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Abstract

Markov–modulated queueing systems are those in which the primary arrival and service mechanisms are influenced by changes of phase in a secondary Markov process. This influence may be external or internal, and may represent factors such as changes in environment or service interruptions. An important example of such a model arises in packet switching, where the calls generating packets are identified as customers being served at an infinite server system. In this paper we first survey a number of different models for Markov–modulated queueing systems. We then analyze a model in which the workload process and the secondary process together constitute a Markov compound Poisson process. We derive the properties of the waiting time, idle time and busy period, using techniques based on infinitesimal generators. This model was first investigated by G.J.K. Regterschot and J.H.A. de Smit using Wiener–Hopf techniques, their primary interest being the queue–length and waiting time.

Keywords: *Markov–modulated queues, changes of phase, Markov–additive process, infinitesimal generators, waiting time, idle period, busy period, matrix–functional equations.*

1. **Introduction**. Markov–modulated queueing systems are those in which the principal process of interest such as the queue–length or waiting time is influenced by a secondary process which happens to be Markovian. Two different classes of models have been studied in the literature. In the first of these the secondary process represents an extraneous influence in the sense that the input process or service mechanism is subject to changes caused by changes of state in the process. Such systems are special cases of queues in random environment, the randomness being Markovian in nature (Eisen and Tainiter [3], Neuts [7], Purdue [10], Yechiali and Naor [15]). A simple example of this type is provided by a system in which the input may vary according to seasons, which is a special case of the N–type input described by Ramaswami [11]. In the second class of models the Markov process is actually a component of the queueing system under consideration. A practical example of this type is the packet–switching model investigated by Burman and Smith ([1], [2]), in which the input process in the single server system studied is a non–homogeneous Poisson process with rate proportional to the queue–length process of an infinite server queue. A second example is provided by models with pre–emptive priorities, breakdowns and in general, service interruptions (Mitrani and Avi–Itzhak [6], White and Christie [13]). Many of the models referred to here may be characterized as simple Markov–modulated systems giving rise to a Markov–modulated birth and death process, described as follows. Let $(Q,J) = \{Q(t), J(t), t \geq 0\}$ be a Markov process on the state space $N_+ \times E$, where $N_+ = \{0,1,2,\ldots\}$ and E is a countable set. Assume that

$$\begin{aligned} &P\{Q(t+h) = j,\ J(t+h) = n \,|\, Q(t) = i,\ J(t) = m\} \\ &\qquad = a(i,m;\, j,n)h + o(h), \end{aligned} \tag{1}$$

where the transition rates $a(i,m;\, j,n)$ for $(i,m) \neq (j,n)$ are given by

$$a(i,m;\, i+1,m) = \lambda_{im},\ a(i,m;\, i-1,m) = \mu_{im}\ \ (i \geq 0,\ \ m \in E)$$
$$a(i,m;\, i,n) = \nu_{mn}\ \ (m,n \in E,\ \ i \geq 0). \tag{2}$$

Here $\mu_{0m} = 0$ $(m \in E)$. Note that the rates $a(i,m;\, i,n)$ do not depend on i, indicating that the J–component of the process evolves by itself. We denote

$$b(i,m) = \sum_{(j,n)} a(i,m;\, j,n) = \lambda_{im} + \mu_{im} - \nu_{mm} \tag{3}$$

where $\nu_{mm} = -\sum_{n} \nu_{mn}$. If the steady state probabilities $u(j,n)$ exist, they satisfy the equations

$$b(j,n)u(j,n) = \sum_{(i,m)} u(i,m)a(i,m;\, j,n). \tag{4}$$

In applications Q represents the queue–length in the system under consideration, while J is the modulating Markov chain. The model represented by (Q,J) is versatile enough to cover the case where J represents the extraneous influence or is a component of the system itself. Some special cases are reviewed in section 2, the main topic of study being the steady state distribution of (Q,J). Particularly interesting is the case in which for $m \in E$,

$$\lambda_{im} = \lambda_{m}\ \ (i \geq 0),\ \ \mu_{im} = \mu_{m}\ \ (i \geq 1),\ \ \mu_{0m} = 0. \tag{5}$$

Here the arrivals form a Markov–modulated Poisson (or simply, a Markov–Poisson) process, while the service is offered at an exponential rate that fluctuates with the state of the underlying Markov chain (Purdue [10]). Neuts [7] investigates an extension of this

model in which the arrivals are Markov–Poisson, but the service time of a customer depends only on the Markov state at the commencement of his service. In section 3 we review the Neuts model in detail, and consider the busy period problem investigated by him by postulating it as a problem involving the Markov–modulated branching process. The class of Markov subordinators of which the Markov–Poisson is a member, is reviewed in section 4.

Regterschot and de Smit [12] consider a queueing model that is identical with the one studied by Neuts [7], with the exception that the service time of a customer depends only on the Markov state at his arrival epoch. Using Wiener–Hopf techniques these authors analyze the waiting time and queue–length processes. A more comprehensive model of this type may be described as follows. Denote by $\{u_k,\ k \geq 1\}$ the sequence of interarrival times, $\{v_k,\ k \geq 1\}$ the sequence of successive service times, $U_n = u_1 + u_2 + \ldots + u_n$, $V_n = v_1 + v_2 + \ldots + v_n$ $(n \geq 1)$ and $U_0 = V_0 = 0$. In addition, there is an underlying Markov chain $J = \{J_n,\ n \geq 0\}$ such that $(U,V,J) = \{(U_n,V_n,J_n),\ n \geq 0\}$ is defined on a given probability space. We assume that (U,V,J) is a time–homogeneous Markov–renewal process (MRP) on the state space $\mathbb{R}_+ \times \mathbb{R}_+ \times E$, where $\mathbb{R}_+ = [0,\infty)$ and E is a countable set. Thus, for any Borel subset A of $\mathbb{R}_+ \times \mathbb{R}_+$, $j,k \in E$, we have

$$\begin{aligned} &P\{(U_{m+n},V_{m+n}) \in A,\ J_{m+n} = k \mid (U_i,V_i,J_i)\ \ (0 \leq i \leq m)\} \\ &\qquad = P\{(U_n,V_n) \in A - (U_m,V_m),\ J_n = k \mid J_0\}\ \text{a.s.} \end{aligned} \tag{6}$$

We shall denote by

$$Q_{jk}\{A\} = P\{(u_1,v_1) \in A,\ J_1 = k \mid J_0 = j\} \tag{7}$$

the one–step transition distribution measure of the MRP under consideration. We are thus assuming that each customer's interarrival time and service time are jointly dependent on the state of the Markov chain J at his arrival epoch and that the service time is not influenced by subsequent changes of state in J.

A second model is obtained if we assume that the underlying Markov chain is defined in continuous time, and the assumption (6) holds with J_n being the state of this Markov chain at the epoch U_n. This model is formally identical with the first, except that the calculation of Q_{jk} will depend on the evolution of the Markov chain during the interval $[U_n,U_{n+1})$. Thus in the Regterschot–Smit [12] model the transition distribution measure (7) is given by

$$Q_{jk}\{C_1 \times C_2\} = A_{jk}\{C_1\}B_j\{C_2\}, \tag{8}$$

where B_j is the service time distribution and j the computation of A_{jk} is based on the assumption of Markov–Poisson arrivals.

Knessl, Matkowsky, Schuss and Tier [5] consider an M/G/1 queueing model whose arrival rate and service time density depend on a two–state Markov chain $\{J(t)\}$, which in turn depends on $W(t)$, the residual workload in the system. Specifically they assume that

$$\begin{aligned} &P\{\text{an arrival during } (t,t+h] \mid J(t) = j,\ W(t) = w\} \\ &\qquad = \lambda_j(w)h + o(h) \end{aligned} \tag{9}$$

and, further, that the service time of a customer depends on the state of the process $\{J(t),W(t)\}$ at the time of his arrival. The dependence of the transition rates of the J–chain on $W(t)$ indicates a feedback mechanism that can be used to control $W(t)$.

The dependence on $W(t)$ indicates possibly discouraged arrivals or service demands, while the dependence on $J(t)$ may indicate two classes of arrivals with different service requirements. The authors use singular perturbation methods to compute asymptotic approximations to the stationary distribution of the workload and other quantities of interest.

In this paper we investigate a single server queueing system where the customers arrive in a Markov–modulated Poisson process, their service times having a distribution depending on the state of the Markov chain at the time of arrival. The resulting workload process is then Markov compound Poisson, as shown in section 5. the busy period problem for this model is investigated in section 6, the result involving a certain matrix–functional equation. The waiting time and idle time processes are studied in section 7. In section 8 we consider the case where the underlying Markov chain has a finite state space. We establish the existence and uniqueness of the functional equation obtained in section 6. The properties of this solution are then used to derive the limit distribution of the waiting time and idle time processes.

