GROUP-VALUED IMPLOSION AND CONJUGATION SPACES

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This thesis consists of two independent parts.

In the first part we discuss group-valued moment maps. Using *group-valued implosion*, introduced by Hurtubise, Jeffrey and Sjamaar, we construct a new class of examples of quasi-Hamiltonian spaces. Associated to each compact Lie group G there is a universal imploded space $D(G)_{impl}$. For $G = \mathbf{Sp}(n)$ we show that there is a stratum of $D(G)_{impl}$ which has a smooth closure diffeomorphic to $\mathbf{HP}^n$ - a quaternionic projective space. We show that $\mathbf{HP}^n$ and S^{2n} exhaust all examples arising from this construction.

The second part is concerned with "conjugation spaces". In particular we study conjugation spaces with a compatible Lie group action. For Lie groups of type A and C, we show that there is a degree halving ring isomorphism from equivariant cohomology of the space to equivariant cohomology of its fixed point set under an involution.

BIOGRAPHICAL SKETCH

Alimjon Eshmatov was born in Tashkent, Uzbekistan in June 1983. Since his father is mathematician, he was exposed to mathematics at an early age. While in school he successfully participated in various mathematical contests. After high school he entered to National University of Uzbekistan formerly known as Tashkent State University. During his undergraduate studies, he was awarded the Presidential Fellowship. After completing his undergraduate studies, he was accepted to Cornell University to pursue a Ph.D. degree. In Cornell he studied under the supervision of Reyer Sjamaar. He completed his thesis in the summer of 2009.

Dedicated to my parents Khasan Eshmatov and Toshbibi Norbabaeva.

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CHAPTER 1

A NEW CLASS OF EXAMPLES OF GROUP-VALUED MOMENT MAPS

1.1 Introduction

Hamiltonian geometry is the geometry of symplectic manifolds equipped with a moment map, a certain collection of quantities conserved by symmetries. Strictly speaking, it is a smooth map from a symplectic manifold M to the dual of the Lie algebra of a group G acting on M , whose components are Hamiltonian functions for the infinitesimal action on M of elements of the Lie algebra. In recent years moment maps have become an important tool in geometry and topology of symplectic manifolds, to name few: symplectic reductions, convexity theorems, symplectic cutting etc (see also [MS]).

For the last two decades, there have been several attempts to extend the notion of the moment map to a more general framework. Based on Drinfeld's Poisson-Lie group [D], Lu and Weinstein study actions of such groups where the moment map takes values in the dual Poisson-Lie group [LW]. Later Alekseev showed that in most interesting cases they were equivalent to usual Hamiltonian actions [A].

The notion of a S^1-valued moment map for a circle action has been considered in [M1] as a natural generalization of Hamiltonian S^1-manifolds. The notion of group-valued moment map for an arbitrary compact Lie group was introduced by Alekseev, Malkin and Meinrenken [AMM]. In contrast to its classical counterpart, the moment map takes values in a Lie group instead of the dual of the Lie algebra. Smooth manifolds equipped with group-valued moment maps are called quasi-Hamiltonian manifolds. Quasi-Hamiltonian manifolds and their moment maps share many features of the Hamiltonian ones, such as reduction, cross-section and implosion. In fact there is a one-to-one correspondence between compact quasi-Hamiltonian G-spaces and infinite-dimensional Hamiltonian LG-spaces with a proper moment map, where LG is the loop group of G.

The motivation of [AMM] for developing a theory of group-valued moment maps comes from one particularly important result. They show that the moduli space $M(\Sigma)$ of flat connections on a closed Riemann surface Σ of genus k is a quasi-Hamiltonian quotient of G^{2k}, the product of $2k$-copies of G, which possesses a natural quasi-Hamiltonian G-structure. Hence it is a symplectic manifold, a result earlier obtained by M. Atiyah and R. Bott [AB]. They go further generalizing it to the case $M(\Sigma, C)$, the moduli space of flat connection on a Riemann surface with boundaries, with fixed conjugacy classes of holonomies associated to the boundary components. One should remark that an analogous description has been obtained by W. Goldman [G] but unlike [AMM] Goldman constructs the symplectic structure using an infinite-dimensional description of [AB].

Due to its somewhat complicated definition it is hard to check whether a given G-manifold possesses a quasi-Hamiltonian structure. The main examples in the original paper [AMM] included conjugacy classes of G and $D(G)$ the product of two copies of a Lie group. On the other hand using the *exponentiation* operation one can build up a quasi-Hamiltonian manifold from a Hamiltonian one. The first non-trivial example appeared in [AMW] and [HJ], where authors constructed a quasi-Hamiltonian structure on S^4. In both constructions it was not clear whether one could find a quasi-Hamiltonian structure on S^{2n} - the $2n$-dimensional sphere. Later it became apparent that these were in fact special cases of more general construction of *imploded spaces* [HJS]. Like most constructions of quasi-Hamiltonian manifolds, it was first defined for Hamiltonian spaces. Recall a *symplectic implosion* is an "abelianization" operation which transforms a Hamiltonian G-manifold into a Hamiltonian T-space preserving some properties of the manifold, but at the expense of producing singularities [GJS]. The singularities are however not completely arbitrary. In particular, the imploded spaces stratify naturally into symplectic manifolds.

By imitating symplectic implosion [GJS], J. Hurtubise, L. Jeffrey, and R. Sjamaar introduced the notion of group-valued implosion [HJS]. Let $D(G)_{\mathrm{impl}}$ be an imploded space of $D(G)$. It was observed in [HJS] that there are certain strata in $D(G)_{\mathrm{impl}}$ where it is singular but whose closure is smooth. This observation led them to a new class of examples of quasi-Hamiltonian manifolds. In particular, when G is A-type, $G = \mathbf{SU}(n)$, there is a one dimensional face of the alcove whose corresponding stratum has a smooth closure diffeomorphic to S^{2n}. As a result, they showed that S^{2n} is a quasi-Hamiltonian $\mathbf{U}(n)$ - space. In particular for $n = 2$ as expected this coincides with a quasi-Hamiltonian structure on S^4 defined in [HJS, AMW]. Motivated by this example, we study the implosion for Lie groups of an arbitrary type. In particular we have shown that for type C, i.e. $G = \mathbf{Sp}(n)$, a unitary quaternionic group, there is a certain stratum of $D(G)_{\mathrm{impl}}$ which has a smooth closure diffeomorphic to $\mathbf{HP}^n$. Unlike in [HJS] for S^{2n}, which they obtained by gluing two copies of $\mathbf{C}^n$, a covering given by strata in our case is not affine. This makes computations more difficult, without referring to Hamiltonian spaces. On the other hand, it also gives a new class of examples of multiplicity-free quasi-Hamiltonian spaces with a non-effective

$G \times T$ action. It is interesting to observe that both of these classes of examples have neither a symplectic nor complex structure. We also study $D(G)_{\mathrm{impl}}$ for other type of Lie groups. In particular, we have shown that the above examples exhaust all possible examples which can be obtained using this construction.

1.2 Quasi-Hamiltonian manifolds

In this section we briefly review some of the basic definitions and results on quasi-Hamiltonian manifolds. First we recall the definition of usual Hamiltonian spaces. Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. Given a G-manifold M, there is an induced infinitesimal Lie algebra action

$$\xi_M(x) = \frac{d}{dt}|_{t=0} \exp(-t\xi).x \quad \text{for} \quad \xi \in \mathfrak{g},\ x \in M. \tag{1.2.1}$$

A *Hamiltonian G-manifold* is a symplectic G-manifold (M, ω) with an equivariant map, called moment map, $\Phi : M \to \mathfrak{g}^*$ satisfying the relation

$$\iota(\xi_M)\omega = d\langle \Phi, \xi \rangle. \tag{1.2.2}$$

for all $\xi \in \mathfrak{g}$. Let $\theta_L, \theta_R \in \Omega^1(G, \mathfrak{g})$ be the Maurer-Cartan forms defined by $\theta_{L,g}(L(g)_*\xi) = \xi$ and $\theta_{R,g}(R(g)_*\xi) = \xi$ for $\xi \in \mathfrak{g}$, where $L(g)$ denotes the left multiplication and $R(g)$ the right multiplication by g. Let $(\cdot\,,\cdot)$ be some choice of invariant inner product on $\mathfrak{g}$. Then there is a closed bi-invariant three-form on G given by

$$\chi = \frac{1}{12}(\theta_L, [\theta_L, \theta_L]) = \frac{1}{12}(\theta_R, [\theta_R, \theta_R])\,.$$

Definition 1. *[AMM] A quasi-Hamiltonian G-manifold is a triple (M, Φ, ω) where M is G-manifold equipped with a G-invariant two-form ω and a G-equivariant map*

$\Phi : M \to G$, *called the group-valued moment map, such that the following properties hold:*

(i) $d\omega = -\Phi^* \chi$

(ii) $\mathtt{Ker}\omega_x = \{\xi_M | \xi \in \mathtt{Ker}(\mathtt{Ad}_{\Phi(x)} + 1)\}$ *for all* $x \in M$

(iii) $\iota(\xi_M)\omega = \frac{1}{2}\Phi^*(\theta_L + \theta_R, \xi)$

The basic examples of quasi-Hamiltonian manifolds are conjugacy classes of the Lie group G, and its "double" $D(G)$. One can think of them as analogs of coadjoint orbits and cotangent bundle respectively.

Conjugacy classes

Let C be a conjugacy class in G. Define a G-invariant 2-form ω on C such that for each $g \in C$, the value of this form on fundamental vector fields v_ξ, v_η is given by

$$\omega(v_\xi, v_\eta) = \frac{1}{2}(\xi, (\mathtt{Ad}_g - \mathtt{Ad}_g^{-1})\eta).$$

Let $\iota : C \to G$ be an inclusion map . One can show

Proposition 1. *[AMM, Proposition 3.1]* (C, ι, ω) *is a quasi-Hamiltonian G-manifold.*

The double D(G)

As a space $D(G)$ is defined by

$$D(G) := G \times G. \tag{1.2.3}$$

A $G \times G$ action on $D(G)$ is given by

$$(g_1, g_2).(u, v) = (g_1 u g_2^{-1}, \mathtt{Ad}_{g_2} v). \tag{1.2.4}$$

Define a moment map $\Phi : D(G) \longrightarrow G \times G$ by $\Phi = \Phi_1 \times \Phi_2$ where

$$\Phi_1(u, v) = \mathtt{Ad}_u v^{-1}, \quad \Phi_2(u, v) = v, \tag{1.2.5}$$

and the two-form

$$\omega = -\frac{1}{2}(\mathtt{Ad}_v u^* \theta_L, u^* \theta_L) - \frac{1}{2}(u^* \theta_L, v^*(\theta_L + \theta_R). \tag{1.2.6}$$

The following statement is proved in [AMM, Proposition 3.2].

Proposition 2. *$(D(G), \Phi, \omega)$ is a quasi-Hamiltonian $G \times G$-manifold.*

Remark 1. *The original definition of quasi-Hamiltonian spaces in [AMM] which we stated above is given only for compact Lie group actions. Recently, using so called Dirac structures A. Alekseev, H. Bursztyn and E. Meinrenken in [ABM] have proposed a new definition of quasi-Hamiltonian G-spaces which is applicable for a wide class of non-compact Lie group actions.*

1.3 Relation between Hamiltonian and quasi-Hamiltonian manifolds

In this section we will describe a construction of a quasi-Hamiltonian manifold from a Hamiltonian one. Let (M, ω_0, Φ_0) be a Hamiltonian G-manifold. Using an invariant inner product, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then we can regard Φ_0 as a map into $\mathfrak{g}$, and composing it with the exponential map we obtain $\Phi = \exp \circ \Phi_0$. Let $\exp_s \lambda := \exp(s\lambda)$ for $\lambda \in \mathfrak{g}$ and $s \in \mathbf{R}$. Consider the 2-form on $\mathfrak{g}$ defined by

$$\varpi = \frac{1}{2} \int_0^1 (\exp_s^* \theta_R, \frac{\partial}{\partial s} \exp_s^* \theta_R) ds$$

which is a G-invariant primitive of the 3-form $\exp^* \chi$. The *exponentiation* operation changes the 2-form and the moment map on M into $\omega = \omega_0 + \Phi_0^* \varpi$ and $\Phi = \exp \circ \Phi_0$. One can show that if the exponential map is regular on the image $\Phi_0(M)$ then (M, Φ, ω) with the same G-action is a quasi-Hamiltonian space.

The inverse of the exponentiation operation is called *linearization*. Let (M, ω, Φ) be a quasi-Hamiltonian G-manifold. Suppose there exists an $\mathtt{Ad}$-invariant open U in $\mathfrak{g}$ such that $\exp : U \to G$ is a diffeomorphism onto an open subset containing $\Phi(M)$ (with inverse denoted by $\log : \exp U \to U$). The linearization of M is the Hamiltonian G-manifold (M, Φ_0, ω_0) where $\Phi_0 = \log \circ \Phi$ and $\omega_0 = \omega - \Phi_0^* \varpi$ (See [AMM, §3.3]).

There is another remarkable relation between Hamiltonian and quasi-Hamiltonian spaces. It is shown in [AMM, §8] that there exists a one-to-one correspondence between quasi-Hamiltonian G-spaces and Hamiltonian LG-spaces with proper moment map. So, one has always a choice either to work with infinite-dimensional but more conventional object such as LG-Hamiltonian spaces or to use finite-dimensional but somewhat peculiar objects as quasi-Hamiltonian G-spaces.

1.4 Properties of quasi-Hamiltonian manifolds

In this section we will consider some natural operations on quasi-Hamiltonian manifolds. In view of the previous section it is not surprising that many constructions for Hamiltonian manifolds have analogous for quasi-Hamiltonian ones.

1.4.1 Fusion

Recall that Hamiltonian G-spaces have a nice functorial property: namely if K is a Lie subgroup, then by restricting the action we obtain a Hamiltonian K-space with a moment map given by composing the moment of G with the projection $i^* : \mathfrak{g}^* \to \mathfrak{k}^*$ induced from the inclusion $i : \mathfrak{k} \to \mathfrak{g}$. A certain kind of functoriality for group-valued moment maps holds for restriction to a diagonal subgroup.

