\left[\frac{\pi/\rho}{\sin(\pi/\rho)}\right]^{\eta}, \tag{9.7}$$

and

$$a = 2^{1-\rho},$$

resulting in

$$A = [\eta(4\pi)^{\rho-1}]^{1/2}, \tag{9.8}$$

and

$$I(t) \tilde{=} \frac{t^{\eta}}{z_0}(1 + 2^{\rho-1} z_0 t^{-\eta})(1 + \tfrac{1}{2} z_0 t^{-\eta}) A t^{-1/2}$$

$$\times \exp\{-(\rho-1) z_0 t^{-1/(\rho-1)}\}. \tag{9.9}$$

The approximations

$$W(t) \tilde{=} 1 - \exp\{-I(t)\},$$

$$W_9(t), \text{ and } W^{[9]}(t),$$

with $I(t)$ as given in (9.9), are given in Table 1. Table 2 gives a similar comparison for $\rho = 5$ and $\rho = 20$. It is seen that the approximation is extremely accurate. The

discrepency at $\rho = 5$ and $\log t = -8$ is due to the lack of convergence of $W_9(t)$ to $W(t)$ in this region, so that the approximation provided by $I(t)$ is the more accurate figure.

10. The effect of load sharing. Encouraged by the accuracy of the $I(t)$ approximation in the case of local load sharing, we give the corresponding formulas for the load sharing rules of (6.5) to (6.7). In these cases, it is necessary to distinguish between ρ as it appears in the power law breakdown rate, and the exponent, which we will call $\tilde{\rho}$, which appears in the birth process rates θ_j.

(1) The linear crack in a planar network. The stress concentration factors in (6.5) give the birth process rates

$$\theta_j = 2c^{\rho}(1 + j)^{\rho/2}, \quad \text{for} \quad j = 1,2,\ldots,$$

whence

$$a = 2c^{\rho}, \quad \text{and} \quad \tilde{\rho} = \rho/2.$$

This leads to

$$\eta = \tilde{\rho}/(\tilde{\rho}-1), \tag{10.1}$$

$$A = [\eta(2\pi)^{\tilde{\rho}-1}/a]^{1/2} , \tag{10.2}$$

and,

$$z_0 = a^{-1/(\tilde{\rho}-1)} \left[\frac{\pi/\tilde{\rho}}{\sin(\pi/\tilde{\rho})}\right]^{\eta}. \tag{10.3}$$

Then

$$I(t) \sim \frac{t^{\eta}}{z_0}(1 + z_0 t^{-\eta}/a)At^{-1/2} \exp\{-(\tilde{\rho}-1)z_0 t^{-1/(\tilde{\rho}-1)}\}. \tag{10.4}$$

(2) The symmetric planar crack. When the stress concentration factors are given by (6.6), the birth process rates are

$$\theta_j = 4c^{\rho}(1 + j)^{[(1/2) + (\rho/4)]}, \text{ for } j = 1,2,\ldots,$$

whence,

$$a = 4c^{\rho} \quad \text{and} \quad \tilde{\rho} = (\tfrac{1}{2} + \tfrac{\rho}{4}),$$

and then $I(t)$ is given by (10.1) to (10.4).

(3) Diffuse load sharing. When the stress concentration factors are given by (6.7), then the birth process rates are

$$\theta_j = 4c^{\rho}(1 + j)^{[(1/2) + (\rho/8)]}, \text{ for } j = 1,2,\ldots,$$

whence

$$a = 4c^{\rho} \quad \text{and} \quad \tilde{\rho} = (\tfrac{1}{2} + \tfrac{\rho}{8}),$$

and $I(t)$ is given by (10.1) to (10.4).

These three load sharing rules differ in two ways. Firstly, they become more diffuse in that lower peak stress concentrations are spread over more elements as we go from (1) to (3). Secondly, they differ in the total amount of load that they redistribute. A load sharing rule is said to be **conservative** if the total load summed over all nonfailed elements in the bundle equals the total bundle load nL. The local load sharing of (6.4) is conservative. The load sharing rules of (6.5) to (6.7) are not since they ignore the load carried by all but the most highly stressed elements. The fraction of load redistributed by the rules (6.5) to (6.7) is

$$\omega = \frac{n_j(\varphi_j - 1)}{j}, \tag{10.5}$$

which in the three cases becomes

$$\omega_1 = \frac{2[c(1+j)^{1/2} - 1]}{j},$$

$$\omega_2 = \frac{4(1+j)^{1/2}[c(1+j)^{1/4} - 1]}{j}$$

and

$$\omega_3 = \frac{4(1+j)^{1/2}[c(1+j)^{1/8} - 1]}{j}.$$

The choice of $c = 1$ in case (1), $c = .90$ in case (2), and $c = 1.05$ in case (3) all lead to an average fraction of load redistributed by the system of about 68% for $j = 1,\ldots,5$, and these values will be used in later computations. The effects of the load sharing assumptions on the stress concentration factors φ_j and birth process rates θ_j are shown in Figure 7.