2. Some simple Markov–modulated systems

(a) Markov–modulated M/M/1 queues. The earliest work on this class of models is due to Eisen and Tainiter [3], who used the term stochastic variation to describe the influence of J. Evidently unaware of this work, Yechiali and Naor [15] investigated the same model, calling their system an M/M/1 queue with heterogeneous arrivals and service. Here J is a two state Markov chain and the arrival and departure rates are given by

$$\lambda_{im} = \lambda_m \ (i \geq 0), \ \mu_{im} = \mu_m \ (i \geq 1), \ \mu_{0m} = 0 \qquad (10)$$

for $m = 0,1$. Yechiali [14] formulated a special case of the (Q,J) process defined by

the parameters (2), with J defined over a finite state space. As a specific example, he investigated the extension of the two state case (10) to the case where J has a finite state space $\{0,1,...,N\}$, so that the parameters (10) are defined for $m = 0,1,...,N$. Yechiali focussed on the solution of the steady state equations (4) which can be written as

$$(z-1)(\mu_n-\lambda_n z)U_n(z) - z \sum_{m=0}^{\infty} U_m(z)\nu_{mn} = \mu_n(z-1)U_n(0) \qquad (n = 0,1,2,...), \tag{11}$$

where $U_n(z) = \Sigma_{j=0}^{\infty} u(j,n)z^j$ $(0 < z < 1)$. Letting $z \to 1$ in this we obtain

$$\sum_{m=j}^{\infty} U_m(1)\nu_{mn} = 0, \tag{12}$$

which shows that the $U_m(1)$ are proportional to the steady state probabilities π_m of the Markov chain J, as is to be expected. We shall take $U_m(1) = \pi_m$. Next, summing (11) over $n = 0,1,2,...$ we obtain

$$\sum_{n=0}^{\infty} U_n(z)(\mu_n-\lambda_n z) = \sum_{n=0}^{\infty} \mu_n U_n(0).$$

Letting $z \to 1$ in this we obtain

$$\hat{\mu} - \hat{\lambda} = \sum_{0}^{\infty} \mu_n U_n(0) \geq 0 \tag{13}$$

where

$$\hat{\lambda} = \sum_0^\infty \pi_n \lambda_n, \quad \hat{\mu} = \sum_0^\infty \pi_n \mu_n. \tag{14}$$

These may be viewed as the steady state values of the arrival and service rates. From (13) we see that for the steady state distribution of the (Q,J) to exist, it is necessary that $\hat{\lambda} < \hat{\mu}$. Closed form solutions of (11) are difficult to obtain even for the finite state space case and Yechiali advocates the use of numerical methods to solve specific cases.

The busy period in this Markov–modulated system was investigated by Purdue [10], who derived the equation (16) below and established the uniqueness of its solution. The following approach is much simpler, being based on the fact that the busy period is actually a first passage time of the underlying Markov–modulated birth and death process. Denote by T_r the busy period initiated by r (≥ 1) customers and let $T_0 = 0$. For convenience we write $J_r = J(T_r)$ ($r \geq 0$). We have then the following.

Theorem 1. The process $\{(T_r, J_r), r \geq 0\}$ is an MRP on the state space $\mathbb{R}_+ \times E$. For $\theta > 0$ let

$$\hat{Q}_{mn}(\theta) = E[e^{-\theta T_1}; J_1 = n \,|\, J_0 = m], \quad \hat{Q} = (\hat{Q}_{mn}(\theta)) \tag{15}$$

Then the matrix $\hat{Q}$ satisfies the equation

$$(\lambda_m \delta_{mn})\hat{Q}^2 - [(\theta + \lambda_m + \mu_m)\delta_{mn} - N]\hat{Q} + (\mu_m \delta_{mn}) = 0, \tag{16}$$

where $N = (\nu_{mn})$.

Proof: We identify T_r as T_{r0}, where the random variable T_{rs} is the first passage time of the Q–process from the state r to state s ($r > s \geq 0$). We have

$T_{r0} = T_{r,r-1} + T_{r-1,r-2} + \cdots + T_{10}$, so that $T_{r+1,0} - T_{r,0} = T_{r+1,r}$. It is easily seen that $\{T_r, J_r\}$ is an MRP. Let us denote its transition distribution as

$$Q_{mn}^{(r)}(t) = P\{T_r \leq t, J_r = n \mid J_0 = m\}.$$

It is known that

$$\int_0^\infty e^{-\theta t} dQ_{mn}^{(r)}(t) = \hat{Q}^r \quad (r \geq 0),$$

with $\hat{Q}$ defined by (15). Considering the epoch of the first jump in the (Q,J) process, namely, an arrival, a departure or a transition in the J–process, we find that

$$\begin{aligned} Q_{mn}^{(r)}(t) = &\int_0^t e^{-(\lambda_m + \mu_m - \nu_{mm})s} \lambda_m Q_{mn}^{(r+1)}(t-s)ds \\ &+ \int_0^t e^{-(\lambda_m + \mu_m - \nu_{mn})s} \mu_m Q_{mn}^{(r-1)}(t-s)ds \\ &+ \sum_{\ell \neq m} \int_0^t e^{-(\lambda_m + \mu_m - \nu_{mm})s} \nu_{m\ell} Q_{\ell n}^{(r)}(t-s)ds \end{aligned}$$

Upon taking transforms in this and simplifying we arrive at the desired result (16). □

(b) Models with breakdowns. The model with parameters given by (10) is an extension of the M/M/1 model with breakdowns, proposed by White and Christie [13]. Assume that breakdowns of service mechanism occur in a Poisson process at a rate α and repair times have exponential density with parameter β. Let $J(t) = 1$ or 0 according as the service mechanism is working or under repair. If breakdowns occur whether service

mechanism is in progress or not, then it is seen that (Q,J) is a process of the type described by (5) with

$$\lambda_0 = \lambda_1 = \lambda, \ \mu_0 = 0, \ \mu_1 = \mu \tag{17}$$

and the transition rates of the Markov chain J given by $\nu_{10} = \alpha, \nu_{01} = \beta$. If breakdowns occur only when service is in progress, then the transition rates a(i,m; i,n) depend on i and the process (Q,J) is not of the type described by (2) and is in fact non–Markovian.

Mitrani and Avi–Itzhak [6] investigated an M/M/s queueing system with parameters (λ,μ), where breakdowns of service mechanism at each counter occur in a Poisson process at a rate α, while the repair times have exponential density with parameter β, independently of the arrival process and service mechanisms at the other counters. Clearly, the service rate offered by the system fluctuates with the number of available (busy or idle) servers. Denotes by J(t) the number of available servers at time t. Then (Q,J) is a process of the type defined by (2) with

$$\begin{gathered} \lambda_{im} = \lambda_m, \ \mu_{im} = \mu \min(i,m) \quad (i \geq 0, 0 \leq m \leq s) \\ \nu_{m,m+1} = (s-m)\beta \quad (0 \leq m \leq s), \ \nu_{m,m-1} = m\alpha \quad (1 \leq m \leq s). \end{gathered} \tag{18}$$

(c) Pre–emptive priorities. Consider a system with two (low and high) classes of priorities, arriving in independent Poisson processes at rates λ_0 and λ_1, their service times having exponential densities with parameters μ_0 and μ_1, respectively. From the point of view of low priority customers, the high priority customers' presence appears as a secondary factor affecting their service. If Q(t) and J(t) denote

respectively the low and high priority queue–lengths, then (Q,J) is a process of the type described by (2) with

$$\lambda_{im} = \lambda_0 \ (i \geq 0, m \geq 0), \quad \mu_{i0} = \mu_0 \ (i \geq 1), \quad \mu_{im} = 0 \ (i \geq 0, m \geq 1)$$
$$\nu_{m,m+1} = \lambda_1 \ (m \geq 0), \quad \nu_{m,m-1} = \mu_1 \ (m \geq 1). \tag{19}$$

3. Single server queues with Markov–modulated Poisson arrivals. The input in the model (a) of section 2 may be characterized as a Markov–modulated Poisson process: that is, during a time–interval spent by the underlying Markov chain in state m, customers arrive in a Poisson process at a rate λ_m $(m = 0,1)$. The service times have an exponential density whose parameter fluctuates with the state of the Markov chain. As a partial generalization of this model, Neuts [7] proposed a single server model with Markov–modulated Poisson arrivals and the service time of a customer depending only on the Markov state at the commencement of his service, the successive service times being conditionally independent, given the states of the Markov chain, and also independent of the arrival process. In terms of concepts and techniques this paper represents perhaps the most comprehensive investigation of a model of this type and the significance of its results go beyond queueing theory. We review two important results of this paper. Denote by $A(t)$ the number of arrivals during the interval $(0,t]$ and $J(t)$ the underlying Markov chain. The process $\{A(t),J(t)\}$ is a Markov–Poisson process on the state space $\mathcal{N}_+ \times E$, satisfying the following properties:

(i) For $0 \leq t_1 \leq \ldots \leq t_n$ $(n \geq 2)$ the increments $A(t_1) - A(0)$, $A(t_2) - A(t_1),\ldots,A(t_n) - A(t_{n-1})$ are conditionally independent, given $J(0),J(t_1),\ldots,J(t_n)$.

(ii) The conditional distribution of $A(t_p) - A(t_{p-1})$, given $J(t_{p-1})$ and $J(t_p)$, depends only on $t_p - t_{p-1}$.

Without loss of generality we assume $A(0) = 0$ and denote

$$P_{jk}(n;t) = P\{A(t) = n,\ J(t) = k \mid J(0) = j\}. \tag{20}$$

(iii) We have

$$P_{jk}(n;h) = a_{jk}(n)h + o(h) \tag{21}$$

where

$$a_{jj}(1) = \lambda_j,\ a_{jk}(0) = \nu_{jk}\ \ (k \neq j), \tag{22}$$

ν_{jk} being the transition rates of the J–chain.

Making additional regularity assumptions concerning the uniformity of the transition rates (21) we find that

$$P'_{jk}(n;t) = -\lambda_k P_{jk}(n;t) + \lambda_k P_{jk}(n-1;t) + \sum_{\ell \in E} P_{j\ell}(n;t)\nu_{\ell k} \tag{23}$$

where $\nu_{jj} = -\Sigma_{k \neq j} \nu_{jk}$. The solution of these equations can be expressed in terms of the transforms

$$\hat{P}_{jk}(s,z) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-st} P_{jk}(n;t)dt,\ \ \hat{P} = (P_{jk}(s,z)) \tag{24}$$

$$(0 < z < 1,\ s > 0).$$

We arrive at the following result, where $\Phi(z) = (\nu_{jk}) - (1-z)(\lambda_j \delta_{jk})$.

Theorem 2. For the Markov–Poisson process we have

$$\hat{P}[sI - \Phi(z)] = I. \tag{25}$$

In particular if J is a finite Markov chain, then

$$\hat{P} = [sI - \Phi(z)]^{-1}. \quad \square \tag{26}$$

In section 4 we shall review the class of Markov–additive processes with the additive component having non–decreasing sample functions (Markov subordinators), of which the process considered above is a special case.

