A TWO-FLUID ACTUARIAL MODEL WITH AN ALTERNATING PAYOFF POLICY

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Abstract. In this paper we consider the model for an actuarial problem dealing with two types of claims and payoffs subject to seasonal switching. Claims are assumed to occur in a fluid fashion whereas payoffs are made at a unit rate so long as claims remain to be paid.

The distribution properties of the accumulated claim sizes $\{Z_1(t),\, Z_2(t)\}$ are derived at finite time as well as in steady state. We first investigate this process embedded at the successive switching points. This process is Markovian with independent components. In continuous time the components $\{Z_1(t),\, Z_2(t)\}$ are also independent for each finite t, but are dependent in steady state.

Extensions are possible to the case of three or more inputs with release over a sequence of intervals of fixed or variable lengths.

1. Introduction.

In this paper we propose the following model for an actuarial problem dealing with two types of claims that arise steadily, but are covered according to seasonal switching as follows. During an interval of length r there is no need for immediate payoff of claims of the first type, and during an interval of length g for immediate payoff of claims of the second, these intervals alternating and forming a cycle of length $c = r + g$. As an example, in an agribusiness there may not be a need to carry out repair work during a part of the year (say, winter) if the production season starts only later (in summer). During the remaining part of the year, another type of repairs may need attention. The claims for these two types of repairs are caused by damages that occur throughout the year. When repair work begins it requires continuous financing at a constant rate, which we take to be a unit. This possibility of gradual and eventually delayed payoffs is the motivation to suggest more economical policies.

The model is subject to the following assumptions.

(i) Let $X_i(t)$ be the accumulated damage of type i that occurs during the time-interval $(0, t]$ for $i = 1, 2$. We assume that $\{X_1(t),\ t \geq 0\}$ and $\{X_2(t),\ t \geq 0\}$ are two independent Lévy processes on the state space $[0, \infty)$ with zero drift (subordinates).

(ii) In the nth cycle ($n = 0, 1, \ldots$) the payoff for the first type of claim is made only during the interval $(n\,c + r,\, n\,c + c]$ while the payoff for the second type is made only during the interval $(n\,c,\, n\,c + r]$. Here $r > 0$, $g > 0$. For convenience we refer to $(n\,c,\, n\,c + r]$ as the red period and to $(n\,c + r,\, n\,c + c]$ as the green period.

(iii) In each case the payoff occurs at a unit rate so long as claims of that type remain to be paid.

We denote by $Z_i(t)$ the total claim of type i that remains to be paid at time t. Of main interest is the study of the process $\{Z_1(t),\, Z_2(t),\ t \geq 0\}$.

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The assumption regarding the accumulated claims implies that small as well as large claims occur in time in a steady fashion. As a particular case we have the compound Poisson process in which damages occur in a simple Poisson process and the amounts of successive damages are independent and identically distributed random variables. This classical case is still important, but recent research in this general area more general Lévy processes have been considered, leading to the fluid model.

A few historical remarks may be in order. In Moran's model for a dam (see [5]), inputs of water occur during the *wet season* and are stored for use until the *dry season*, when it is released. Dam models with ordered inputs were considered by Gani and Pyke in [2]. See also [6] for additional references.

The red and green periods of our model may be generalized to intervals of random lengths, that are either independent of the state of the claim process or dependent. Dependency arises, for example, when dividends are paid or not paid depending of the state of the risk reserve fund. In this case the term $\delta_i(s)\mathbf{1}_{\{Z_i(s)>0\}}$ in (2.3) is to be replaced by $d_i \circ Z_i(s)$, for $i = 1, 2$. Also, the model will have to reflect the need to incorporate the present value of the dividends, so instead of $d_i \circ Z_i(s)$ we need to use $e^{-\delta s} d_i \circ Z_i(s)$ where δ is the interest force for valuation (see for example [3]).

In section 2 we derive the equations that describe the dynamics of the claim process $\{Z_1(t),\, Z_2(t)\}$. We prove a result concerning the effect of the interchange of red and green periods and the two types of damages. This result provides a certain amount of simplicity and elegance in the proofs of the main results. In section 3 we present a preliminary result leading to the independence of the component processes $Z_1(t)$, $Z_2(t)$ for finite t. In the next section we derive the limit behavior of the claim process as $t \to \infty$. In order to do this we first study the process embedded at the sequence of switching points, and then extend this study to the continuous time process $\{Z_1(t),\, Z_2(t)\}$. In the first case the independence of the components remains intact, but in the general case we find that the limit random variables Z_1 and Z_2 are negatively dependent. The final section 5 contains some further remarks on the model and possible variants and extensions of the model.

