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MARGINAL DISTRIBUTIONS OF SELF-SIMILAR PROCESSES WITH STATIONARY INCREMENTS

by

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Abstract

Marginal distributions of self-similar processes with stationary increments

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Dedicated to Professor John Lamperti in recognition of his pioneering work in this field.

Let $X = (X_t)_{t \geq 0}$ be a real-valued stochastic process which is self-similar with exponent $H > 0$ and has stationary increments. Several results about the marginal distribution of X_1 are given. For $H \neq 1$, there is a universal bound, depending only on H, on the concentration function of $\log X_1^+$. For all $H > 0$, X_1 cannot have any atoms except in certain trivial cases. Some lower bounds are given for the tails of the distribution of X_1 in case $H > 1$. Finally, some results are given concerning the connectedness of the support of X_1.

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Marginal Distributions of Self-Similar Processes with Stationary Increments

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1. Introduction. A real-valued stochastic process $X = (X(t))_{t \geq 0} = (X_t)_{t \geq 0}$ is said to be self-similar with parameter $H > 0$ if

$$X(a\cdot) \stackrel{d}{=} a^H X(\cdot) \quad \text{for all} \quad a > 0 \; , \tag{1}$$

where $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions of the two processes. We will say X is H-sssi if X is self-similar with parameter H and has stationary increments, that is,

$$X(b + \cdot) - X(b) \stackrel{d}{=} X(\cdot) - X(0) \quad \text{for all} \quad b > 0 \; . \tag{2}$$

The importance of H-sssi processes arises from the fact that if Y has stationary increments and if, for some positive function c, the finite-dimensional distributions of $(c(a))^{-1} Y(a\cdot)$ converge to those of a process X as $a \to \infty$, then X is H-sssi for some H. A result of this kind was shown by Lamperti (1962). Lamperti used the name "semi-stable" instead of "self-similar", reflecting the fact that all strictly stable processes of index α are H-sssi with $H = \alpha^{-1}$.

Many articles on H-sssi processes have been published in the last decade. Most of these are listed by at least one of Major (1981), Taqqu (1982) or Vervaat (1982). Lamperti (1972) and Kiu (1975) consider self-similar Markov processes while O'Brien, Torfs and Vervaat (1982) consider self-similar extremal processes. Most of the work on H-sssi processes deals with special classes of H-sssi processes. The present paper on marginal distributions and Vervaat's (1982) paper on sample path properties seem to be the first articles dealing with general properties of H-sssi processes.

We consider several aspects of the (one-dimensional) marginal distributions of an H-sssi process X. In Section 2, we consider the concentration function Q of $\log X_t^+$ ($= \log X_t$ if $X_t > 0$ and $= -\infty$ otherwise), defined by

$$Q(y) = \sup_{b \in \mathbb{R}} P[b < \log X_t^+ < b + y], \quad y > 0, \tag{3}$$

or equivalently by

$$Q(y) = \sup_{b \in \mathbb{R}} P[e^b < X_t < e^{b+y}], \quad y > 0 \tag{4}$$

(cf. Hengartner and Theodorescu (1973) or Petrov (1975)). We show that Q is independent of $t > 0$ and that, for each $H \neq 1$, $Q(y)$ has a universal upper bound $Q_H(y)$ such that $Q_H(y) \to 0$ as $y \downarrow 0$. In Section 3 we show that the distribution of $X(t)$ has no atoms for $t > 0$, except in certain trivial cases. In Section 4, we present a lower bound on $P[X(t) > x]$ for large x, for the case $H > 1$. Finally, in Section 5, we give some conditions under which the distribution of X_t has no "gaps", that is, if $P[X_t > b] > 0$ and $P[X_t < a] > 0$ where

$0 < a < b$, then $P[a < X_t < b] > 0$.