We next review the busy period problem for the queueing system under consideration. Let S_r denote the number of arrivals during r uninterrupted service periods and J_r the state of the Markov chain at the end of these r periods. Then it is clear that $\{(S_r,J_r),\ r \geq 0\}$ is an MRP with the transition probabilities given by

$$\begin{aligned} q_{jk}(n) &= P\{S_{r+1} - S_r = n,\ J_{r+1} = k \mid J_r = j\} \\ &= \int_0^\infty P_{jk}(n,v)B_j\{dv\}, \end{aligned} \tag{27}$$

where B_j is the distribution of a service time that commences when the Markov state is j. Following the branching process analogy as in the standard case let us denote by $\{S_r^{(t+1)}, J_r^{(t+1)}\}$ the MRP corresponding to the descendants of the t–th generation $(t \geq 0)$. Its transition probabilities are given by

$$q_k^{(r)}(n) = P\{S_r^{(t+1)} = n,\ J_r^{(t+1)} = k \mid J_0^{(t+1)} = j\} \tag{28}$$

where $q_{jk}^{(1)}(n) = q_{jk}(n)$ given by (27). It is known that for $r \geq 0$

$$\left(\sum_{n=0}^{\infty} q_{jk}^{(r)}(n) z^n \right) = Q(z)^r, \quad \text{where} \quad Q(z) = \left(\sum_{n=0}^{\infty} q_{jk}(n) z^n \right) \quad (0 < z < 1) \tag{29}$$

Finally, let us denote by X_{t+1} the number of arrivals during the service time of members of the t–th generation and J_{r+1} the state of the Markov chain at the end of this service completion. Then $\{(X_t, J_t),\ t \geq 0\}$ is a Markov–modulated branching process. For this process the set $\{0\} \times E$ is absorbing in the sense that $X_\tau = 0$, $J_\tau = k$ implies that $X_t = 0$, $J_t = k$ for all $t \geq \tau$. Absorption means extinction of the population (end of the busy period). The number of generations before extinction is given by

$$T = \min\{t: X_t = 0\} \tag{30}$$

and the total progeny before extinction (total number of customers served during a busy period) is given by $X_0 + X_1 + \ldots + X_{T-1}$. If we are interested in the duration of the busy period, then we need only to replace the distribution B_j by $e^{-\theta v} B_j\{dv\}$ in (27). Below we give a somewhat more elaborate definition of the branching process and derive the main results. We need the notion of a function of a matrix. Suppose

$$A(z) = \left(\sum_{n=0}^{\infty} A_{jk}(n) z^n \right) \tag{31}$$

where the power series converges for $|z| \leq 1$, and $B = (B_{jk})$ is a complex matrix with $\|B\| \leq 1$, where

$$\|B\| = \sup_j \sum_k |B_{jk}|.$$

Then we define

$$A(B) = (\sum_{n} \sum_{\ell} A_{j\ell}(n)(B^n)_{\ell k}). \tag{32}$$

In applications, the elements of A and B are probability generating functions as in (29) and the question of convergence is easily settled. We prefer the notation $A \circ B$ over $A(B)$, so that we can write (31) as $A(z) = A \circ (zI)$.

Definition 1. Given a family of random sequences $\{(S_r^{(t)}, J_r^{(t)}),\ r \geq 0\}$ $(t \geq 1)$ on the state space $\mathcal{N}_+ \times E$, we define a process $\{(X_t, J_t),\ t \geq 0\}$ as follows: (X_0, J_0) is given, and

$$X_{t+1} = S_{X_t}^{(t+1)},\ J_{t+1} = J_{X_t}^{(t+1)} \quad (t \geq 0). \tag{33}$$

We assume that given $\{(S_r^{(\tau)}, J_r^{(\tau)}),\ 0 \leq r \leq X_{\tau-1}\}$ $(1 \leq \tau \leq t)$, the family $\{(S_r^{(t+1)}, J_r^{(t+1)}),\ r \geq 0\}$ depends only on $J_{X_{t-1}}^{(t)}$ and is an MRP, with $S_0^{(t+1)} = 0$, $J_0^{(t+1)} = J_{X_{t-1}}^{(t)}$ and the transition probabilities given by (28). The process $(X,J) = \{(X_t, J_t)\}$ so defined is a Markov–modulated branching process.

Theorem 3. The Markov–modulated branching process (X,J) is a Markov process with one–step transition probabilities given by

$$P_{jk}(r;n) = P\{X_{t+1} = n,\ J_{t+1} = k \mid X_0 = r, J_0 = j\} = q_{jk}^{(r)}(n), \tag{34}$$

where $q_{jk}^{(r)}(n)$ is given by (28).

Proof: For $t \geq 1$ we have

$$P\{X_{t+1} = n, J_{t+1} = k \mid X_0, J_1, X_1, J_1, \ldots, X_t, J_t\}$$

$$= P\{S_{X_t}^{(t+1)} = n,\ J_{X_t}^{(t+1)} = k \mid X_0, J_0, S_{X_0}^{(1)}, J_{X_0}^{(1)}, \ldots, S_{X_{t-1}}^{(t)}, J_{X_{t-1}}^{(t)}\}$$

$$= P\{S_{X_t}^{(t+1)} = n,\ J_{X_t}^{(t+1)} = k \mid S_{X_{t-1}}^{(t)}, J_{X_{t-1}}^{(t)}\}$$

because of our assumptions. This establishes the Markov property. Moreover,

$$P\{X_{t+1} = n,\ J_{t+1} = k \mid X_t = r, J_\tau = j\}$$

$$= P\{S_r^{(t+1)} = n,\ J_r^{(t+1)} = k \mid J_0^{(t+1)} = j\} = q_{jk}^{(r)}(n),$$

as desired. □

Theorem 4. For the process (X,J) let

$$F_t^{(r)}(z) = (E[z^{X_t};\ J_t = k \mid X_0 = r,\ J_0 = j]) \tag{35}$$

for $r \geq 0, t \geq 0$. Then

$$F_t^{(r)}(z) = [F_t(z)]^r \tag{36}$$

where $F_0(z) = zI$, $F_1(z) = Q(z)$ and in general

$$F_{t+s}(z) = F_t \circ F_s(z). \tag{37}$$

Proof: We have

$$F_{t+1}^{(r)}(z) = \sum_{\ell}\sum_{\nu} P\{X_t = \nu,\ J_t = \ell \mid X_0 = r,\ J_0 = j\}$$
$$\cdot\ E[z^{X_{t+1}};\ J_{t+1} = k \mid X_t = \nu,\ J_t = \ell]$$
$$= \sum_{\ell}\sum_{\nu} P\{X_t = \nu,\ J_t = \ell \mid X_0 = r,\ J_0 = j\}[Q(z)]_{\ell k}^{\nu}$$
$$= F_t^{(r)} \circ Q(z) \quad (t \geq 0).$$

Also, $F_0^{(r)}(z) = z^r I = (zI)^r = [F_0(z)]^r$ and $F_1^{(r)}(z) = Q(z)^r = [F_1(z)]^r$, so that (36) is true for $t = 0$ and 1. Assume that (36) is true up to some t. Then

$$F_{t+1}^{(r)}(z) = F_t^{(r)} \circ Q = [F_t \circ Q]^r = [F_{t+1}(z)]^r.$$

Thus (36) holds for all $t \geq 0$. Again

$$F_{t+s}^{(r)}(z) = \sum_{\ell}\sum_{\nu} P\{X_t = \nu,\ J_t = \ell \mid X_0 = 1,\ J_0 = j\}$$
$$\cdot\ E[z^{X_{t+s}};\ J_{t+s} = k \mid X_t = \nu,\ J_t = \ell]$$
$$= \sum_{\ell}\sum_{\nu} P\{X_t = \nu,\ J_t = \ell \mid X_0 = r,\ J_0 = j\}[F_s(z)]_{\ell k}^{\nu}$$
$$= F_t \circ F_s(z),$$

as was required to be proved. □

Theorem 5. For $0 < w < 1$ let

$$\xi_{jk}(w) = E[w^{X_0+X_1+\ldots+X_T};\ J_T = k \mid X_0 = 1,\ J_0 = j]. \tag{38}$$

Then $\xi = (\xi_{jk}(w))$ satisfies the matrix functional equation

$$\xi = wQ \circ \xi. \tag{39}$$

Proof: For $t \geq 1,\ r \geq 0,\ 0 < w < 1,\ 0 < z < 1$ let

$$G_t^{(r)}(w,z) = E[w^{X_0+X_1+\ldots+X_{t-1}} z^{X_t};\ J_t = k \mid X_0 = r,\ J_0 = j]. \tag{40}$$

Then proceeding as in the proof of Theorem 4 we obtain

$$G_{t+1}^{(r)}(w,z) = G_t^{(r)}(w,wQ). \tag{41}$$

We have $G_1^{(r)}(w,z) = w^r Q(z)^r = [G_1^{(1)}(w,z)]^r$ and by induction

$$G_t^{(r)}(w,z) = [G_t^{(1)}(w,z)]^r \quad (t \geq 1). \tag{42}$$

Therefore it suffices to consider $G_t(w,z) \equiv G_t^{(1)}(w,z)$. It turns out that

$$G_{t+1}(w,z) = wQ \circ G_t(w,z). \tag{43}$$

Now

$$G_t(w,0) = E[w^{X_0+X_1+\ldots+X_{t-1}};\ X_t = 0,\ J_t = k \mid X_0 = 1,\ J_0 = j]$$

$$= E[w^{X_0+X_1+\ldots+X_T};\ T \le t,\ J_T = k \mid X_0 = 1,\ J_0 = j]$$

so that

$$\lim_{t\to\infty} G_t(w,0) = E[w^{X_0+X_1+\ldots+X_T};\ J_T = k \mid X_0 = 1,\ J_0 = j]$$

$$= \xi_{jk}(w). \tag{44}$$

From (43) we find that ζ satisfies (39). □

In the case where J is a finite Markov chain, Neuts [7] has established the existence and uniqueness of the solution of (39). The probability of ultimate extinction is given by

$$P\{T < \infty,\ J_T = k \mid J_0 = j\} = \lim_{w\to 1-} \xi_{jk}(w) = \zeta_{jk} \tag{45}$$

where $\zeta = (\zeta_{jk})$ satisfies the equation

$$\zeta = Q \circ \zeta. \tag{46}$$

4. Markov subordinators. The Markov–modulated Poisson process described in section 3 belongs to the class of Markov subordinators. The input process of the queueing model investigated in this paper (see section 5), namely the Markov–modulated compound Poisson process, also belongs to this class. In this section we give a constructive definition of a Markov subordinator and describe its important

properties. The presentation here follows Prabhu [9], who cites other relevant references on Markov renewal theory and Markov–additive processes.