2. The claim process.

For the subordinator $X_i(t)$ we denote the Laplace transform

$$\mathbf{E}[e^{-\theta_i X_i(t)}] = e^{-t\phi_i(\theta_i)}\,, \quad (\theta_i > 0) \tag{2.1}$$

where

$$\phi_i(\theta_i) = \int_0^\infty (1 - e^{-\theta_i x})\, \nu_i(dx)\,, \tag{2.2}$$

ν_i being a Lévy measure ($i = 1, 2$). We assume that $X_i(t)$ has a finite mean $\rho_i\, t$, where $\rho_i = \phi_i'(0+)$ ($i = 1, 2$). Our assumptions imply that the process of remaining claims $\{Z_1(t),\, Z_2(t)\}$ satisfies the integral equations

$$Z_i(t) = Z_i(0) + X_i(t) - \int_0^t \delta_i(s)\, \mathbf{1}_{\{Z_i(s)>0\}} ds\,, \tag{2.3}$$

for $i = 1, 2$, where for any $n \geq 0$

$$\delta_1(t) = \begin{cases} 0 & \text{if} \quad n\,c < t \leq n\,c + r\,, \\ 1 & \text{if} \quad n\,c + r < t \leq n\,c + c\,, \end{cases}$$

and $\delta_2(t) = 1 - \delta_1(t)$. We can rewrite (2.3) as

$$Z_i(t) = Z_i(0) + Y_i(t) + \int_0^t \delta_i(s)\,\zeta_i(s)\,ds\,, \tag{2.4}$$

where $\zeta_i(s) = \mathbf{1}_{\{Z_i(s)=0\}}$ and

$$Y_i(t) = X_i(t) - D_i(t)\,, \quad D_i(t) = \int_0^t \delta_i(s)\,ds\,,$$

for $i = 1, 2$. We may refer to $Y_i(t)$ as the net claim of type i ($i = 1, 2$).

The integral equation (2.4) has the unique non-negative measurable solution given by

$$Z_i(t) = \max\{Z_i(0) + Y_i(t)\,,\ \sup_{0\le s\le t}[Y_i(t) - Y_i(s)]\}\,, \tag{2.5}$$

for $i = 1, 2$ (see [4, Th.1]), which hold almost surely (a.s.). It is more convenient to write (2.5) as

$$Z_i(t) = Z_i(0) + Y_i(t) + I_i(t)\,,$$

where

$$I_i(t) = [Z_i(0) + \inf_{0\le s\le t} Y_i(s)]^-\,,$$

for $i = 1, 2$.

Considerable simplification is achieved in the analysis of the claim process by noting the following.

Lemma 2.1. *Choose* $Z_1(0) \overset{d}{=} Z_2(r)$. *Then the permutation*

$$(r,\, g,\, X_1,\, X_2) \;\to\; (g,\, r,\, X_2,\, X_1)\,, \tag{2.6}$$

results in the permutation

$$\begin{aligned}&[Z_1(t),\, I_1(t),\, Z_2(t+r),\, I_2(t+r) - I_2(r)] \;\to\\ &\qquad [Z_2(t+r),\, I_2(t+r) - I_2(r),\, Z_1(t),\, I_1(t)]\,.\end{aligned} \tag{2.7}$$

Proof. It can be easily verified that the permutation (2.6) results in the permutation

$$[\delta_1(s),\, \delta_2(s+r)] \;\to\; [\delta_2(s+r),\, \delta_1(s)]\,.$$

Consequently

$$Y_1(t) = X_1(t) - \textstyle\int_0^t \delta_1(s)\,d\,s$$

$$\to\; X_2(t) - \textstyle\int_0^t \delta_2(s+r)\,d\,s \overset{d}{=} X_2(t+r) - X_2(r) - \int_r^{t+r} \delta_2(s)\,d\,s = Y_2(t+r) - Y_2(r)\,,$$

and similarly $Y_2(t+r) - Y_2(r) \;\to\; Y_1(t)$. Thus

$$[Y_1(t),\, Y_2(t+r) - Y_2(r)] \;\to\; [Y_2(t+r) - Y_2(r),\, Y_1(t)]\,. \tag{2.8}$$

Now

$$I_1(t) = [Z_1(0) + \inf\nolimits_{0\le s\le t} Y_1(s)]^- \;\to\; [Z_1(0) + \inf\nolimits_{0\le s\le t}\{Y_2(s+r) - Y_2(r)\}]^-$$

$$\overset{d}{=} I_2(t+r) - I_2(r)\,,$$

and similarly $I_2(t+r) - I_2(r) \;\to\; I_1(t)$. Finally, using (2.8)

$$Z_1(t) = Z_1(0) + Y_1(t) + I_1(t) \;\to\; Z_2(r) + [Y_2(t+r) - Y_2(r)] + [I_2(t+r) - I_2(r)] = Z_2(t+r)\,,$$

and similarly $Z_2(t+r) \;\to\; Z_1(t)$. Thus (2.7) is completely proved. $\square$

Remark 2.2. The permutation (2.7) and others in the proof are in the sense of equality in distribution of random vectors on each side.

3. Main results.

Lemma 3.1. *For $n\,c < t \le n\,c + c$, $(n = 0, 1, \dots)$, $\theta_1 > 0$, $\theta_2 > 0$ the identity*

$$(3.1)\quad e^{-\theta_1[I_1(t)-I_1(nc)]-\theta_2[I_2(t)-I_2(nc)]}$$
$$= \left\{1-\theta_1 \int_{nc+r}^{t\vee nc+r} e^{-\theta_1[I_1(s)-I_1(nc)]}\, d\,I_1(s)\right\}\left\{1-\theta_2 \int_{nc}^{t\wedge nc+r} e^{-\theta_2[I_2(s)-I_2(nc)]}\, d\,I_2(s)\right\},$$

holds a.s.

