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A GENERALIZED METHOD OF ANALYSIS OF FACTORIAL EXPERIMENT WITH UNEQUAL NUMBER OF OBSERVATIONS*

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Abstract

A calculus of factorials was developed by Kurkjian and Zelen [1962]. A series of papers on the application and extensions were presented by M. Zelen and co-workers [1963-66]. One of the remaining problems associated with the application of the calculus for factorials is to relate standard contrasts in factorials to the calculus for both equal and unequal numbers of observations on each treatment or combination. This problem is resolved in the present paper. A computing procedure using the calculus is presented for any estimable linear contrast or any set of estimable linear contrasts. A Kronecker product representation of the v-1 single degree of freedom contrasts is given wherein the linear contrasts of the levels of each effect are utilized. A new operation is introduced which simplifies the method of construction of contrasts and computation thereof. A numerical example is used to illustrate the procedure.

A second unsolved problem in the analysis of factorials pertains to less than full model situations. E.g., consider the situation wherein a set of factorial treatments is designed in a randomized complete block design. Also,

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suppose that no observations are present for a treatment in some but not all of the blocks. This problem is resolved in the present paper and a numerical example is used to illustrate the computations. Other related results are obtained in connection with the above two main problems.

A GENERALIZED METHOD OF ANALYSIS OF FACTORIAL EXPERIMENT WITH UNEQUAL NUMBER OF OBSERVATIONS*

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1. Introduction

A calculus of factorials was developed by Kurkjian and Zelen [1962]. A series of papers on the application and extensions was presented by M. Zelen and co-workers. The first paper on applications (Kurkjian and Zelen [1963]) was devoted to analyses of a large class of experimental designs with one-way elimination of heterogeneity which included b.i.b. and p.b.i.b. designs and designs obtained as direct products of these designs. The second paper on application (Zelen and Federer [1964]) dealt with the analyses of a large class of experimental designs for two-way elimination where the row-treatment and the column-treatment associations were each of the type discussed by Kurkjian and Zelen [1963]. The third paper (Zelen and Federer [1965]) dealt with the analysis of an n-factor factorial treatment design with unequal numbers (non-zero) of observations. The principal theoretical result of this paper enables the sums of squares for any main effect or interaction to be written as a simplified explicit form. Utilizing this form the necessary calculations for interaction sums of squares in an analysis of variance may be performed by inverting relatively small matrices. The main effect sums of squares are obtained without inverting a matrix. A two-factor interaction sum of squares associated with q_1 levels of one factor and q_2 levels of the second factor for

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$q_1 \geq q_2$, requires the inversion of a $(q_2-1) \times (q_2-1)$ matrix. A three factor interaction sum of squares with the three factors at levels q_1, q_2, and q_3 $(q_1 \geq q_2 \geq q_3)$ requires the inversion of $q_1 + q_2$ $(q_3-1) \times (q_3-1)$ matrices and an additional square matrix of side $(q_2-1)(q_3-1)$; etc. In a fourth paper Federer and Zelen [1966] applied the theory of the third paper to the analysis of multifactor (factorial) experiments; a $4 \times 3 \times 2$ factorial with unequal numbers of observations and a method of procedure were utilized to illustrate the computations.

One of the remaining problems associated with the application of the calculus for factorials is to relate standard contrasts in factorials to the calculus for both equal and unequal numbers of observations on each treatment or combination. This problem is resolved in the present paper. A computing procedure using the calculus is presented for any estimable linear contrast or any set of estimable linear contrasts. (The single degree of freedom contrast procedure is given by Zelen and Federer [1965]). In presenting the procedure, heavy use was made of the results and notations in the third and fourth papers. A Kronecker product representation of the v-1 single degree of freedom contrasts is given wherein the linear contrasts of the levels of each effect are utilized. A new operation is introduced which simplifies the method of construction of contrasts and computation thereof. A numerical example is used to illustrate the procedure.

A second unsolved problem in the analysis of factorials pertains to less than full model situations. E.g., consider the situation wherein a set of factorial treatments is designed in a randomized complete block design. Also, suppose that no observations are present for a treatment in some but not all of the blocks. This problem is resolved in the present paper and a numerical example is used to illustrate the computations. Other related results are obtained in connection with the above two main problems.

2. Notations and Operations

2.1. Introduction.

Consider a factorial experiment with n factors $\{A_h\}$ such that the h^{th} factor A_h has q_h levels. Then the number of treatment combinations is $v = \prod_{h=1}^{n} q_h$. Let the i^{th} treatment combination be denoted by the n-tuple

$$i = (i_1, i_2, \cdots, i_n) \quad ,$$

where i_h denotes a parameter level from factor A_h and $i_h = 0, 1, \cdots, q_h - 1$.

Let $y_{ji} = y_{j(i_1, i_2, \cdots, i_n)}$ be the j^{th} observation made on the treatment combination $(i_1, i_2, \cdots, i_n)$, where $j = 1, 2, \cdots r_i$, $(r_i \geq 1)$ and let N be the total number of observations. Furthermore, we assume that the $\{y_{j(i_1, i_2, \cdots, i_n)}\}$ are to be independently distributed following a normal distribution with

$$Ey_{j(i_1, i_2, \cdots, i_n)} = \tau(i_1, i_2, \cdots, i_n)$$

$$\text{var } y_{j(i_1, i_2, \cdots, i_n)} = \sigma^2$$

We shall denote the main effect, two factor interaction, $\cdots$, n-factor interaction parameter by

$$\alpha_h(i_h),\ \alpha_{hk}(i_h, i_k),\ \cdots,\ \alpha_{1,2,\cdots,n}(i_1, i_2, \cdots, i_n) \quad ,$$

furthermore, because of the factorial structure of the experiment for i^{th} treatment combination $i=(i_1,i_2,\cdots,i_n)$, we have

$$\tau_{(i_1,i_2,\cdots,i_n)} = \mu + \sum_{h=1}^{n} \alpha_h(i_h) + \sum_{h} \sum_{\substack{k \\ 1\le h<k\le n}} \alpha_{hk}(i_h,i_k) + \cdots$$

$$+ \alpha_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n)$$

We may assume the following without loss in generality:

$$\sum_{i_h=0}^{q_h-1} \alpha_h(i_h) = 0$$

$$\sum_{i_h=0}^{q_h-1} \alpha_{hk}(i_h,i_k) = \sum_{i_k=0}^{q_k-1} \alpha_{hk}(i_h,i_k) = 0 \tag{2.1}$$

$$\sum_{i_1=0}^{q_1-1} \alpha_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n) = \sum_{i_2=0}^{q_2-1} \alpha_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n) = \cdots$$

$$= \sum_{i_n=0}^{q_n-1} \alpha_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n) = 0$$