Let $J = \{J(t),\ t \geq 0\}$ be a time–homogeneous Markov chain on the state space E, all of whose states are stable. Let $T_0 = 0$, T_n $(n > 1)$ the epochs of successive jumps in J and denote $J_n = J(T_n)$ $(n \geq 0)$. We define a sequence of continuous time processes $\{X_n^{(1)},\ n > 1\}$ and a sequence of random variables $\{X_n^{(2)},\ n > 1\}$ as follows:

(i) On $\{T_n < t < T_{n+1},\ J_n = j\}$, $X_{n+1}^{(1)}(\tau)$ is a subordinator with Lévy measure μ_{jj}.

(ii) Given $X_m^{(1)}, X_m^{(2)}, J_m$ $(1 \leq m \leq n)$, J_0, the increment $X_{n+1}^{(1)}(t) - X_{n+1}^{(1)}(T_n)$ and the random variables $(X_{n+1}^{(2)}, T_{n+1}, J_{n+1})$ depend only on J_n.

(iii) Given J_n, $X_{n+1}^{(1)}(t) - X_{n+1}^{(1)}(T_n)$ and $(X_{n+1}^{(2)}, T_{n+1}, J_{n+1})$ are conditionally independent, with respective distribution measures

$$H_j\{s;A\} \text{ and } e^{-q_{jj}s} q_{jk} M_{jk}\{A\}ds \quad (j \neq k) \tag{47}$$

for any Borel subset A of $\mathbb{R}_+$. Here the M_{jk} are concentrated on $[0,\infty)$, while H_j is concentrated on $[0,\infty)$; q_{jk} are the transition rates of the J–process $(j \neq k)$ and $q_{jj} = \Sigma_{k \neq j}\, q_{jk}$ $(0 < q_{jj} < \infty)$. Let

$$S_0 = 0,\ S_n = \sum_{m=1}^{n} [X_m^{(1)}(T_m) - X_m^{(1)}(T_{m-1}) + X_m^{(2)}] \quad (n \geq 1). \tag{48}$$

From the above conditions it follows that $\{(S_n, T_n, J_n),\ n \geq 0\}$ is an MRP on the state space $\mathbb{R}_+ \times \mathbb{R}_+ \times E$, whose transition distribution measure

$$P\{S_{n+1} \in dy,\ T_{n+1} \in dt,\ J_{n+1} = k \mid S_n = x,\ T_n = s,\ J_n = j\}$$

$$= Q_{jk}\{dy-x,\ dt-s\}$$

is given by

$$Q_{jk}\{dy,dt\} = e^{-q_{jj}t} q_{jk} dt \int_0^y H_j\{t;dx\} M_{jk}\{dy-x\} \quad (j \neq k) \tag{49}$$

and $Q_{jj} = 0$. We denote by $U_{jk}\{A\times B\}$ the Markov renewal measure associated with this process, so that

$$U_{jk}\{A\times B\} = \sum_{n=0}^{\infty} P\{(S_n,T_n) \in A\times B,\ J_n = k \mid J_0 = j\}. \tag{50}$$

Let

$$L' = \sup_{n\geq 0} T_n, \quad L = \sup_{n\geq 0} S_n. \tag{51}$$

We construct a process $(X,J) = \{(X(t), J(t)), t \geq 0\}$ as follows:

$$\begin{aligned} \{X(t), J(t)\} &= \{S_n + X_{n+1}^{(1)}(t) - X_{n+1}^{(1)}(T_n), J_n\} \text{ for } T_n \leq t < T_{n+1} \\ &= (L,\Delta) \text{ for } t \geq L' \end{aligned} \tag{52}$$

where Δ is a point of compactification of the set E. It is easily seen that (X,J) is a Markov additive process on the state space $\mathbb{R}_+ \times E$. We shall call it a Markov subordinator. Let us denote by

$$F_{jk}\{t;A\} = P\{X(t) \in A,\ J(t) = k \mid J(0) = j\} \tag{53}$$

the transition distribution measure of this process. We have the following.

Theorem 6. We have

$$F_{jk}\{t;A\} \geq \int_0^\infty \int_0^t U_{jk}\{dx,ds\} e^{-q_{kk}(t-s)} H_k\{t-s;A-x\}, \tag{54}$$

where the equality holds iff $L = \infty$ a.s.

Proof: We have

$$F_{jk}\{t;A\} = P\{X(t) \in A,\ J(t) = k,\ T_1 > t \mid J(0) = j\}\delta_{jk}$$

$$+ \sum_{\ell \in E} \int_0^\infty \int_0^t P\{S_1 \in dx,\ T_1 \in ds,\ J_1 = \ell \mid J_0 = j\} \cdot$$

$$P\{X(t) \in A, J(t) = k \mid S_1 = x,\ T_1 = x,\ J_1 = \ell\}$$

$$= e^{-q_{jj}t} H_j\{t;A\}\delta_{jk} + \sum_{\ell \in E} \int_0^\infty \int_0^t Q_{j\ell}\{dx,ds\} F_{\ell k}\{t-s;\ A-x\}.$$

Thus $F_{jk}\{t;A\}$ satisfies the integral equation

$$F_{jk}\{t;A\} = f_{jk}\{t;A\} + \sum_{\ell \in E} \int_0^\infty \int_0^t Q_{j\ell}\{dx,ds\} F_{\ell k}\{t-s;\ A-x\} \tag{55}$$

where

$$f_{jk}\{t;A\} = e^{-q_{jj}t} H_j\{t;A\}\delta_{jk}.$$

We seek a solution of (55) for F_{jk} such that for $j,k \in E$, a Borel subset A of $\mathbb{R}_+$, F_{jk} is bounded over finite intervals, and for each $t \in \mathbb{R}_+$, F_{jk} is bounded. The inequality (54) follows from the fact that the minimal solution of (55) is given by

$$\sum_{\ell \in E} \int_0^\infty \int_0^t U_{j\ell}\{dx,ds\} f_{\ell k}\{t-s;\ A-x\}$$

$$= \int_0^\infty \int_0^t U_{jk}\{x,ds\} e^{-q_{kk}(t-s)} H_k\{t-s;\ A-x\}.$$

From Markov renewal theory we also know that this solution is unique iff $L = \infty$. □

If J is a finite Markov chain, then $L = \infty$ a.s. and the solution of (55) is unique, being given by

$$F_{jk}\{t;A\} = \int_0^\infty \int_0^t U_{jk}\{dx,ds\} e^{-q_{kk}(t-s)} H_k\{t-s;\ A-x\}. \tag{56}$$

We can express this solution in terms of transforms as follows. For $\theta > 0$ let

$$\int_0^\infty e^{-\theta x} H_j\{t;dx\} = e^{-t\phi_{jj}(\theta)}, \qquad \int_0^\infty e^{-\theta x} M_{jk}\{dx\} = \hat{M}_{jk}(\theta), \tag{57}$$

where ϕ_{jj} is the Laplace exponent of the subordinator $X_m^{(1)}$. Also, let $\Phi(\theta) = (\phi_{jk}(\theta))$, where

$$\Phi_{jk}(\theta) = q_{jk}\hat{M}_{jk}(\theta) \ \ (k \neq j), \ \ \Phi_{jj}(\theta) = -q_{jj} - \phi_{jj}(\theta). \tag{58}$$

We have then the following.

Theorem 7. Suppose that the J–chain is finite and for $\theta > 0$, $s > 0$ denote

$$\hat{F}_{jk}(s,\theta) = \int_0^\infty \int_0^\infty e^{-\theta x - st} F_{jk}\{t;dx\}dt, \quad \hat{F} = (\hat{F}_{jk}(s,\theta). \tag{59}$$

Then

$$\hat{F} = [sI - \Phi(\theta)]^{-1}. \tag{60}$$

Proof: From (49) we obtain

$$\hat{Q}_{jk}(\theta,s) = \int_0^\infty \int_0^\infty e^{-\theta y - st} Q_{jk}\{dy,dt\} = \frac{\Phi_{jk}(\theta)}{s - \Phi_{jj}(\theta)}.$$

Denoting

$$\hat{U}_{jk}(\theta,s) = \int_0^\infty \int_0^\infty e^{-\theta y - st} U_{jk}\{dy,dt\}$$

and $\hat{Q} = (Q_{jk}(\theta,s))$, $\hat{U} = (U_{jk}(\theta,s))$ we find from Markov renewal theory that $\hat{U} = (I–\hat{Q})^{-1}$. Now (56) gives

$$\hat{F}_{jk}(s,\theta) = \frac{\hat{U}_{jk}(\theta,s)}{s - \Phi_{kk}(\theta)}$$

or

$$\hat{F} = \hat{U}\left(\frac{\delta_{jk}}{s - \Phi_{kk}(\theta)}\right)$$

Therefore

$$\hat{F}^{-1} = ((s - \Phi_{kk}(\theta))\delta_{jk})\left(\delta_{jk} - \frac{\Phi_{jk}(\theta)}{s - \Phi_{jj}(\theta)}\right).$$

This leads to the desired result. □

Example 1. (Markov–modulated simple Poisson process). Here $M_{jk}\{0\} = 1$ and H_j is an atomic measure with weight $h_j(n;t)$ at the atom n given by

$$h_j(n;t) = e^{-\lambda_j t}\frac{(\lambda_j t)^n}{n!} \quad (n = 0,1,2,\ldots). \tag{61}$$

With a slight change of notation we write

$$\sum_{n=0}^{\infty} z^n h_j(n;t) = e^{-t\phi_{jj}(z)} \quad (0 < z < 1) \tag{62}$$

where $\phi_{jj}(z) = \lambda_j(1-z)$. Also, let $\Phi_{jk}(z) = q_{jk}$ $(k \neq j)$ and $\Phi_{jj}(z) = -q_{jj} - \lambda_j + \lambda_j z$. Finally, let

$$\hat{F}_{jk}(s,z) = \int_0^{\infty} e^{-st}\, E[z^{X(t)};\ J(t) = k \mid J(0) = j]dt \tag{63}$$

$(s > 0)$. Then Theorem 7 gives $\hat{F} = (sI - \Phi(z))^{-1}$, which agrees with Theorem 2. □

To understand the significance of the constructive definition of the process (X,J) given above, it is instructive to derive its infinitesimal generator. For this purpose let $f(x,j)$ be a bounded function on $\mathbb{R}_+ \times E$ such that for each fixed j, f is continuous and has a bounded continuous derivative $\frac{\partial f}{\partial t}$. Then we have the following.