All the proofs of these results use only elementary probability theory. Furthermore, they almost never use the full strength of (1) and (2) but only the one-dimensional versions:

$$X(at) \stackrel{d}{=} a^H X(t) \quad \text{for all} \quad a > 0 \quad \text{and} \quad t \geq 0 ; \tag{5}$$

and

$$X(b+t) - X(b) \stackrel{d}{=} X(t) - X(0) \quad \text{for all} \quad b > 0 \text{ and } t > 0 . \tag{6}$$

Thus, it may very well be possible to obtain better results by a more sophisticated use of (1) and (2). We will indicate several open problems as they arise (cf. Remarks 2,3 and 4).

We conclude this section with several preliminary observations. First, we have restricted attention to the case $H > 0$ in (1) because the situation is transparent for $H \leq 0$, at least if the process X is assumed to be measurable and separable. If H=0, these assumptions imply that X_t is a constant function of t with probability one (wp1) , but X_t can have any distribution. If $H < 0$, then separability implies that $X \equiv 0$ wp1 . These results are proved by Vervaat (1982).

Note that $X_0 \stackrel{d}{=} 2^H X_0$ by (1) so that $P[X_0 = 0] = 1$ since $H > 0$. Thus we may restrict our attention to the distribution of X_t for $t > 0$. Since $X_t \stackrel{d}{=} t^H X_1$, we may narrow our focus further by looking only at the distribution of X_1 . Since $-X$ is H-sssi iff X is H-sssi, we mainly consider the part of the marginal distributions on $[0,\infty)$.

The process $X \equiv 0$ wp1 is obviously H-sssi for all H and any process X for which $X_t \equiv tX_1$ wp1 is 1-sssi. These two special cases must be taken into account when we consider whether the distribution of X_1 has any atoms.

2. <u>A bound on the concentration function</u>. By (4) and (1) we see that

$$Q(y) = \sup_{b\in\mathbb{R}} P[e^{b-H\log t} < X_1 < e^{b+y-H\log t}]$$

which is obviously independent of $t > 0$. The main result of this section is Theorem 1 below, which shows that for each $H \neq 1$, there is a universal bound on the concentration function of $\log X_t^+$ for all H-sssi processes X. We first need the following simple lemma.

<u>Lemma 1</u>. Let $A_1, A_2, \ldots, A_n$ be events such that $P(A_i) \geq p$ for all i. If $n = [2p^{-1}]$, where $[\cdot]$ denotes the greatest integer function, then

$$\max_{1\leq i<j\leq n} P(A_iA_j) \geq \tfrac{1}{2} p^2 . \tag{7}$$

<u>Proof</u>. We have

$$1 \geq P(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \Sigma P(A_i) - \sum_{i>j} P(A_iA_j)$$

$$\geq np - \binom{n}{2} \max_{i\neq j} P(A_iA_j) .$$

Let $\delta = 2p^{-1} - [2p^{-1}]$ (so that $0 \leq \delta < 1$). Then

$$\max_{i\neq j} P(A_iA_j) \geq (np - 1)\left(\binom{n}{2}\right)^{-1}$$

$$= \frac{\frac{1}{2} p^2(1 - p\delta)}{1 - p\delta - (\frac{1}{2}p - \frac{1}{4}p^2\delta(1 + \delta))} \tag{8}$$

$$\geq \frac{1}{2} p^2 . \quad \square$$

Remark 1. One can improve the lower bound in (7) somewhat by choosing n so as to maximize the right side of the first inequality in (8) (e.g. with $p = \frac{1}{2}$, take $n = 3$ to get the bound $1/6$). The bound in (7) cannot be more than doubled, as can be seen from the independent case. The value of n cannot be improved (i.e. decreased) to less than p^{-1}, as can be seen from the disjoint case.

Theorem 1. For each $H \neq 1$, there is a function $Q_H: (0,\infty) \to (0,1]$ such that

$$\lim_{y\downarrow 0} Q_H(y) = 0 \tag{9}$$

and such that, for any H-sssi process X,

$$Q(y) \leq Q_H(y) \quad \text{for all} \quad y > 0 , \tag{10}$$

where Q is the concentration function of $\log X_t^+$, $t > 0$.