2.2. Vectors, Matrices and Kronecker product.

We shall write the model in matrix notation. For this purpose define

$$Y' = (y_{11}, y_{21}, \cdots, y_{r_1,1}, \cdots, y_{r_v,v})$$

$$\underline{\tau}' = (\tau_1, \tau_2, \cdots, \tau_v)$$

where $\tau_i = \tau_{(i_1, i_2, \cdots, i_n)}$. Then the model may be written as

$$EY = X\underline{\tau}$$
$$\text{var } Y = I\sigma^2 \tag{2.2}$$

where X is an $N \times v$ design matrix and I is an $N \times N$ identity matrix. Let

$$\underline{\alpha}'_h = [\alpha_h(0), \alpha_h(1), \cdots, \alpha_h(q_h-1)]$$

$$\underline{\alpha}'_{hk} = [\alpha_{hk}(0,0), \alpha_{hk}(0,1), \cdots, \alpha_{hk}(q_h-1, q_k-1)] \tag{2.3}$$

$$\underline{\alpha}'_{1,2,\cdots,n} = [\alpha_{1,2,\cdots,n}(0,0,\cdots,0), \alpha_{1,2,\cdots,n}(0,0,\cdots,1), \cdots,$$
$$\alpha_{1,2,\cdots,n}(q_1-1, \cdots, q_n-1)]$$

and let $E_h(i_h)$ be a unit coordinate row vector in a q_h-dimensional vector space such that

$$
\begin{aligned}
E_h(0) &= (1,0,\cdots,0) \\
E_h(1) &= (0,1,\cdots,0) \\
&\cdots \\
E_h(q_h-1) &= (0,0,\cdots,1)\ ,
\end{aligned}
\tag{2.4}
$$

then

$$
\begin{aligned}
\tau_{(i_1,i_2,\cdots,i_n)} &= \mu + \sum_{h=1}^{n} E_h(i_h)\underline{\alpha}_h + \sum_{h}\sum_{\substack{k \\ 1\le h<k\le n}} E_{hk}(i_h,i_k)\underline{\alpha}_{hk} + \cdots \\
&\quad + E_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n)\underline{\alpha}_{1,2,\cdots,n}
\end{aligned}
\tag{2.5}
$$

where

$$
\begin{aligned}
E_{hk}(i_h,i_k) &= E_h(i_h) \otimes E_k(i_k) \\
&\cdots \\
E_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n) &= \prod_{h=1}^{n} \otimes\, E_h(i_h)
\end{aligned}
$$

where $\otimes$ refers to the Kronecker product and define the product order as follows:

$$
\begin{aligned}
\prod_{h=1}^{n} \otimes\, E_h(i_h) &= E_1(i_1) \otimes \left(\prod_{h=2}^{n} \otimes\, E_h(i_h)\right) \\
&= E_1(i_1) \otimes \left(E_2(i_2) \otimes \left(\prod_{h=3}^{n} \otimes\, E_h(i_h)\right)\right)
\end{aligned}
$$

Assumption (2.1) may be represented as

$$\sum_{i_h=0}^{q_h-1} E_h(i_h)\underline{\alpha}_h = 0$$

$$\sum_{i_h=0}^{q_h-1} E_{hk}(i_h,i_k)\underline{\alpha}_{hk} = \sum_{i_k=0}^{q_k-1} E_{hk}(i_h,i_k)\underline{\alpha}_{hk} = 0$$

$$\cdots$$

$$\sum_{i_1=0}^{q_1-1} E_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n)\underline{\alpha}_{1,2,\cdots,n}$$

$$= \sum_{i_2=0}^{q_2-1} E_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n)\underline{\alpha}_{1,2,\cdots,n}$$

$$= \cdots = \sum_{i_n=0}^{q_n-1} E_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n)\underline{\alpha}_{1,2,\cdots,n} = 0$$

2.3. Orthogonal Contrast Matrix

If we denote the contrast matrix K_h for the h^{th} factor having q_h levels, the representation of contrast matrix among n factors is:

$$K = \prod_{h=1}^{n} \otimes K_h \tag{2.6}$$

where

$$K_h = \begin{bmatrix} 1 & \gamma_{01} & \cdots & \gamma_{0,q_h-1} \\ 1 & \gamma_{11} & \cdots & \gamma_{1,q_h-1} \\ & & \cdots & \\ 1 & \gamma_{q_h-1,1} & \cdots & \gamma_{q_h-1,q_h-1} \end{bmatrix} \tag{2.7}$$

where

$$\sum_{i_h=0}^{q_h-1} \gamma_{i_h,j_h} = 0 \quad \text{for} \quad j_h=1,2,\cdots,q_h-1$$

and

$$\sum_{i_h=0}^{q_h-1} \gamma_{i_h,j_h}\gamma_{i_h,k_h} = 0 \quad \text{for} \quad j_h \neq k_h \quad \text{and} \quad j_h,k_h=1,2,\cdots,q_h-1 \quad .$$

The treatment combination order corresponding to the row order of K is regarded as an n-tuple $i=(i_1,i_2,\cdots,i_n)$, the ordering is to fix all levels at the first level ($i_h=0$, $h=1,2,\cdots,n$) and run through the q_n levels A_n; then put $i_{n-1}=1$ and run through the levels of A_{n-1}; repeat for $i_{n-1}=2,3,\cdots,q_n-1$. After running through these $q_{n-1}q_n$ combinations, change $i_{n-2}=1$ and continue as before. After disposing of $q_{n-2}q_{n-1}q_n$ combinations, change $i_{n-3}=1$ and this process continues until all combinations have been enumerated.

In (2.7), let

$$K_h = (\underline{k}_h^0 \quad \underline{k}_h^1 \quad \cdots \quad \underline{k}_h^{q_h-1})$$

$$= \begin{bmatrix} \underline{t}_h^0 \\ \underline{t}_h^1 \\ \vdots \\ \underline{t}_h^{q_h-1} \end{bmatrix}$$

where $\underline{k}_h^{i_h}$ and $\underline{t}_h^{i_h}$ are $q_h \times 1$ column vector and $1 \times q_h$ row vector respectively. Then the i^{th} column vector $\underline{f}^i = \underline{f}^{(i_1, i_2, \cdots, i_n)}$ and i^{th} row vector $\underline{t}_i = \underline{t}_{(i_1, i_2, \cdots, i_n)}$ in K may be represented respectively as:

$$\underline{f}^i = \underline{f}^{(i_1, i_2, \cdots, i_n)} = \prod_{h=1}^{n} \otimes \underline{k}_h^{i_h}$$

$$\underline{t}_i = \underline{t}_{(i_1, i_2, \cdots, i_n)} = \prod_{h=1}^{n} \otimes \underline{t}_h$$

Particularly,

$$\underline{f}_h^{i_h} = \underline{f}^{(0, 0, \cdots, i_h, \cdots, 0)}$$

$$= \mathbf{1}_t \otimes \underline{k}_h^{i_h} \otimes \mathbf{1}_u$$

$$= \mathbf{1}_t \otimes \begin{bmatrix} \gamma_{0i_h} & \mathbf{1}_u \\ \gamma_{1i_h} & \mathbf{1}_u \\ \vdots & \\ \gamma_{q_h-1, i_h} & \mathbf{1}_u \end{bmatrix}$$

where $\mathbf{1}_t$ and $\mathbf{1}_u$ are $t \times 1$ and $u \times 1$ column vectors with all elements equal to one and where $t = \sum_{j=1}^{h-1} q_j$, $u = \sum_{j=h+1}^{n} q_j$.