Theorem 8. The generator of the process (X,J) is given by $\mathscr{A}$, where

$$\begin{aligned}\mathscr{A}f = d_j\frac{\partial f}{\partial x} &+ \int_{0-}^{\infty}[f(x+y,j) - f(x,j)]\mu_{jj}\{dy\} \\ &+ \sum_{k\neq j}\int_{0-}^{\infty}[f(x+y,k) - f(x,j)]q_{jk}M_{jk}\{dy\}\end{aligned} \tag{64}$$

where $d_j \geq 0$ is the drift and μ_{jj} the Lévy measure of the process $X_{n+1}^{(1)}$ given $J_n = j$.

Proof: We have

$$F_{jk}\{h;dx\} = (1 - q_{jj}h)H_j\{h;dx\}\delta_{jk} + q_{jk}hM_{jk}\{dx\}(1-\delta_{jk}) + o(h). \tag{65}$$

Therefore

$$\begin{aligned} & h^{-1} \sum_{k\in E} \int_{0-}^{\infty+} [f(x+y,k) - f(x,j)]F_{jk}\{h;dy\} \\ & = h^{-1}(1-q_{jj}h) \int_{0-}^{\infty+} [f(x+y,j) - f(x,j)]H_j\{h:dy\} \\ & \qquad + \sum_{k\neq j} q_{jk} \int_{0-}^{\infty} [f(x+y,k) - f(x,j)]M_{jk}\{dy\} + o(1) \\ & \to \mathscr{A}_0 f + \sum_{k\neq j} \int_{0-}^{\infty} [f(x+y,k) - f(x,j)]q_{jk}M_{jk}\{dy\} \quad \text{as} \quad h \to 0+, \end{aligned}$$

where $\mathscr{A}_0$ is the generator of the process $X_{n+1}^{(1)}$, given by

$$\mathscr{A}_0 f = d_j \frac{\partial f}{\partial x} + \int_{0-}^{\infty+} [f(x+y,j) - f(x,j)]\mu_{jj}[dy].$$

we are thus led to the desired result (64). □

Theorem 8 shows the presence of Markov–modulated jumps with distribution $q_{jk}M_{jk}$ in the additive component of the process (X,J), in addition to the jumps in the Lévy process $X_{n+1}^{(1)}$.

We need the following result for application to the queueing model investigated in the next section.

Theorem 9. For fixed $\theta > 0$ if

$$\sup_{j \in E} (q_{jj} + \phi_{jj}(\theta)) < \infty, \tag{66}$$

then

$$(E[e^{-\theta X(t)};\ J(t) = k \mid J(0) = j]) = e^{t\Phi(\theta)} \tag{67}$$

where the matrix $\Phi(\theta) = (\Phi_{jk}(\theta))$ is given by (58).

Proof: Denote

$$P_{jk}(t;\theta) = E[e^{-\theta X(t)};\ J(t) = k \mid J(0) = j]. \tag{68}$$

We have

$$P_{jk}(t;\theta) \geq 0,\ P_{jk}(0;\theta) = \delta_{jk}$$

$$\sum_{k \in E} P_{jk}(t;\theta) = E[e^{-\theta X(t)} \mid J(0) = j] \leq 1.$$

$$P_{jk}(t+s;\ \theta) = E[e^{-\theta X(t+s)};\ J(t+s) = k \mid J(0) = j]$$

$$= \sum_{\ell \in E} E[e^{-\theta X(t)};\ J(t) = \ell \mid J(0) = j] \cdot$$

$$E[e^{-\theta[X(t+s)-X(t)]};\ J(t+s) = k \mid J(t) = \ell]$$

$$= \sum_{\ell \in E} P_{j\ell}(t;\theta) P_{\ell k}(s;\theta).$$

Denoting $P(t;\theta) = (P_{jk}(t;\theta))$, we can write the last identity as

$$P(t+s;\ \theta) = P(t;\theta)P(s;\theta). \tag{69}$$

Thus the family $\{P(t;\theta),\ t \geq 0\}$ of substochastic matrices forms a semigroup with the norm

$$\|P\| = \sup_j \sum_{k\in E} |P_{jk}(t)| \leq 1. \tag{70}$$

From (65) we find that

$$\begin{aligned} P_{jk}(h;\theta) &= \int_0^\infty e^{-\theta x} F_{jk}\{h;dx\} \\ &= \delta_{jk}(1-q_{jj}h)e^{-h\phi_{jj}(\theta)} + (1-\delta_{jk})q_{jk}h\hat{M}_{jk}(\theta) + o(h) \\ &= \delta_{jk}(1-q_{jj}h-\phi_{jj}h) + (1-\delta_{jk})q_{jk}h\hat{M}_{jk}(\theta) + o(h). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{k\in E} |P_{jk}(h;\theta) - \delta_{jk}| &= h\left[\sum_{k\neq j} q_{jk}\hat{M}_{jk}(\theta) + q_{jj} + \phi_{jj}(\theta) + o(1)\right] \\ &\leq h\left[\sum_{k\neq j} q_{jk} + q_{jj} + \phi_{jj}(\theta) + o(1)\right] \leq 2h[A + o(1), \end{aligned}$$

where $\sup_{j\in E} (q_{jj} + \phi_{jj}(\theta)) = A < \infty$. Thus

$$\|P(h;\theta) - I\| = \sup_j \sum_{k\in E} |P_{jk}(h;\theta) - \delta_{jk}| \to 0 \ \text{ as } \ h \to 0.$$

From Hille and Phillips ([4], page 635) we find that $P(t;\theta) = e^{t\Phi(t)}$, where $\Phi(\theta)$ is the generator of the semigroup. Our calculations also show that Φ is given by (58). □

Example 2 (Markov–modulated compound Poisson process). Suppose that on $\{T_n \leq t < T_{n+1}, J_n = j\}$, $X^{(1)}_{n+1}$ is a compound Poisson process with drift $d_j \geq 0$ and jumps occurring at a rate λ_j with distribution B_j. The Levy measure is then $\mu_{jj}\{dx\} = \lambda_j B_j\{dx\}$ and by Theorem 8 the generator of the process $\{X,J\}$ is given by

$$\begin{aligned}\mathcal{A}f = d_j \frac{\partial f}{\partial x} &+ \lambda_j \int_{0-}^{\infty+} [f(x+y,j) - f(x,j)]B_j\{dy\} \\ &+ \sum_{k\neq j} \int_{0-}^{\infty} [f(x+y,k) - f(x,j)]q_{jk}M_{jk}\{dy\}.\end{aligned} \tag{71}$$

Also, the Laplace exponent of $X^{(1)}_{n+1}$ is given by $\phi_{jj}(\theta) = \lambda_j - \lambda_j\psi_j(\theta) + d_j\theta$, where ψ_j is the Laplace transform of B_j. We have $\phi_{jj}(\theta) \leq \lambda_j + d_j\theta$, so that if

$$\sup_{j\in E} (q_{jj} + \lambda_j + d_j\theta) < \infty \tag{72}$$

then the (66) is satisfied, and Theorem 9 holds with

$$\begin{aligned}\Phi(\theta) &= (q_{jk}\hat{M}_{jk}(\theta) - (q_{jj} + \lambda_j - \lambda_j\psi_j(\theta) + d_j(\theta)\delta_{jk}). \\ &= Q(\theta) - \Lambda + \Lambda\Psi(\theta) - D\theta\end{aligned} \tag{73}$$

where $Q(\theta)$ has diagonal elements $-q_{jj}$ and nondiagonal elements $q_{jk}\hat{M}_{jk}(\theta)$, $\Lambda = (\lambda_j\delta_{jk})$, $\Psi(\theta) = (\psi_j(\theta)\delta_{jk})$ and $D = (d_j\delta_{jk})$. □

5. Single server queues with Markov compound Poisson input. The queueing model investigated in this paper may be characterized as one with Markov compound Poisson input. This may be described as follows. There is an underlying Markov chain $J = \{J(t), t \geq 0\}$ on the state space E and with all states stable, such that during a time–interval in which $J(t) = j$, customers arrive in a Poisson process at a rate λ_j $(1 < \lambda_j < \infty)$ and their service times have a distribution B_j. Given the states of the Markov chain, the service times are conditionally independent and also independent of the arrival process. There is a single server and the queue discipline is first come, first served.

Let $A(t)$ denote the number of arrivals during a time–interval $(0,t]$, and $v_1, v_2, \ldots$ the successive service times. Then the total workload submitted to the server during $(0,t]$ is given by $X(t) = v_1 + v_2 + \ldots + v_{A(t)}$. From Example 2 of section 4 we know that $(X,J) = \{X(t), J(t), t \geq 0\}$ is a Markov compound Poisson process with zero drift and no Markov–modulated jumps. The generator of the process given by (71) reduces to

$$\mathscr{A}f(x,j) = \int_{0-}^{\infty} [f(x+y, j) - f(x,j)]\lambda_j B_j\{dy\} + \sum_{k \neq j} q_{jk}[f(x,k) - f(x,j)], \tag{74}$$

where q_{jk} $(k \neq j)$ are the transition rates of the J–chain. Let $q_{jj} = \Sigma_{k \neq j} q_{jk}$. We denote by Q the generator matrix of the J–chain, with diagonal element $-q_{jj}$ and non–diagonal elements q_{jk}. We shall assume that

$$\sup_{j \in E} (q_{jj} + \lambda_j) < \infty. \tag{75}$$

Then (72) is satisfied and (with a slight change of notation) we have for $\theta > 0$

$$(E[e^{-\theta X(t)}; J(t) = k \mid J(0) = j]) = e^{-t\Phi} \tag{76}$$

where

$$\Phi(\theta) = -Q + \Lambda - \Lambda\Psi(\theta) \tag{77}$$

as in (73) with $\Lambda = (\lambda_j \delta_{jk})$ and $\Psi(\theta) = (\psi_j(\theta)\delta_{jk})$, ψ_j being the Laplace transform of the service time distribution B_j.