In discussing the effect of the different load sharing assumptions, let us first emphasize that the exponent in the scale factor $K(L)$ that is used to translate failure times under $K(L) = 1$ to failure times under arbitrary loads remains unchanged by the load sharing assumptions. What does change is the exponent in the characteristic distribution $W(t)$, from ρ to $\tilde{\rho}$. This can be a significant reduction in value of an extremely important parameter. The effect of lowering the exponent in the characteristic distribution is to increase the critical crack size, and correspondingly, to slow the convergence of $W_n(t)$ and $W^{[k]}(t)$ to $W(t)$. Because the convergence is slowed, the asymptotic formulas of (6.21) and (10.1) to (10.4) become the preferred means of analysis.

We have computed the characteristic distribution under local load sharing and the three load sharing assumptions of cases (1)

to (3) and the results are plotted in Figure 8. Since the load sharing rules are not conservative, in order to keep the share of the load that is not assigned about the same for each of the three rules, the constant c was chosen to be 1, .90, 1.05, respectively, in the three cases. We used $\rho = 10$. As can be seen, as the load sharing becomes more diffuse, the characteristic distribution is lowered and the median correspondingly increases.

11. The effect of initial failures. Suppose that at time $t = 0$ there are distributed randomly throughout the bundle N initial failures, where N has a Poisson distribution with mean δn, so that δ measures the mean fraction of initial failures. Such initial failures might represent defects scattered throughout the system, or molecule ends where no load can be carried [Bauer (1982), Smith (1980)]. As before, all initial failures generate cracks that grow independently to an explosion time Y. Thus, given that $N = i$, the probability that no initial failure has caused system failure by time z is

$$[P(Y > z)]^i = [1 - F(z)]^i. \tag{11.1}$$

We multiply this by the probability that $N = i$ and sum to obtain the unconditional probability that no initial failure has caused system failure by time z:

$$E[\{1 - F(z)\}^N] = \Sigma_{i=0}^{\infty} \frac{e^{-\delta n}(\delta n)^i}{i!} [1 - F(z)]^i \qquad (11.2)$$

$$= \exp\{-n\delta F(z)\}.$$

Finally, the probability that the bundle survives to time z is the probability that no initial failure present at time 0 has caused system failure, and none of the microcracks initiated after time 0 has caused system failure. Since all microcracks are assumed to grow independently, this probability is

$$P(Z_n > z) = \exp\{-n\delta F(z)\}\exp\{-n(1-\delta)\int_0^z F(y)dy\} \qquad (11.3)$$

$$= \exp\{-n\delta F(z) - n(1-\delta)\int_0^z F(y)dy\}.$$

The characteristic function of the system has changed to

$$W(z) = 1 - \exp\{-\delta F(z) - (1-\delta)\int_0^z F(y)dy\}. \qquad (11.4)$$

12. Summary and conclusions. We have set forth a model that describes the failure process of a system having many discrete elements that fail individually, shifting load onto nearby neighbors and thereby increasing their failure rates. The model is explicit and tractable. While some modern methods were used, they are at a relatively introductory level and, the development

of the model does not require a deep and advanced background in probability theory. The model captures many of the essential features of the simulation models of Dobrodumov and El'yashevich (1973), and Termonia, Meakin and Smith (1985). The major difference is that stress concentration factors calculated from a spring model were used in the simulations, while explicit, but approximate, stress concentration formulas were used by us. One must remember, however, that the spring models themselves are idealizations, so that what is exact in the spring model is still an approximation of reality.

Many things are gained by having a model that is simple enough to be amenable to analysis. First, we demonstrate the difficulties of extending the results of a simulation, where the system size is on the order of thousands, to systems of realistic size. Secondly, we are able to develop the useful notion of a critical crack size in a precise manner, and to study its complex behavior as a function of material size and load. Thirdly, the equivalence of the model that allows microcrack interaction with the model that precludes such interaction tells us something about how these materials behave: Initial failures occur throughout the system, that, while they may be large in absolute number, are small in number relative to the material size. Then, these initial failures grow into microcracks which extend, but the system effectively fails before the microcracks consume any significant volume of the material.

These three qualitative conclusions would seem to hold under

quite general conditions. Further conclusions await the assignment of numbers to the model parameters, and the comparison of model predictions with known material behavior.