If we define a product of two matrices $A_{m\times n} = (a_{ij})$ and $B_{m\times n} = (b_{ij})$ such as

$$A:B = \begin{bmatrix} a_{11} \otimes b_{11} & a_{12} \otimes b_{12} & \cdots & a_{1m} \otimes b_{1m} \\ a_{21} \otimes b_{21} & a_{22} \otimes b_{22} & \cdots & a_{2n} \otimes b_{2n} \\ & & \cdots & \\ a_{m1} \otimes b_{m1} & a_{m2} \otimes b_{m2} & \cdots & a_{mn} \otimes b_{mn} \end{bmatrix} \qquad (2.8)$$

where unit is a_{ij} and b_{ij} then $\underline{f}_{h,h+k}^{(i_h,i_{h+k})} = \underline{f}^{(0,\cdots,i_h,\cdots,i_{h+k},\cdots,0)}$ may be represented as

$$\underline{f}_{h,h+k}^{(i_h,i_{h+k})} = \underline{f}_h^{i_h} : \underline{f}_{h+k}^{i_{h+k}} \quad , \qquad (2.9)$$

where unit is $j_{i_h j_h}$. Equation (2.9) follows from the fact that

$$\underline{f}_{h,k+k}^{(i_h,i_{h+k})} = 1_s \otimes \underline{k}_h^{i_h} \otimes 1_t \otimes \underline{k}_{h+k}^{i_{h+k}} \otimes 1_u$$

where

$$s = \sum_{j=1}^{h-1} q_j \quad , \quad t = \sum_{j=h+1}^{h+k-1} q_j \quad \text{and} \quad u = \sum_{j=h+k+1}^{n} q_j \quad .$$

While

$$1_s \otimes \underline{k}_h^{i_h} \otimes 1_t \otimes \underline{k}_{h+k}^{i_{h+k}} \otimes 1_u = 1_s \otimes \underline{k}_h^{i_h} 1_{u'} : 1_{t'} \otimes \underline{k}_{h+k}^{i_{h+k}} \otimes 1_u$$

where

$$u' = v/sq_h \quad \text{and} \quad t' = v/uq_{h+k} \quad .$$

In general,

$$\underline{f}^{(i_1,i_2,\cdots,i_n)} = \underline{f}_1^{i_1} : \underline{f}_2^{i_2} : \cdots : \underline{f}_n^{i_n} \quad , \tag{2.10}$$

where unit is $\gamma_{i_h j_h}$. The representation in (2.10) is useful for constructing the contrast matrix K.

Using (2.4) and (2.8) defining

$$E_h' = [E_h(0), E_h(1), \cdots, E_h(q_h-1)]$$

and defining

$$Z_h = \mathbf{1}_t \otimes E_h \otimes \mathbf{1}_u \quad ,$$

$$Z_{hk} = Z_h : Z_k \quad ,$$

and

$$Z_{1,2,\cdots,n} = Z_1 : Z_2 : \cdots : Z_n \quad ,$$

where unit is E_h; then (2.5) may be represented as

$$\underline{\tau} = \mu \mathbf{1}_v + \sum_{h=1}^{n} Z_h \underline{\alpha}_h + \sum_{h} \sum_{k}_{1 \le h < k \le n} Z_{hk} \underline{\alpha}_{hk} + \cdots + Z_{1,2,\cdots,n} \underline{\alpha}_{1,2,\cdots,n}$$

Now, define the $q_h \times (q_h-1)$, $q_h q_k \times (q_h-1)(q_k-1), \cdots, v \times \prod_{h=1}^{n}(q_h-1)$ and

$v \times \left(\prod_{h=1}^{n} q_h - 1\right)$ submatrices P_h, P_{hk}, $\cdots$, $P_{1,2,\cdots,n}$ and C of K, $K_h \otimes K_k$, $\cdots$, and K respectively as follows:

$$P_h = [\underline{k}_h^1, \underline{k}_h^2, \cdots, \underline{k}_h^{q_h-1}]$$

$$P_{hk} = [\underline{k}_h^1 \otimes \underline{k}_h^1, \underline{k}_h^1 \otimes \underline{k}_h^2, \cdots, \underline{k}_h^{q_h-1} \otimes \underline{k}_h^{q_h-1}]$$

$$\begin{aligned} P_{1,2,\cdots,n} = [&\underline{k}_1^1 \otimes \underline{k}_2^1 \otimes \cdots \otimes \underline{k}_n^1, \underline{k}_1^1 \otimes \underline{k}_2^1 \otimes \cdots \otimes \underline{k}_n^2, \cdots, \\ &\underline{k}_1^{q_1-1} \otimes \underline{k}_2^{q_2-1} \otimes \cdots \otimes \underline{k}_n^{q_n-1}] \end{aligned} \qquad (2.11)$$

$$\begin{aligned} C &= [\underline{\ell}_1(1), \underline{\ell}_1(2), \cdots, \underline{\ell}_1(q_1-1), \underline{\ell}_2(1), \cdots, \\ &\qquad \underline{\ell}_{1,2,\cdots,n}(q_1-1,\cdots,q_n-1)] \\ &= [c_1, c_2, \cdots, c_{1,2,\cdots,n}] \end{aligned}$$

where

$$\underline{\ell}_h(i_h) = \underline{f}_{h,k}^{(i_h, i_k)}, \quad i_h = 1,2,\cdots,q_h-1$$

$$\underline{\ell}_{hk}(i_h, i_k) = \underline{f}_{h,k}^{(i_h, i_k)}, \quad h < k; \quad i_h = 1,2,\cdots,q_h-1; \quad i_k = 1,2,\cdots,q_k-1$$

$$\cdots$$

$$\underline{\ell}_{1,2,\cdots,n}(q_1-1,\cdots,q_n-1) = \underline{f}^{(q_1-1, q_2-1, \cdots, q_n-1)}$$

and

$$C_h \quad = [\underline{\ell}_h(1), \underline{\ell}_h(2), \cdots, \underline{\ell}_h(q_h-1)]$$

$$C_{hk} \quad = [\underline{\ell}_{hk}(1,1), \underline{\ell}_{hk}(1,2), \cdots, \underline{\ell}_{hk}(q_h-1, q_k-1)]$$

$$\cdots$$

$$C_{1,2,\cdots,n} = [\underline{\ell}_{1,2,\cdots,n}(1,1,\cdots,1), \underline{\ell}_{1,2,\cdots,n}(1,1,\cdots,2), \cdots$$
$$\underline{\ell}_{1,2,\cdots,n}(q_1-1,\cdots,q_n-1)] \ .$$

It is understood that $C'C$ is $(v-1) \times (v-1)$ diagonal matrix from the definition of the contrast matrix K_h ; from (2.9) and 2.10) we obtain

$$\begin{aligned} &\underline{\ell}_{hk}(i_h, i_k) = \underline{\ell}_h(i_h) : \underline{\ell}_k(i_k) \\ &\qquad \cdots \\ &\underline{\ell}_{1,2,\cdots,n}(i_1, i_2, \cdots, i_n) = \underline{\ell}_1(i_1) : \underline{\ell}_2(i_n) : \cdots : \underline{\ell}_n(i_n) \ , \end{aligned} \tag{2.12}$$

where unit is $\gamma_{i_h j_k}$ and $i_h = 1,2,\cdots,q_h-1$.

3. Estimation of Interactions and Variances of Contrasts

From (2.2) and the least squares method

$$X'X\hat{\underline{\tau}} = X'Y \ .$$

Since X'X is invertible

$$\hat{\underline{\tau}} = S^{-1}X'Y$$

$$\text{var } \hat{\underline{\tau}} = S^{-1}\sigma^2$$

where S = X'X.