Proof. By integration by parts we obtain

$$(3.2)\quad \int_{t_1}^{t_2} e^{-\theta_1[I_1(s)-I_1(t_1)]-\theta_2[I_2(s)-I_2(t_1)]}[\theta_1\, d\,I_1(s) + \theta_2\, d\,I_2(s)]$$
$$= 1 - e^{-\theta_1[I_1(t_2)-I_1(t_1)]-\theta_2[I_2(t_2)-I_2(t_1)]},$$

for $0 \le t_1 < t_2$. For $t_1 = n\,c$ and $t_2 = t$ the left side of (3.1) equals

$$(3.3)\quad 1 - \int_{nc}^{t} e^{-\theta_1[I_1(s)-I_1(nc)]-\theta_2[I_2(s)-I_2(nc)]}\,[\theta_1\, d\,I_1(s) + \theta_2\, d\,I_2(s)].$$

For $n\,c < t \le n\,c + r$ this last expression is

$$1 - \theta_2 \int_{nc}^{t} e^{-\theta_2[I_2(s)-I_2(nc)]}\, d\,I_2(s),$$

since $I_1(s) = I_1(n\,c)$ and $dI_1(s) = 0$ for $n\,c < s \le t$. We thus obtain the desired factorization (3.1) in this case since the second factor on the right side reduces to unity.

For $n\,c + r < t \le n\,c + c$ the expression (3.3) is

$$1 - \theta_2 \int_{nc}^{nc+r} e^{-\theta_2[I_2(s)-I_2(nc)]}\, d\,I_2(s) - \theta_1\, e^{-\theta_2[I_2(nc+r)-I_2(nc)]} \int_{nc+r}^{t} e^{-\theta_1[I_1(s)-I_1(nc)]}\, d\,I_1(s),$$

which leads to the desired factorization in this case, since

$$e^{-\theta_2[I_2(nc+r)-I_2(nc)]} = 1 - \theta_2 \int_{nc}^{nc+r} e^{-\theta_2[I_2(s)-I_2(nc)]}\, d\,I_2(s),$$

on account of (3.2). The proof is thus completed. $\square$

Theorem 3.2. *For each $t > 0$ the random variables $Z_1(t)$ and $Z_2(t)$ are independent. Furthermore,*

$$\mathbf{E}[e^{-\theta_1 Z_1(t)-\theta_2 Z_2(t)}] = \mathbf{E}[e^{-\theta_1 Z_1(t)}]\,\mathbf{E}[e^{-\theta_2 Z_2(t)}],$$

where

$$(3.4)\quad \mathbf{E}[e^{-\theta_1 Z_1(t)}] = \begin{cases} \mathbf{E}[e^{-\theta_1 Z_1(nc)}]e^{-(t-nc)\phi_1(\theta_1)}, & (n\,c < t \le n\,c + r) \\ \mathbf{E}[e^{-\theta_1 Z_1(nc+r)}]e^{-(t-nc-r)[\phi_1(\theta_1)-\theta_1]} & \\ \quad -\theta_1 \int_0^{t-nc-r} e^{-s[\phi_1(\theta_1)-\theta_1]}\, F_1(0, t-s)\, ds, & (n\,c + r < t \le n\,c + c) \end{cases}$$

and

$$
\mathbf{E}[e^{-\theta_2 Z_2(t)}] = \begin{cases} \mathbf{E}[e^{-\theta_2 Z_2(nc)}]e^{-(t-nc)[\phi_2(\theta_2)-\theta_2]} \\ \quad -\theta_2 \int_0^{t-nc} e^{-s[\phi_2(\theta_2)-\theta_2]}\, F_2(0, t-s)\, ds\,, & (n\,c < t \le n\,c + r) \\ \mathbf{E}[e^{-\theta_2 Z_2(nc+r)}]e^{-(t-nc-r)\phi_2(\theta_2)}\,, & (n\,c + r < t \le n\,c + c) \end{cases} \tag{3.5}
$$

and $F_i(0, s) = P[Z_i(s) = 0]\,,\ i = 1, 2.$

Proof. For $n\,c < t \le n\,c + c\,,\ (n \ge 0)$ we have a.s.

$$
\begin{aligned}
& e^{-\theta_1 Z_1(t)-\theta_2 Z_2(t)} = e^{-\theta_1 Z_1(nc)-\theta_2 Z_2(nc)-\theta_1[Z_1(t)-Z_1(nc)]-\theta_2[Z_2(t)-Z_2(nc)]} \\
= \;& e^{-\theta_1 Z_1(nc)-\theta_2 Z_2(nc)} e^{-\theta_1[Y_1(t)-Y_1(nc)+I_1(t)-I_1(nc)]} e^{-\theta_2[Y_2(t)-Y_2(nc)+I_2(t)-I_2(nc)]} \\
= \;& e^{-\theta_1 Z_1(nc)-\theta_2 Z_2(nc)} e^{-\theta_1[Y_1(t)-Y_1(nc)]-\theta_2[Y_2(t)-Y_2(nc)]} \times
\end{aligned}
$$

$$
\left\{1 - \theta_2 \int_{nc}^{t\wedge nc+r} e^{-\theta_2[I_2(s)-I_2(nc)]}\, d\,I_2(s)\right\} \left\{1 - \theta_1 \int_{nc+r}^{t\vee nc+r} e^{-\theta_1[I_1(s)-I_1(nc)]}\, d\,I_1(s)\right\}\,, \tag{3.6}
$$