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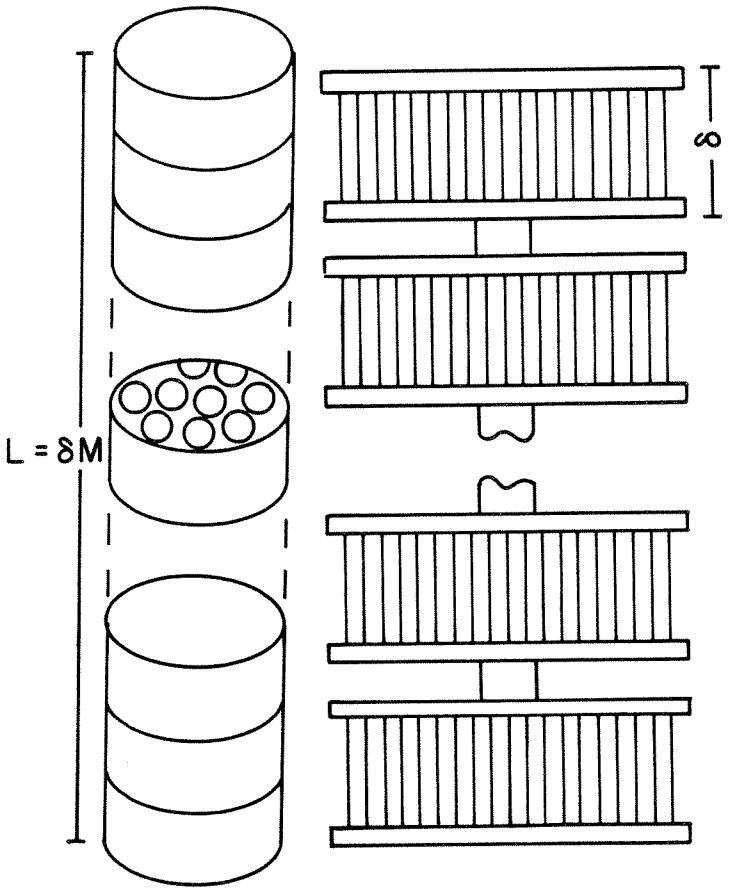

Figure 1: The chain-of-bundles model. The material or system is partitioned into m short sections, called, bundles, arranged in series along the axis of the tensile force. The bundles are assumed to be statistically independent. However, the elements within a bundle share the total load, and therefore the failure of each element in the bundle affects the rate of failure of the remeining elements in that same bundle.

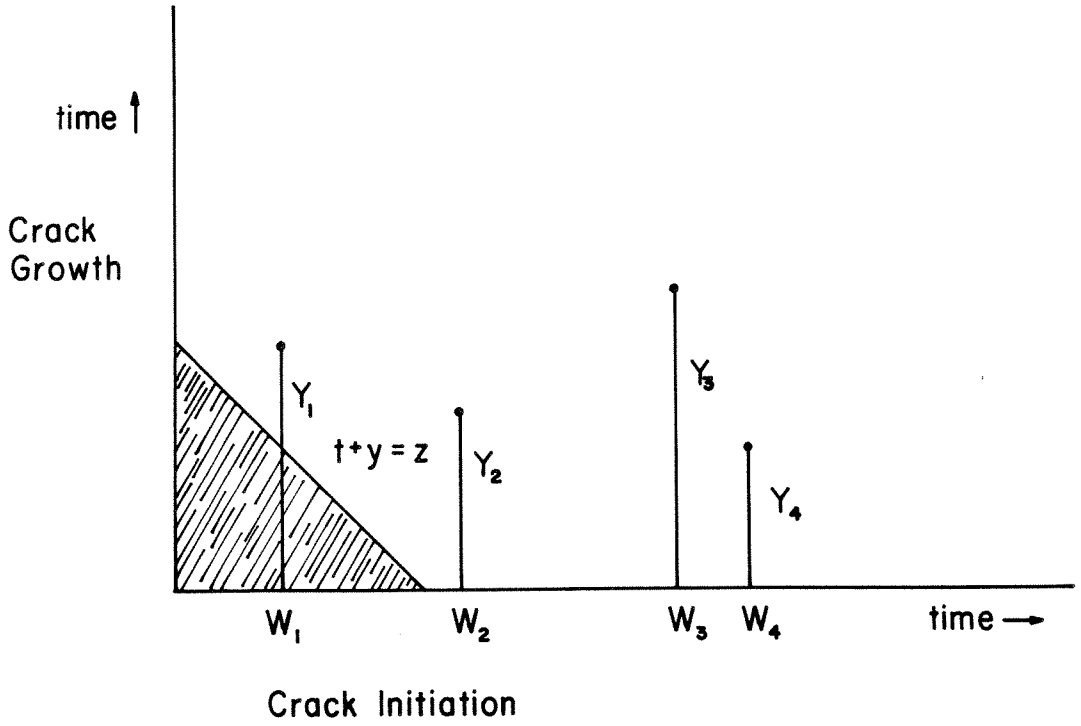

Figure 2: The crack initiation-crack growth process. The time of the jth initial element failure is W_j, and Y_j is the additional time that it takes for this jth initial failure to grow to a crack of infinite size. The bundle failure time is $Z = \min W_1+Y_1, W_2+Y_2, \ldots$, the earliest of the crack initiation plus crack growth times.