Let (M, ω, Φ) be a quasi-Hamiltonian $G \times G \times H$-manifold, with moment map $\Phi = \Phi_1 \times \Phi_2 \times \Phi_3 : M \to G \times G \times H$. Let $G \times H$ act on M via the embedding $(g, h) \mapsto (g, g, h)$. Then M with 2-form

$$\tilde{\omega} = \omega + (\Phi_1^* \theta_L, \Phi_2^* \theta_R)$$

and the moment map

$$(\Phi_1 \cdot \Phi_2 \times \Phi_3) : M \to G \times H$$

is a quasi-Hamiltonian $G \times H$-manifold [AMM, Theorem 6.1]. This restriction process from $G \times G \times H$ to $G \times H$ is called the *internal fusion*. An important special case is the Cartesian product $M = M_1 \times M_2$ of two quasi-Hamiltonian $G \times H_i$-spaces (M_i, ω_i, Φ_i) which is a quasi-Hamiltonian $G \times G \times H_1 \times H_2$-space in a natural way. We define their fusion product $M_1 \circledast M_2$ to be the quasi-Hamiltonian $G \times H_1 \times H_2$-space obtained by fusing two copies of G.

1.4.2 Quasi-symplectic reduction

In this subsection we discuss reduction, another important operation, which behaves well for quasi-Hamiltonian spaces.

Let (M, ω, Φ) be a quasi-Hamiltonian G-manifold such that G is a product $G_1 \times G_2$. Let $\Phi = (\Phi_1, \Phi_2)$ be the corresponding components of the moment map Φ. We want to reduce the space with respect to the first factor. Let $g \in G_1$ be regular value so that $\Phi_1^{-1}(g)$ is a smooth manifold. Then the centralizer $(G_1)_g$ acts locally freely on the submanifold $\Phi^{-1}(g)$ and we define the *quasi-symplectic quotient* at g to be a topological space

$$M//_g G_1 = \Phi_1^{-1}(g)/(G_1)_g. \tag{1.4.1}$$

Under above assumptions, we have

Theorem 1. *[AMM, Theorem 5.1] The restriction of ω to $\Phi_1^{-1}(g)$ descends to $M//_g G_1$ and makes it a quasi-Hamiltonian G_2-space where the map $M//_g G_1 \to G_2$ induced by Φ_2 is a moment map for the induced G_2-action on $M//_g G_1$.*

Remark 2. *In case when G_2 is abelian, $M//_g G_1$ is a symplectic orbifold.*

Let G_2 be an abelian Lie group. In the singular case, the quotient space stratifies into symplectic manifolds according to orbit type. Let g be an arbitrary element of G_1. For each subgroup H define a $(G_1)_g$-invariant submanifold $M_{(H)}$ consisting of all points such that the stabilizer $(G_1)_g \cap (G_1)_x$ is conjugate to H. Put $Z = \Phi_1^{-1}(g)$ and $Z_{(H)} = Z \cap M_{(H)}$. Let Z_i be the collection of connected components of $Z_{(H)}$, where H ranges over all conjugacy classes of $(G_1)_g$. Then we have the following decomposition:

$$M//_g G_1 = \bigsqcup_{i \in I} Z_i/(G_1)_g. \tag{1.4.2}$$

Theorem 2. *[HJS, Theorem 2.9] The decomposition* (1.4.2) *is a locally normally trivial stratification of $M//_g G_1$ into symplectic submanifolds. Moreover, the stratification is G_2-invariant and the continuous map $\bar{\Phi}_2 : M//_g G_1 \to G_2$ induced by Φ_2 restricts to a moment map for the G_2 action on each stratum.*

1.5 Imploded cross-section.

1.5.1 Symplectic implosion

We start by reviewing this notion in the Hamiltonian case. Symplectic implosion is an "abelianization functor", which transforms a Hamiltonian G-manifold into a Hamiltonian T-space preserving some of properties of the manifold, but at the expense of producing singularities [GJS]. However, like singular symplectic quotients it stratifies naturally into symplectic manifolds in such a way that the T action preserves the stratification.

Fix a maximal torus T of G and an open chamber C in $\mathfrak{t}^*$, the dual of $\mathfrak{t} = Lie(T)$. The closed chamber $\overline{C}$ is a polyhedral cone, which is a disjoint union of relatively open faces (or *walls*). We define a partial order on the faces by putting $\sigma \leq \tau$ if $\sigma \subseteq \overline{\tau}$.

Let (M, ω_0, Φ_0) be a connected Hamiltonian G-manifold. The *principal face* σ_{prin} is the smallest face σ of C such that the Kirwan polytope $\Phi_0(M) \cap \overline{C}$ is contained in the closure of σ. In many cases σ_{prin} is just the whole of C. The cross-section of M is $\Phi_0^{-1}(\sigma_{prin})$. This is a T-invariant connected symplectic submanifold of M. The torus action on the cross-section is a Hamiltonian with moment map equal to the restriction of the G-moment map, and the G-invariant subset $\Phi_0^{-1}(\sigma_{prin})$ is open and dense in M. The imploded cross-section can be thought of as "completion" of the cross-section to a stratified space with symplectic strata. It is obtained by taking the preimage of the closed chamber, $\Phi_0^{-1}(\overline{C})$, which stratifies into smooth manifolds in a natural way, and by quotienting out the null-foliation of the form ω_0 on each stratum. Namely, we define two points m_1

and m_2 in $\Phi_0^{-1}(\overline{C})$ to be equivalent if there exists g in the commutator group $[G_{\Phi_0(m_1)}, G_{\Phi_0(m_1)}]$ such that $m_2 = g.m_1$. (Here G_ξ denotes the centralizer of $\xi \in \mathfrak{g}^*$ under the coadjoint action). Then the imploded cross-section is the quotient space

$$M_{impl} = \Phi_0^{-1}(C)/\sim$$

equipped with the quotient topology. Set-theoretically it is a disjoint union

$$M_{impl} = \coprod_{\sigma \leq C} \Phi_0^{-1}(\exp\sigma)/[G_\sigma, G_\sigma], \tag{1.5.1}$$

over the faces of C. Here K_σ is the centralizer of the face σ. The pieces in the decomposition (1.5.1) are usually not manifolds. However, the decomposition can be refined into a stratification of M_{impl} with symplectic strata. The *imploded moment map* is the continuous map $(\Phi_0)_{impl} : M_{impl} \to \mathfrak{t}^*$ induced by Φ_0. The restriction of $(\Phi_0)_{impl}$ to each stratum is a moment map for the T-action.

Example. Let $G = \mathbf{SU}(2)$ with a maximal torus $T = S^1$. Recall that T^*G is a Hamiltonian $G\times G$-space equipped with canonical symplectic form. Let $(T^*G)_{impl}$ be an implosion with respect to the right action. Then $(T^*G)_{impl}$ is smooth Hamiltonian $G \times T$ and symplectomorphic to $\mathbb{C}^2$. The left action of G on $(T^*G)_{impl}$ is given by the standard representation of $\mathbf{SU}(2)$ on $\mathbb{C}^2$, whereas the right T-action is given by $t.z = t^{-1}z$ (see [GJS]).

1.5.2 Quasi-symplectic implosion

Along the same lines, in [HJS] the implosion was defined for quasi-Hamiltonian spaces. Let (M, ω, Φ) be a quasi-Hamiltonian G-space. In [AMM], it is shown that like in the Hamiltonian case one can prove a convexity theorem. But one has to consider the image of moment map in an alcove instead of a Weyl chamber. Here the assumption of simply connectedness of the group is crucial, since otherwise a description of the space of conjugacy classes is quite complicated. Therefore in the rest of chapter we will assume that G is simply connected.

Let $C^\vee$ be a dual of Weyl chamber C in $\mathfrak{t}$. Let $\mathcal{A}$ be the unique (open) alcove contained in $C^\vee$ such that $0 \in \bar{\mathcal{A}}$. Using the exponential map one can identify $\bar{\mathcal{A}}$ with space of conjugacy classes $T/W \cong G/Ad\,G$, where W is the corresponding Weyl group. Let denote by G_g the centralizer of g in G. For points $m_1, m_2 \in \Phi^{-1}(\exp\bar{\mathcal{A}})$ define $m_1 \sim m_2$ if $m_2 = gm_1$ for some $g \in [G_{\Phi(m_1)}, G_{\Phi(m_1)}]$. One can check that $\sim$ is indeed an equivalence relation.

Definition 2. *The imploded cross-section of M is the quotient space $M_{\mathrm{impl}} = \Phi^{-1}(\exp\bar{\mathcal{A}})/\sim$, equipped with the quotient topology. The imploded moment map Φ_{impl} is the continuous map $M_{\mathrm{impl}} \to T$ induced by Φ.*

The space M_{impl} has many nice properties that smooth manifolds posses. It is Hausdorff, locally compact and second countable. The action of T preserves $\Phi^{-1}(\exp\bar{\mathcal{A}})$ and descends to a continuous action on M_{impl}.

We have a decomposition of M_{impl} into orbit spaces

$$M_{\mathrm{impl}} = \coprod_{\sigma\leq\mathcal{A}} \Phi^{-1}(\exp\sigma)/[G_\sigma, G_\sigma], \tag{1.5.2}$$

where σ ranges over the faces of the alcove $\mathcal{A}$ and G_σ is the centralizer of $\exp\sigma$. Let us denote the piece $\Phi^{-1}(\exp\sigma)/[G_\sigma, G_\sigma]$ by X_σ. It is proved in [HJS] that each X_σ stratifies into symplectic manifolds. Let $\{X_i | i \in I\}$ be the collection of all strata of all pieces X_σ. Then the imploded cross-section M_{impl} is the disjoint union

$$M_{\mathrm{impl}} = \coprod_{i\in I} X_i \tag{1.5.3}$$

such that each piece X_i is a symplectic manifold.

Theorem 3. *[HJS, Theorem 3.17] The decomposition* (1.5.3) *of the imploded cross-section is a locally finite partition into locally closed subspaces, each of which is a symplectic manifold. There is a unique open stratum, which is dense in M_{impl} and symplectomorphic to the principal cross section of M. The action of the maximal torus T on M_{impl} preserves the decomposition and the imploded moment map $\Phi_{\mathrm{impl}} : M_{\mathrm{impl}} \to T$ restricts to a moment map for the T-action on each stratum.*

Therefore we call M_{impl} a *stratified quasi-Hamiltonian T-space*.

1.5.3 Implosion of the double.

In the example of quasi-Hamiltonian manifolds we have seen that $D(G) := G \times G$ possesses a natural quasi-Hamiltonian $G \times G$-structure. By imploding with respect to, say, the right G-action and the moment map Φ_2 yields a $G \times T$-space $D(G)_{\mathrm{impl}}$. Strictly speaking we form the quotient topological space of $\Phi_2^{-1}(\exp \mathcal{A}) = G \times \exp \mathcal{A}$ by the equivalence relation $(u, v)\ (ug^{-1}, v)$ for $g \in [G_v, G_v]$. Similarly as in (1.5.3) the resulting space is a union of subspaces,

$$D(G)_{\mathrm{impl}} = \coprod_{\sigma \leq \mathcal{A}} G/[G_\sigma, G_\sigma] \times \exp \sigma \tag{1.5.4}$$

The moment map Φ_2 is transverse to all faces of the alcove, therefore each piece $X_\sigma = G/[G_\sigma, G_\sigma] \times \exp \sigma$ in this partition is a smooth manifold. One can show that X_σ is a quasi-Hamiltonian $G \times T$-manifold by writing it as a quasi-symplectic quotient up to a covering [HJS, Lemma 4.5].

Let M be an arbitrary quasi-Hamiltonian G-space. By the fusion operation, which we discussed in section 1.4.1, one obtains a quasi-Hamiltonian $G \times G$-manifold $M \circledast D(G)$. Now define embedding $j : M \to M \circledast D(G)$ by $j(m) =$

$(m, 1, \Phi(m))$. Then one of the main results of [HJS] states:

Theorem 4 (universality of imploded double)**.** *Let M be a quasi-Hamiltonian G-manifold. The map j induces a homeomorphism*

$$j_{\text{impl}} : M_{\text{impl}} \xrightarrow{\cong} (M \circledast D(G)_{\text{impl}})//G$$

which maps strata to strata and whose restriction to each stratum is an isomorphism of quasi-Hamiltonian T-manifolds.

This result implies that any imploded cross-section M_{impl} can be constructed as a symplectic quotient of the product $M \times D(G)_{\text{impl}}$ by the diagonal action of G Hence, the study of imploded spaces reduces to the implosion of the double of the corresponding Lie group.

1.5.4 Smoothness criterion and quasi-Hamiltonian structure on S^{2n}

The implosion of the double is a singular space, however the singularities on certain strata are removable. In order to show that one has to use the explicit correspondence between $D(G)$ and T^*G.

Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using a bi-invariant inner product on $\mathfrak{g}$. Trivializing T^*G in a left-invariant manner, define $G\times G$-equivariant map $\mathcal{H} = id\times\exp : T^*G \to D(G)$. Let (T^*G, ω_0, Ψ_0) be a Hamiltonian $G \times G$ manifold, where ω_0 is the canonical symplectic form on the cotangent bundle and a moment map $\Psi_0(g, \lambda) = (-Ad_g\lambda, \lambda)$. Let O be the set of all $\xi \in \mathfrak{t}$ with $(2\pi i)^{-1}\alpha(\xi) < 1$ for all positive roots α and $U = (AdG)O$.

Lemma 1. *[HJS, Proposition 4.15] The triple* $(T^*G, \mathcal{H}^*\omega, \mathcal{H}^*\Psi)$ *is the exponentiation of* (T^*G, ω_0, Ψ_0)*. In particular,* $G \times U$ *is a quasi-Hamiltonian* $G \times G$*-manifold.*

Now using a local diffeomorphism given by $\mathcal{H}$ and of [GJS, Proposition 6.15] we have

Theorem 5 (Smoothness criterion)**.** *[HJS, Theorem 4.20] Let* σ *be a face of* $\mathcal{A}$ *satisfying* $[G_\sigma, G_\sigma] \cong \mathbf{SU}(2)^k$ *(resp.* $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma] \cong \mathfrak{su}(2)^k$*) for some* $k \geq 0$ *and possessing a vertex* ξ *such that* $\exp \xi$ *is central. Then* $D(G)_{\mathrm{impl}}$ *is a smooth quasi-Hamiltonian* $G \times T$*-manifold(resp. orbifold) in a neighborhood of the stratum corresponding to* σ*.*

A partial converse of this result is also true. Suppose that σ contains a vertex ξ such that $\exp \xi$ is central and $D(G)_{\mathrm{impl}}$ is smooth in a neighborhood of the corresponding stratum. Then $[G_\sigma, G_\sigma] \cong \mathbf{SU}(2)^k$.