Proof. Fix $H \neq 1$. We will show that for any $p \in (0,1)$ there exists a positive real number $C = C(H,p)$ such that, for any H-sssi process X and any $b \in \mathbb{R}$,

$$P[b < \log X_1^+ < b + C] \leq p . \tag{11}$$

Define $Q_H : (0,\infty) \to [0,1]$ by

$$Q_H(y) = \inf \{p > 0 : y \leq C(H,p)\} .$$

Then (9) follows from the fact that $C(H,p) > 0$ for all $p > 0$. By (11), $Q(y) \leq p$ if $y \leq C(H,p)$, which implies (10).

In order to prove (11) for some $C > 0$, fix $p \in (0,1)$ and set $m = [2p^{-2}] + 1$ and $n = [2p^{-1}]$. Consider the function $g: (0,1) \to \mathbb{R}$ given by

$$g(u) = (1 - u^H)(1 - u)^{-H} .$$

Since $g'(u) = H(1 - u^{H-1})(1 - u)^{-H-1}$, we see that g is strictly increasing if $H > 1$ and strictly decreasing if $0 < H < 1$. It follows that the numbers $g(e^{-1})$, $g(e^{-2})$, $g(e^{-3})$, ... are distinct. For $u \in (0,1)$ and $r > 1$, consider the intervals

$$B(u,r) = ((1 - ru^H)(1 - u)^{-H} , (r - u^H)(1 - u)^{-H}) .$$

It is clear that $g(u) \in B(u,r)$ for any $r > 1$ and that $B(u,r)$ shrinks to the singleton set $\{g(u)\}$ as r decreases to 1. Thus, there exists a real number $r > 1$, depending only on H and p, such that the intervals $B(e^{-i}, r)$, $i = 1,2,\ldots,n^m$, are disjoint.

For $x > 0$ and r as just described, consider the events

$$A(i) = \{xe^{iH} < X(e^i) < rxe^{iH}\}, \quad i = 0,1,2,\ldots \qquad (12)$$

We will show that (11) holds with $C = \log r$. If this is not so for some

$b \in \mathbb{R}$, then fix $x = e^b$ in (12) . Then $P(A(0)) > p$, and, by self-similarity, $P(A(i)) > p$ for all $i \geq 0$. Let $\beta \in \{0,1,\ldots,m-1\}$. By Lemma 1, there exist i and j with $0 \leq i < j < n$ for which $P(A(i')A(j')) \geq \frac{1}{2} p^2$, where $i' = in^\beta$ and $j' = jn^\beta$. By (1) and (2), we then have (for each β)

$$
\begin{aligned}
&P[x^{-1} X_1 \in B(e^{i'-j'}, r)] \\
&\quad = P[x(e^{j'H} - re^{i'H})(e^{j'} - e^{i'})^{-H} < X_1 < x(re^{j'H} - e^{i'H})(e^{j'} - e^{i'})^{-H}] \\
&\quad = P[x(e^{j'H} - re^{i'H}) < X(e^{j'}) - X(e^{i'}) < x(re^{j'H} - e^{i'H})] \\
&\quad \geq P(A(i')\, A(j')) \geq \tfrac{1}{2} p^2 > m^{-1} .
\end{aligned}
$$

This is impossible since the m events $\{x^{-1} X_1 \in B(e^{i'-j'}, r)\}$ are disjoint. Thus (11) must hold with $C = \log r$. □

Remark 2. We have not given an explicit expression for Q_H in Theorem 1 . It would be difficult to calculate and, in any case, is probably gigantic compared to the best possible bound. In particular, our bound is insufficient for providing a positive answer to the following open question. Is the distribution of X_1 outside 0 absolutely continuous if $H \neq 1$?

3. Continuity of marginal distribution functions. In this section, we will show that if X is H-sssi then X_1 has no atoms, except for some trivial cases. We begin with three lemmas.

Lemma 2. Let X be H-sssi for some $H \neq 1$. Then X_1 has no atoms except possibly at zero.

Proof. This is an immediate consequence of Theorem 1 .