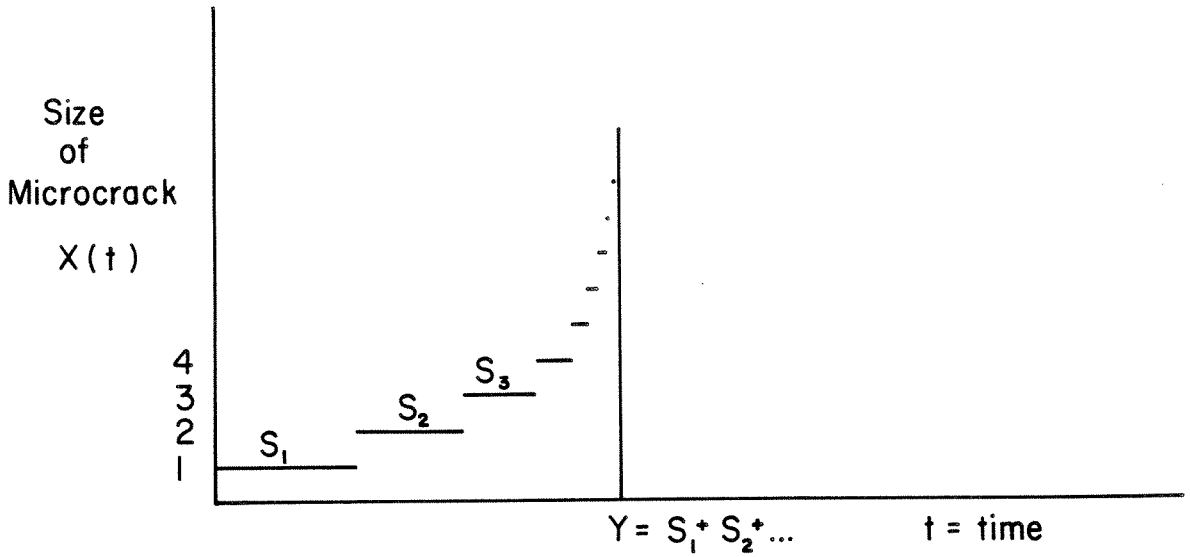

Figure 3: The size $X(t)$ of a crack as a function of time t. The crack is assumed to grow from size j to size $j+1$ at a probabilistic rate $\lambda_j = n_j K(\phi_j L)$, where L is the nominal load per element, ϕ_j is the maximal stress concentration factor for nonfailed elements neighboring the crack, and n_j is the number of elements carrying this maximal stress. The breakdown rule is $K(\sigma)$.