where we have used the Lemma 3.1. For $t = n\,c + c$ it follows from (3.6) that

$$
\begin{aligned}
& e^{-\theta_1[Z_1(nc+c)-Z_1(nc)]-\theta_2[Z_2(nc+c)-Z_2(nc)]} \\
= \;& e^{-\theta_1[Y_1(nc+c)-Y_1(nc)]} \left\{1 - \theta_1 \int_{nc+r}^{nc+c} e^{-\theta_1[I_1(s)-I_1(nc+r)]}\, d\,I_1(s)\right\} \times \\
& e^{-\theta_2[Y_2(nc+c)-Y_2(nc)]} \left\{1 - \theta_2 \int_{nc}^{nc+r} e^{-\theta_2[I_2(s)-I_2(nc)]}\, d\,I_2(s)\right\}\,.
\end{aligned} \tag{3.7}
$$

Taking expectations in (3.7) we find that the increments

$$
Z_1(n\,c + c) - Z_1(n\,c)\,, \quad Z_2(n\,c + c) - Z_2(n\,c)\,,
$$

are independent. Therefore $Z_1(n\,c)$ and $Z_2(n\,c)$ are independent and so are $Z_1(t)$ and $Z_2(t)$ on account of (3.6).

From (3.6) we obtain

$$
e^{-\theta_1 Z_1(t)} = e^{-\theta_1 Z_1(nc)}\, e^{-\theta_1[Y_1(t)-Y_1(nc)]}\,,
$$

for $n\,c < t \le n\,c + r$ and

$$
\begin{aligned}
e^{-\theta_1 Z_1(t)} &= e^{-\theta_1[Z_1(nc)+Y_1(t)-Y_1(nc)]} - \theta_1 \int_{nc+r}^{t} e^{-\theta_1[Z_1(nc)+Y_1(t)-Y_1(nc)+Z_1(s)-Z_1(nc)-Y_1(s)-Y_1(nc)]}\zeta_1(s)ds \\
&= e^{-\theta_1[Z_1(nc)+Y_1(t)-Y_1(nc)]} - \theta_1 \int_{nc+r}^{t} e^{-\theta_1[Y_1(t)-Y_1(s)]}\, \zeta_1(s)\, ds\,,
\end{aligned}
$$

for $n\,c + r < t \le n\,c + c$, since $Z_1(s) > 0$ if and only if $\zeta_1(s) = 1$. Taking expectations in these we arrive at the results of (3.4).

Again from (3.6) we obtain similarly for $n\,c < t \le n\,c + r$

$$
e^{-\theta_2 Z_2(t)} = e^{-\theta_2[Z_2(nc)+Y_2(t)-Y_2(nc)]} - \theta_2 \int_{nc}^{t} e^{-\theta_2[Y_2(t)-Y_2(s)]}\, \zeta_2(s)\, ds\,.
$$

Taking expectations we obtain the first result in (3.5). For $n\,c+r<t\le n\,c+c$, we have

$$e^{-\theta_2 Z_2(t)} = e^{-\theta_2[Z_2(nc)+Y_2(t)-Y_2(nc)]} - \theta_2 \int_{nc}^{nc+r} e^{-\theta_2[Y_2(t)-Y_2(s)]}\,\zeta_2(s)\,ds\,.$$

Taking expectations in this we obtain

$$\begin{aligned}\mathbf{E}[e^{-\theta_2 Z_2(t)}] &= \mathbf{E}[e^{-\theta_2 Z_2(nc)}]\,e^{-(t-nc-r)\phi_2(\theta_2)-r[\phi_2(\theta_2)-\theta_2]}\\ &- \theta_2\,e^{-(t-nc-r)\phi_2(\theta_2)}\int_0^r e^{-s[\phi_2(\theta_2)-\theta_2]}\,F_2(0,nc+r-s)\,ds,\end{aligned}$$

since $Y_2(t)-Y_2(s)\stackrel{d}{=}X_2(t-n\,c-r)+X_2(n\,c+r-s)-(n\,c+r-s)$ when $n\,c<s\le n\,c+r$. This agrees with (3.5).

□

4. Limit distributions.

We are now in a position to derive the limit distribution of $\{Z_1(t),\,Z_2(t)\}$ as $t\to\infty$. Our plan is to first consider the embedded sequence $\{Z_1(n\,c),\,Z_2(n\,c)\}$ and then extend the results to $\{Z_1(t),\,Z_2(t)\}$ using Theorem 3.2. We first note that according to the model description

$$\begin{aligned}Z_1(n\,c+r) &= Z_1(n\,c)+X_1(n\,c+r)-X_1(n\,c)\stackrel{d}{=}Z_1(n\,c)+X_1(r)\,,\\ Z_2(n\,c+c) &= Z_2(n\,c+r)+X_2(n\,c+c)-X_2(n\,c+r)\stackrel{d}{=}Z_2(n\,c+r)+X_2(g)\,.\end{aligned}$$

Thus it is more convenient to express the results for $\{Z_1(n\,c),\,Z_2(n\,c)\}$ equivalently in terms of $\{Z_1(n\,c),\,Z_2(n\,c+r)\}$.