Lemma 3. Let X be a separable H-sssi process for some $H > 0$. Then

$$P[X_1 = 0] = P[X \equiv 0] .$$

Proof. By (1), the quantity $p := P[X_t = 0]$ is independent of $t > 0$. By (2), we have

$$P[X_s = X_t] = P[|X_s - X_t| = 0] = P[X_{|t-s|} = 0] = p \tag{13}$$

for all $s,t \geq 0$ with $s \neq t$. Again applying (1) we have, for fixed t and for M and then u sufficiently large, that

$$\begin{aligned} P[X_t = X_u \neq 0] &\leq P[|X_t| \geq M] + P[0 < |X_u| < M] \\ &= P[|X_t| \geq M] + P[0 < |X_1| < Mu^{-H}] \\ &< 2\varepsilon \end{aligned}$$

for any $\varepsilon > 0$, so that

$$\lim_{u\to\infty} P[X_t = X_u \neq 0] = 0 . \tag{14}$$

Combining (13) and (14) we have for $x \neq t$

$$\begin{aligned} P[X_s = 0 , X_t \neq 0] &\leq P[X_s = 0 , X_u \neq 0] + P[X_u = 0 , X_t \neq 0] \\ &= P[X_s = 0] + P[X_s = X_u \neq 0] - P[X_s = X_u] \\ &+ P[X_u = 0] + P[X_u = X_t \neq 0] - P[X_u = X_t] \\ &\to 0 , \end{aligned}$$

as $u \to \infty$. Thus $P[X_s = 0 \mid X_t = 0] = 1$ if $p > 0$. Since X is separable, $P[X \equiv 0 \mid X_t = 0] = 1$. □

Lemma 4. Let X be a separable 1-sssi process. For $x \in \mathbb{R}$,

$$P[X_1 = x] = P[X_t \equiv tx] .$$

Proof. The process $Y_t := X_t - tx$, $t \geq 0$, is also separable and 1-sssi. The result follows by applying Lemma 3 to Y. □

Let X be a separable H-sssi process. The event $\{X_t \equiv tX_1\}$, which for $H \neq 1$ differs from the event $\{X_t \equiv 0\}$ by a set of probability 0, is invariant under the transformations in (1) and (2). If this event has probability less than 1, then conditioning on its complement leads to a new separable H-sssi process. Combining the last three Lemmas, we obtain:

Theorem 2. Let X be a separable H-sssi process for which $P[X_t \equiv tX_1] < 1$. For $x \in \mathbb{R}$,

$$P[X_1 = x \mid X_t \not\equiv tX_1] = 0 .$$

4. The tails of the marginal distributions. In this section, we assume that X is H-sssi with $H > 1$. If X is strictly stable then

$$\lim_{x\to\infty} x^{1/H} P[X_1 > x] = c \tag{15}$$

where c is a positive constant if $P[X_1 > 0] > 0$ (cf. Feller (1971), Lemma XVII.7.1) As will be shown in Remark 3 below, (15) is not valid for general H-sssi processes, but it is possible that the lower bound on the tails does extend to the general case, i.e., that

$$\liminf_{x\to\infty} x^{1/H} P[X_1 > x] > 0 \text{ in case } P[X_1>0] > 0. \tag{16}$$

We give partial results in this direction in Theorems 3 and 4 below. We will have occasion to use the following lemma.

Lemma 5. Let Y be a real-valued random variable such that

$$P[Y > x] \leq r P[Y > r^d x] \quad \text{for all} \quad x \geq \beta \tag{17}$$

where $r > 1$, $d > 0$ and $\beta > 0$. Then

$$P[Y > x] > \beta^{1/d} x^{-1/d} P[Y > r^d \beta] \quad \text{for all} \quad x \geq r^d \beta \quad .$$

Proof. Let n be the positive integer such that

$$\beta r^{nd} \leq x < \beta r^{(n+1)d} \quad .$$

Iterating (17), we obtain

$$\begin{aligned} P[Y > x] &\geq r^{-1} P[Y > r^{-d} x] \\ &\geq r^{-n} P[Y > r^{-nd} x] \\ &\geq (\beta x^{-1})^{1/d} P[Y > r^d \beta] \ . \quad \square \end{aligned}$$