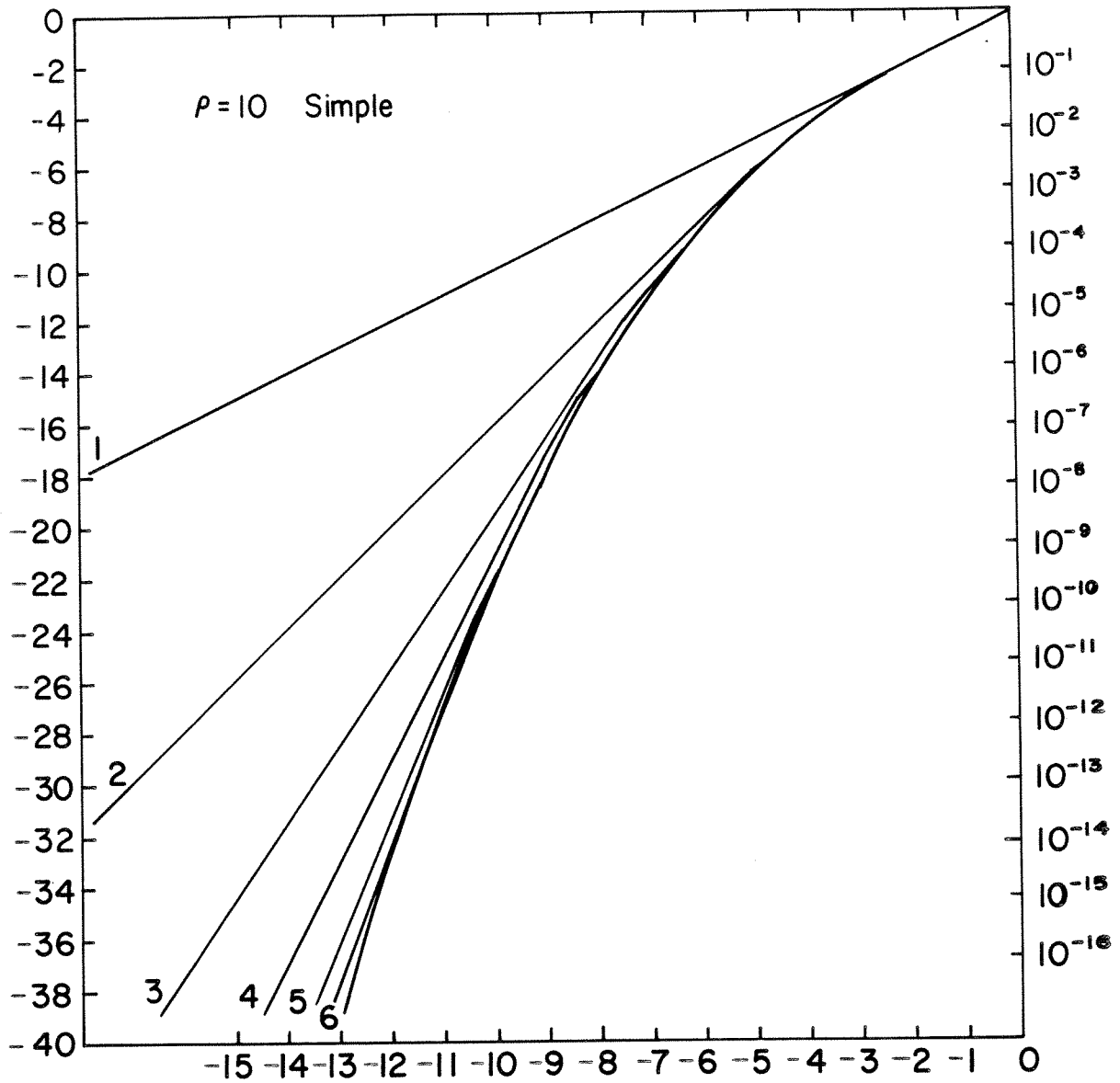

Figure 4: The cumulative distribution function of failure time in the simple model. The right ordinate is labeled with the probabilities of failure $W^{[k]}(t)$, the left ordinate is $\log \log 1/[1 - W^{[k]}(t)]$, and the abcissa is $\log t$. These coordinates are such that a Weibull distribution plots as a straight line.