Let $W(t)$ be the virtual waiting time in this model. As in the standard case, $W(t)$ satisfies the integral equation

$$W(t) = W(0) + X(t) - t + \int_0^t 1_{\{W(s)=0\}} ds. \tag{78}$$

Here the integral

$$I(t) = \int_0^t 1_{\{W(s)=0\}} ds \tag{79}$$

represents the duration of the idle time during $(0,t]$. We shall also be interested in the busy period initiated by a workload $x \geq 0$, which is denoted as $T(x)$, where

$$T(x) = \inf\{t \geq 0: W(t) = 0\} \text{ on } \{W(0) = x\}. \tag{80}$$

We need a new notion, which is an extension of the notion of function of a matrix used in section 3. Suppose that for $j,k \in E$, $A_{jk}\{dx\}$ is a probability distribution on $[0,\infty]$ and for $\theta > 0$ let

$$A(\theta) = (\int_0^\infty e^{-\theta x} A_{jk}\{dx\}). \tag{81}$$

Now if $-\eta$ is the generator matrix of a semigroup of the type described in section 4, then we define

$$A \circ \eta = (\int_0^\infty \sum_{\ell \in E} A_{j\ell}\{dx\}(e^{-x\eta})_{\ell k}). \tag{82}$$

We investigate the busy period in section 6 and the waiting time and idle time in section 7.

6. The busy period process. We shall call $(T,J) = \{T(x), J \circ T(x), x \geq 0\}$ the busy period process. Clearly, $T(0) = 0$ a.s. We note that for each $j \in E$, $x > 0$,

$$P\{T(x) \in \mathbb{R}_+, J \circ T(x) \in E \mid J(0) = j\} \leq 1, \tag{83}$$

so that the distribution of $\{T(x), J \circ T(x)\}$ has possibly an atom at (∞,Δ). For the busy period $T_1 = T(v)$ initiated by a new arrival we denote

$$G_{jk}\{A\} = P\{T_1 \in A, J \circ T_1 = k \mid J(0) = j\} \tag{84}$$

for any Borel subset A of $\mathbb{R}_+$, and for $s > 0$

$$\Gamma_{jk}(s) = \int_0^\infty e^{-st} G_{jk}\{dt\}, \quad \Gamma(s) = (\Gamma_{jk}(s)). \tag{85}$$

The distribution G_{jk} is concentrated on $(0,\infty)$. For simplicity we shall attach the atom at (∞,Δ) to the distribution G_{jj}, with weight $G_{jj}\{\infty\}$, so that

$$\sum_{k\neq j} G_{jk}\{(0,\infty)\} + G_{jj}\{(0,\infty]\} = 1. \tag{86}$$

We have then the following.

Theorem 10. $\{T(x),\ J \circ T(x),\ x \geq 0\}$ is a Markov compound Poisson process with unit drift, whose generator is given by

$$\begin{aligned}\mathscr{B}f(t,j) = \frac{\partial f}{\partial t} &+ \int_0^{\infty+} [f(t+s,j) - f(t,j)]\tilde{\lambda}_j\tilde{B}_j\{ds\} \\ &+ \sum_{k\neq j} \int_0^{\infty} [f(t+s,k) - f(t,j)]\tilde{q}_{jk}\hat{M}_{jk}\{ds\}\end{aligned} \tag{87}$$

where

$$\tilde{\lambda}_j\tilde{B}_j\{s\} = \lambda_j G_{jj}\{ds\}\ \ (0 < s < \infty),\ \ \tilde{\lambda}_j\tilde{B}_j\{\infty\} = \lambda_j G_{jj}\{\infty\} \tag{88}$$

and for $k \neq j$

$$\tilde{q}_{jk}\tilde{M}_{jk}\{ds\} = q_{jk}\epsilon_0\{ds\} + \lambda_j G_{jk}\{ds\}. \tag{89}$$

ϵ_0 being a measure concentrated at the origin.

Proof: As in the standard case $T(x+y) - T(x)$ has the same distribution as $T(y)$ but in our model $T(x+y) - T(x)$ and $T(y)$ are conditionally independent, given $J(0)$ and $J \circ T(y)$. Therefore

$$P\{T(x+y) \in A,\ J \circ T(x+y) = k \mid T(x'),\ J \circ T(x')\ \ (0 \le x' \le x)\}$$

$$= P\{T(x+y) - T(x) \in A - T(x),\ J \circ T(x+y) = k \mid J \circ T(x)\}$$

$$= P\{T(y) \in A - T(x),\ J \circ T(y) = k \mid J(0)\} \quad \text{a.s.}$$

This proves the Markov–additive property. Again, we have $T(x) = x + T[X(x)]$ as in the standard case. Therefore

$$P\{T(h) \in A,\ J \circ T(h) = k \mid J(0) = j]$$

$$= P\{h + T[X(h)] \in A,\ J \circ (h + T[X(h)]) = k \mid J(0) = j\}$$

$$= \sum_{\ell \in E} \int_{0-}^{\infty} F_{j\ell}\{h;dy\} P\{T(y) \in A-h,\ J \circ (h+T(y)) = k \mid J(h) = \ell\}$$

$$= \sum_{\ell \in E} \int_{0-}^{\infty} F_{j\ell}\{h;dy\} P\{T(y) \in A-h,\ J \circ T(y) = k \mid J(0) = \ell\} \tag{90}$$

where $F_{jk}\{t;A\}$ is the distribution of (X,J). Now from (65) we obtain

$$F_{jk}\{h;dx\} = (1-q_{jj}h)H_j\{h;dx\}\delta_{jk} + q_{jk}h\epsilon_0\{dx\}(1-\delta_{jk}) + o(h)$$

where

$$H_j\{h;dx\} = (1 - \lambda_j h)\epsilon_0\{dx\} + \lambda_j h B_j\{dx\} + o(h).$$

Therefore

$$F_{jk}\{h;dx\} = (1 - \lambda_j h - q_{jj}h)\epsilon_0\{dx\}\delta_{jk} \tag{91}$$

$$+ \lambda_j h B_j\{dx\}\delta_{jk} + q_{jk}h\epsilon_0\{dx\}(1-\delta_{jk}) + o(h).$$

Substituting (91) in (90) we find that

$$\begin{aligned} &P\{T(h) \in A,\ J \circ T(h) = k \mid J(0) = j\} \\ &\quad = (1 - \lambda_j h - q_{jj}h)1_{\{0\in A-h\}}\delta_{jk} + \lambda_j h G_{jk}\{A-h\} \\ &\qquad\qquad + q_{jk}h1_{\{0\in A-h\}}(1-\delta_{jk}) + o(h). \end{aligned} \tag{92}$$

The infinitesimal generator of the process is given by $\mathscr{B}$, where

$$\begin{aligned} \mathscr{B}f(t,j) = \lim_{h\to 0} \frac{1}{h} \sum_{k\in E} \int_{0-}^{\infty+} &\{f(t+s,k) - f(t,j)\} \\ &\cdot P\{T(h) \in ds,\ J \circ T(h) = k \mid J(0) = j\}. \end{aligned}$$

Using (92) in this we find that $\mathscr{B}$ is indeed given by (87), as desired. □

Theorem 11. Under the condition (75) we have

$$(E[e^{-sT(x)};\ J \circ T(x) = k \mid J(0) = j]) = e^{-x\eta(s)} \tag{93}$$

where the matrix $\eta = \eta(s)$ satisfies the matrix–functional equation

$$\eta = sI + \Phi \circ \eta. \tag{94}$$

Also

$$\Gamma = \Psi \circ \eta \tag{95}$$

Proof: From (88) and (89) we find that

$$\tilde{\lambda}_j = \lambda_j G_{jj}\{(0,\infty]\},\ \tilde{q}_{jk} = q_{jk} + \lambda_j G_{jk}\{(0,\infty)\}\ \ (k \neq j)$$

so that

$$\tilde{q}_{jj} = \sum_{k\neq j} \tilde{q}_{jk} = q_{jj} + \lambda_j \sum_{k\neq j} G_{jk}\{(0,\infty)\}.$$