We have the following results.

Theorem 4.1. *As* $n\to\infty$, $\{Z_1(n\,c),\,Z_2(n\,c{+}r)\}\stackrel{d}{\to}\{Z_1(\infty),\,Z_{2r}(\infty)\}$, *where* $Z_1(\infty)$, $Z_{2r}(\infty)$ *are finite if and only if* $\rho_1<\dfrac{g}{c}<1-\rho_2$, *in which case*

$$\mathbf{E}[e^{-\theta_1 Z_1(\infty)-\theta_2 Z_{2r}(\infty)}] = \mathbf{E}[e^{-\theta_1 Z_1(\infty)}]\,\mathbf{E}[e^{-\theta_2 Z_{2r}(\infty)}]\,,$$

where

$$\mathbf{E}[e^{-\theta_1 Z_1(\infty)}] = \frac{F_1(0)\,\theta_1}{\theta_1-\phi_1(\theta_1)}\,\frac{1-e^{-g[\phi_1(\theta_1)-\theta_1]}}{1-e^{-c\phi_1(\theta_1)+g\theta_1}}\,, \tag{4.1}$$

$$\mathbf{E}[e^{-\theta_2 Z_{2r}(\infty)}] = \frac{F_2(0)\,\theta_2}{\theta_2-\phi_2(\theta_2)}\,\frac{1-e^{-r[\phi_2(\theta_2)-\theta_2]}}{1-e^{-c\phi_2(\theta_2)+r\theta_2}}\,, \tag{4.2}$$

and

$$F_1(0)=1-\frac{c}{g}\,\rho_1\,,\quad F_2(0)=1-\frac{c}{r}\,\rho_2\,.$$

Proof. The independence of $Z_1(\infty)$ and $Z_{2r}(\infty)$ follows from Theorem 3.2 and the remarks at the beginning of this section.

From Theorem 3.2 we obtain

$$\mathbf{E}[e^{-\theta_1 Z_1(nc+c)}] = \mathbf{E}[e^{-\theta_1 Z_1(nc)}]\,e^{-c\phi_1(\theta_1)+g\theta_1} - \theta_1\int_0^g e^{-s[\phi_1(\theta_1)-\theta_1]}\,F_1(0,nc+c-s)\,ds\,,$$

and for $0 < \alpha < 1$

$$\sum_{n=0}^{\infty} \alpha^n \, \mathbf{E}[e^{-\theta_1 Z_1(nc)}] = \frac{e^{-\theta_1 Z_1(0)} - \theta_1 \int_0^g e^{-s[\phi_1(\theta_1)-\theta_1]} \, J_1(s,\alpha)\, ds}{1 - \alpha\, e^{-c\phi_1(\theta_1)+g\theta_1}}\,, \tag{4.3}$$

where $J_1(s,\alpha) = \sum_{n=0}^{\infty} \alpha^{n+1}\, F_1(0, n\,c + c - s)$.

Denote $\alpha^{-1} = e^{c\beta}$, $\beta > 0$. Then the denominator in (4.3) vanishes at η_1 where $\eta_1 \equiv \eta_1(\beta)$ is the unique positive root of the equation

$$\frac{g}{c}\,\eta_1 = \beta + \phi_1(\eta_1)\,,$$

with $\eta_1(\infty) = \infty$. Since the left side of (4.3) is a bounded analytic function of θ_1 we must have

$$\int_0^g e^{-s[\phi_1(\eta_1)-\eta_1]}\, J_1(s,\alpha)\, ds = \frac{e^{-\eta_1 Z_1(0)}}{\eta_1}\,. \tag{4.4}$$

Here (4.4) is an integral equation of J_1. We shall not solve for J_1 explicitly since we are interested only in the limit behavior of $Z_1(nc)$. Proceeding as in the proof of Lemma 2 of [7, page. 114], we find that

(i) as $\beta \to 0+$, $\eta_1(\beta) \to \eta_1(0)$, where $\eta_1(0)$ is the least positive root of the equation

$$\frac{g}{c}\,\eta_1(0) = \phi_1(\eta_1(0))\,,$$

and $\eta_1(0) > 0$ if and only if $\rho_1 > g/c$,

(ii)

$$\eta_1'(0+) = \begin{cases} \left(\dfrac{g}{c} - \rho_1\right)^{-1} & \text{if } \rho_1 < g/c\,, \\ \infty & \text{if } \rho_1 = g/c\,. \end{cases}$$

Using these results in (4.4) we obtain

$$\lim_{\alpha\to 1-} (1-\alpha) \int_0^g e^{-s[\phi_1(\eta_1)-\eta_1]}\, J_1(s,\alpha)\, ds = \lim_{\beta\to 0+} (1 - e^{-c\beta})\, \frac{e^{-\eta_1 Z_1(0)}}{\eta_1}\,,$$

which reduces to

$$\int_0^g e^{-s[\phi_1(\eta_1(0))-\eta_1(0)]}\, F_1(0)\, ds = \begin{cases} c\left(\dfrac{g}{c} - \rho_1\right) & \text{if } \rho_1 < g/c\,, \\ 0 & \text{if } \rho_1 \ge g/c\,. \end{cases}$$

We conclude that

$$F_1(0) = \begin{cases} 1 - \dfrac{c}{g}\,\rho_1 & \text{if } \rho_1 < g/c\,, \\ 0 & \text{if } \rho_1 \ge g/c\,. \end{cases}$$

Collecting all these results and using them in (4.3) we arrive at the desired result (4.1).