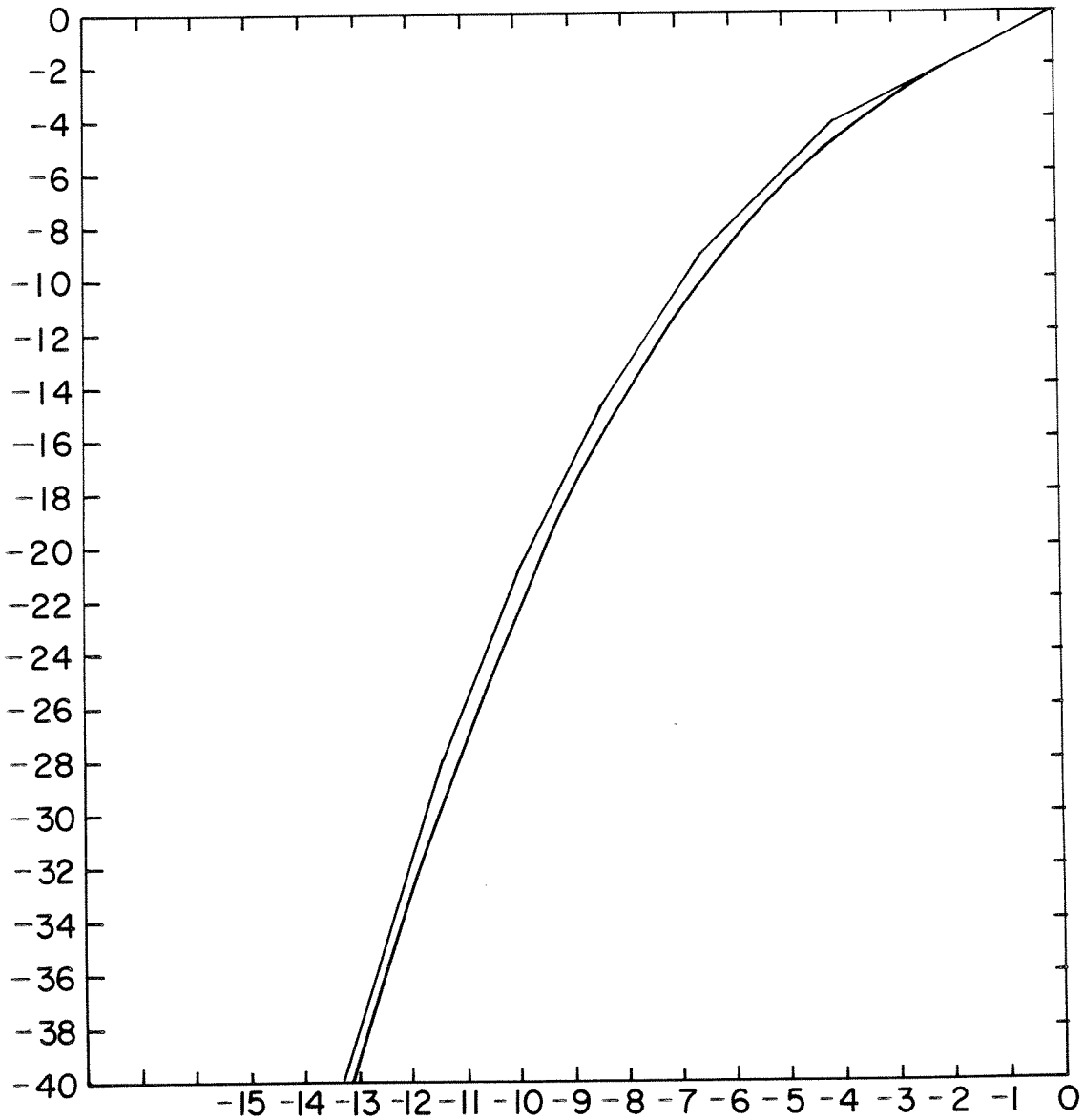

Figure 5: The approximation to the cumulative distribution of failure time that is associated with the concept of critical crack size. The upper curve is the approximation. It is a series of straight line segments, each segment corresponding to a different critical crack size.

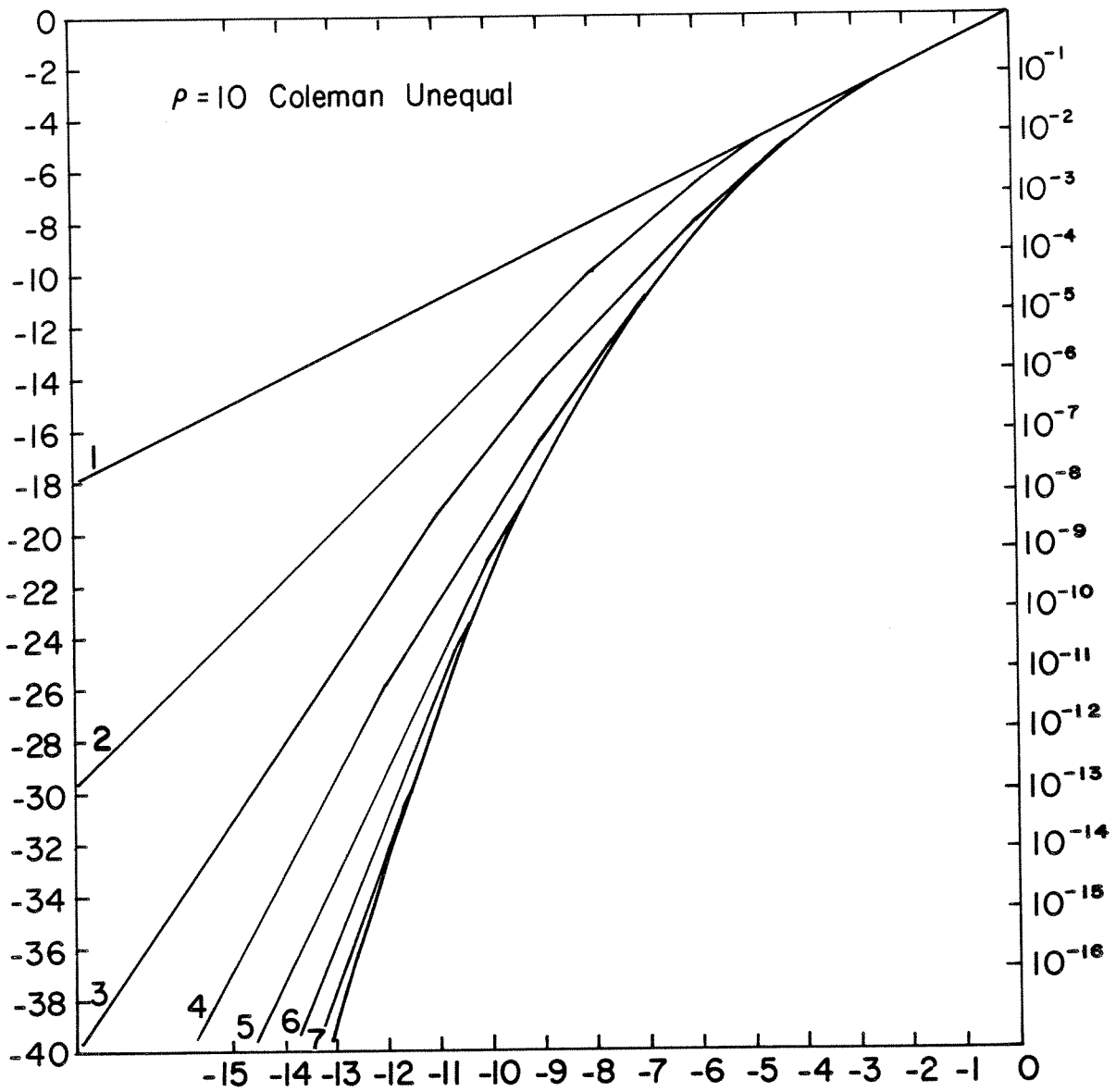

Figure 6: The cumulative distribution function of failure time in the Coleman model under local load sharing. The coordinates are the same as those in Figure 4. The Coleman model has smaller cracks coalescing to form larger ones, while the simple model assumes that all cracks grow independently.