We know that $C'\underline{\tau}$ is estimable and its unbiased linear estimator is $C'\hat{\underline{\tau}}$ and also

$$\text{var } C'\hat{\underline{\tau}} = C'S^{-1}C\sigma^2$$

Theorem 1. Under assumption (2.1), using notations (2.3) and (2.11), the linear contrast $C'\underline{\tau}$ may be represented as

$$C'\underline{\tau} = \begin{bmatrix} C_1'\underline{\tau} \\ C_2'\underline{\tau} \\ \vdots \\ C_n'\underline{\tau} \\ C_{1,2}'\underline{\tau} \\ \vdots \\ C_{1,2,\cdots,n}'\underline{\tau} \end{bmatrix} = \begin{bmatrix} (v/q_1)\,P_1'\underline{\alpha}_1 \\ (v/q_2)\,P_2'\underline{\alpha}_2 \\ \vdots \\ (v/q_n)\,P_n'\underline{\alpha}_n \\ (v/q_1q_2)\,P_{1,2}'\underline{\alpha}_{1,2} \\ \vdots \\ P_{1,2,\cdots,n}'\underline{\alpha}_{1,2,\cdots,n} \end{bmatrix}$$

Proof:

$$C_h' \, \underline{\tau} = \begin{bmatrix} \underline{\ell}'_h(1) \, \underline{\tau} \\ \underline{\ell}'_h(2) \, \underline{\tau} \\ \vdots \\ \underline{\ell}'_h(q_h-1) \cdot \underline{\tau} \end{bmatrix}$$

$$\underline{\ell}_h'(i_h)\underline{\tau} = \underline{\ell}_h'(i_h)[\mu \mathbf{1}_v + \sum_{h=1}^{n} Z_h \underline{\alpha}_h + \sum_{\substack{h \\ 1 \le h < k \le n}} \sum_{k} Z_{hk} \underline{\alpha}_{hk} + \cdots$$

$$+ Z_{1,2,\cdots,n} \underline{\alpha}_{1,2,\cdots,n}]$$

$\underline{\ell}_h'(i_h)(\mu \mathbf{1}_v) = 0$ from the definition of $\underline{C}_h'(i_h)$.

$$\underline{\ell}_h'(i_h) Z_j \underline{\alpha}_j = (\mathbf{1}_{t_h} \otimes \underline{k}_h^{i_h} \otimes \mathbf{1}_{u_h})'(\mathbf{1}_{t_j} \otimes \underline{\alpha}_j \otimes \mathbf{1}_{u_j})$$

$$= \begin{cases} \left(\frac{v}{q_h}\right)\left(\underline{k}_h^{i_h}\right)' \underline{\alpha}_h & \text{if } j = h \\[2ex] b_1 \sum_{g=0}^{q_h-1} \gamma_{gi_h} \left(\sum_{s=0}^{q_j-1} \alpha_j(s)\right) = 0 & \text{if } j < h \\[2ex] b_2 \sum_{s=0}^{g_j-1} \alpha_j(s) \left(\sum_{g=0}^{q_h-1} \gamma_{gi_h}\right) = 0 & \text{if } j > h \end{cases}$$

where b_1 and b_2 are constants.

$$\underline{\ell}_h'(i_h)Z_{gj}\underline{\alpha}_{gj} = \begin{cases} d_1 \sum\limits_r \gamma_{ri_h} \sum\limits_s \alpha_{gj}(r,s) = 0 & \text{if } h < g \\ d_2 \sum\limits_r \sum\limits_u \gamma_{ui_h} \sum\limits_s \alpha_{gj}(r,s) = 0 & \text{if } g < h < j \\ d_3 \sum\limits_r \sum\limits_s \alpha_{gj}(r,s) \sum\limits_u \gamma_{ui_h} = 0 & \text{if } j < h \end{cases}$$

where d_1, d_2, and d_3 are constants. Similarly

$$\underline{\ell}_h'(i_h)Z_{1,2,\cdots,n}\underline{\alpha}_{1,2,\cdots,n} = 0 \quad .$$

Then

$$\underline{\ell}_h'(i_h)\underline{\tau} = \left(\frac{v}{q_h}\right)\left(\underline{k}_h^{i_h}\right)'\underline{\alpha}_h \quad .$$

Hence

$$C_h'\underline{\tau}\ \left(\frac{v}{q_h}\right)\ P_h'\underline{\alpha}_h \quad .$$

Other situations are similarly proven.

From this Theorem, we obtain the following:

$$\underline{\ell}_h'(i_h)\underline{\tau} = 0 \ \text{ implies } \ \left(\underline{k}_h^{i_h}\right)'\underline{\alpha}_h = 0$$

$C_h'\underline{\tau} = 0$ implies $P_h'\underline{\alpha}_h = 0$, i.e.,

$$\alpha_h(0) = \alpha_h(1) = \cdots = \alpha_h(q_h-1)$$

$$\cdots$$

$C_{1,2,\cdots,n}'\underline{\tau} = 0$ implies $P_{1,2,\cdots,n}' \underline{\alpha}_{1,2,\cdots,n} = 0$, i.e.,

$$\alpha_{1,2,\cdots,n}(0,0,\cdots,0) = \alpha_{1,2,\cdots,n}(0,0,\cdots,1) = \cdots$$
$$= \alpha_{1,2,\cdots,n}(q_1-1,\cdots,q_n-1) \quad .$$

4. Sums of Squares for Factorial Effects

We shall present the following well known Lemma without proof.

Lemma 1. If Y is distributed $N(\mu, I\sigma^2)$, then $Y'A\sigma^{-2}Y$ is distributed as χ^2 with k degrees of freedom if and only if $\mu'A\mu = 0$ and A is an idempotent matrix of rank k.

Theorem 2. The quadratic forms $\tau'\underline{\ell}_h(i_h)[\underline{\ell}_h'(i_h)S^{-1}\underline{\ell}_h(i_h)]^{-1}\sigma^{-2}\underline{\ell}_h'(i_h)\hat{\underline{\tau}}$, $\hat{\underline{\tau}}\ell_{hk}(i_h,i_k)[\underline{\ell}_{hk}'(i_h,i_k)S^{-1}\underline{\ell}_{hk}(i_h,i_k)]^{-1}\sigma^{-2}\underline{\ell}_{hk}'(i_h,i_k)\hat{\underline{\tau}}$, $\cdots$, $\hat{\underline{\tau}}'\underline{\ell}_{1,2,\cdots,n}(i_1,\cdots,i_n)[\underline{\ell}_{1,2,\cdots,n}'(i_1,\cdots,i_n)S^{-1}\underline{\ell}_{1,2,\cdots,n}(i_1,\cdots,i_n)]^{-1}\sigma^{-2}$ $\underline{\ell}_{1,2,\cdots,n}'(i_1,\cdots,i_n)\hat{\underline{\tau}}$ subject to restrictions $\underline{\ell}_h'(i_h)\underline{\tau} = 0$, $\underline{\ell}_{hk}'\underline{\tau} = 0$, $\cdots$, $\underline{\ell}'_{1,2,\cdots,n}(i_1,\cdots,i_n)\underline{\tau} = 0$ are distributed as χ^2 with one degree of freedom respectively and quadratic forms $\hat{\underline{\tau}}'C_h[C_h'S^{-1}C_h]^{-1}\sigma^{-2}C_h'\hat{\underline{\tau}}$, $\hat{\underline{\tau}}'C_{hk}[C_{hk}'S^{-1}C_{hk}]^{-1}\sigma^{-2}C_{hk}'\hat{\underline{\tau}}$, $\cdots$, $\hat{\underline{\tau}}'C_{1,2,\cdots,n}[C_{1,2,\cdots,n}'S^{-1}C_{1,2,\cdots,n}]^{-1}\sigma^{-2}C_{1,2,\cdots,n}'\hat{\underline{\tau}}$ subject to restrictions

$C_h'\underline{\tau} = 0$, $C_{hk}'\underline{\tau} = 0$, $\cdots$, $C_{1,2,\cdots,n}'\underline{\tau} = 0$ are distributed as χ^2 with q_h-1, $(q_h-1)(q_k-1)$, $\cdots$, $\prod_{h=1}^{n}(q_h-1)$ degrees of freedom respectively.

Proof: We shall prove only one case, e.g., $\hat{\underline{\tau}}'C_h[C_h'S^{-1}C_h]^{-1}\sigma^{-2}C_h'\hat{\underline{\tau}}$. The other cases can be proven similarly.