One can naturally wonder what can we say about strata where it is not smooth. In [HJS] authors have made a very interesting observation, which is in some sense one of the main motivations for defining imploded spaces. Namely, they have shown that there are certain strata where $D(G)_{\mathrm{impl}}$ is singular, but whose closure is a smooth quasi-Hamiltonian manifold.

Let G be $\mathbf{SU}(n)$. Consider the edge σ_{01} of an alcove with centralizer $G_{01} = \mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(n-1))$. By Theorem 5 we know that for $n > 3$ the corresponding stratum X_{01} in X consists of genuine singularities. Nevertheless the following result asserts that it is a smooth quasi-Hamiltonian manifold and in fact diffeomorphic to S^{2n}.

Theorem 6. *[HJS, Theorem 4.26] The closure of the stratum* X_{01} *of* $X = D\mathbf{SU}(n)_{\mathrm{impl}}$ *is a smooth quasi-Hamiltonian* $\mathbf{U}(n)$*-manifold diffeomorphic to* S^{2n}*. Furthermore antipodal* *map of S^{2n} corresponds to an involution of X_{01} obtained by lifting symmetry of the alcove $\mathcal{A}$ that reverses the edge σ_{01}.*

The proof of this result relies on two main facts. First of all, a symmetry of the alcove of $\mathbf{SU}(n)$. Namely, the center $\mathrm{Z}(\mathbf{SU}(n))$ acts transitively on the vertices which makes possible to shift any vertex of the alcove to the origin. The second one is that the closure of the big open stratum around each vertex is an affine space, which readily gives an affine cover. In special case $n = 2$, it was first constructed in [HJ] and in [AMW]

One can ask whether there are cases where one can make similar construction to obtain new examples of quasi-Hamiltonian spaces. The answer is yes and we will carry out this construction for type C Lie groups. As a result we will show that $\mathbf{HP}^n$ has a quasi-Hamiltonian structure. We will discuss the general case afterwards.

1.6 Imploded cross-section of $\mathbf{Sp}(n)$

In the previous section we have constructed of a quasi-Hamiltonian structure on a sphere using an imploded cross-section. In this section we show using a different approach that a quaternionic projective space has a quasi-Hamiltonian structure.

Preliminaries

Let $\mathbf{H}$ be the set of quaternionic numbers. Then $\mathbf{H}^*$, the set of nonzero quaternions, acts on $\mathbf{H}^{n+1} \setminus \{0\}$ by multiplication on the right. The quotient space of this action is known as a quaternionic projective space, denoted by $\mathbf{HP}^n$.

Let $G = \mathbf{Sp}(n)$, the group of unitary $n \times n$ matrices over the quaternions, with maximal torus $T = \{\mathtt{diag}(e^{2\pi i x_1}, ..., e^{2\pi i x_n})\}$. As an invariant inner product on $\mathfrak{sp}(n)$ we take

$$(\xi, \eta) = -(4\pi^2)^{-1}\mathrm{Re}(\mathtt{tr}(\xi \cdot \eta)) \quad \text{for} \quad \xi, \eta \in \mathfrak{sp}(n). \tag{1.6.1}$$

If we identify $\mathfrak{t}$ with $\mathbf{R}^n$ via the map $x \mapsto 2\pi i\, \mathtt{diag}(x_1, ..., x_n)$, then the simple roots have the form

$$(2\pi i)^{-1}\alpha_k(x) = x_k - x_{k+1} \quad \text{for} \quad k = 1, ..., n-1 \quad \text{and} \quad (2\pi i)^{-1}\alpha_n(x) = 2x_n$$

with maximal root $(2\pi i)^{-1}\tilde{\alpha}(x) = 2x_1$. The corresponding alcove $\mathcal{A}$ is the n-simplex $0 < x_n < ... < x_1 < 1/2$. By slightly abusing our notation, we denote by σ_{01} the edge of the simplex with vertices σ_0, σ_1 which exponentiate to torus elements of the form $\mathtt{diag}(t_1, 1, ..., 1)$ with vertices $I = \mathtt{diag}(1, 1, ..., 1)$ and $\mathtt{diag}(-1, 1, ..., 1)$ correspondingly. Their centralizers are $G_{01} = \mathbf{U}(1) \times \mathbf{Sp}(n-1)$, $G_0 = \mathbf{Sp}(n)$ and $G_1 = \mathbf{Sp}(1) \times \mathbf{Sp}(n-1)$ respectively. One can immediately see that $\exp(\sigma_1)$ is not central, which implies that there is no Weyl group element shifting it to the origin. Define a map $\mathcal{H} : G \times \mathcal{A} \to \coprod_{\sigma \leq \mathcal{A}} G/[G_\sigma, G_\sigma]$ by $\mathcal{H}(g, x) = g[G_{\exp x}, G_{\exp x}]$. By (1.5.4) the stratum corresponding to a general face is given by $X_\sigma = G/[G_\sigma, G_\sigma] \times \exp \sigma$ and therefore we have

$$X_0 = \mathcal{H}(I, 0) \times \{I\} \cong \{\mathtt{pt}\}, \tag{1.6.2}$$

$$X_{01} = \mathbf{Sp}(n)/\mathbf{Sp}(n-1) \times \{\mathtt{diag}(e^{2\pi i x_1}, 1, ..., 1) | x_1 \in (0, \frac{1}{2})\} \cong S^{4n-1} \times (0, 1), \tag{1.6.3}$$

$$X_1 = \mathbf{Sp}(n)/(\mathbf{Sp}(n-1)\times\mathbf{Sp}(1))\times \mathtt{diag}(-1,1,...,1) \cong \mathbf{HP}^{n-1}. \tag{1.6.4}$$

Consider the closure of the stratum corresponding to σ_{01}, which is $\bar{X}_{01} = X_0 \bigsqcup X_{01} \bigsqcup X_1$. Notice that we have bijections $X_0 \bigsqcup X_{01} \cong \mathbf{H}^n$ and $X_1 \cong \mathbf{HP}^{n-1}$, and one would expect that $\bar{X}_{01}$ and $\mathbf{HP}^n$ are homeomorphic. However, the covering obtained from the strata, $U_0 = X_0 \bigsqcup X_{01}$ and $U_1 = X_{01} \bigsqcup X_1$ is not an affine cover, since U_1 is a $\mathbf{H}$-line bundle over $\mathbf{HP}^{n-1}$. Hence, we have to construct directly a homeomorphism from $\bar{X}_{01}$ to $\mathbf{HP}^n$.

Homeomorphism between $\bar{X}_{01}$ and $\mathbf{HP}^n$

Define a map

$$\mathcal{G} : X_{01} \to \mathbf{HP}^n \quad , \quad (\mathcal{H}(g,x), \exp x) \mapsto \left[\sqrt{(1-2x_1)},\ \sqrt{2x_1} g.v\right], \tag{1.6.5}$$

where $v = (1, 0, ..., 0) \in \mathbf{H}^n$. One can easily check that it is well-defined, i.e. does not depend on the equivalence class of g in $\mathbf{Sp}(n)/\mathbf{Sp}(n-1)$. Moreover, $\mathcal{G}$ is a continuous, injective map on X_{01}(or $0 < x_1 < \frac{1}{2}$) and can be extended continuously to a bijective map on $\widehat{X}_{01}$. Indeed, on X_0 (or $x_1 = 0$) we have $\mathcal{G}(\mathcal{H}(g,x), x) = [1, 0, ..., 0]$ and on X_1 (or $x_1 = \frac{1}{2}$) we have $\mathcal{G}(\mathcal{H}(g,x), x) = [0, g.v]$.

It is known (see [HJS]) that an imploded space is Hausdorff, locally compact and second countable. Thus, we have proved

Theorem 7. *The map $\mathcal{G} : \bar{X}_{01} \to \mathbf{HP}^n$ is a homeomorphism.*

This statement allows us to define a smooth structure on $\bar{X}_{01}$ by pulling back the smooth structure on $\mathbf{HP}^n$ via $\mathcal{G}$. Now we would like to give a description of the inverse of $\mathcal{G}$. For this we use the following decomposition $Y_0 \bigsqcup Y_{01} \bigsqcup Y_1$ of $\mathbf{HP}^n$, where

$$Y_0 = \{[1, 0, ..., 0]\},$$
$$Y_{01} = \{Z \in \mathbf{HP}^n | \, Z_1 \neq 0 \,\text{and}\, \sum_{l=2}^{n+1} |Z_l|^2 \neq 0\},$$
$$Y_1 = \{Z \in \mathbf{HP}^n | \, Z_1 = 0\}.$$

We define a map $\mathcal{F} : \mathbf{HP}^n \to \bar{X}_{01}$ stratawise as follows. First, we map

$$Y_0 \mapsto X_0.$$

Second, the restriction of $\mathcal{F}$ to Y_{01}

$$\begin{tikzcd} & X_{01} \arrow[d] \\ Y_{01} \arrow[ur, dashed, "\mathcal{F}|_{Y_{01}}"] \arrow[r] & S^{4n-1} \times \exp \sigma_{01} \end{tikzcd} \tag{1.6.6}$$

is given through the homeomorphisms $Y_{01} \to S^{4n-1} \times \exp \sigma_{01}$ and $X_{01} \to S^{4n-1} \times \exp \sigma_{01}$, defined respectively by

$$[Z_1, Z_2, ..., Z_{n+1}] \mapsto \left(\frac{Z_2 \bar{Z}_1}{|Z_1| \sqrt{\sum_{l=2}^{n+1} |Z_l|^2}}, ..., \frac{Z_{n+1} \bar{Z}_1}{|Z_1| \sqrt{\sum_{l=2}^{n+1} |Z_l|^2}} \right) \times \mathtt{diag}(e^{\lambda \pi i}, 1, ..., 1)$$

and

$$\Big(\mathcal{H}(g, x), \exp x \Big) \mapsto (g.v, \exp x),$$

where

$$\lambda = \frac{\sum_{l=2}^{n+1} |Z_l|^2}{\sum_{l=1}^{n+1} |Z_l|^2}. \tag{1.6.7}$$

Therefore $\mathcal{F}|_{Y_{01}}$ is of the form

$$[Z_1, Z_2, ..., Z_{n+1}] \mapsto \Big(\mathcal{H}(g, x), \exp x \Big)$$

where g is a matrix $(f_{p,q}(Z))_{p,q=1}^n$ whose first column is

$$f_{p,1}(Z) = \left\{ \; \frac{Z_{p+1} \bar{Z}_1}{|Z_1| \sqrt{\sum_{l=2}^{n+1} |Z_l|^2}} \quad \text{for} \quad p = 1, ..., n \right. \tag{1.6.8}$$

and $x = (\lambda/2, 0, ...0)$. Third, for the stratum Y_1 we have a commutative diagram

$$\begin{tikzcd} & X_1 \arrow[d] \\ Y_1 \arrow[ur, dashed, "\mathcal{F}|_{Y_1}"] \arrow[r] & \mathbf{HP}^{n-1} \times \mathtt{diag}(-1, 1, ..., 1) \end{tikzcd} \tag{1.6.9}$$

where $\mathcal{F}_{Y_1}$ is determined by homeomorphisms $Y_1 \to \mathbf{HP}^{n-1} \times \mathtt{diag}(-1, 1, ..., 1)$ and $X_1 \to \mathbf{HP}^{n-1} \times \mathtt{diag}(-1, 1, ..., 1)$ given by

$$[0, Z_2, ..., Z_{n+1}] \mapsto ([Z_2, ..., Z_{n+1}], \mathtt{diag}(-1, 1, ..., 1))$$

and

$$\left(\mathcal{H}(g, x), \exp x\right) \mapsto (g.v, \exp x)$$

respectively. Therefore $\mathcal{F}|_{Y_1}$ is of the form

$$[0, Z_2, ..., Z_{n+1}] \mapsto \mathcal{H}(g, x) \times \mathtt{diag}(-1, 1, ..., 1) \quad \text{in} \quad X_1,$$

where g is a matrix $f_{p,q}(Z)$ whose first column is

$$f_{p,1}(Z) = \frac{Z_{p+1}}{\sqrt{\sum_{l=2}^{n+1} |Z_l|^2}} \quad \text{for} \quad p = 1, ..., n$$

and $x = (1/2, 0, ..., 0)$. So we have

Lemma 2. *The map $\mathcal{F}$ is the inverse of $\mathcal{G}$ and hence smooth.*

Now if we define an action of $\mathbf{Sp}(n) \times T$ on $\mathbf{HP}^n$ by

$$(g, t)[Z_1, ..., Z_{n+1}] = [Z_1 t_1, g.(Z_2, ..., Z_{n+1})]\ , \tag{1.6.10}$$

where $t = \mathtt{diag}(t_1, ..., t_n)$ then one can easily prove

Lemma 3. *The map $\mathcal{G}$, and hence $\mathcal{F}$, is $\mathbf{Sp}(n) \times T$-equivariant.*

Quasi-Hamiltonian structure on $\mathbf{HP}^n$

First, let us recall the quasi-Hamiltonian structure on a stratum X_σ. Since the moment map Φ_2 defined as in (1.2.5) is transversal to all faces of the alcove, using quasi-symplectic reduction one can show

Lemma 4. *[HJS, Lemma 4.5] For every $\sigma \leq \mathcal{A}$, the subspace $X_\sigma = G/[G_\sigma, G_\sigma] \times \exp\sigma$ of $D(G)_{\mathrm{impl}}$ is a quasi-Hamiltonian $G\times T$-manifold. The moment map $X_\sigma \to G\times T$ is the restriction to X_σ of the continuous map $\Phi_{\mathrm{impl}} \to G\times T$ induced by $\Phi : D(G) \to G\times G$.*