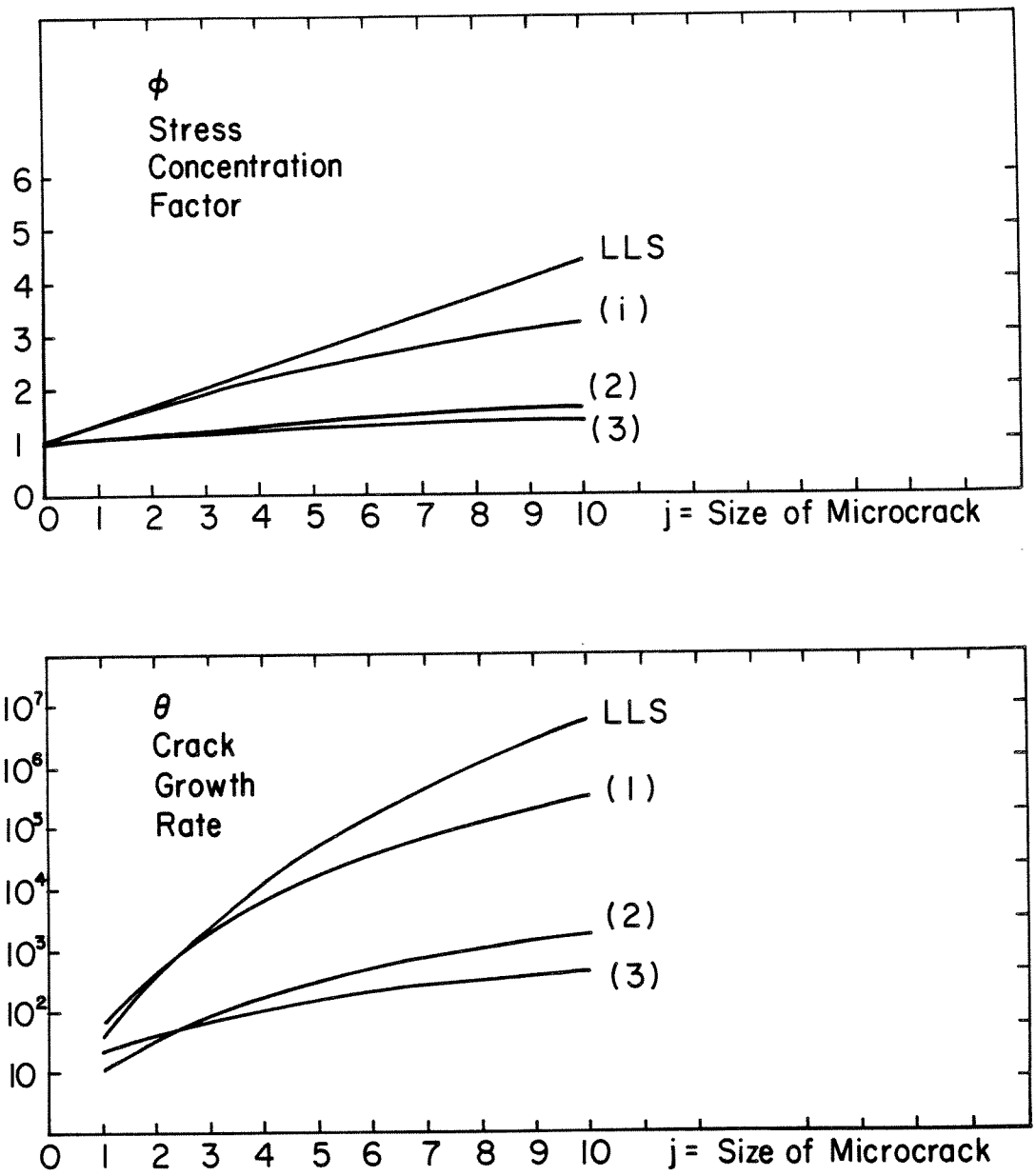

Figure 7: The stress concentration factors and crack growth rates under a variety of load sharine assumptions. Here, LLS denotes local load sharing, (1) is the load sharing under a linear crack in a planar network, (2) is the load sharing under a symmetric planar crack, and (3) is a diffuse load sharing rule.