Since

$$\hat{\underline{\tau}} = S^{-1}X'Y$$

$$\hat{\underline{\tau}}'C_h[C_h'S^{-1}C_h]^{-1}\sigma^{-2}C_h'\hat{\tau} = Y'XS^{-1}C_h[C_h'S^{-1}C_h]^{-1}C_h'S^{-1}X'\sigma^{-2}Y$$

Let

$$A = XS^{-1}C_h[C_h'S^{-1}C_h]^{-1}C_h'S^{-1}X' \quad .$$

then

$$AA = XS^{-1}C_h]C_h'S^{-1}C_h]^{-1}C_h'S^{-1}X'XS^{-1}C_h[C_h'S^{-1}C_h]^{-1}C_h'S^{-1}X'$$

$$= XS^{-1}C_h[C_h'S^{-1}C_h]^{-1}[C_h'S^{-1}C_h][C_h'S^{-1}C_h]^{-1}C_h^{-1}S^{-1}X'$$

$$= XS^{-1}C_h[C_h'S^{-1}C_h]^{-1}C_h'S^{-1}X$$

$$= A$$

and

$$r(A) = q_h-1 \; .$$

Next, since $EY = X\underline{\tau}$

$$(EY)'A(EY) = \underline{\tau}'X'XS^{-1}[C_h'S^{-1}C_h]^{-1}C_h'S^{-1}X'X\underline{\tau}$$

$$= \underline{\tau}'C_h[C_h'S^{-1}C_h]^{-1}C_h'\underline{\tau}$$

$$= 0$$

because

$$C_h'\underline{\tau} = 0 \quad .$$

This proves that the quadratic form $\hat{\underline{\tau}}'C_h[C_h'S^{-1}C_h]^{-1}\sigma^{-2}C_h'\hat{\underline{\tau}}$ subject to restriction $C_h'\underline{\tau} = 0$ is distributed as χ^2 with (q_h-1) degrees of freedom.

From this theorem, the sums of squares for each factorial effect may be represented as follows:

	d.f.
$SS(A_h) = \hat{\underline{\tau}}'C_h[C_h'S^{-1}C_h]^{-1}C_h'\hat{\underline{\tau}}$	q_h-1
$SS(A_h^{i_h}) = \hat{\underline{\tau}}'\underline{\ell}_h(i_h)[\underline{\ell}_h'(i_h)S^{-1}\underline{\ell}_h(i_h)]^{-1}\underline{\ell}_h'(i_h)\hat{\underline{\tau}}$	1
$SS(A_hA_k) = \hat{\underline{\tau}}'C_{hk}[C_{hk}'S^{-1}C_{hk}]^{-1}C_{hk}'\hat{\underline{\tau}}$	$(q_h-1)(q_k-1)$
$SS(A_h^{i_h}\ A_k^{i_k}) = \hat{\underline{\tau}}'\underline{\ell}_{hk}(i_h,i_k)[\underline{\ell}_{hk}'(i_h,i_k)S^{-1}\underline{\ell}_{hk}(i_h,i_k)]^{-1}\underline{\ell}_{hk}'(i_h,i_k)\hat{\underline{\tau}}$	1
$\cdots$	
$SS(A_1\ A_2,\cdots A_n) = \hat{\underline{\tau}}'C_{1,2,\cdots,n}[C_{1,2,\cdots,n}'S^{-1}C_{1,2,\cdots,n}]^{-1}C_{1,2,\cdots,}'\hat{\underline{\tau}}$	$\prod_{h=1}^{n}(q_h-1)$
$SS(A_1^{i_1}\ A_2^{i_2}\ A_n^{i_n}) = \hat{\underline{\tau}}'\underline{\ell}_{1,2,\cdots,n}(i_1,\cdots,i_n)[\underline{\ell}_{1,2,\cdots,n}'(i_1,\cdots,i_n)S^{-1}\underline{\ell}_{1,2,\cdots,n}(i_1,\cdots,i_n)]^{-1} \times \underline{\ell}_{1,2,\cdots,n}'(i_1,\cdots,i_n)\hat{\underline{\tau}}$	1

5. Numerical Example 1

Table 1

Data for Example

A 4 × 3 × 2 factorial experiment for four levels of lysine, A_1, three levels of methionine, A_2, and two levels of protein, A_3. The data (average daily gains, in pounds, of swine) are selected from Table 1 of Federer and Zelen [1966].

	Level of Methionine					
	$a_2(0),0$		$a_2(1),0.025$		$a_2(2),0.050$	
	Level of Protein		Level of Protein		Level of Protein	
Level of Lysine	$a_3(0),12$	$a_3(1),14$	$a_3(0),12$	$a_3(1),14$	$a_3(0),12$	$a_3(1),14$
$a_1(0),0$	1.11	1.52	1.09	1.27	-	-
	0.97	1.45	0.99	1.22	1.21	1.24
$a_1(1),0.05$	1.30	1.55	1.03	1.24	1.12	-
	1.00	1.53	1.21	1.34	0.96	1.27
$a_1(2),0.10$	1.22	1.38	1.34	1.40	1.34	1.46
	1.13	1.08	1.41	1.21	1.19	1.39
$a_1(3),0.15$	1.19	-	1.36	1.42	1.46	1.62
	1.03	1.29	1.16	1.39	1.03	-

In this example

$$S^{-1} = \text{diag}\ \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},1,1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},1\right)$$

$$\hat{\underline{\tau}}' = (S^{-1}X'Y)' = (1.040,\ 1.485,\ 1.040,\ 1.245,\ 1.210,\ 1.240,\ 1.150,\ 1.540$$

1.120, 1.290, 1.040, 1.270, 1.175, 1.230, 1.375, 1.305

1.265, 1.425, 1.110, 1.290, 1.260, 1.405, 1.245, 1.620)

Define the following orthogonal contrast matrices (in the sense of $K_h' K_h$ = diagonal) for A_1, A_2, A_3.

$$K_1 = \begin{bmatrix} 1 & -3 & 1 & -1 \\ 1 & -1 & -1 & 3 \\ 1 & 1 & -1 & -3 \\ 1 & 3 & 1 & 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

From (2.11) and (2.12), we obtain the following orthogonal contrast matrix C:

$$\begin{array}{c}
\begin{array}{ccccccc} C_1(\cdot) & C_2(\cdot) & C_3(\cdot) & C_{12}(\cdot) & C_{13}(\cdot) & C_{23}(\cdot) & C_{123}(\cdot) \end{array} \\
\left[\begin{array}{rrrrrrrrrrrrrrrrrrrrrrr}
-3 & 1 & -1 & -1 & 1 & -1 & 3 & -3 & -1 & 1 & 1 & -1 & 3 & -1 & 1 & 1 & -1 & -3 & 3 & 1 & -1 & -1 & 1 \\
-3 & 1 & -1 & -1 & 1 & 1 & 3 & -3 & -1 & 1 & 1 & -1 & -3 & 1 & -1 & -1 & 1 & 3 & -3 & -1 & 1 & 1 & -1 \\
-3 & 1 & -1 & 0 & -2 & -1 & 0 & 6 & 0 & -2 & 0 & 2 & 3 & -1 & 1 & 0 & 2 & 0 & -6 & 0 & 2 & 0 & -2 \\
-3 & 1 & -1 & 0 & -2 & 1 & 0 & 6 & 0 & -2 & 0 & 2 & -3 & 1 & -1 & 0 & -2 & 0 & 6 & 0 & -2 & 0 & 2 \\
-3 & 1 & -1 & 1 & 1 & -1 & -3 & -3 & 1 & 1 & -1 & -1 & 3 & -1 & 1 & -1 & -1 & 3 & 3 & -1 & -1 & 1 & 1 \\
-3 & 1 & -1 & 1 & 1 & 1 & -3 & -3 & 1 & 1 & -1 & -1 & -3 & 1 & -1 & 1 & 1 & -3 & -3 & 1 & 1 & -1 & -1 \\
-1 & -1 & 3 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -3 & 3 & 1 & 1 & -3 & 1 & -1 & -1 & 1 & -1 & 1 & 3 & -3 \\
-1 & -1 & 3 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -3 & 3 & -1 & -1 & 3 & -1 & 1 & 1 & -1 & 1 & -1 & -3 & 3 \\
-1 & -1 & 3 & 0 & -2 & -1 & 0 & 2 & 0 & 2 & 0 & -6 & 1 & 1 & -3 & 0 & 2 & 0 & -2 & 0 & -2 & 0 & 6 \\
-1 & -1 & 3 & 0 & -2 & 1 & 0 & 2 & 0 & 2 & 0 & -6 & -1 & -1 & 3 & 0 & -2 & 0 & 2 & 0 & 2 & 0 & -6 \\
-1 & -1 & 3 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 1 & 1 & -3 & -1 & -1 & 1 & 1 & 1 & 1 & -3 & -3 \\
-1 & -1 & 3 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 3 & 3 & -1 & -1 & 3 & 1 & 1 & -1 & -1 & -1 & -1 & 3 & 3 \\
1 & -1 & -3 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 3 & -3 & -1 & 1 & 3 & 1 & -1 & 1 & -1 & -1 & 1 & -3 & 3 \\
1 & -1 & -3 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 3 & -3 & 1 & -1 & -3 & -1 & 1 & -1 & 1 & 1 & -1 & 3 & -3 \\
1 & -1 & -3 & 0 & -2 & -1 & 0 & -2 & 0 & 2 & 0 & 6 & -1 & 1 & 3 & 0 & 2 & 0 & 2 & 0 & -2 & 0 & -6 \\
1 & -1 & -3 & 0 & -2 & 1 & 0 & -2 & 0 & 2 & 0 & 6 & 1 & -1 & -3 & 0 & -2 & 0 & -2 & 0 & 2 & 0 & 6 \\
1 & -1 & -3 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -3 & -3 & -1 & 1 & 3 & -1 & -1 & -1 & -1 & 1 & 1 & 3 & 3 \\
1 & -1 & -3 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -3 & -3 & 1 & -1 & -3 & 1 & 1 & 1 & 1 & -1 & -1 & -3 & -3 \\
3 & 1 & 1 & -1 & 1 & -1 & -3 & 3 & -1 & 1 & -1 & 1 & -3 & -1 & -1 & 1 & -1 & 3 & -3 & 1 & -1 & 1 & -1 \\
3 & 1 & 1 & -1 & 1 & 1 & -3 & 3 & -1 & 1 & -1 & 1 & 3 & 1 & 1 & -1 & 1 & -3 & 3 & -1 & 1 & -1 & 1 \\
3 & 1 & 1 & 0 & -2 & -1 & 0 & -6 & 0 & -2 & 0 & -2 & -3 & -1 & -1 & 0 & 2 & 0 & 6 & 0 & 2 & 0 & 2 \\
3 & 1 & 1 & 0 & -2 & 1 & 0 & -6 & 0 & -2 & 0 & -2 & 3 & 1 & 1 & 0 & -2 & 0 & -6 & 0 & -2 & 0 & -2 \\
3 & 1 & 1 & 1 & 1 & -1 & 3 & 3 & 1 & 1 & 1 & 1 & -3 & -1 & -1 & -1 & -1 & -3 & -3 & -1 & -1 & -1 & -1 \\
3 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 3 & 3 & 1 & 1 & 1 & 1
\end{array}\right]
\end{array}$$

$SS(A_1)$, $SS(A'$ or $L_{A_1})$, $SS(A_1A_2)$ and $SS(A_1^2\ A_2^2\ A_3'$ or $Q_{A_1} \times Q_{A_2} \times L_{A_3})$, for example, are as follows:

$SS(A_1)$:

$$\underline{\hat{\tau}}'C_1 = (2.375,\ 0.005,\ -0.425$$

$$C_1'S^{-1}C_1 = \begin{bmatrix} 78.5 & 0.5 & 4.5 \\ 0.5 & 14.5 & -1.5 \\ 4.5 & -1.5 & 66.5 \end{bmatrix}$$

$$[C_1'S^{-1}C_1]^{-1} = \begin{bmatrix} 0.0127925 & -0.0005319 & -0.0008777 \\ -0.0005319 & 0.0691489 & 0.0015957 \\ -0.0008777 & 0.0015957 & 0.01513298 \end{bmatrix}$$

Then

$$SS(A_1) = \underline{\hat{\tau}}'C_1[C_1S^{-1}C_1]^{-1}C_1'\underline{\hat{\tau}}$$

$$= 0.076645 \quad .$$

$SS(L_{A_1})$:

$$\underline{\hat{\tau}}'\underline{\ell}_1(1) = 2.375$$

$$\underline{\ell}_1'(1)S^{-1}\underline{\ell}_1(1) = 78.5$$

$$[\underline{\ell}_1'(1)S^{-1}\underline{\ell}_1(1)]^{-1} = 0.0127389 \quad .$$

Then

$$SS(L_{A_1}) = \underline{\hat{\tau}}'\underline{\ell}_1(1)[\underline{\ell}_1(1)S^{-1}\underline{\ell}_1(1)]^{-1}\underline{\ell}_1'(1)\underline{\hat{\tau}}$$

$$= 0.071855 \quad .$$

$SS(A_1A_2)$:

$$\underline{\hat{\tau}}'C_{12} = (2.285, -1.855, 0.485, 0.425, -1.455, 0.865)$$

$$C_{12}'S^{-1}C_{12} = \begin{bmatrix} 58.5 & 9.5 & 0.5 & -2.5 & 4.5 & 1.5 \\ 9.5 & 138.5 & -2.5 & 0.5 & 1.5 & 4.5 \\ 0.5 & -2.5 & 10.5 & 1.5 & -1.5 & -2.5 \\ -2.5 & 0.5 & 1.5 & 26.5 & -2.5 & -1.5 \\ 4.5 & 1.5 & -1.5 & -2.5 & 46.5 & 5.5 \\ 1.5 & 4.5 & -2.5 & -1.5 & 5.5 & 126.5 \end{bmatrix}$$

$$[C_{12}'S^{-1}C_{12}]^{-1} = \begin{bmatrix} 0.0175000 & -0.0012143 & -0.0016071 & 0.0016071 & -0.0016071 & -0.0001071 \\ -0.0012.43 & 0.0073449 & 0.0018010 & -0.0003724 & -0.0000561 & -0.0002132 \\ -0.0016071 & 0.0018010 & 0.0973023 & -0.0053380 & 0.0027487 & 0.0016952 \\ 0.0016071 & -0.0003724 & -0.0053380 & 0.0383737 & 0.0017156 & 0.0002691 \\ -0.0016071 & -0.0000561 & 0.0027487 & 0.0017155 & 0.0219452 & -0.0008584 \\ -0.0001071 & -0.0002133 & 0.0016952 & 0.0002691 & -0.0008584 & 0.0079880 \end{bmatrix}$$

Then

$$SS(A_1A_2) = \underline{\hat{\tau}}'C_{12}[C_{12}'S^{-1}C_{12}]^{-1}C_{12}'\underline{\hat{\tau}}$$

$$= 0.212323$$

$SS(Q_{A_1} \times Q_{A_2} \times L_{A_3})$:

$$\hat{\tau}'\underline{\ell}_{123}(2,2,1) = -0.305$$

$$\underline{\ell}_{123}'(2,2,1)S^{-1}C_{123}(2,2,1) = 26.5$$

$$[\underline{\ell}_{123}'(2,2,1)S^{-1}\underline{\ell}_{123}(2,2,1)]^{-1} = 0.0377358$$

Then

$$SS(Q_{A_1} \times Q_{A_2} \times L_{A_3}) = \underline{\hat{\tau}}'\underline{\ell}_{123}(2,2,1)[\underline{\ell}_{123}'(2,2,1)S^{-1}\underline{\ell}_{123}(2,2,1)]^{-1} \underline{\ell}_{123}'(2,2,1)\underline{\hat{\tau}}$$

$$= 0.003510 \quad .$$

Thus we obtain the following Table 2.