Next we compute the corresponding 2-form ω_σ on X_σ. Let $(g, \exp x)$ be an arbitrary point on X_σ. A tangent vector at $(g, \exp x)$ is of the form $((L(g)_*\xi, (L(\exp x))_*\eta)$ where $\xi \in \mathfrak{g}$ and $\eta \in \zeta + \mathfrak{z}(\mathfrak{g}_\sigma)$ for some $\zeta \in \mathfrak{z}(\mathfrak{g}_\sigma)^\perp = [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$ [HJS, Lemma A.3]. Then a simple calculation yields

$$\begin{aligned}(\omega_\sigma)_{(g,\exp x)}&((L(g)_*\xi_1, L(\exp x)_*\eta_1), (L(g)_*\xi_2, L(\exp x)_*\eta_2)) \\ &= -\frac{1}{2}((\mathtt{Ad}_{\exp x} - \mathtt{Ad}_{\exp(-x)})\xi_1, \xi_2) - (\xi_1, \eta_2) + (\xi_2, \eta_1).\end{aligned} \tag{1.6.11}$$

One can check that it does not depend on the equivalence class of ξ_i in $\mathfrak{g}/[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$. Consider the 2-form ω_{01} on an open stratum X_{01}. In what follows we compute the pull back of this 2-form via $\mathcal{F}$ and show that it extends smoothly to all of $\mathbf{HP}^n$. Since ω_{01} is $\mathbf{Sp}(n)\times T$-invariant, it suffices to consider vectors of the form $z_0 = [s, 1, 0, ..., 0]$, where

$$s = \frac{|Z_1|}{\sqrt{\sum_{l=2}^{n+1} |Z_l|^2}}.$$

The tangent space at z_0 is

$$T_{z_0}\mathbf{HP}^n = \{(w_1, ..., w_{n+1}) \in \mathbf{H}^{n+1} | sw_1 + w_2 = 0\}, \tag{1.6.12}$$

where $w_l = w_{l1} + w_{l2}i + w_{l3}j + w_{l4}k$. Let us first find the corresponding tangent vectors at $\mathcal{F}(z_0)$, or more precisely corresponding pull-backs ξ_i, η_i to elements of a Lie algebra as in (1.6.11). Let v and w be tangent vectors of form (1.6.12). Note, since

$$\mathcal{F}(z_0) = (\mathcal{H}(I, (\lambda/2, 0, ..., 0)), \mathtt{diag}(\exp(\lambda\pi i), 1, ..., 1)) =: (\mathcal{H}(g, x), x), \tag{1.6.13}$$

the first component of the image is already an element of the Lie algebra, while the second one has to be translated by an appropriate element of the Lie group (that is $\exp x$). Denote by $(A^v_{p,q})^n_{p,q=1}$ and $(B^v_{p,q})^n_{p,q=1}$ ($(A^w_{p,q})^n_{p,q=1}$ and $(B^w_{p,q})^n_{p,q=1}$) the matrix representation of ξ_1 and η_1 (correspondingly ξ_2 and η_2). Then substituting these to the first term of (1.6.11) and using (1.6.13) expression for $\mathcal{F}(z_0)$ we have

$$\begin{aligned}((\mathtt{Ad}_{\exp x} - \mathtt{Ad}_{\exp(-x)})\xi_1, \xi_2) = (4\pi^2)^{-1}\mathrm{Re}\Big(\big[\exp(\lambda\pi i)A^v_{11}\exp(-\lambda\pi i) -\\ \exp(-\lambda\pi i)A^v_{11}\exp(\lambda\pi i)\big]\bar{A}^w_{11} - \big[\exp(\lambda\pi i) - \exp(-\lambda\pi i)\big]\sum_{p=2}^{n} A^v_{1p}A^w_{p1} + \\ \sum_{p=2}^{n} A^v_{1p}\big[\exp(-\lambda\pi i) - \exp(\lambda\pi i)\big]\bar{A}^w_{p1}\Big),\end{aligned} \tag{1.6.14}$$

where the inner product is given by (1.6.1) and

$$A^v_{11} = -(s + s^{-1})(v_{12}i + v_{13}j + v_{14}k), \tag{1.6.15}$$

and by the skew-symmetry

$$A^v_{p1} = -A^v_{1p} = v_{(p+1)1} + v_{(p+1)2}i + v_{(p+1)3}j + v_{(p+1)4}k. \tag{1.6.16}$$

There are similar relations to (1.6.15) and (1.6.16) if we replace v by w. Thus we can rewrite (1.6.14) in the following form

$$\begin{aligned}((Ad_{\exp x} - Ad_{\exp(-x)})\xi_1, \xi_2) = \frac{\sin(2\pi\lambda)}{2\pi^2}(s + s^{-1})(v_{13}w_{14} - w_{13}v_{14}) - \\ \frac{\sin(\pi\lambda)}{\pi^2}\sum_{p=3}^{n+1}(v_{p1}w_{p2} - w_{p1}v_{p2} - v_{p3}w_{p4} + w_{p3}v_{p4}).\end{aligned} \tag{1.6.17}$$

Hence, the corresponding two-form will be

$$\frac{\sin(2\pi\lambda)}{2\pi^2}(s + s^{-1})dx_{13}dx_{14} - \frac{\sin(\pi\lambda)}{\pi^2}\sum_{p=3}^{n+1}(dx_{p1}dx_{p2} - dx_{p3}dx_{p4}), \tag{1.6.18}$$

where x's are just real coordinates for Z's, such that $Z_l = x_{l1} + x_{l2}i + x_{l3}j + x_{l4}k$. For the remaining part of (1.6.11) we have:

$$-(\xi_1, \eta_2) + (\xi_2, \eta_1) = (4\pi^2)^{-1}\mathrm{Re}(-A^v_{11}\bar{B}^w_{11} + A^w_{11}\bar{B}^v_{11}), \tag{1.6.19}$$

where

$$B^v_{11} = -\frac{2\pi i s}{s^2+1}v_{11}, \tag{1.6.20}$$

and the corresponding two-form is

$$\frac{1}{2\pi}dx_{11}dx_{12}. \tag{1.6.21}$$

Combining (1.6.18) with (1.6.21) yields

$$\mathcal{F}^*\omega_{01} = \tfrac{1}{2\pi}dx_{11}dx_{12} - \tfrac{\sin(2\pi\lambda)}{2\pi^2}(s+s^{-1})dx_{13}dx_{14} + \tag{1.6.22}$$

$$\tfrac{\sin(\pi\lambda)}{\pi^2}\textstyle\sum_{p=3}^{n+1}(dx_{p1}dx_{p2} - dx_{p3}dx_{p4}).$$

It is a smooth two-form defined on open dense subset Y_{01} of $\mathbf{HP}^n$. Moreover we can show

Lemma 5. *The two-form $\mathcal{F}^*\omega_{01}$ extends smoothly to all of $\mathbf{HP}^n$.*

Proof. It suffices to check two critical cases $Z_1 = 0$, a line at infinity, and $[1, 0, ..., 0]$, a point at infinity. As $|Z_1|$ approaches to 0, λ tends to 1 and therefore the third expression on the right hand side of (1.6.22) vanishes. Now since $\lambda = 1 - s^2$, we have $s \to 0$ and hence

$$\frac{\sin(2\pi\lambda)}{2\pi^2}(s+s^{-1}) \longrightarrow -\frac{1}{\pi}.$$

So in the neighborhood of $Z_1 = 0$ the two-form $\mathcal{F}^*\omega_{01}$ can be written as

$$\frac{1}{2\pi}dx_{11}dx_{12} + \frac{1}{\pi}dx_{13}dx_{14}.$$

In a similar fashion one can show that in the neighborhood of $[1, 0, ..., 0]$ it is given by:

$$\frac{1}{2\pi}dx_{11}dx_{12} - \frac{1}{\pi}dx_{13}dx_{14}.$$

This finishes the proof of this lemma. □

Notice that the obtained 2-form is given in dehomogenized coordinates by

$$\left[\frac{|Z_1|}{\sqrt{\sum_{l=2}^{n+1}|Z_l|^2}}, \frac{Z_2\bar{Z}_1}{|Z_1|\sqrt{\sum_{l=2}^{n+1}|Z_l|^2}}, ..., \frac{Z_{n+1}\bar{Z}_1}{|Z_1|\sqrt{\sum_{l=2}^{n+1}|Z_l|^2}}\right]. \tag{1.6.23}$$

For $Q = q_1 + q_2 i + q_3 j + q_4 k$ we define $\mathrm{Im}_i(Q) = q_2$, then we have

$$\mathrm{Im}_i(d\bar{Z}_p dZ_p) = 2(dx_{p1}dx_{p2} - dx_{p3}dx_{p4}). \tag{1.6.24}$$

Now using (1.6.23) and (1.6.24) in homogeneous coordinates the first two terms vanishes, our 2-form will take the form

$$\begin{aligned}&\frac{\sin(\lambda\pi)}{\pi^2}\left(|Z_1|^2\sum_{l=2}^{n+1}|Z_l|^2\right)^{-1}\left[\sum_{p=3}^{n+1}|Z_p|^2\mathrm{Im}_i(dZ_1 d\bar{Z}_1) - \right.\\ &\mathrm{Im}_i(Z_1 d\bar{Z}_p dZ_p \bar{Z}_1) + \left(\sum_{p=3}^{n+1}|Z_p|^2\mathrm{Im}_i((Z_1 d\bar{Z}_1)) - \mathrm{Im}_i(Z_1\bar{Z}_p dZ_p\bar{Z}_1)\right)\times \\ &\left.\left(\frac{Z_1 d\bar{Z}_1 + dZ_1\bar{Z}_1}{|Z_1|^2} + \frac{\sum_{l=2}^{n+1}(Z_l d\bar{Z}_l + dZ_l\bar{Z}_l)}{\sum_{l=2}^{n+1}|Z_l|^2}\right)\right].\end{aligned} \tag{1.6.25}$$

The last thing we need to show that there is a well-defined smooth moment map. Define a map $\Phi : \mathbf{HP}^n \to \mathbf{Sp}(n) \times T$ stratawise, so that the following diagram commutes

$$\begin{tikzcd} & X_\sigma \arrow[dl, "\mathcal{G}|_{X_\sigma}"'] \arrow[dr, "\Phi_\sigma"] & \\ Y_\sigma \arrow[rr, dashed, "\Phi"] & & \mathbf{Sp}(n)\times T \end{tikzcd} \tag{1.6.26}$$

for each face σ in the closure. Then on each stratum it has the form

$$[Z_1, ..., Z_{n+1}] \mapsto (AB^{-1}A^{-1}, B) \tag{1.6.27}$$

where $B = (B_{pq})_{p,q=1}^n$

$$B = \mathtt{diag}(\exp(\lambda\pi i), 1, ..., 1) \tag{1.6.28}$$

and $A = (A_{pq})_{p,q=1}^n$ is any representative of $\mathcal{H}(g, x)$. One can easily check that it does not depend on the representative of $\mathcal{H}(g, x)$. Evidently, Φ is uniquely determined and $\mathbf{Sp}(n) \times T$-equivariant. We have to show that it is smooth. From the construction one can see that B_{pq} are smooth. As for the first component of Φ, on Y_{01} we have

$$AB^{-1}A^{-1} = \mathrm{Id}_n + C,$$

where $C = (C_{pq})_{p,q=1}^n$:

$$C_{pq} = A_{p1}\bar{B}_{11}\bar{A}_{q1} - A_{p1}\bar{A}_{q1},$$

or to be more precisely

$$C_{pq} = \left(|Z_1|^2 \sum_{l=2}^{n+1} |Z_l|^2\right)^{-1} Z_{p+1}\left[\bar{Z}_1 \exp(\pi i\lambda)Z_1 - |Z_1|^2\right]\bar{Z}_{q+1}. \tag{1.6.29}$$

We can easily see that it is smooth for $Z_1 \neq 0$ and $\sum_{l=2}^{n+1} |Z_l|^2 \neq 0$. Hence it is smooth on Y_{01}. Using almost the same argument as in Lemma 5 we can show it is smooth in these two cases as well. Now summarizing these facts we have

Theorem 8. *The closure of the stratum X_{01} of $X = D\mathbf{Sp}(n)_{\mathrm{impl}}$ is a smooth quasi-Hamiltonian $\mathbf{Sp}(n) \times T$-manifold diffeomorphic to n-dimensional quaternionic projective space with the 2-form and the moment map determined by* (1.6.25) *and* (1.6.27) *respectively.*

Remark. Notice that the homomorphism $p : \mathbf{Sp}(n) \times T \to \mathbf{Sp}(n) \times \mathbf{U}(1)$ defined by $p(g, t) = (g, t_1)$ is surjective; its kernel is the kernel of the action on $\mathbf{HP}^n$ which one can immediately see from (1.6.10). Therefore $\mathbf{HP}^n$ is in fact a quasi-Hamiltonian $\mathbf{Sp}(n) \times \mathbf{U}(1)$-manifold where the second component of moment map is $\Phi_2(Z) = \exp(\lambda\pi i)$. By reduction theorem [AMM, Theorem 5.1], the reduction of $\mathbf{HP}^n$ with respect to the circle action $\mathbf{U}(1)$ is a quasi-Hamiltonian $\mathbf{Sp}(n)$-manifold. Reduction at $\lambda = 0$ and $\lambda = 1$ give quotients consisting of a single point and $\mathbf{HP}^{n-1}$ respectively, while the reduction at an intermediate level $0 < \lambda < 1$ gives a projective space $\mathbf{CP}^{2n-1}$. In fact using [HJS, Addendum 3.18] one can show that this quasi-Hamiltonian structure is the same as the one obtained by considering $\mathbf{HP}^{n-1}$ and $\mathbf{CP}^{2n-1}$ as conjugacy classes of $\mathbf{Sp}(n)$.

1.7 Smoothness Criterion for other type of Lie groups

The main result of this section is to show that the example we have discussed in the last section, $\mathbf{HP^n}$ and the "spinning" sphere of [HJS] are in some sense only examples which can be constructed using universal imploded spaces.