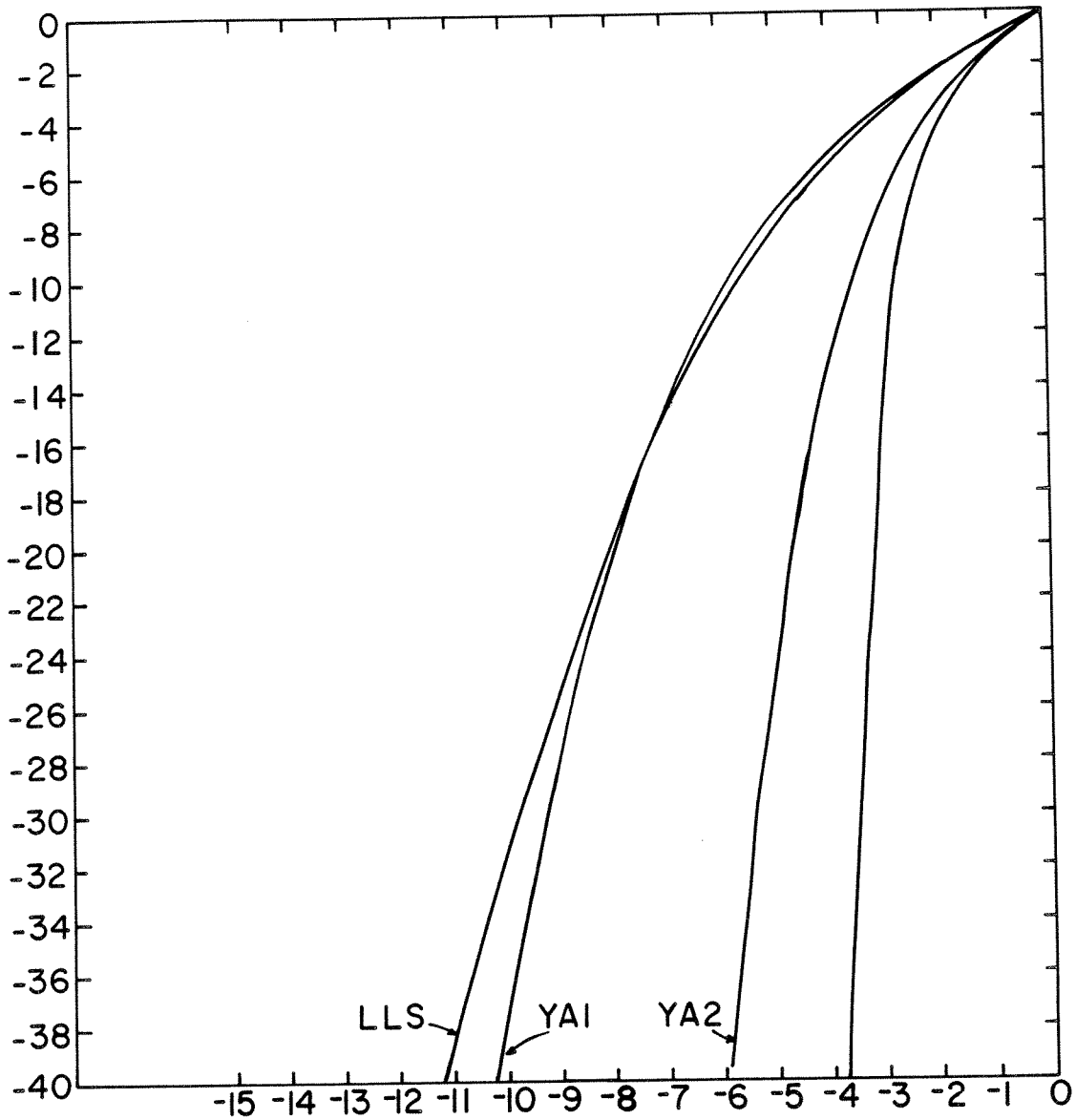

Figure 8: The distribution function of failure time under a variety of load sharing rules. The notation is the same as that in Figure 7.

TABLE 1

log t	log I(t)	log log $\frac{1}{1 - W_9(t)}$	log log $\frac{1}{1 - W^{[9]}(t)}$
-1	-0.897	-1.026	-1.025
-2	-1.873	-2.071	-2.071
-3	-3.019	-3.205	-3.204
-4	-4.438	-4.584	-4.584
-5	-6.195	-6.330	-6.336
-6	-8.341	-8.456	-8.466
-7	-10.93	-11.026	-11.04
-8	-14.01	-14.093	-14.10
-9	-17.64	-17.709	-17.72
-10	-21.88	-21.939	-21.96
-11	-26.81	-26.846	-26.87
-12	-32.52	-32.485	-32.54
-13	-39.08	-38.87	-39.03

TABLE 2

log t	log I(t)	$\log \log \frac{1}{1 - W_9(t)}$	$\log \log \frac{1}{1 - W_{[9]}(t)}$
		$\rho = 5$	
-2	-2.73	-2.84	-2.86
-4	-8.60	-8.45	-8.61
-6	-20.44	-19.10	-19.57
-8	-42.22	-33.93	-35.37
		$\rho = 20$	
-5	-4.64	-5.02	-5.02
-10	-11.83	-12.01	-12.01
-15	-23.51	-23.64	-23.64
-20	-41.05	-41.14	-41.14