Therefore

$$\begin{aligned}\tilde{q}_{jj} + \tilde{\lambda}_j &= q_{jj} + \lambda_j \sum_{k\neq j} G_{jk}\{(0,\infty)\} + \lambda_j G_{jj}\{(0,\infty]\} \\ &= q_{jj} + \lambda_j\end{aligned}$$

in view of (86). Thus Theorem 9 holds and from equation (73) of Example 2 we find that the generator matrix is given by $-\eta(s)$, where

$$\eta(s) = sI - Q + \Lambda - \Lambda\Gamma(s). \tag{96}$$

However, we have

$$\begin{aligned}\Gamma_{jk}(s) &= \int_0^\infty B_j\{dv\}E\{e^{-sT(v)};\ J \circ T(v) = k \mid J(0) = j\} \\ &= \int_0^\infty B_j\{dv\}(e^{-v\eta(s)})_{jk}\end{aligned}$$

so that

$$\Gamma(s) = (\psi_j \delta_{jk}) \circ \eta(s) = \Psi \circ \eta.$$

This gives (95) and consequently (96) reduces to

$$\eta = sI - Q + \Lambda - \Lambda\Psi \circ \eta = sI + \Phi \circ \eta$$

which is the desired result. □

We have thus characterized (T,J) as a process rather than considering $\{T(x), J \circ T(x)\}$ merely as a pair of random variables for each $x \geq 0$. This characterization is useful because of the following connection between the process and I(t), the idle period. We note that I(t) is the local time at zero of the process $\{W(t)\}$ and in the standard case its right continuous inverse is the subordinator $\{T(x)\}$. In our model, we need to consider $\{I(t), J(t)\}$ and the corresponding subordinator is then precisely (T,J). This observation leads to the following result, where we denote

$$\zeta(t) = 1_{\{W(t)=0\}} \tag{97}$$

so that

$$I(t) = \int_{T(x)}^{t} \zeta(s)ds \quad (t \geq T(x) \geq 0). \tag{98}$$

Theorem 12. For $\theta > 0$, $s > 0$ we have

$$\left(\int_0^\infty e^{-st}\, E[e^{-\theta I(t)}\zeta(t);\ J(t) = k \mid W(0) = x,\ J(0) = j]dt\right)$$

$$= e^{-x\eta(s)}[\theta I + \eta(s)]^{-1}. \tag{99}$$

Proof: The matrix element on the left side of (99) can be written as

$$E\int_0^\infty [e^{-st-\theta I(t)}\zeta(t);\ J(t) = k \mid J(0) = j]dt.$$

Now we carry out the transformation $I(t) = \tau$. Then (98) gives $t - T(x) = T(\tau)$ and $dI(t) = \zeta(t)dt$. Therefore the last expression becomes

$$E\int_0^\infty [e^{-sT(x)-sT(\tau)-\theta\tau};\ J\circ[T(x)+T(\tau)]=k \mid J(0)=j]d\tau$$

$$=\sum_{\ell\in E} E[e^{-sT(x)};\ J\circ T(x)=\ell \mid J(0)=j]$$

$$\cdot\int_0^\infty E[e^{-\theta\tau-sT(\tau)};\ J\circ T(\tau)=k \mid J(0)=\ell]d\tau$$

$$=\sum_{\ell\in E}(e^{-x\eta(s)})_{j\ell}\int_0^\infty (e^{-\theta\tau-\tau\eta(s)})_{\ell k}d\tau$$

$$=\sum_{\ell\in E}(e^{-x\eta(s)})_{j\ell}[\theta I+\eta(s)]^{-1}_{\ell k}.$$

This leads to the desired result (99). □

Corollary 1. For $s>0$ we have

$$(\int_0^\infty e^{-st}P\{W(t)=0,\ J(t)=k \mid W(0)=x,\ J(0)=j\}dt)=e^{-x\eta(s)}\eta(s)^{-1}. \quad (100)$$

Proof: Letting $\theta\to 0$ in (99) we obtain the desired result. □

7. **The waiting time and idle time**. The process $\{W(t), J(t)\}$ is clearly a Markov process on the state space $\mathbb{R}_+ \times E$, for which an integro–differential equation can be derived as in the standard case. However, following Prabhu ([8], chapter 3), we can investigate $\{W(t), I(t), J(t)\}$ directly by using the integral equation (78). Details of the derivation are left out as they follow closely those of the standard case. We denote the net input as $Y(t)=X(t)-t$. Then $\{Y(t), J(t)\}$ is a Markov additive process on $\mathbb{R}\times E$, for which

$$(E[e^{-\theta Y(t)};\ J(t)=k \mid J(0)=j])=e^{-t(\Phi(\theta)-\theta I)} \quad (101)$$

for $\theta > 0$. We have the following.

Theorem 13. For $\theta_1 > 0$, $\theta_2 > 0$, $s > 0$ we have

$$\left(\int_0^\infty e^{-st} E[e^{-\theta_1 W(t) - \theta_2 I(t)}; J(t) = k \mid W(0) = x,\ J(0) = j]dt\right)$$
$$= \{e^{-\theta_1 x} I - (\theta_1+\theta_2)e^{-x\eta(s)}[\theta_2 I + \eta(s)]^{-1}\}[sI - \theta_1 I + \Phi(\theta_1)]^{-1}. \qquad (102)$$

Proof: From (78) we find that

$$E[e^{-\theta_1 W(t) - \theta_2 I(t)}; J(t) = k \mid W(0) = x,\ J(0) = j]$$
$$= E[e^{-\theta_1 x - \theta_1 Y(t)}; J(t) = k \mid J(0) = j] - (\theta_1+\theta_2)$$
$$\cdot E\int_0^t [e^{-\theta_1[Y(t) - Y(\tau)] - \theta_2 I(\tau)} \zeta(\tau); J(t) = k \mid J(0) = j]d\tau.$$

Now, denoting the left side of (102) as $a(x; \theta_1, \theta_2, s) = (a_{jk}(x; \theta_1, \theta_2, s)$, we find that

$$a_{jk}(x; \theta_1, \theta_2, s) = e^{-\theta_1 x} \int_0^\infty e^{-st} E[e^{-\theta_1 Y(t)}; J(t) = k \mid J(0) = j]dt$$
$$- (\theta_1+\theta_2) \sum_{\ell \in E} \int_0^\infty E[e^{-s\tau - \theta_2 I(\tau)} \zeta(\tau); J(\tau) = \ell \mid J(0) = j]d\tau \cdot$$
$$\int_\tau^\infty e^{-s(t-\tau)} E[e^{-\theta_1[Y(t) - Y(\tau)]}; J(t-\tau) = k \mid J(0) = \ell]dt.$$

This leads to the desired result, in view of (101) and Theorem 12. □

For the marginal processes $(W,J) = \{W(t), J(t)\}$ and $(I,J) = \{I(t), J(t)\}$ we have the following.

Corollary 2. For $\theta > 0$, $s > 0$ we have

$$\begin{aligned}(i)\ &\Big(\int_0^\infty e^{-st}E[e^{-\theta W(t)};\ J(t) = k \mid W(0) = x,\ J(0) = j]dt\Big) \\ &= [e^{-\theta x}I - \theta e^{-x\eta(s)}\eta(s)^{-1}][sI - \theta I + \Phi(\theta)]^{-1}\end{aligned} \tag{103}$$

and

$$\begin{aligned}(ii)\ &\Big(\int_0^\infty e^{-st}E[e^{-\theta I(t)};\ J(t) = k \mid W(0) = x,\ J(0) = j]dt\Big) \\ &= \{I - \theta e^{-x\eta(s)}[\theta I + \eta(s)]^{-1}\}(sI - Q)^{-1}.\ \square\end{aligned} \tag{104}$$

8. **The case of finite E**. We shall now assume that $E = \{1,2,...,m\}$. Our first task is to establish the existence and uniqueness of the matrix functional equation (94), namely

$$\eta = sI + \Phi \circ \eta \quad (s \geq 0) \tag{105}$$

Here $\eta(s)$ is the negative of the generator of a (strictly) substochastic matix, so for its eigenvalues $\eta_r(s)$ we must have $\mathrm{Re}\ \eta_r(s) \geq 0$ $(r = 1,2,...,m)$. We shall assume that these eigenvalues are all distinct for $s \geq 0$. Actually, this assumption will always be true except for countably many values of $s \geq 0$. We seek a solution of (105) that belongs to the class of matrices so described.

Theorem 14. The matrix functional equation (105) has a unique solution, with eigenvalues given by the roots with $\mathrm{Re}(\theta) \geq 0$ of the equation

$$|sI - \theta I + \Phi(\theta)| = 0 \tag{106}$$

and the corresponding eigenvectors also uniquely determined thereby.

Proof: Denote $f(x) = xI - \Phi(x)$. We need to solve for the matrix X such that

$$f \circ X = sI. \tag{107}$$

By assumption X has distinct eigenvalues $x_r \equiv x_r(s)$ $(r = 1,2,...,m)$, so we have the spectral representation

$$X = \sum_{r=1}^{m} x_r Z_r$$

where $Z_r \equiv Z_r(s)$ is the idempotent matrix corresponding to the eigenvalue x_r. It can be easily proven that

$$f \circ X = \sum_{r=1}^{m} f(x_r) Z_r$$

so that

$$sI - f \circ X = \sum_{r=1}^{m} [sI - f(x_r)] Z_r.$$

Since X satisfies (107) we must have

$$\sum_{r=1}^{m} [sI - f(x_r)] Z_r = 0.$$

Multiplying this identity by Z_t we obtain

$$[sI - f(x_t)] Z_t = 0 \quad (t = 1,2,...,m). \tag{108}$$

Since Z_t is nonnull (being of rank one) we must have

$$|sI - f(x_t)| = 0 \quad (t = 1,2,...,m).$$

This means that x_t $(t = 1,2,...,m)$ are the roots of the equation

$$|sI - f(x)| = 0$$

which reduces to (106). Our argument also shows that the Z_t are also uniquely determined. The desired result follows since a matrix is uniquely determined by its eigenvalues and eigenvectors. □

We shall next investigate the limit behaviour of the processes (W,J) and (I,J). For this purpose we shall assume that the J–chain is irreducible and aperiodic, with all states persistent. Then a stationary distribution $\pi = (\pi_1,\pi_2,...,\pi_m)$ exists, with the properties

$$\pi_j \geq 0, \quad \sum_1^m \pi_j = 1, \quad \pi Q = 0 \tag{109}$$

where Q is the generator matrix of the J–chain. We define the traffic intensity of the system as ρ, where

$$\rho = \sum_1^m \pi_j \rho_j, \quad \text{with} \quad \rho_j = \lambda_j \int_0^\infty v B_j\{dv\}. \tag{110}$$

Since Theorem 14 identifies the eigenvalues of $\eta(s)$ as the roots of the equation (106), we have the following result due to Regterschot and de Smit ([12], Lemma 3.1), which has been extended to the case $\rho = 1$.

Corollary 3. (i) If $\rho < 1$, then the matrix $\eta(0)$ has a simple eigenvalue $\eta_1(0) = 0$ and m−1 eigenvalues $\eta_2(0), \eta_3(0),\ldots,\eta_m(0)$, with Re $\eta_r(0) > 0$ $(r = 2,3,\ldots,m)$. If $\rho = 1$, then $\eta_1(0) = 0$. If $\rho > 1$, then Re $\eta_r(0) > 0$ $(r = 1,2,\ldots,m)$.