Table 2.

Analysis of Variance

Source of variation	d.f.		s.s.	
Total	43		69.3586	
CFM	1		68.115684	
Among groups	23		0.936266	
Within groups	19		0.306650	
A_1	3		0.076645	
L_A		1		0.071855
Q_{A_1}		1		0.000002
C_{A_1}		1		0.002716
A_2	2		0.010012	
L_{A_2}		1		0.008288
Q_{A_2}		1		0.002454
A_3	1		0.369602	
$A_1 \times A_2$	6		0.212323	
$L_{A_1} \times L_{A_2}$		1		0.089252
$L_{A_1} \times Q_{A_2}$		1		0.024845
$Q_{A_1} \times L_{A_2}$		1		0.022402
$Q_{A_1} \times Q_{A_2}$		1		0.006816
$C_{A_1} \times L_{A_2}$		1		0.045527
$C_{A_1} \times Q_{A_2}$		1		0.005915
$A_1 \times A_3$	3		0.080971	
$L_{A_1} \times L_{A_3}$		1		0.004360
$Q_{A_1} \times L_{A_3}$		1		0.013657
$C_{A_1} \times L_{A_3}$		1		0.057474

Table 2. Continued

Source of variation	d.f.	s.s.
$A_2 \times A_3$	2	0.045573
$L_{A_2} \times L_{A_3}$	1	0.007202
$Q_{A_2} \times L_{A_3}$	1	0.035141
$A_1 \times A_2 \times A_3$	6	0.083617
$L_{A_1} \times L_{A_2} \times L_{A_3}$	1	0.075026
$L_{A_1} \times Q_{A_2} \times L_{A_3}$	1	0.003290
$Q_{A_1} \times L_{A_2} \times L_{A_3}$	1	0.002593
$Q_{A_1} \times Q_{A_2} \times L_{A_3}$	1	0.003510
$C_{A_1} \times L_{A_2} \times L_{A_3}$	1	0.000736
$C_{A_1} \times Q_{A_2} \times L_{A_3}$	1	0.000005

6. Randomized Complete Block Design with Missing Plots

Let $y_{jgi} = y_{jg(i_1,i_2,\cdots,i_n)}$ be the j^{th} observation made on the treatment combination $(i_1,i_2,\cdots,i_n)$ in the g^{th} block, where $j=1,2,\cdots,r_{gi}$ ($r_{gi} \neq 0$ for some g); $g=1,2,\cdots,b$; $i=1,2,\cdots,v$ and let N be the total number of observations.

6.1. Block Effect is Fixed

Assume the $\{y_{jgi}\}$ are assumed to be independently distributed following a normal distribution with

$$E\, y_{jg(i_1,i_2,\cdots,i_n)} = \beta_g + \tau_{(i_1,i_2,\cdots,i_n)} \tag{6.1}$$

$$\text{var}\ y_{jgi} = \sigma^2 \ . \tag{6.2}$$

Using the same notation as section 2

$$\tau_{(i_1,i_2,\cdots,i_n)} = \mu + \sum_{h=1}^{n} \alpha_h(i_h) + \sum_{h}\sum_{\substack{k \\ 1\le h<k\le n}} \alpha_{hk}(i_h,i_k) + \cdots$$

$$+ \alpha_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n)$$

where

$$\sum_{i_h=0}^{q_h-1} \alpha_h(i_h) = 0 \ , \quad \sum_{i_h=0}^{q_h-1} \alpha_{hk}(i_h,i_k) = \sum_{i_k=0}^{q_k-1} \alpha_{hk}(i_h,i_k) = 0$$

$$\sum_{i_1=0}^{q_1-1} \alpha_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n) = \sum_{i_n=0}^{q_n-1} \alpha_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n) = 0$$

By matrix notation

$$EY = X \begin{pmatrix} \underline{\beta} \\ \underline{\tau} \end{pmatrix} \tag{6.3}$$

$$\text{var}\ Y = I\sigma^2 \tag{6.4}$$

where X is an $N \times (b+v)$ matrix and I is an $N \times N$ unit matrix.

Let $X = [X_1\ X_2]$, where X_1 is an $N \times b$ matrix and X_2 is an $N \times v$ matrix, then by the least squares method

$$\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \begin{bmatrix} \hat{\underline{\beta}} \\ \hat{\underline{\tau}} \end{bmatrix} = \begin{bmatrix} X_1'Y \\ X_2'Y \end{bmatrix}$$

$$(X_2'X_2 - X_2'X_1(X_1'X_1)^{-1}X_1'X_2)\hat{\underline{\tau}} = X_2'Y - X_2'X_1(X_1'X_1)^{-1}X_1'Y$$

$$X_2'[I - X_1(X_1'X_1)^{-1}X_1']X_2\hat{\underline{\tau}} = X_2'[I - X_1(X_1'X_1)^{-1}X_1']Y$$

Let G be a g-inverse of $X_2'[I - X_1(X_1'X_1)^{-1}X_1']X_2$, then

$$\hat{\tau} = GX_2'[I - X_1(X_1'X_1)^{-1}X_1']Y + [H - I]Z \quad ,$$

where $H = GX_2'[I - X_1(X_1'X_1)^{-1}X_1']X_2$ and Z is an arbitrary $v \times 1$ column vector of components $z_1, z_2, \cdots, z_v$.

Searle [1965] proved the following Lemma.

Lemma 2. Each of $\underline{\ell}_h'(i_h)G\underline{\ell}_h(i_h), \underline{\ell}_{hk}'(i_h,i_k)G\underline{\ell}_{hk}(i_h,i_k), \cdots, \underline{\ell}_{1,2,\cdots,n}'(i_1,\cdots,i_n)G\underline{\ell}_{1,2,\cdots,n}(i_1,i_2,\cdots,i_n), C_h'GC_h, C_{hk}'GC_{hk}, \cdots, C_{1,2,\cdots,n}'GC_{1,2,\cdots,n}$, and $C'GC$ is a non-singular matrix.

Corollary: In Theorem 2 we can replace S^{-1} by G.