In [E] and [HJS], the closure of a certain stratum for imploded spaces of type A and C Lie groups has been studied . It is natural to ask whether there are other examples of quasi-Hamiltonian spaces appearing in this context. Surprisingly, the answer to this question is negative. To be precise, let K be any connected and simply-connected compact Lie group. We show using results of Popov and Vinberg and computation of dimensions of strata that a stratum of DK_{impl} has a smooth closure only in the above mentioned examples. That is, S^{2n} and $\mathbf{HP}^n$ are the only examples where the stratum has a smooth closure. The idea is based on the close relationship between the quasi-Hamiltonian space $D(K)$ and the Hamiltonian space T^*K. Namely, using Lemma 1 one can show that $\mathcal{H} : T^*K \to D(K)$ descends to imploded cross section, that is, there is the equivariant local homeomorphism

$$h : (T^*K)_{\text{impl}} \to D(K)_{\text{impl}}$$

induced by $\mathcal{H}$. Hence instead of showing smoothness criteria for $D(K)_{\mathrm{impl}}$, one can do that for T^*K.

Let K be a semisimple and simply connected compact Lie group. It is known that (see [GJS, Proposition 6.8]) there is an isomorphism of K-Hamiltonian spaces:

$$f : (T^*K)_{\mathrm{impl}} \to G_N$$

where $G = K^{\mathbb{C}}$ and N is a maximal unipotent subgroup of G (Following Kraft [K] for any affine G-variety X we denote by X_N the affine variety with the coordinate ring $\mathbb{C}[X]^N$). Under this isomorphism the strata of $(T^*K)_{\mathrm{impl}}$ coincide with the orbits of G:

$$f((K \times \Sigma_\sigma)//[K_\sigma, K_\sigma]) = G/[P_\sigma, P_\sigma]$$

where σ is a face of Weyl chamber and P_σ the corresponding parabolic subgroup. We will denote this orbit by $O_\sigma = G/[P_\sigma, P_\sigma]$. Note in [GJS] they obtained it as the orbit of vector v_σ which is the sum of all fundamental weights contained in the face σ. This implies that each stratum is a quasi-affine subvariety and its closure in the classical topology is the same as its Zariski closure [K, Appendix 1.7.2]. Hence the stratum $X_\sigma = (K \times \Sigma_\sigma)//[K_\sigma, K_\sigma]$ has a smooth closure if and only if its algebraic closure $Z_\sigma = \overline{G/[P_\sigma, P_\sigma]}$ is a smooth algebraic variety. These varieties also appeared in [PV] in the context of so called *S-varieties*. Let first recall their definition.

Definition 3. *An irreducible affine variety X with a regular action of the group G on it is called an S-variety of the group G if one of the orbits of this action is open in X and if the isotropy subgroup of any point of this orbit contains a maximal unipotent subgroup of G.*

The main example of S-varieties discussed in [PV] is a closure of an orbit of a vector which is a sum of highest weight vectors. In particular Z_σ by their definition are S-varieties [GJS, Lemma 6.2]. They gave explicit description of coordinate ring of Z_σ [PV, Theorem 6]

Theorem 9. *$\mathbb{C}[Z_\sigma] = S_{\Lambda_\sigma}$ where S_{Λ_σ} is a semigroup algebra with the corresponding semigroup $\Lambda_\sigma \subseteq \overline{C}$ generated by fundamental weights contained in face σ.*

They also compute the coordinate ring of O_σ:

Theorem 10. *$\mathbb{C}[O_\sigma] = S_{\mathbb{Z}\Lambda_\sigma \cap \overline{C}}$ where $\mathbb{Z}\Lambda_\sigma \cap \overline{C}$ a semigroup obtained by the intersection group $\mathbb{Z}\Lambda_\sigma$ generated by σ with closure of the Weyl chamber $\overline{C}$.*

The other interesting result regarding properties of these varieties, which is not relevant to what follows [PV, Theorem 13]

Theorem 11. *Every S-variety is rational.*

By the results of Popov and Vinberg [PV, §3], Z_σ consists of finitely many G-orbits which are labeled by the faces of the cone $\mathbb{Q}^+\Lambda_\sigma$. In particular, it always contains a fixed point, strata corresponding to $\{0\}$ face. Using the criteria given in [P, Proposition 10] it is sufficient to check smoothness at the "most singular" stratum $\{0\}$.

Proposition 3. *Let Z be a S-variety of a connected reductive group G which contains a fixed point o, and let Λ be the subsemigroup with zero of the semigroup of leading weights of G which corresponds to the variety Z (see [PV]). Let T_oZ denote the tangent space to Z at o. Then the set of weights $\Lambda_o = \Lambda \backslash \{0\}$ is also a semigroup (with respect to addition); this semigroup is finitely generated and has a unique minimal system (i.e. one which cannot be diminished) of generators $\omega_1, \ldots, \omega_m$. This system can be determined* *from the relation*

$$T_o Z = \bigoplus_{i=1}^{m} V_r^* \tag{1.7.1}$$

where V_r is the contragradient representation corresponding to ω_r. In particular,

$$\mathtt{dim}(T_o Z) = \sum_{i=1}^{m} \mathtt{dim}(V_i)$$

In particular, for Z_σ we have

$$\mathtt{dim}(T_0(Z_\sigma)) = \sum_{i=1}^{m} \mathtt{dim}(V_i)$$

where V_i is an irreducible representation, one for each minimal generator $\{\omega_1, ..., \omega_m\}$ of the face σ. In our case these are just fundamental representations contained in the closure of the face σ. So a necessary condition for smoothness is:

$$\mathtt{dim}(G/[P_\sigma, P_\sigma]) = \sum_{i=1}^{m} \mathtt{dim}(V_i) \tag{1.7.2}$$

In what follows, we will compute these dimensions for each type of Lie groups.

Type A_l

First we compute these dimensions for strata corresponding to 1-dimensional faces σ. There are exactly l number of 1-dimensional faces, one for each fundamental weight $\varpi_r (1 \le r \le l)$ with the dimension of the fundamental representation [B]:

$$\mathtt{dim}(V_r) = \begin{pmatrix} l+1 \\ r \end{pmatrix}$$

whereas the dimension of the corresponding open stratum:

$$\mathtt{dim}(G/[P_\sigma, P_\sigma]) = r(l - r + 1) + 1$$

These are equal if and only if $r = 1$ or $r = l$. In these cases we have $G/[P_\sigma, P_\sigma] \cong \mathbb{C}^{l+1}$.

Now we examine smoothness of closure of stratum corresponding to higher dimensional faces of the Weyl chamber. First note following simple but a very useful dimension formula for an arbitrary semisimple Lie group

$$\mathtt{dim}(G/N) = \frac{1}{2}(\mathtt{dim}(G) + \mathtt{rank}(G)). \tag{1.7.3}$$

which is for type **A** equals to $\frac{l(l+3)}{2}$. On the other hand we have upper bound:

$$\mathtt{dim}(G/[P_\sigma, P_\sigma]) \leq \mathtt{dim}(G/N)$$

Using criteria (1.7.2), by direct examination one can see that there are very few cases. For 2-dimensional faces containing ϖ_i and ϖ_j

$$\mathtt{dim}(V_i) + \mathtt{dim}(V_j) = \begin{pmatrix} l+1 \\ i \end{pmatrix} + \begin{pmatrix} l+1 \\ j \end{pmatrix}$$

and the only case this value is below that upper bound $i = 1,\ j = l$, in which case it is $2(l+1)$. The dimension of the corresponding open stratum:

$$\mathtt{dim}(G/[P_\sigma, P_\sigma]) = 2l + 1 < 2(l+1),$$

hence it is not smooth. In all other cases $\mathtt{dim}(V_i) + \mathtt{dim}(V_j)$ is at least $\frac{(l+1)(l+2)}{2} > \frac{l(l+3)}{2}$. This also implies that there are no smooth closures for higher dimensional faces.

Type B_l.

Like in type **A** we first examine 1-dimensional faces. In this case dimensions of fundamental representations are:

$$\begin{array}{llll} \varpi_r & \text{for} & 1 \le r \le l-1 & \mathtt{dim}(V_r) = \begin{pmatrix} 2l+1 \\ r \end{pmatrix} \\ \varpi_l & & & \mathtt{dim}(V_l) = 2^l. \end{array}$$

Dimensions of corresponding open strata:

$$\mathtt{dim}(G/[P_r, P_r]) = r\Big(2l - \frac{3r}{2} + \frac{1}{2}\Big) + 1$$

and

$$\mathtt{dim}(G/N) = l(l+1)$$

For $2 \le r \le l-1$ we have $\mathtt{dim}(V_r) > \mathtt{dim}(G/N)$ and $\mathtt{dim}(V_1) > \mathtt{dim}(G/[P_1, P_1])$. In case $r = l$

$$\mathtt{dim}(G/[P_l, P_l]) = \frac{l(l+1)}{2} + 1$$

Hence equality holds if and only if $r = l = 2$, in which case $B_2 \cong C_2$ and $G_{[P_2,P_2]} \cong \mathbb{C}^4$. Again by direct examination one can easily check that $\mathtt{dim}(V_i) + \mathtt{dim}(V_j) > l(l+1) = \mathtt{dim}(G/N)$ which implies that there are no smooth closures for higher dimensional strata.

Type C_l.

Dimensions of fundamental representations

$$\varpi_r \quad \text{for} \quad 1 \le r \le l \quad \mathtt{dim}(V_r) = \begin{pmatrix} 2l \\ r \end{pmatrix} - \begin{pmatrix} 2l \\ r-2 \end{pmatrix}$$

The equality holds only in case $r = 1$, for which corresponding closure $G_{[P_1,P_1]} \cong \mathbb{C}^{2l}$. Note that $\mathtt{dim}(V_r) < \mathtt{dim}(V_{r+1})$. On the other hand $l \geq 3$ we have $\mathtt{dim}V_2 = l(2l-1) - 1 > l(l+1) = \mathtt{dim}(G/N)$. This implies that there are no smooth closures for higher dimensional strata.

Type D_l.

In this case dimensions of fundamental representations are :

$$\varpi_r \quad \text{for} \quad 1 \leq r \leq l-2 \quad \mathtt{dim}(V_r) = \begin{pmatrix} 2l \\ r \end{pmatrix}$$

$$\varpi_r \quad \text{for} \quad l-1 \leq r \leq l \quad \mathtt{dim}(V_r) = 2^{l-1}.$$

Dimensions of corresponding open strata:

$$\mathtt{dim}(G/[P_r, P_r]) = r\Big(2l - \frac{3r}{2} - \frac{1}{2}\Big) + 1 \quad \text{for} \quad 1 \leq r \leq l-2$$

Equality happens only in case:

$$2^{l-1} = \frac{l(l-1)}{2} \quad \Rightarrow \quad l = 2 \quad \text{or} \quad l = 3$$

For $l = 2$, we have $Spin(4) \cong SU(2) \times SU(2)$. For $l = 3$, we have $Spin(6) \cong SU(4)$. These cases have already been considered in Type A.

For $l \geq 4$, we have $\mathtt{dim}(G/N) = l^2 \leq l(2l+1) \leq \mathtt{dim}(V_i) + \mathtt{dim}(V_j)$, therefore we can deduce there are no smooth closures for higher dimensional faces.

Type E_6.

First one can note that $\mathtt{dim}(E_6) = 78$. There are six fundamental weights with dimensions of fundamental representation given as

Fundamental weights	Dimensions of Representations
ϖ_1	$\texttt{dim}(V_1) = 27$
ϖ_2	$\texttt{dim}(V_2) = 78$
ϖ_3	$\texttt{dim}(V_3) = 351$
ϖ_4	$\texttt{dim}(V_4) = 2925$
ϖ_5	$\texttt{dim}(V_5) = 351$
ϖ_6	$\texttt{dim}(V_6) = 27$

The only possible smooth closures for one dimensional faces can happen for ϖ_1 and ϖ_6. For which we have:

$$\texttt{dim}(G/[P_1, P_1]) = 17 < 27 = \texttt{dim}(V_1)$$

$$\texttt{dim}(G/[P_6, P_6]) = 17 < 27 = \texttt{dim}(V_6)$$

.

Therefore we conclude that there are no smooth closures for strata corresponding to one dimensional faces. Likewise, since $\texttt{dim}(G/N) = 42 < 54 \leq \texttt{dim}(V_i) + \texttt{dim}(V_j)$ we conclude that in fact there are no smooth closures for any higher dimensional faces.

Type E_7.

The dimension of group is $\texttt{dim}(E_7) = 133$. There are seven fundamental weights with dimensions of fundamental representation given as

Fundamental weights	Dimensions of Representations
ϖ_1	$\mathtt{dim}(V_1) = 133$
ϖ_2	$\mathtt{dim}(V_2) = 912$
ϖ_3	$\mathtt{dim}(V_3) = 8645$
ϖ_4	$\mathtt{dim}(V_4) = 365750$
ϖ_5	$\mathtt{dim}(V_5) = 27664$
ϖ_6	$\mathtt{dim}(V_6) = 1539$
ϖ_7	$\mathtt{dim}(V_7) = 56$

There is only one possible one dimensional face for which we have:

$$\mathtt{dim}(G/[P_7, P_7]) = 28 < 56 = \mathtt{dim}(V_7).$$

Also one can see from dimensions of fundamental representations that there can not be any smooth closure for higher dimensional faces.

Type E_8.

The dimension of group is $\mathtt{dim}(E_8) = 248$. There are eight fundamental weights with dimensions of fundamental representation given as

Fundamental weights	Dimensions of Representations
ϖ_1	$\mathtt{dim}(V_1) = 3875$
ϖ_2	$\mathtt{dim}(V_2) = 147250$
ϖ_3	$\mathtt{dim}(V_3) = 6696000$
ϖ_4	$\mathtt{dim}(V_4) = 6899079264$
ϖ_5	$\mathtt{dim}(V_5) = 146325270$
ϖ_6	$\mathtt{dim}(V_6) = 2450240$
ϖ_7	$\mathtt{dim}(V_7) = 30380$
ϖ_8	$\mathtt{dim}(V_8) = 248$

From the table one can easily imply that there are no smooth closures for any strata.

Type F_4.