To prove (4.2) we note from Lemma 2.1 that the permutation

$$(r,\, g,\, X_1,\, X_2) \;\to\; (g,\, r,\, X_2,\, X_1)\,,$$

results in the permutation

$$\{Z_1(n\,c),\, Z_2(n\,c + r)\} \to \{Z_2(n\,c + r),\, Z_1(n\,c)\}\,,$$

and in the limit as $n \to \infty$

$$\{Z_1(\infty),\, Z_{2r}(\infty)\} \to \{Z_{2r}(\infty),\, Z_1(\infty)\}\,.$$

Thus (4.1) leads to (4.2). □

Remark 4.2. In order to derive the steady state behaviour of $\{Z_1(t),\, Z_2(t)\}$ we note that $t \to \infty$ either through red periods or through green periods (namely, the sets $(n\,c,\, n\,c + r]$, $n = 0, 1, \ldots$ and $(n\,c + r,\, n\,c + c]$, $n = 0, 1, \ldots$ respectively). To be specific, let $N \equiv N(t) = [t/c]$ so that $(N\,c,\, N\,c+r]$ is the red period and $(N\,c+r,\, N\,c+c]$ is the green period that correspond to the given t. Clearly, $N(t) \sim t\,c^{-1}$, $(t \to \infty)$. It turns out that $t - N\,c$ (t red) and $t - N\,c - r$ (t green) both have limit distributions as $t \to \infty$. In fact

$$t - N\,c \xrightarrow{d} U_r\,, \qquad t - N\,c - r \xrightarrow{d} U_g\,, \tag{4.5}$$

where for any $a > 0$, U_a is a random variable having uniform density in $(0, a)$. In particular, in steady state the probability of t red is r/c and the probability of t green is g/c. For proof see the Appendix.

Theorem 4.3. *As $t \to \infty$, $\{Z_1(t),\, Z_2(t)\} \xrightarrow{d} \{Z_1,\, Z_2\}$, with*

$$\begin{aligned}\mathbf{E}\left(e^{-\theta_1 Z_1 - \theta_2 Z_2}\right) &= \mathbf{E}[e^{-\theta_1 Z_1(\infty) - \theta_2 Z_{2r}(\infty)}]\,\mathbf{E}[e^{-\theta_1 W_1 - \theta_2 W_2}] \\ &-\frac{g}{c}\,\theta_1\,F_1(0)\,\mathbf{E}[e^{-\theta_2 Z_{2r}(\infty)}]\,\mathbf{E}\left[e^{-U_g \phi_2(\theta_2)}\,\int_0^{U_g} e^{-s(\phi_1(\theta_1)-\theta_1)}\,ds\right] \\ &-\frac{r}{c}\,\theta_2\,F_2(0)\,\mathbf{E}[e^{-\theta_1 Z_1(\infty)}]\,\mathbf{E}\left[e^{-U_r \phi_1(\theta_1)}\,\int_0^{U_r} e^{-s(\phi_2(\theta_2)-\theta_2)}\,ds\right]\,,\end{aligned} \tag{4.6}$$

where the random vector $\{W_1,\, W_2\}$ is defined by

$$\{W_1,\, W_2\} \stackrel{d}{=} \begin{cases} \{X_1(U_r),\, X_2(U_r) - U_r + X_2(g)\} & \text{with probability } \dfrac{r}{c}\,, \\[2ex] \{X_1(U_g) - U_g + X_1(r),\, X_2(U_g)\} & \text{with probability } \dfrac{g}{c}\,. \end{cases} \tag{4.7}$$

Proof. From Theorem 3.2 we find that

$$\begin{aligned}\mathbf{E}\left(e^{-\theta_1 Z_1(t) - \theta_2 Z_2(t)}\right) &= \mathbf{E}\left(e^{-\theta_1 Z_1(Nc) - \theta_2 Z_2(Nc)}\right)\, e^{-(t-Nc)[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_2]} \\ &- \theta_2\,\mathbf{E}\left(e^{-\theta_1 Z_1(Nc)}\right)\, e^{-(t-Nc)\phi_1(\theta_1)} \int_0^{t-Nc} e^{-s[\phi_2(\theta_2)-\theta_2]}\,F_2(0, t-s)\,ds\,,\end{aligned}$$

for $N\,c < t \le N\,c + r$ and

$$\begin{aligned}\mathbf{E}\left(e^{-\theta_1 Z_1(t) - \theta_2 Z_2(t)}\right) &= \mathbf{E}\left(e^{-\theta_1 Z_1(Nc+r) - \theta_2 Z_2(Nc+r)}\right)\, e^{-(t-Nc-r)[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_1]} \\ &- \theta_1\,\mathbf{E}\left(e^{-\theta_2 Z_2(Nc+r)}\right)\, e^{-(t-Nc-r)\phi_2(\theta_2)} \int_0^{t-Nc-r} e^{-s[\phi_1(\theta_1)-\theta_1]}\,F_1(0, t-s)\,ds\,,\end{aligned}$$