Proof: Since $C_h'\underline{\tau}$ is estimable $C_h'H = C_h'$ then

$$C_h'(H-I) = 0$$

then

$$C_h'\underline{\tau} = C_h'GX_2'[I - X_1(X_1'X_1)^{-1}X_1']Y \quad .$$

Hence

$$\underline{\hat{\tau}}'C_h[C_h'GC_h]^{-1}\sigma^{-2}C_h'\underline{\hat{\tau}}$$

$$= Y'[I - X_1'(X_1'X_1)^{-1}X_1]X_2GC_h(C_h'GC_h)^{-1}C_h'GX_2'[I - X_1(X_1'X_1)^{-1}X_1']\sigma^{-2}Y \quad .$$

Let

$$A = [I - X_1'(X_1'X_1)^{-1}X_1]X_2GC_h(C_h'GC_h)^{-1}C_h'GX_2'[I - X_1(X_1'X_1)^{-1}X_1'] \quad ,$$

then

$$AA = [I - X_1'(X_1'X_1)^{-1}X_1]X_2GC_h(C_h'GC_h)^{-1}C_h'GC_h(C_h'GC_h)^{-1}C_hGX_2'[I - X_1(X_1'X_1)^{-1}X_1']$$

$$= [I - X_1'(X_1'X_1)^{-1}X_1]X_2GC_h(C_h'GC_h)^{-1}C_hGX_2[I - X_1(X_1'X_1)^{-1}X_1']$$

$$= A$$

$$(EY)'A(CY) = (\underline{\beta}'X_1' + \underline{\tau}'X_2')A(X_1\underline{\beta} + X_2\underline{\tau})$$

$$= \underline{\beta}'X_1'AX_1\underline{\beta} + \underline{\beta}X_1'AX_2\underline{\tau} + \underline{\tau}'X_2'AX_1\underline{\beta} + \underline{\tau}'X_2'AX_2\underline{\tau}$$

$$\underline{\beta}'X_1'AX_1\underline{\beta} = \underline{\beta}'X_1'[I - X_1'(X_1'X_1)^{-1}X_1]X_2GC_h(C_h'GC_h)^{-1}C_h'GX_2'[I - X_1(X_1'X_1)^{-1}]X_1\underline{\beta}$$

$$= \underline{\beta} X_1'[I - X_1'(X_1'X_1)^{-1}X_1]X_2GC_h(C_h'GC_h)^{-1}C_hGX_2'[X_1-X_1]\underline{\beta}$$

$$= 0$$

Similarly

$$\underline{\tau}'X_2'AX_2\beta \quad = \quad 0$$

$$\underline{\beta}'X_1'AX_2\underline{\tau} \quad = \underline{\beta}'X_1[I - X_1'(X_1'X_1)^{-1}X_1]X_2GC_h(C_h'GC_h)^{-1}C_h'H\tau$$

$$= \underline{\beta}'X_1(I - X_1'(X_1'X_1)^{-1}X_1]X_2GC_h(C_h'GC_h)^{-1}C_h'\tau$$

$$= 0$$

since $C_h'\tau = 0$. Similarly

$$\underline{\tau}'X_2'AX_2\underline{\tau} \quad = \quad 0 \ .$$

Hence

$$(EY)'A(EY) = 0$$

and

$$r(A) \ = q_h-1 \ .$$

This proves the corollary.

6.2. Block Effect is Random

Assuming $E\beta_g = 0$ and var $\beta_g = \sigma_\beta^2$ for $g=1,2,\cdots,b$, we obtain

$$E\, y_{jgi} = \tau_i$$

$$\text{var}\, y_{jgi} = \sigma_\beta^2 + \sigma^2$$

$$\text{cov}(y_{jgi}, y_{j'gi'}) = \sigma_\beta^2 \quad \text{for} \quad j \neq j' \text{ or } i \neq i'$$

and

$$\text{cov}(y_{jgi}, y_{j'g'i'}) = 0 \quad \text{for} \quad g \neq g'$$

By matrix notation

$$EY = X \begin{bmatrix} \underline{0} \\ \underline{\tau} \end{bmatrix} \tag{6.5}$$

$$\text{cov}\, Y = I\sigma^2 + X_1 X_1' \sigma_\beta^2 \quad . \tag{6.6}$$

Now we need the following Lemma 3.

Lemma 3: If Y is distributed $N(\mu, V)$, then $Y'BY$ is distributed as χ^2 with k degrees of freedom if and only if $\mu' B \mu = 0$ and BV is an idempotent matrix of rank k.

In our case,

$$V = I\sigma^2 + X_1 X_1' \sigma_\beta^2 \tag{6.7}$$

$$B = A\sigma^{-2} \quad . \tag{6.8}$$

Then, since BV = A, clearly the corollary in 6.1 holds in our case.

7. Numerical Example 2

Suppose in the Example 1 the factor A_1 is a block factor, and suppose we missed the observation $y_{1,4(3,2,1)}$; then we shall obtain

$$X_1'X_1 = \begin{bmatrix} 10 & & & \\ & 11 & & 0 \\ & 0 & 12 & \\ & & & 9 \end{bmatrix} \qquad X_1'X_2 = \begin{bmatrix} 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

$$X_2'X_2 = \begin{bmatrix} 8 & & & & & \\ & 7 & & & 0 & \\ & & 8 & & & \\ & 0 & & 8 & & \\ & & & & 7 & \\ & & & & & 4 \end{bmatrix}$$

$$X_1'Y = \begin{bmatrix} 12.07 \\ 13.55 \\ 15.55 \\ 11.33 \end{bmatrix} \qquad X_2'Y = \begin{bmatrix} 8.95 \\ 9.80 \\ 9.59 \\ 10.49 \\ 8.31 \\ 5.36 \end{bmatrix}$$

$$X_2'[I - X_1(X_1'X_1)^{-1}X_1']X_2$$

$$= \begin{bmatrix} 6.458586 & -1.319192 & -1.541414 & -1.545414 & -1.341414 & -0.715152 \\ -1.319192 & 5.791919 & -1.319192 & -1.319192 & -1.119192 & -0.715152 \\ -1.541414 & -1.319192 & 6.458586 & 01.541414 & -1.341414 & -0.715152 \\ -1.541414 & -1.319192 & -1.541414 & 6.458586 & 01.341414 & -0.715752 \\ -1.341414 & -1.119192 & -1.341414 & -1.341414 & 5.758586 & -0.615152 \\ -0.715152 & -0.715152 & -0.715152 & -0.715152 & -0.615152 & 3.475758 \end{bmatrix}$$

$$X_2'[I - X_1(X_1'X_1)^{-1}X_1']Y = \begin{bmatrix} -1.037081 \\ 1.071808 \\ -0.397081 \\ 0.502919 \\ -0.470081 \\ 0.329515 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.269550 & 0.125 & 0.125 & 0.124394 & 0.128915 \\ 0 & 0.125 & 0.25 & 0.125 & 0.125 & 0.125 \\ 0 & 0.125 & 0.125 & 0.25 & 0.125 & 0.125 \\ 0 & 0.124394 & 0.125 & 0.125 & 0.269387 & 0.124710 \\ 0 & 0.128915 & 0.125 & 0.125 & 0.124710 & 0.387742 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & & & 0 & \\ -1 & & 1 & & & \\ -1 & & & 1 & & \\ -1 & 0 & & & 1 & \\ -1 & & & & & 1 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccccc} \multicolumn{2}{c}{\overbrace{\underline{\ell}_1(1) \quad \underline{\ell}_1(2)}^{C_1}} & \overbrace{\underline{\ell}_2(1)}^{C_2} & \multicolumn{2}{c}{\overbrace{\underline{\ell}_{12}(1,1) \quad \underline{\ell}_{12}(2,1)}^{C_{12}}} \end{array} \\ C = & \begin{bmatrix} -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \\ 0 & -2 & -1 & 0 & 2 \\ 0 & -2 & 1 & 0 & -2 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Then we obtain the following analysis of variance.

Table 3.

Analysis of Variance

Source of variation	d.f.	s.s.
Total	42	66.7342
C.F.M.	1	65.625
Block (ignoring treatment effect)	3	0.04805
Among treatment (eliminating block effect)	5	0.415723
Remainder	33	0.645327
A_1	2	0.000369
L_{A_1}	1	0.000031
Q_{A_1}	1	0.000306
A_2	1	0.332748
$A_1 \times A_2$	2	0.060747
$L_{A_1} \times L_{A_2}$	1	0.023990
$Q_{A_1} \times L_{A_2}$	1	0.028877

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