The dimension of group is $\mathtt{dim}(F_4) = 52$. There are four fundamental weights with dimensions of fundamental representation given as

Fundamental weights	Dimensions of Representations
ϖ_1	$\mathtt{dim}(V_1) = 52$
ϖ_2	$\mathtt{dim}(V_2) = 1274$
ϖ_3	$\mathtt{dim}(V_3) = 273$
ϖ_4	$\mathtt{dim}(V_4) = 26$

Notice that for $1 \leq r \leq 3$, we have $\mathtt{dim}(F_4) \leq \mathtt{dim}(V_r)$. For the face containing last fundamental weight we have:

$$\mathtt{dim}(G/[P_4, P_4]) = 16 < 26 = \mathtt{dim}(V_4).$$

From the table it is obvious that there are no smooth closures for other strata as well.

Type G_2.

First one can note that $\mathtt{dim}(G_2) = 14$. There are two fundamental weights with dimensions of fundamental representation given as

Fundamental weights	Dimensions of Representations
ϖ_1	$\mathtt{dim}(V_1) = 7$
ϖ_2	$\mathtt{dim}(V_2) = 14$

Dimensions of corresponding strata have dimensions:

$$\mathtt{dim}(G/[P_1, P_1]) = 6 < 7 = \mathtt{dim}(V_1)$$
$$\mathtt{dim}(G/[P_2, P_2]) = 6 < 14 = \mathtt{dim}(V_2)$$

.

It is clear from the table and dimension of group that there are no smooth closure for any strata.

Conclusion

Now summarizing all our computations we have

Theorem 1. *Let K be a compact, connected, simply connected Lie group. The stratum X_σ in $D(K)_{\mathrm{impl}}$ has a smooth closure if and only if K is type A or C with stratum X_σ defined as in Theorem 6 or 8.*

This result implies that S^{2n} and $\mathbf{HP}^n$ are the only class examples which can be constructed using imploded spaces.

According to [AMM] to every quasi-Hamiltonian manifold corresponds naturally a loop group manifold with a proper moment map. It is at present unclear what the loop group analogues of the spinning $2n$-sphere or the quaternionic projective space are. In particular we do not know whether these loop group manifolds possess Kahler structure. If that is case whether there is quasi-Hamiltonian analogue of it. These and related problems we would like to pursue in our future projects.

CHAPTER 2

EQUIVARIANT COHOMOLOGY OF CONJUGATION SPACES

2.1 Introduction

Consider a topological space X with an action of a compact connected Lie group G. It is a well-known that the rational equivariant cohomology $H^*_G(X, \mathbf{Q})$ is isomorphic to the subalgebra of Weyl group invariants of $H^*_T(X, \mathbf{Q})$, where T is a maximal torus of G. To be precise

Theorem 12. *[B1] Let G be a compact connected Lie group, T be a maximal torus of G, $W = N(T)/T$ be the Weyl group of G and X be a G-space. Then*

$$H^*_G(X, Q) \cong H^*_T(X, Q)^W,$$

$$H^*_T(X, Q) \cong H^*_G(X, Q) \otimes_{H^*(B_G, Q)} H^*(B_T, Q).$$

If we specialize $X = \{\mathtt{pt}\}$, as corollary we obtain a classical result of Borel

Corollary 1. *[B1] The cohomology of a complex flag variety G/T can be described as*

$$H^*(G/T) \cong H^*_T(\mathtt{pt}, Q)/H^*_T(\mathtt{pt}, Q)^W_+.$$

where $H^*_T(\mathtt{pt}, Q)^W_+$ is an ideal generated by Weyl group invariants in positive degrees.

In his later paper [B2], he obtained a similar description of the cohomology ring of real flags in $\mathbf{R}^n$. Namely, for $K = \mathbf{SO}(n)$ with diagonal subgroup $T_2 \cong (\mathbf{Z}_2)^{n-1}$, he showed that

$$H^*(K/T_2) \cong H^*_{T_2}(\mathtt{pt}, \mathbf{F}_2)/H^*_{T_2}(\mathtt{pt}, \mathbf{F}_2)^W_+.$$

where W the *restricted Weyl group*. Also if one takes $G = \mathbf{SU}(n)$ with a maximal torus T, we have a degree halving isomorphism from the cohomology ring of G/T to the cohomology ring of K/T_2. On the other hand Bott and Samelson [BS] noted that if G is a also simply connected with the *Chevalley involution* then we have:

$$\mathtt{dim}\, H^{2i}(G/T, \mathbf{F}_2) = \mathtt{dim}\, H^{i}(K/T_2, \mathbf{F}_2)$$

where K and T_2 are fixed point subgroups, under the involution, of G and T respectively.

Much earlier, A. Borel and A. Haefliger had studied the degree-halving isomorphism between the cohomology rings of complex and real projective spaces and Grassmannians from a different point of view (and without using equivariant cohomology) [BH].

Hausmann, Holm and Puppe have put these observation in the framework of equivariant cohomology, and come up with the concept of *conjugation spaces*, where the ring homomorphisms arise naturally from the existence of what they call *cohomology frames* [HHP]. Later, Hamel using ideas in [BH] gave a purely geometrical description of cohomological frames which explains topologically the origin of the degree halving isomorphisms [Ha].

In the thesis we mainly interested in topology of conjugation space with a group action. The important invariant of a space with a group action is an equivariant cohomology. This is a cohomology theory, pioneered by Borel, effectively reflects both behavior of the space and the action. In [HHP] it has been shown if X is a conjugation space with a compatible T action then we have a degree halving isomorphism on the equivariant cohomology level

$$\overline{\kappa} : H_T^{2*}(X) \to H_{T_2}^{*}(X^{\sigma})$$

where X^σ is a fixed point set under involution. Here we study conjugation spaces with an arbitrary compact Lie group actions G. Let K be a fixed point subgroup of G under an involution. In particular, under some restriction on a group G we have an isomorphism between the G-equivariant cohomology of the space and the K-equivariant cohomology of its fixed point set. As a corollary we show that the K-equivariant cohomology of its fixed set is isomorphic to an certain subalgebra of its T_2-equivariant cohomology. One can think of last stated result similar to recent results obtained by T. Holm and R. Sjamaar [HS]. In fact we use some of their results in this paper.

For the remainder of this chapter, all cohomology groups will be understood to have coefficients in $\mathbf{F}_2$ and we will drop the coefficient group from the notation.

2.2 Equivariant Cohomology

In this section we will closely follow V. Guillemin and S. Sternberg [GS] and lecture notes by W. Fulton [F]. Let G be a compact Lie group acting continuously on a topological space X. If G acts on X freely, then the quotient space X/G is usually as nice a topological space as X is. In particular, if X is a manifold then so is X/G. The definition of the equivariant cohomology group is motivated by principle that if G acts freely then the equivariant cohomology of X should be just the cohomology of X/G:

$$H_G^*(X) = H^*(X/G)$$

If the action is not free, the space X/G might be pathological and the right substitute for $H^*(X/G)$ is $H_G^*(X)$. Let E_G be a contractible space with free G action and denote by $X_G = E_G \times_G X$ the *Borel homotopy quotient*.

Definition 4. *The equivariant cohomology of X is $H^*_G(X) = H^*(X_G)$ singular cohomology of X_G.*

One can show that such space E_G exists and that $H^*_G(X)$ is independent of the choice of E_G. For the special case of a point, we have

$$H^*_G(\mathtt{pt}) = H^*(B_G)$$

where $B_G = E_G/G$.

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$.

Example 1. *Let $G = \mathbb{F}^*$ (or S^1 and $\mathbf{Z}_2$ respectively) and take $E_G = \mathbb{F}^\infty \backslash 0$. Then $B_G = \mathbb{FP}^\infty$ and $H^*(B_G) = \mathbf{F}_2[u]$. Here $u = c_1(L)$ $(w_1(L))$ is the first Chern (Stieffel-Whitney) class of the tautological complex (real) line bundle L on $\mathbb{FP}^\infty$.*

Similarly for $G = \mathbf{SU}(n)$ (or $\mathbf{SO}(n)$) we can take $E_G = F_n(\mathbb{F})$ - the space of n-frames in $\mathbb{F}^n$. Then $B_G = Gr(n, \mathbb{F}^\infty)$ is the Grassmanian of n-planes and $H^*(B_G) = \mathbf{F}_2[u_1, u_2, \ldots, u_n]$ where u_i is the i-th Chern (Stiefel-Whitney) class of the tautological vector bundle of rank n (see [B1, B2]).

Much of what we do involve fiber bundles

$$X_G \to B_G$$

with the fiber X. One can think of equivariant geometry as "spread-out geometry". These bundles are spread-out versions of X, in the same spirit as the passages from vector space to vector bundle.

One of the main problems in the equivariant cohomology states:

Problem. Let X be a given G-space and K be a closed subgroup of G. Then the restriction of G action to K makes X into K-space. What is the relationship between $H^*_G(X)$ and $H^*_K(X)$?

We may take $E_K = E_G$ with the restricted K-action. Then, we have the following commutative diagram

$$\begin{tikzcd} & X \arrow[r] \arrow[d] & X \arrow[d] \\ G/K \arrow[r] \arrow[d] & X_K \arrow[r] \arrow[d] & X_G \arrow[d] \\ G/K \arrow[r] & B_K \arrow[r] & B_G. \end{tikzcd}$$

For such a geometric setting, there is a Eilenberg-Moore spectral sequence $\{E_n, d_n\}$ such that

$$E_n \Rightarrow H^*(X_K) = H^*_K(X),$$

$$E_2^{p,q} = \mathtt{Tor}^{p,q}_{H^*(B_G)}(H^*(B_K), H^*(X_G)).$$

In particular for $K = \{\mathtt{id}\}$ the spectral sequence reduces to

$$E_2^{p,q} = \mathtt{Tor}^{p,q}_{H^*(B_G)}(H^*(\mathtt{pt}), H^*(X_G)), \quad E_n \Rightarrow H^*(X).$$

2.3 Conjugation Spaces

We will closely follow construction given in [Sj]. Let X be a topological space with a continuous involution σ. This gives rise to a continuous action of the cyclic group $\Gamma = \{1, \sigma\}$ of order 2. We will denote by X^σ the fixed point set under involution. Recall from Example 1 the space $E_\Gamma = S^\infty$ with classifying space

$B_\Gamma = \mathbb{RP}^\infty$. Since Γ action on X^σ is trivial, its homotopy quotient is $X^\sigma_\Gamma = B_\Gamma \times X^\sigma$. We have a commutative diagram

$$\begin{tikzcd}
X_\Gamma & B_\Gamma \times X^\sigma \arrow[l, "i_\Gamma"'] \\
X \arrow[u, hook, "j"] & X^\sigma \arrow[l, hook', "i"] \arrow[u, hook, "j_\sigma"']
\end{tikzcd}$$

where i is the inclusion of the real locus, i_Γ is its equivariant counterpart, and j and j_σ are the inclusions of the fibre. The associated diagram in cohomology is

$$\begin{tikzcd}
H_\Gamma(X) \arrow[r, "i_\Gamma^*"] \arrow[d, "j^*"'] & H_\Gamma^*(X^\sigma) \arrow[d, "j_\sigma^*"'] \\
H^*(X) \arrow[r, "i^*"'] & H^*(X^\sigma)
\end{tikzcd}$$

and using Kunneth theorem we have $H^*_\Gamma(X^\sigma) \cong H^*(X^\sigma) \otimes \mathbf{F}_2[u] \cong H^*(X^\sigma)[u]$. Suppose that $H^{odd}(X) = 0$. A *cohomological frame* is a pair (s, κ), where

$$s : H^*(X) \to H^*_\Gamma(X)$$

is an additive section of j^* and

$$\kappa : H^{2*}(X) \to H^*(X^\sigma)$$

is an additive isomorphism which divides the degrees in half. In addition, these are required to satisfy the conjugation equation:

$$i^*_\Gamma s(a) = \kappa(a)u^d + \omega_{d+1}u^{d-1} + \ldots + \omega_{2d-1}u + \omega_{2d}$$

for each $a \in H^{2d}(X)$ and $d \in \mathbb{N}$, where $\omega_i \in H^i(X^\sigma)$. If a cohomological frame exists, we call the involution σ a *conjugation* and the manifold X a *conjugation space*.

Conjugation spaces have a number of remarkable properties

- such a pair (κ, s) is unique ;
- the real locus X^σ is nonempty and if X is connected, then so is X^σ;
- the additive homomorphisms κ and s are ring homomorphisms;
- the coefficients $\omega_i \in H^i(X^\sigma)$ in the conjugation equation are uniquely determined by $\kappa(a)$, namely $\omega_i = Sq^{i-d}(\kappa(a))$, the $(i-d)$-th Steenrod square of $\kappa(a)$ (see [FP])

The following example illustrates general feature of conjugation spaces.

Example 2. *Let $\mathbb{CP}^n$ be a complex projective space equipped with an involution given by conjugation. Then the real locus is just real projective space $\mathbb{RP}^n$. Their cohomology rings are $H^*(\mathbb{CP}^n) = \boldsymbol{F}_2[a]/(a^n)$ and $H^*(\mathbb{RP}^n) = \boldsymbol{F}_2[b]/(b^n)$ with $\mathtt{deg}(a) = 2$ and $\mathtt{deg}(b) = 1$ respectively. Then one can show that there is a cohomological frame (κ, s) is given by*

$$\kappa : H^*(\mathbb{CP}^n) \to H^*(\mathbb{RP}^n), \quad a \mapsto b$$

and

$$i_\Gamma^* s(a^k) = (b^2 + bu)^k. \tag{2.3.1}$$

In fact equation (2.3.1) was one of the motivations for [FP]. Namely, if rewrite $b^2 + bu = Sq^1(b) + Sq^0(b)u$ one can show

$$(b^2 + bu)^k = \sum_{i=0}^{k} Sq^i(b^k)u^{k-i}.$$

The main example in [HHP] of a conjugation space is a so-called *spherical conjugation complex*, which include many known examples as special cases.

Definition 5.

1. *A conjugation cell (of dimension* $2k$*) is a space with involution which is equivariantly homeomorphic to the closed unit disk in* $\mathbb{C}^k$ *, equipped with the involution corresponding to complex conjugation.*
2. *A spherical conjugation complex is a space (with involution) obtained from the empty set by countably many successive adjunction of collections of conjugation cells.*

It has been shown that a spherical conjugation complex is a conjugation space [HHP, Proposition 5.2].