for $N\,c + r < t \le N\,c + c$. Letting $t \to \infty$ and using (4.5)

$$\begin{aligned}\mathbf{E}\left(e^{-\theta_1 Z_1 - \theta_2 Z_2}\right) &= \mathbf{E}\left(e^{-\theta_1 Z_1(\infty) - \theta_2 Z_2(\infty)}\right)\,\mathbf{E}e^{-U_r[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_2]} \\ &- \theta_2 F_2(0)\,\mathbf{E}\left(e^{-\theta_1 Z_1(\infty)}\right)\,\mathbf{E}e^{-U_r\phi_1(\theta_1)} \int_0^{U_r} e^{-s[\phi_2(\theta_2)-\theta_2]}\,ds\,,\end{aligned} \tag{4.8}$$

with probability r/c and

$$\begin{aligned} \mathbf{E}\left(e^{-\theta_1 Z_1-\theta_2 Z_2}\right) &= \mathbf{E}\left(e^{-\theta_1 Z_{1r}(\infty)-\theta_2 Z_{2r}(\infty)}\right)\,\mathbf{E}e^{-U_g[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_1]} \\ &- \theta_1\,F_1(0)\,\mathbf{E}\left(e^{-\theta_2 Z_{2r}(\infty)}\right)\,\mathbf{E}e^{-U_g\phi_2(\theta_2)}\int_0^{U_g} e^{-s[\phi_1(\theta_1)-\theta_1]}\,ds\,, \end{aligned} \tag{4.9}$$

with probability g/c. Adding the limit results (4.8) and (4.9) with appropriate weights we arrive at the result (4.6) since

$$\begin{aligned} \mathbf{E}\left(e^{-\theta_1 W_1-\theta_2 W_2}\right) &= \frac{r}{c}\,\mathbf{E}[e^{-\theta_1 X_1(U_r)-\theta_2 X_2(U_r)+\theta_2 U_r-\theta_2 X_2(g)}] + \frac{g}{c}\,\mathbf{E}[e^{-\theta_1 X_1(r)-\theta_1 X_1(U_g)+\theta_1 U_g-\theta_2 X_2(U_g)}] \\ &= \frac{r}{c}\,\mathbf{E}[e^{-U_r[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_2]}]\,e^{-g\phi_2(\theta_2)} + \frac{g}{c}\,\mathbf{E}[e^{-U_g[\phi_1(\theta_1)+\phi_2(\theta_2)-\theta_1]}]\,e^{-r\phi_1(\theta_1)}\,. \end{aligned}$$

□

Letting $\theta_2 \to 0+$ and $\theta_1 \to 0+$ respectively in (4.6) we obtain the following for the marginal distributions of Z_1 and Z_2.

Corollary 4.4. *(i) We have*

$$\mathbf{E}\left(e^{-\theta_1 Z_1}\right) = \mathbf{E}\left(e^{-\theta_1 Z_1(\infty)}\right)\,\mathbf{E}[e^{-\theta_1 W_1}] - \frac{g}{c}\,\theta_1\,F_1(0)\,\mathbf{E}\int_0^{U_g} e^{-s[\phi_1(\theta_1)-\theta_1]}\,ds\,, \tag{4.10}$$

and

$$\mathbf{E}\left(e^{-\theta_2 Z_2}\right) = \mathbf{E}\left(e^{-\theta_1 Z_{2r}(\infty)}\right)\,\mathbf{E}[e^{-\theta_2 W_2}] - \frac{r}{c}\,\theta_2\,F_2(0)\,\mathbf{E}\int_0^{U_r} e^{-s[\phi_2(\theta_2)-\theta_2]}\,ds\,. \tag{4.11}$$

(ii) Assume that $\mathrm{var}[X_i(1)] = \sigma_i^2 < \infty\,,\ i = 1,2.$ *Then*

$$\mathbf{E}[Z_1] = \frac{\frac{c}{g}\,\sigma_1^2}{2\left(1-\frac{c}{g}\,\rho_1\right)}\,,\qquad \mathbf{E}[Z_2] = \frac{\frac{c}{r}\,\sigma_2^2}{2\left(1-\frac{c}{r}\,\rho_2\right)}\,, \tag{4.12}$$

and

$$\mathrm{cov}[Z_1,\,Z_2] = -\frac{1}{12}\,(r\,\rho_1)\,(g\,\rho_2) < 0\,. \tag{4.13}$$

Proof. The results (4.10) and (4.11) follow easily from (4.6). The result (4.12) follow from (4.1), (4.2), (4.10) and (4.11) by differentiation. Also, from (4.6) we obtain by differentiation

$$\mathbf{E}[Z_1\,Z_2] = \mathbf{E}[Z_1]\,\mathbf{E}[Z_2] - \frac{1}{12}\,(r\,\rho_1)\,(g\,\rho_2)\,.$$

□

Remark 4.5. From Lemma 2.1 we find that the permutation (2.7) results in the permutation

$$\{W_1,\,W_2\} \to \{W_2,\,W_1\}\,,$$

as can be seen from (4.7). Theorem 4.3 shows that

$$\{Z_1,\,Z_2\} \to \{Z_2,\,Z_1\}\,.$$

5. Concluding remarks.

The result (4.13) indicates the dependence of the limit random variables Z_1 and Z_2. This dependence is not surprising. In the equations (2.3) formulating the processes $Z_i(t)$, $(i = 1,2)$ the inputs $X_1(t)$ and $X_2(t)$ are independent, so the components $Z_1(t)$ and $Z_2(t)$ are independent, as was claimed in Theorem 3.2. However, in steady state $X_1(t)$ and $X_2(t)$ have to be considered over the intervals of random lengths U_r and U_g. This is evident from the proof of Theorem 4.3, where the random vector $\{W_1,\, W_2\}$ is defined in terms of $\{X_1(U_r),\, X_2(U_r)\}$ and $\{X_1(U_g),\, X_2(U_g)\}$ (see equation (4.7)).