Theorem 3. Let X be H-sssi with $X > 1$ and suppose $P[X_1 > 0] > 0$. Then there are positive constants C and β such that

$$P[X_1 > x] \geq Cx^{-1/(H-1)} \quad \text{for all} \quad x \geq \beta \ .$$

<u>Proof</u>. Observe that

$$\begin{aligned} P[X_1 > x] &\leq P[X_{\frac{1}{2}} > \tfrac{1}{2}x] + P[X_1 - X_{\frac{1}{2}} > \tfrac{1}{2}x] \\ &= 2P[X_{\frac{1}{2}} > \tfrac{1}{2}x] \qquad (18) \\ &= 2P[X_1 > 2^{H-1}x] \end{aligned}$$

for all $x > 0$. Now apply Lemma 5 with $Y = X_1$, $r = 2$, and $d = H - 1$. □

We can do a bit better with the help of an extra regularity assumption which holds in many cases including that of stable processes.

<u>Theorem 4</u>. Let X be X-sssi with $H > 1$ and suppose $P[X_1 > 0] > 0$. Assume

$$P[X_2 - X_1 > x \mid X_1 > x] \to 0 \quad \text{as} \quad x \to \infty \ . \tag{19}$$

Then for all $\alpha > H^{-1}$,

$$\lim_{x\to\infty} x^{\alpha} P[X_1 > x] = \infty \ . \tag{20}$$

<u>Proof</u>. Note first that the left side of (19) is well-defined since $P[X_1 > x] > 0$ for all $x > 0$ by Theorem 3. Fix $\delta > 0$, let $\varepsilon = 2^H - 2^{H-\delta}$ and let k be an integer such that $2^{k(H-1)}\varepsilon > 2^{H-\delta}$.

We have

$$
\begin{aligned}
P[X_1 > x] &= P[X_2 > 2^H x] \\
&\le P[X_1 > 2^{H-\delta}x] + P[X_2 - X_1 > 2^{H-\delta}x] \\
&\qquad + P[X_1 > \varepsilon x \ , \ X_2 - X_1 > \varepsilon x] \\
&= 2P[X_1 > 2^{H-\delta}x] + o(P[X_1 > \varepsilon x])
\end{aligned}
\tag{21}
$$

as $x \to \infty$, by (19). By the argument at (18) and iteration,

$$
\begin{aligned}
P[X_1 > \varepsilon x] &\le 2P[X_1 > 2^{H-1}\varepsilon x] \\
&\le 2^k \, P[X_1 > 2^{k(H-1)}\varepsilon x] \\
&\le 2^k \, P[X_1 > 2^{H-\delta}x] \quad .
\end{aligned}
$$

Combining this with (21), we find

$$P[X_1 > x] \le (2 + \delta) \, P[X_1 > 2^{H-\delta}x]$$

for x sufficiently large. Applying Lemma 5 with $Y = X_1$, $r = 2 + \delta$ and $d = (H-\delta)(\log 2)(\log(2 + \delta))^{-1}$, we have

$$P[X_1 > x] > Cx^{-1/d} \tag{22}$$

for some positive constant C and for x sufficiently large. Choosing δ sufficiently small that $H > d > \alpha^{-1}$, we obtain (20) from (22). □

<u>Remark 3</u>. (a) $H > 1$. It is an open question whether (19) must always hold. It is also open whether (16) or even (20) must always hold.

On the other hand, the tails of the distribution may be thicker than in the stable case (cf. (15)). Consider for example the processes which Kesten and Spitzer (1979) obtain as limits of their so-called random walks in random scenery. Their processes are H-sssi with $H = 1 - \alpha^{-1} + \alpha^{-1}\beta^{-1}$, for any α and β with $1 \leq \alpha \leq 2$ and $0 < \beta \leq 2$, whereas the marginal distributions are strictly stable with index β (cf. their Lemma 5) so that, for $0 < \beta < 2$, $x^{\beta} P[X_1 > x] \to c > 0$ as $x \to \infty$. If $0 < \beta < 1$ and $1 < \alpha \leq 2$ we have $H > 1 > \beta H$, thereby giving a counterexample to (15).