Example 3. *Let* G *be a compact connected Lie group. A Chevalley involution of* G *is an involution satisfying* $\sigma(g) = g^{-1}$ *for* g *in some maximal torus* T *of* G *and* $\sigma(\alpha) = -\alpha$ *for all roots* α *of* G*. It is known that Chevalley involutions exist for all* G *and are unique up to conjugation (see [Sa]). In particular, for* $G = \mathbf{SU}(n)$*, it is given by the complex conjugation:* $\sigma(g) = \overline{g}$*. It has been proved that with respect to the Chevalley involution, every coadjoint orbit* X *is conjugation space [HHP]. For instance, all the well-known examples of complex projective spaces, complex Grassmannians with the involution given by complex conjugation are conjugation spaces (hence the fixed point sets are real projecive spaces and real Grassmannians).*

2.4 Compatible group actions

Let X be a space together with an involution σ . Let G be a compact connected Lie group with involution τ. Suppose that a group G acts continuously on X. We say that σ is *compatible* with the group action if $\sigma(g.x) = \tau(g).\sigma(x)$ for any $g \in G$. In case G is a torus with an involution $\tau(g) = g^{-1}$, it was carefully been studied in the original paper [HHP]. In particular they have shown

Theorem 2. *[HHP, Corollary 7.6] Let X be a conjugation space with a compatible T-action. Then X_T is a conjugation space. In particular, there is a ring isomorphism*

$$\overline{\kappa} : H_T^{2*}(X) \to H_{T_2}^*(X^\sigma).$$

2.4.1 Statement of result

Motivated by this result we will consider a problem in slightly more general setting. Namely, we will be interested with simply connected G actions where an involution is the Chevalley involution. Denote by K the fixed point group under this involution. It is known that K is connected (i.e. see [BS]). The main result of this chapter

Theorem 3. *Let X be a conjugation space with a compatible G-action. Let G be of type A or C. Then X_G is also a conjugation space and we have a degree-halving isomorphism:*

$$\overline{\kappa} : H_G^{2*}(X) \to H_K^*(X^\sigma).$$

We will see that the restriction on type of groups is necessary. First we will recall some notions.

2.4.2 Equivariant fiber bundles over spherical conjugation complexes

In this section we only require G to be compact. Let (B, γ) be a space with involution.

Definition 6. *(G, τ)-principal bundle is a G-principal bundle $p : E \to B$ equipped with an involution $\tilde{\sigma}$ such that the right action of G is compatible, i.e. $\widetilde{\gamma}(x.g) = \widetilde{\gamma}(x).\tau(g)$*

Let (X, σ) be a space with involution together with a compatible G-action. The space $E \times_G X$ has an induced involution (which also be called γ) and the associated bundle $E \times_G X \to B$, with fiber X, is a γ -equivariant locally trivial bundle. The following result is especially useful for computing equivariant cohomology [HHP, Proposition 5.3]

Theorem 4. *Let X be a conjugation space and B be a spherical conjugation complex. Then $E \times_G X$ is a conjugation space.*

2.4.3 Milnor construction

For a given Lie group G we will present a construction of the universal G-bundle due to J. Milnor. The affine n-simplex Δ^n is the compact subset of $\mathbb{R}^{n+1}$ consisting of $t = (t_0, \ldots, t_n) \in \mathbb{R}^{n+1}$ satisfying $\sum_{i=0}^{n} t_i = 1$ and $t_i \geq 0$. We form the product $\Delta^n \times G^{n+1}$ where elements are written as double $n + 1$ tuples $(t_0 : g_0, \ldots, t_n : g_n)$ with $t \in \Delta^n$ and $g_i \in G$. Let $E_G(n)$ be the quotient of $\Delta^n \times G^{n+1}$ defined by the following equivalence relation:

$$(t'_0 : g'_0, \ldots, t'_n : g'_n) = (t''_0 : g''_0, \ldots, t''_n : g''_n)$$

provided $t'_i = t''_i$ for each i, and $g'_i = g''_i$ for all i with $t'_i = t''_i > 0$. If $t'_i = t''_i = 0$ different g'_i and g''_i define the same equivalence class.

We have an action $E_G(n) \times G \to E_G(n)$ given by the formula $(t_0 : g_0, \ldots, t_n : g_n)g = (t_0 : g_0 g, \ldots, t_n : g_n g)$. One can easily see it is free. The natural inclusion of the products $\Delta^n \times G^{n+1} \subset \Delta^{n+1} \times G^{n+2}$ induces an inclusion on quotients spaces

$E_G(n) \subset E_G(n+1)$, where $(t_0 : g_0, \ldots, t_n : g_n)$ sent to $(t_0 : g_0, \ldots, t_n : g_n, 0 : e)$ for $e \in G$, the identity element of G. The inclusion is G-equivariant.

The *Milnor universal principal G-bundle* is $E_G = \varinjlim E_G(n)$ equipped with inverse limit topology, that is a subset $M \subset E_G$ is closed if and only if $M \subset E_G(n)$ is closed in $E_G(n)$ for each n. The Milnor classifying space is the quotient $E_G/G = B_G$. For short we will denote elements of E_G by $(t_i : g_i)$ (see [Hu]).

Under the right diagonal action of G on E_G, each $(t_i : g_i)$ is equivalent to a unique element $(t_i : \tilde{g}_i)$ for which $\tilde{g}_j = e$, where j is the minimal integer for which $t_j = 0$. Therefore, each class in B_G has a unique such representative which we call the *minimal*.

2.4.4 Fixed point set of X_G

Let X and G be as above. Consider X_G the Borel homotopy quotient of X. Note that any involution τ on G induces the involution on E_G, namely $\tau(t_i : g_i) = (t_i : \tau(g_i))$, which is G-compatible. One can check that E_G equipped with such involution is a (τ, G)-principal bundle. There is a natural involution on $E_G \times X$ which we denote by σ. Similarly one can check that it is (τ, G)-principal bundle. The following proposition gives a description of fixed point set under this involution.

Proposition 4. *Let K be a fixed point subgroup of G under an involution τ then*

$$(X_G)^\sigma = (X^\sigma)_K.$$

Proof. Basically we follow the proof given in [HHP]. In the same way as for B_G, each class in X_G has a unique *minimal representative* $(w, x) \in E_G \times X$ for which w is minimal. Note that the inclusion $i : X^\sigma \to X$ induces an inclusion $i_K : (X^\sigma)_K \to X_K$ while the group inclusion $K \subset G$ induces a projection $i_X : X_K \to X_G$. On the other hand, using the Milnor construction we have

$$E_K \times X^\sigma \hookrightarrow E_G \times X$$

which is equivariant with respect to diagonal actions of K action on the LHS and G action on the RHS. Hence there is an induced injective map $(X^\sigma)_K \to X_G$. One can easily check that the image is invariant under σ and therefore there is an injective map $(X^\sigma)_K \to (X_G)^\sigma$ which we will denote by β. Hence, we have a commutative diagram

$$\begin{tikzcd}
(X^\sigma)_K \arrow[r, hook, "i_K"] \arrow[drr, "\beta"'] & X_K \arrow[r, two heads, "i_X"] & X_G \\
& & (X_G)^\sigma \arrow[u, hook]
\end{tikzcd}$$

To see that β is surjective, consider (w, x) minimal representative in X_G. Then $\sigma(w, x)$ is also minimal. On the other hand if $\sigma(w, x) = (w, x)$ then $\sigma(x) = x$ and $\tau(g_i) = g_i$ that is $g_i \in K$. Hence (w, x) is in the image of β which implies that β is surjective. $\square$

In view of this proposition and Theorem 4 we only need to prove that B_G is a spherical conjugation complex. Now we restrict ourselves to the Chevalley involution.

Proposition 5. *Let G be a compact, simply connected, simple Lie group of type A or C. Then B_G is a spherical conjugation complex.*

Proof. Let G be of type A, namely $\mathbf{SU}(n)$. Let $F_n(\mathbb{C}^{n+m})$ be the space of orthonormal families of n vectors in $\mathbb{C}^{n+m}$. The group $\mathbf{SU}(n)$ acts freely on $F_n(\mathbb{C}^{n+m})$ with the quotient space $Gr(n, \mathbb{C}^{n+m})$, the Grassmanian of n planes. $F_n(\mathbb{C}^{n+m})$ is $(m-1)$-connected. Therefore one can take as $E_G(m) = F_n(\mathbb{C}^{n+m+1})$ and for the base $B_G(m) = Gr(n, \mathbb{C}^{n+m+1})$. We define a structure of spherical conjugation complex as follows. Denote by X_m the Schubert cell decomposition of $Gr(n, \mathbb{C}^{n+m})$. It is known that it is a spherical conjugation complex and X_{m+1} is obtained from X_m by adjunction of a finite collection of conjugation cells. Hence, $B_G = \bigcup_{i=-1}^{\infty} X_m$ is a spherical conjugation complex equipped with involution given by conjugation. Note in this case the fixed point set is just $(B_{\mathbf{SU}(n)})^{\sigma} = Gr(n, \mathbb{R}^{\infty}) = B_{\mathbf{SO}(n)}$

Similarly for type C, i.e. $G = \mathbf{Sp}(n)$ we can take for $E_G(m) = F_n(\mathbb{H}^{n+m})$, the space of orthogonal frames and $B_G(m) = Gr(n, \mathbb{H}^{n+m})$ the Grassmanian of n planes in $\mathbb{H}^{n+m}$. Note that the Chevalley involution for complex simple Lie algebra of type C is just the complex conjugation for $\mathbf{Sp}_{2n}(\mathbb{C})$, a complex symplectic group. On the other hand

$$\mathbf{Sp}(n) = \mathbf{U}(n) \cap \mathbf{Sp}_{2n}(\mathbb{C})$$

Therefore the fixed point set for $\mathbf{Sp}(n)$ under the Chevalley involution is

$$\mathbf{U}(n) = \mathbf{O}(2n) \cap \mathbf{Sp}_{2n}(\mathbb{R}) = \mathbf{U}(n) \cap \mathbf{Sp}_{2n}(\mathbb{R})$$

Hence the fixed point set for the classifying space is $Gr(n, \mathbb{C}^{\infty})$. By the similar argument as for type we conclude that $B_{\mathbf{Sp}(n)}$ is spherical conjugation complex.

□

2.4.5 Other type of Lie groups

Unfortunately, B_G is not a conjugation space for any other type of Lie groups, possibly except for type G_2. The main problem is that except for type A, C and G_2, the $\mathbf{F}_2$-cohomologies of the corresponding classifying spaces always contain nontrivial odd degree cocycles. For type B and D the corresponding simply connected groups are $\mathbf{Spin}(n)$. For $n \leq 6$ we have

$$\mathbf{Spin}(3) \cong \mathbf{SU}(2) \quad \mathbf{Spin}(4) \cong \mathbf{SU}(2) \times \mathbf{SU}(2)$$
$$\mathbf{Spin}(5) \cong \mathbf{Sp}(2) \qquad \mathbf{Spin}(6) \cong \mathbf{SU}(4)$$

which might be included for type A case. For $n > 9$, it is known that $H^*(B_{\mathbf{Spin}(n)})$ is not even a polynomial ring. For $n \leq 10$ it was computed by Borel [B3]. In particular,

$$H^*(B_{\mathbf{Spin}(10)}) = \mathbf{F}_2[y_4, y_6, y_7, y_8, y_{10}, y_{32}]/(y_7 y_{10})$$

Now consider a sequence

$$w_2, Sq^1 w_2, \ldots, Sq^{2^{h-2}} Sq^{2^{h-3}} \ldots Sq^1 w_2 \tag{2.4.1}$$

where w_i's are universal Stieffel-Whitney classes and h certain parameter depending on n. The following remarkable result due to D. Quillen gives an explicit description of the cohomology ring of the classifying space of spin groups [Q].

Theorem 5. *Let J be the ideal in $H^*(B_{\mathbf{SO}(n)})$ generated by the regular sequence* (2.4.1). *Then the canonical homomorphism*

$$H^*(B_{\mathbf{SO}(n)}) \otimes \boldsymbol{F}_2[w_{2^h}] \to H^*(B_{\mathbf{Spin}(n)}) \tag{2.4.2}$$

is an isomorphism.

In particular, if we rewrite the sequence more explicitly, the ideal has form

$$J = \langle w_2, w_3, w_5, w_9, w_7 w_{10}, \ldots \rangle$$

which implies that for $n \geq 7$, the cohomology ring $H^*(B_{\mathbf{Spin}(n)})$ always contains a nontrivial 7-cocycle. Hence, it is not a conjugation space. For exceptional groups computations of cohomologies of classifying spaces were carried out mostly by Japanese school A. Kono, M. Mimura, H. Toda, etc. For type E:

$$H^*(B_{E_6}) = \mathbf{F}_2[y_4, y_6, y_7, \bar{y}_{10}, \bar{y}_{18}, y_{32}, \bar{y}_{34}, y_{48}]/(ideal)$$

$$H^*(B_{E_7}) = \mathbf{F}_2[x_4, x_6, x_7, x_{10}, x_{11}, x_{18}, x_{19}, x_{34}, x_{35}, x_{66}, x_{67}, y_{64}, y_{96}, y_{112}]/(ideal)$$

It is somewhat surprising that the cohomology ring of B_{E_8} is still not determined. For some further results and related conjectures on $H^*(B_{E_8})$ one can see a recent paper by M. Mimura, T. Nishimoto [MN]. One should also remark that these computations heavily based on computations on spin groups. However by Remark 3 we can conclude that $H^*(B_{E_8})$ also contains odd dimensional cohomology classes. On the other hand for type F and G it was computed long before by A. Borel [B1, Theorem 19.1],

$$H^*(B_{F_4}) = \mathbf{F}_2[y_4, y_6, y_7, y_{16}, y_{24}]$$

$$H^*(B_{G_2}) = \mathbf{F}_2[y_4, y_6, y_{10}].$$

based on the fact that the generators of $H^*(G_2)$ and $H^*(F_4)$ are *universally transgressive*.

Definition 7. *Let*

$$F \overset{i}{\hookrightarrow} E \xrightarrow{p} B$$

be a fibration with fibre F. We say that F is totally non-homologous to zero in E with respect to ring R if the homomorphism $i^* : H^*(E.R) \to H^*(F, R)$ is onto.