(ii) If $\rho < 1$, then as $s \to 0+$, $\eta_1(s) \to 0$ and $\eta_1'(s) \to (1-\rho)^{-1}$. If $\rho \geq 1$, then as $s \to 0+$, $s\eta_r(s)^{-1} \to 0$ $(r = 1,2,\ldots,m)$. □

Remark 1. Denoting by $Z_r(s)$ the idempotent matrix corresponding to the eigenvalue $\eta_r(s)$ we have the spectral representation for $\eta(s)$:

$$\eta(s) = \sum_{r=1}^{m} \eta_r(s)Z_r(s) \quad (s \geq 0), \tag{111}$$

where $Z_r(s)$ is of rank one. In the limit as $s \to 0+$ this gives

$$\eta(0) = \sum_{r=1}^{m} \eta_r(0)Z_r(0). \tag{112}$$

Here for $\rho \leq 1$, $\eta_1(0) = 0$ by Corollary 3. To find $Z_1(0)$ we proceed as follows. For $t = 1$, equation (108) gives

$$[sI - \eta_1(s)I + \Phi(\eta_1(s))]Z_1(s) = 0.$$

Letting $s \to 0+$ in this we obtain for $\rho \leq 1$

$$QZ_1(0) = 0 \tag{113}$$

since $\Phi(0) = -Q$. Again, the proof of Theorem 14 also shows that $\eta_1^{-1}(\theta)$ is an eigenvalue of $\theta I - \Phi(\theta)$, where η_1^{-1} is the inverse function of η_1. The corresponding idempotent matrix is $Z_1(\eta_1^{-1}(\theta))$. Therefore

$$Z_1(\eta_1^{-1}(\theta))[\theta I - \Phi(\theta)] = \eta_1^{-1}(\theta) Z_1(\eta_1^{-1}(\theta)).$$

Letting $\theta \to 0+$ in this we find that for $\rho \leq 1$, by Corollary 3,

$$Z_1(0)Q = 0. \tag{114}$$

Since Q is the generator of the Markov chain $J = \{J(t), t \geq 0\}$, which is stochastic, equations (113)–(114) show that $Z_1(0)$ is the idempotent matrix corresponding to the eigenvalue 0 of Q. The existence of the unique solution π in (109) implies that

$$Z_1(0) = \Pi \qquad (\rho \leq 1) \tag{115}$$

where Π is the matrix with m identical rows $(\pi_1, \pi_2, \ldots, \pi_m)$. □

Theorem 15. For each fixed $x > 0$ we have

$$P\{T(x) \in \mathbb{R}_+, J \circ T(x) \in E \mid J(0) = j\} = 1 \text{ if } \rho \leq 1 \tag{116}$$
$$< 1 \text{ if } \rho > 1.$$

Proof. Let e be the column vector with elements $(1,1,\ldots,1)$. Then $Z_1(0)e = e$ from (115). For $t \neq 1$, since $Z_t(0)Z_1(0) = 0$, we obtain $Z_t(0)Z_1(0)e = 0$ or $Z_t(0)e = 0$. Thus for $\rho \leq 1$

$$Z_1(0)e = e, \;\; Z_t(0)e = 0 \;\; (t = 2,3,\ldots,m),$$

so that from (112),

$$\eta(0)e = \sum_{2}^{m} \eta_r(0)Z_r(0)e = 0. \tag{117}$$

Now letting $s \to 0+$ in (93) we find that

$$(P\{T(x) \in \mathbb{R}_+, \;\; J \circ T(x) = k \mid J(0) = j\}) = e^{-x\eta(0)}, \tag{118}$$

so that $-\eta(0)$ is the generator matrix of the Markov chain $\{J \circ T(x), \;\; x \geq 0\}$. If $\rho \leq 1$, then (117) shows that this chain is stochastic. If $\rho > 1$, it is strictly substochastic, as otherwise $\eta(0)$ will have an eigenvalue equal to 0, which is not the case, by Corollary 3. Thus we arrive at the desired result, already anticipated in equation (83). □

Remark 2. The Markov chain $\{J \circ T(x), \;\; x \geq 0\}$, is subordinate to the original chain $J = \{J(t), \; t \geq 0\}$. Here as $x \to \infty$, $T(x) \to \infty$ a.s. and so it is easily seen that this subordinate process has also a limit distribution, which we denote as $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_m)$. Now we have the representation

$$e^{-x\eta(0)} = \sum_{r=1}^{m} e^{-x\eta_r(0)} Z_r(0). \tag{119}$$

In view of Corollary 3 we find from this that if $\rho < 1$,

$$\lim_{x \to \infty} e^{-x\eta(0)} = Z_1(0) = \Pi. \tag{120}$$

However, this limit should be the matrix $\tilde{\Pi}$ with m identical rows $(\tilde{\pi}_1,\tilde{\pi}_2,\ldots,\tilde{\pi}_m)$. This shows that $\Pi = \tilde{\Pi}$. □

Remark 3. We have

$$s\eta(s)^{-1} = \sum_{r=1}^{m} \frac{s}{\eta_r(s)} Z_r(s)$$

Applying Corollary 3 we find that

$$\begin{aligned} \lim_{s\to 0+} s\eta(s)^{-1} &= (1-\rho)\Pi \quad \text{if } \rho < 1 \\ &= 0 \qquad\quad \text{if } \rho \geq 1. \end{aligned} \tag{121}$$

For $m = 1$ this agrees with the known result. □

The following result will be used in the derivation of limit results for (W,J) and (I,J) processes. The existence of the limits follows as in the standard case (Prabhu [8], chapter 3).

Theorem 16. We have

$$\begin{aligned} &\lim_{t\to\infty} P\{W(t) = 0,\ J(t) = k \mid W(0) = x,\ J(0) = j\} \\ &\qquad = (1-\rho)\pi_k \text{ if } \rho < 1 \\ &\qquad = 0 \qquad \text{if } \rho \geq 1. \end{aligned} \tag{122}$$

Proof: By a Tauberian theorem the required limit equals the element of the matrix

$$\lim_{s\to 0+} \left(s \int_0^\infty e^{-st} P\{W(t) = 0,\ J(t) = k \mid W(0) = x,\ J(0) = j\}dt\right)$$

which by Corollary 1 equals

$$\lim_{s\to 0+} e^{-x\eta(s)}s\eta(s)^{-1} = \lim_{s\to 0+} \sum_{r=1}^{m} e^{-x\eta_r(s)} \frac{s}{\eta_r(s)} Z_r(s).$$

Applying Corollary 3 and (121) we find the limit as $(1-\rho)\Pi$ if $\rho < 1$ and $= 0$ if $\rho \geq 1$. This leads to the desired result. □

Theorem 17. As $t \to \infty$, $\{W(t), J(t)\} \to (\infty,\Delta)$ in distribution if $\rho \geq 1$; otherwise $\{W(t), J(t)\} \to (W,J)$, where

$$(E[e^{-\theta W}; J = k]) = \theta(1-\rho)\pi[\theta I - \Phi(\theta)]^{-1}. \tag{123}$$

Proof: From Corollary 2(i) we have

$$\begin{aligned}
&\lim_{t\to\infty} (E[e^{-\theta W(t)}; J(t) = k \mid W(0) = x,\ J(0) = j]) \\
&\quad = \lim_{s\to 0+} (s \int_0^{\infty} e^{-st} E[e^{-\theta W(t)};\ J(t) = k \mid W(0) = x,\ J(0) = j]dt) \\
&\quad = \lim_{s\to 0+} (e^{-x\eta(s)}s\eta(s)^{-1}) \cdot \theta[\theta I - \Phi(\theta)]^{-1}
\end{aligned}$$

where by (121) the limit is $(1-\rho)\Pi$ or 0 according as $\rho < 1$ or $\rho \geq 1$. This is identical with the desired result. □

Theorem 18. As $t \to \infty$, $\{I(t), J(t)\} \to (\infty.\Delta)$ in distribution if $\rho \leq 1$; otherwise $\{I(t), J(t)\} \to (\mathcal{I},J)$, where

$$(P\{\mathcal{I} \le y,\ J = k \mid W(0) = x,\ J(0) = j\}) = [I - e^{-(x+y)\eta(0)}]\Pi. \tag{124}$$

Proof: Using Corollary 2(ii) we find that

$$\begin{aligned}
&\lim_{t\to\infty} (E[e^{-\theta I(t)}; J(t) = k \mid W(0) = x,\ J(0) = j]) \\
&\quad = \lim_{s\to 0+} (s \int_0^\infty e^{-st} E[e^{-\theta I(t)}; J(t) = k \mid W(0) = x,\ J(0) = j]dt) \\
&\quad = \{I - \theta e^{-x\eta(0)} [\theta I + \eta(0)]^{-1}\} \lim_{s\to 0+} s(sI - Q)^{-1}
\end{aligned}$$

where the limit equals Π by our assumption. It remains to simplify the first matrix in the last expression. We have

$$\begin{aligned}
I - \theta e^{-x\eta(0)}[\theta I + \eta(0)]^{-1} &= \sum_{r=1}^{m} [1 - e^{-x\eta_r(0)} \frac{\theta}{\theta + \eta_r(0)}] Z_r(0) \\
&= \int_0^\infty e^{-\theta y} \sum_{r=1}^{m} \mu_r\{x;dy\} Z_r(0) \qquad (125)
\end{aligned}$$

where

$$\mu_r\{x; [0,y]\} = 1 - e^{-(x+y)\eta_r(0)}$$

so that

$$\sum_{r=1}^{m} \mu_r\{x; [0,y]\} Z_r(0) = I - e^{-(x+y)\eta(0)}.$$

In view of the proof of Theorem 15 the desired result thus follows. □

Remark 4. The result (124) shows that the limit random variables $\mathcal{J}$ and J are conditionally independent, given W(0) and J(0). □

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