(b) $0 < H < 1$. In this case, the tails may either thicker or thinner than indicated by (15). For any $H \in (0,1)$ and $\beta \in (0,2)$, there is a fractional stable process (cf. Maejima (1982), Taqqu and Wolpert (1981), or Vervaat (1982), § 5.5) that is H-sssi and whose marginals are strictly stable with index β. Other examples with thin tails are fractional Brownian motion (Taqqu (1982)) which has normal marginals (this includes regular Brownian motion) and the aforementioned processes of Kesten and Spitzer (1979) with $\beta \geq 1$ and $1 < \alpha \leq 2$ (here $H \leq 1 \leq \beta H$).

(c) $H = 1$. Excluding the linear case $X_t \equiv tX_1$, the only example for which we have computed the tails of the marginals is the case of the symmetric Cauchy process, for which (15) does hold.

5. <u>Support of the marginal distributions</u>. Let X be H-sssi for some $H > 0$. To avoid trivialities, we assume throughout this section that X is separable and $P[X_t \equiv tX_1]=0$. We wish to consider what sets are possible as the support of X_1, which is defined to be the smallest closed set S for which

$P[X_1 \in S] = 1$. We startwith the following result for the case $H > 1$.

Theorem 5. Let X be H-sssi where $H > 1$. If the support S of X_1 has the property that $S \cap (0,\infty)$ and $S \cap (-\infty,0)$ are both connected, then S is one of $\mathbb{R}$, $[b,\infty)$, $(-\infty,a]$ or $(-\infty,a] \cup [b,\infty)$ where $a \leq 0 \leq b$. All these cases are possible.

Proof. The fact that S must have one of the above forms follows immediately from the connectedness requirement and Theorem 3. If X is generated by a dyadic lattice process according to the scheme developed in O'Brien and Vervaat (1982), § 4 , then X_1 has support $[b,\infty)$ for some $b > 0$. Multiplication of X by any constant $r \neq 0$ shows that any set $[b,\infty)$ or $(-\infty,a]$ is possible for $a < 0 < b$. The one-sided strictly stable processes (with index $\alpha = H^{-1} < 1$) show that $[0,\infty)$ and $(-\infty,0]$ are possible. If Y has support $(-\infty,b]$ and Z has support $[a,\infty)$, then the process X , obtained by tossing a coin which is independent of (Y,Z) and letting $X = Y$ or $X = Z$ on the basis of the outcome of the toss, has support $(-\infty,a] \cup [b,\infty)$. Taking $a = b = 0$ yields $S = \mathbb{R}$. □

We have much less information about the support S of X_1 if X is H-sssi for some $H \leq 1$. If $H < 1$ then S intersects both $(-\infty,0)$ and $(0,\infty)$. This is a consequence of Theorem 3.1a of Vervaat (1982), which shows in particular that

$$\limsup_{h \downarrow 0} h^{-1} X_h = \infty \quad \text{and} \quad \liminf_{h \downarrow 0} h^{-1} X_h = -\infty \quad \text{wp1.}$$

proof of Theorem 6.

Remark 4. The conclusion of Theorems 6 and 7 can be strengthened to the statement that (23) holds for infinitely many positive integers n. To see this, observe that if (23) holds for some n, then for any $t > 1$, either

$$P[X_1 < a , X_t > bt^H] = P[X_1 > b , X_t < at^H] > 0$$

or else the events $\{X_1 \le a\}$ and $\{X_t \le at^H\}$ differ by a null set, which implies that

$$P[X_1 < a , X_{nt} > b(nt)^H] = P[X_t < at^H , X_{nt} > b(nt)^H] > 0 .$$

The theorems of this section leave two main unanswered questions. Is it possible, if $H \le 1$, to have the support of X_1 bounded or at least bounded above? Is it necessarily the case, for general $H > 0$, that the support of X_1 must be connected, except for the two component situation described in Theorem 5?

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