Remark 3. *In the inclusions $G_2 \hookrightarrow F_4 \hookrightarrow E_6 \hookrightarrow E_7 \hookrightarrow E_8$, every subgroup is totally non-homologous to zero in any bigger group containing it. By well-known result of Borel [B1], $H \hookrightarrow G$ is totally non-homologous to zero if and only if $\sigma^*(H, G)$: $H^*(B_G) \to H^*(B_H)$ is an epimorphism.*

2.5 Abelianization in Equivariant Cohomology

In this section motivated by the result of [HS] we would like to find in some sense a simpler description of equivariant cohomology rings of conjugation spaces.

As we have mentioned in the introduction when we consider topological space X with G action, there is a remarkable relationship between cohomology rings of $H^*_G(X, \mathbf{Q})$ and $H^*_T(X, \mathbf{Q})$, where T is a maximal torus. Namely, $H^*_G(X, \mathbf{Q})$ is isomorphic to the subalgebra of Weyl group invariants of the equivariant cohomology ring $H^*_T(X, \mathbf{Q})$. But in general this relationship breaks down for other coefficient rings. To explain this relationship lets recall a general construction.

Let U be a closed, not necessarily connected subgroup of G. The canonical map

$$\mathrm{p}_X : X_U \twoheadrightarrow X_G \tag{2.5.1}$$

is a locally trivial fibre bundle with fibre G/U. We have the induced map $p_X^* : H^*_G(X, \mathbf{k}) \to H^*_U(X, \mathbf{k})$ with coefficients in commutative ring $\mathbf{k}$. In case U is a maximal torus T, the image of p_X^* is W-invariant (where W is the corresponding Weyl group) and

$$p_X^* : H^*_G(X, \mathbf{k}) \to H^*_T(X, \mathbf{k})^W.$$

It is known that for general $\mathbf{k}$ it is neither injective nor surjective. Generalizing observations made in [AC] and [BE], R. Sjamaar and T. Holm have shown that under rather mild conditions on $\mathbf{k}$ one can give similar description of the cohomology ring $H^*_G(X, \mathbf{k})$.

2.5.1 Demazure Algebra

Let R be a root system of (G, T). For each $\alpha \in R$ define an operator δ_α on the polynomial ring $S_{\mathbf{k}} = H_T^*(pt, \mathbf{k})$

$$\delta_\alpha = \frac{1 - s_\alpha}{\alpha}$$

where s_α is a reflection with respect to a hyperplane defined by the root α. There are number of interesting identities on these operators

$$s_\alpha \delta_\alpha = \delta_\alpha = -\delta_\alpha s_\alpha, \quad \delta_{-\alpha} = -\delta_\alpha, \quad w\delta_\alpha w^{-1} = \delta_{w(\alpha)}$$

for any $w \in W$. Note that δ_α is an operator of degree (-2). Let $\mathcal{D}_{\mathbf{k}}$ be the algebra of endomorphisms generated by δ_α for $\alpha \in R$ and $S_{\mathbf{k}}$ (which acts by multiplication). The generators are $S_{\mathbf{k}}^W$-linear, so $\mathcal{D}_{\mathbf{k}}$ is a subalgebra of $\mathtt{End}_{S_{\mathbf{k}}^W}(S_{\mathbf{k}})$. In particular, $\mathcal{D}_{\mathbf{k}}$ contains the Weyl group and hence the group algebra $S_{\mathbf{k}}[W]$. It was shown that $H_T^*(X)$ is a module over $\mathcal{D}_{\mathbf{k}}$ in a natural way [HS, Theorem 1.10].

Define the *augmentation left ideal* of $\mathcal{D}_{\mathbf{k}}$ to be the annihilator of constant polynomial $1 \in S_{\mathbf{k}}$,

$$I(\mathcal{D}_{\mathbf{k}}) = \{\Delta \in \mathcal{D}_{\mathbf{k}} | \Delta(1) = 0\}.$$

Let M be a left $\mathcal{D}_{\mathbf{k}}$ module. We denote by $M^{I(\mathcal{D}_{\mathbf{k}})}$ the set of all elements in M annihilated by $I(\mathcal{D}_{\mathbf{k}})$. Note $M^{I(\mathcal{D}_{\mathbf{k}})}$ is not a module over $\mathcal{D}_{\mathbf{k}}$, but it is a module over the ring $S_{\mathbf{k}}^W$. It is easy to see that,

$$p_X^*(H_G^*(X, \mathbf{k})) \subseteq H_T^*(X, \mathbf{k})^{I(\mathcal{D}_{\mathbf{k}})} \subseteq H_T^*(X, \mathbf{k})^W.$$

Note that there are examples when both of these inclusions are strict. In [HS], they gave sufficient conditions when the first inclusion is an isomorphism. To state the result first let us recall a notion of *torsion*. By well-known result of

Borel [B1], the characteristic homomorphism $i^* : S_{\mathbf{k}} \to H^*(G/T, \mathbf{k})$ is surjective over $\mathbf{k} = \mathbf{Q}$ and therefore has a finite cokernel over $\mathbf{Z}$. The *torsion index* $t(G)$ of G is the order of the cokernel of $i^* : S^N \to H^{2N}(G/T, \mathbf{Z})$, where $2N$ is the degree of top cohomology class of G/T. One can show that torsion index always divides the order of Weyl group. The main result in [HS, Theorem 2.5, 2.6] states

Theorem 6. *Assume that $t(G)$ is unit in $\mathbf{k}$. Then*

1. $\mathcal{D}_{\mathbf{k}} = \mathrm{End}_{S_{\mathbf{k}}^W}(S_{\mathbf{k}})$.
2. *The map is p_X^*an isomorphism from $H_G^*(X, \mathbf{k})$ onto $H_T^*(X, \mathbf{k})^{I(\mathcal{D}_{\mathbf{k}})}$.*

2.5.2 Back to Conjugation Spaces

Now using the results of the previous subsection we can give another description of equivariant cohomologies of conjugation spaces. Namely, we will relate the cohomology rings of Theorem 2 to that of Theorem 3. Our main result,

Theorem 7. *Let (X, σ) be a conjugation space with compatible G-action, where G is either of type A or C. Then*

$$H_K^*(X^\sigma) \cong H_{T_2}^*(X^\sigma)^{I(\mathcal{D}^\sigma_{\mathbf{F}_2})}.$$

Proof. First note that the torsion index $t(G)$ for type A and C is 1, therefore it is unit in any ring $\mathbf{k}$. Hence by Theorem 6 over $\mathbf{F}_2$ we have

$$H_G^*(X) \cong H_T^*(X)^{I(\mathcal{D}_{\mathbf{F}_2})}.$$

On the other hand by Theorem 2 and Theorem 3 we have degree halving isomorphisms

$$\begin{aligned}\kappa : H_G^{2*}(X) &\to H_K^*(X^\sigma)\\ \kappa : H_T^*(X) &\to H_{T_2}^*(X^\sigma).\end{aligned}$$

Now for each $\delta_\alpha \in \mathcal{D}_{\mathbf{F}_2}$ there is a unique operator δ_α^σ of degree (-1) on $H_{T_2}^*(X^\sigma)$ such that the following diagram commutes

$$\begin{tikzcd} H_T^*(X) \arrow[r, "\delta_\alpha"] \arrow[d, "\kappa"'] & H_T^*(X) \arrow[d, "\kappa"'] \\ H_{T_2}^*(X^\sigma) \arrow[r, dashed, "\delta_\alpha^\sigma"] & H_{T_2}^*(X^\sigma). \end{tikzcd}$$

We have $\delta_\alpha^\sigma = \kappa^{-1} \circ \delta_\alpha \circ \kappa$. Let $\mathcal{D}_{\mathbf{F}_2}^\sigma$ be the algebra generated by δ_α^σ and the symmetric algebra $H_{T_2}^*(pt)$. Hence we have an isomorphism

$$\kappa : H_T^*(X)^{I(\mathcal{D}_{\mathbf{F}_2})} \to H_{T_2}^*(X^\sigma)^{\mathcal{D}_{\mathbf{F}_2}^\sigma}.$$

Summarizing all these facts there is a commutative diagram

$$\begin{tikzcd} H_G^{2*}(X) \arrow[r, "p_X^*"] \arrow[d, "\kappa"'] & H_T^{2*}(X)^{I(\mathcal{D}_{\mathbf{F}_2})} \arrow[d, "\kappa"'] \\ H_K^*(X^\sigma) \arrow[r, dashed, "p_{X^\sigma}^*"] & H_{T_2}^*(X^\sigma)^{I(\mathcal{D}_{\mathbf{F}_2}^\sigma)} \end{tikzcd}$$

where $p_{X^\sigma}^* = \kappa \circ p_X^* \circ \kappa^{-1}$. Since all the maps in diagram are isomorphisms so is $p_{X^\sigma}^*$ which proves theorem. $\square$

BIBLIOGRAPHY

[A] A. Alekseev, *On Poisson actions of compact Lie groups on symplectic manifolds*, J. Differential Geom. **45** (1997), 241–256.

[AB] M. F. Atiyah, R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London **308** (1982), 523–615.

[ABM] A. Alekseev, H. Bursztyn and E. Meinrenken *Pure Spinors on Lie groups*, arXiv:0709.1452.

[AC] E. Akyildiz, J. B. Carrell, *Zeros of holomorphic vector fields and the Gysin homomorphism. Singularities*, Part 1 (Arcata, Calif., 1981), 47–54, Proc. Sympos. Pure Math., **40**, Amer. Math. Soc., Providence, RI, 1983.

[AMM] A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment map*, J. Differential Geom. **48** (1998), 445-495. MR 99k:58062.

[AMW] A. Alekseev, E. Meinrenken and C. and Woodward, *Duistermaat-Heckman measure and moduli spaces of flat bundles over surfaces*, Geom. Funct. Anal. **12** (2002), 1-31. MR 2003d:53151.

[B] N. Bourbaki, *Lie groups and Lie algebras*, Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998, 2002, 2005.

[B1] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, (French) Ann. of Math. **(2)** 57, (1953), 115–207.

[B2] A. Borel, *La cohomologie mod* 2 *de certains espaces homogénes*. (French) Comment. Math. Helv. **27**, (1953), 165–197.

[B3] A. Borel, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. Jour. Math. **76**, (1954), 273-342.

[BH] A. Borel, A. Haefliger, *La classe d'homologie fondamentale d'un espace analytique*, (French) Bull. Soc. Math. France **89** (1961) 461–513.

[BS] R. Bott, H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80**, (1958), 964–1029.

[BE] P. Bressler, S. Evens, *The Schubert calculus, braid relations, and generalized cohomology*, Trans. Amer. Math. Soc. **317** (1990), no. 2, 799–811.

[D] V. G. Drinfeld, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation*, Soviet. Math. Dokl. **27** (1983) 68–71.

[E] A. Eshmatov, *A new class of examples of group-valued moment maps*, arXiv:0710.3844.

[F] W. Fulton, *Equivariant Cohomology in Algebraic Geometry*, notes by D. Anderson, online.

[FP] M. Franz, V. Puppe, *Steenrod squares on conjugation spaces*, C. R. Math. Acad. Sci. Paris **342** (2006), no. 3, 187–190.

[G] W. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. Math. **54** (1984) 200–225.

[GJS] V. Guillemin, L. Jeffrey, and R. Sjamaar, *Symplectic implosion*, Transform. Groups **7** (2002), 155-184. MR 2003b:53090.

[GS] V. Guillemin, S. Sternberg, *Supersymmetry and equivariant de Rham theory*, with an appendix containing two reprints by Henri Cartan , Mathematics Past and Present. Springer-Verlag, Berlin, 1999.

[Ha] J. van Hamel, *Geometric cohomology frames on Hausmann-Holm-Puppe conjugation spaces*, Proc. Amer. Math. Soc. **135** (2007), no. 5, 1557–1564.

[Hu] D. Husemoller, *Fibre bundles*, Third edition. Graduate Texts in Mathematics, 20. Springer-Verlag, New York, 1994.

[HHP] J.-C. Hausmann, T. Holm, V. Puppe, *Conjugation spaces*, Algebr. Geom. Topol. **5** (2005), 923–964.

[HS] T.S. Holm, R. Sjamaar, *Torsion and abelianization in equivariant cohomology*, Transformation Groups, **13** 2008.

[HJ] J. Hurtubise and L. Jeffrey, *Representation with weighted frames and framed parabolic bundles*, Canad.J.Math. **52** (2000), 1235-1268.

[HJS] J. Hurtubise, L. Jeffrey, and R. Sjamaar, *Group-valued implosion and parabolic structures*, Amer.J.Math. **128** (2006), 167-214. 2007d:53141.

[K] H. Kraft, *Geometricheskie metody v teorii invariantov*. (Russian) [Geometric methods in the theory of invariants] Translated from the second German edition by D. I. Panyushev. Translation edited and with a preface by V. L. Popov. "Mir", Moscow, 1987.

[L] M. Libine, *Lecture notes on equivariant cohomology*, preprint, Yale University, 2007, arXiv:0709.3615.

[LW] J.-H. Lu, A. Weinstein, *Poisson Lie groups, dressing transformations and Bruhat decompositions*, J. Differential Geom. **31** (1990) 501–526.

[MS] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford Mathematical Monographs (Oxford: Clarendon, 1995).

[M1] D. McDuff, *The moment map for circle actions on symplectic manifolds*, J. Geom. Phys. **5** (1988) 149–160.

[MN] M. Mimura, T. Nishimoto, *On the Stiefel-Whitney classes of the representations associated with* **Spin**(15), arXiv:0903.4969v1.

[P] Popov, V. L. *Classification of three-dimensional affine algebraic varieties that are quasihomogeneous with respect to an algebraic group* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 3, 566–609, 703.

[PV] V.L. Popov, E. B. Vinberg, *A certain class of quasihomogeneous affine varieties*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 749–764.

[Q] D. Quillen, The mod 2 cohomology ring of extra-special 2-groups and the spinor groups, Math. Ann., **194** (1971), 197-212.

[Sa] H. Samelson, *Notes on Lie Algebras*, Universitext, Springer-Verlag, New York (1990).

[Sj] R. Sjamaar, *Real Symplectic Geometry*, preprint, 2007.

[SL91] R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. (2) **134** (1991), 375-422. MR 92g:58036.