It is possible to consider several variations and extensions of the model analyzed in this paper. Thus, payoffs of claims of each type may be treated as customers and served in a single server queue according to a FIFO discipline. In such a model the integrals in (2.1) and (2.2) will have to be formulated in terms of a suitable redefinition of δ_i, $i = 1, 2$, with intervals of fixed or variable lengths.

Extensions to three or more types of claims may also be considered. These claims may arise independently of each other, or else may depend on a secondary source of randomness (such as an underlaying Markov chain). Thus the claim process may be represented as a so-called Markov-additive process, with the payoffs made in a sequence of fixed or variable lengths.

6. Appendix.

Let $\{X_k\,,\ k \geq 1\}$ and $\{Y_k\,,\ k \geq 1\}$ be independent sequences of i.i.d. non-negative random variables with

$$P[X_k \leq x] = F(x)\,, \quad P[Y_k \leq x] = G(x)\,,$$

and

$$\mathbf{E}[X_k] = \mu_1\,, \quad \mathbf{E}[Y_k] = \mu_2\,,$$

for $0 < \mu_1 \leq \infty\,,\ 0 < \mu_2 \leq \infty$. Denote $S_0 = 0$,

$$S_n = \sum_{k=1}^{n} (X_k + Y_k)\,,$$

for $n \geq 1$. Then $\{S_n\,,\ n \geq 0\}$ is a renewal process with lifespan distribution H, where

$$H(z) = P[X_k + Y_k \leq z]\,,$$

for $k \geq 1$. Let $N(t) = \max\{n : S_n \leq t\}$ and $\mathbf{E}N(t) = \sum_{n=0}^{\infty} P[S_n \leq t] < \infty$. We are concerned with the two probabilities

$$\begin{aligned} Z_x(t) &= P[S_{N(t)} < t \leq S_{N(t)} + X_{N(t)+1}\,,\ t - S_{N(t)} \leq x]\,, \\ Z_y(t) &= P[S_{N(t)} + X_{N(t)+1} < t \leq S_{N(t)+1}\,,\ t - S_{N(t)} - X_{N(t)+1} \leq y]\,. \end{aligned}$$

The following results describe the behavior of these at $t \to \infty$ (see Feller [1]).

Theorem 6.1. *(a) We have*

$$\begin{aligned} \lim_{t\to\infty} Z_x(t) &= \frac{1}{\mu_1 + \mu_2} \int_0^x [1 - F(s)]\, ds\,, \\ \lim_{t\to\infty} Z_y(t) &= \frac{1}{\mu_1 + \mu_2} \int_0^y [1 - G(s)]\, ds\,, \end{aligned}$$

the limit in each case being zero if μ_1 or μ_2 is infinite.

(b) If at least one of μ_1 or μ_2 is finite, then

$$\begin{aligned}\lim_{t\to\infty} P[S_{N(t)} < t \le S_{N(t)} + X_{N(t)+1}] &= \frac{\mu_1}{\mu_1+\mu_2}\,,\\ \lim_{t\to\infty} P[S_{N(t)} + X_{N(t)+1} < t \le S_{N(t)+1}] &= \frac{\mu_2}{\mu_1+\mu_2}\,,\end{aligned}$$

the limits being interpreted as 1 and 0 if $\mu_1 = \infty$ and 0 and 1 if $\mu_2 = \infty$.

□

The sequence of red and green periods represents an alternating renewal process with lifespans $X_k = r$ and $Y_k = g$ with probability one. For this we have $S_n = n\,c\,,\ n \ge 0$ and $N(t) = [t/c]$. Writing $N \equiv N(t)$ we see that

$$\begin{aligned}Z_x(t) &= P[N\,c < t \le N\,c + r\,,\ t - N\,c \le x]\,,\\ Z_y(t) &= P[N\,c + r < t \le N\,c + c\,,\ t - N\,c - r \le y]\,.\end{aligned}$$

Since

$$F(x) = \begin{cases} 0 & \text{for } 0 < x < r\,,\\ 1 & \text{for } x \ge r\,.\end{cases}$$

we have $r^{-1}[1 - F(x)] = r^{-1}$ for $0 < x < r$, which is the uniform density in $(0, r)$. Therefore

$$\lim_{t\to\infty} Z_x(t) = \begin{cases} \dfrac{x}{c} & \text{for } 0 < x < r\,,\\ \dfrac{r}{c} & \text{for } x \ge r\,.\end{cases}$$

Similar results hold for $Z_y(t)$.

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