On the Theory of Variance Balanced

Incomplete Block Designs

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BU-413-M

April, 1972

ABSTRACT

This paper presents a simple method for constructing variance balanced incomplete block designs with many different block sizes and two or three different replications.

The method is essentially a generalization and extension of the unionizing method given by Hedayat and Federer [1971] combined with some ideas from various papers in the area, namely Das [1958], Federer [1961], John [1964], and Rao [1958]. In section 2 a method is given for constructing a variance balanced design with many different block sizes and two different replications. A similar procedure is also given for constructing ternary variance balanced designs with many different block sizes and three different replications.

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1. Introduction

The concept of variance balanceness and connectedness are particularly useful to an experimenter when he has no idea which treatments are important and/or which linear functions of the treatments he may be interested in estimating, but wishes to estimate those he "chooses" with equal precision. Connectedness guarantees estimability of v-1 independent treatment contrasts, where v is the number of treatments and variance balanceness assures that the estimable linear functions of the treatments are estimated with the same variance. A block design is said to be connected if its C-matrix has rank v-1 and it is defined to be variance balanced if all estimable linear functions of the treatments are estimated with the same variance.

In this paper we are primarily concerned with variance balanceness, however, all the designs constructed are also connected. Thus the designs have a C-matrix of the form $C = \lambda I_v - \frac{\lambda}{v} J_v$. In general $C = R - NK^{-1}N'$ where R is a diagonal matrix with the (i,i) entry equal to the number of replications of treatment i, K^{-1} is a diagonal matrix with the (j,j) entry being the reciprocal of the number of elements in block j of the design and N is the incidence matrix of the design with its (i,j) entry equal to the number of times treatment i occurs in block j. The result $C = \lambda I - \frac{\lambda}{v} J$ is the crux of the method of construction.

Most variance balanced block designs given in the literature are the classical block designs (i.e. RCB and BIB) which require equal block sizes and equal replication of all the treatments. In many situations an experimenter may have unequal block sizes and/or unequal replication. Thus there is a need for a practical and simple method of constructing variance balanced designs in general. This problem is extremely difficult and remains unsolved. However, in this paper some methods are given for constructing variance balanced designs with certain types of unequal block sizes and unequal replication.

The method is essentially a generalization and extension of the unionizing method given by Hedayat and Federer [1971] combined with some ideas from various papers in the area, namely Das [1958], Federer [1961], John [1964], and Rao [1958]. In section 2 a method is given for constructing a variance balanced design with many different block sizes and two different replications. A similar procedure is also given for constructing ternary variance balanced designs with many different block sizes and three different replications.

2. Results

This section is divided into two parts. Part A gives the method for constructing a variance balanced design with many different block sizes and two different replications. In part B a procedure which yields tertiary variance balanced designs with many different block sizes and three different replications is presented.

Part A

A balanced incomplete block design with v treatments each replicated r times, and b blocks of size k is denoted by BIBD $(v;b;r;k;\lambda)$ where $\lambda = r(k-1)/v-1$ and is an integer. Let D_i be a BIBD $(v,b_i,r_i,k_i,\lambda_i)$ and D_i^* be D_i with every block augmented a_i times with a new treatment v+1. Define the unionized design $\hat{D}$ as follows:

$$\hat{D} = (\alpha_1 D_1^*) \cup (\alpha_2 D_2^*) \cup \cdots \cup (\alpha_z D_z^*) \cup (\alpha_{z+1} D_{z+1}) \cup \cdots \cup (\alpha_n D_n)$$

where $\alpha_i D_i$ is the union of α_i D_i's, similarly for $\alpha_\ell D_\ell^*$.

$\hat{D}$ has incidence matrix $\hat{N}$;

$$\hat{N} = \left(\begin{array}{c:c:c:c:c:c} N_1 \cdots N_1 & \cdots\cdots & N_z \cdots N_z & N_{z+1} \cdots N_{z+1} & \cdots\cdots & N_n \cdots N_n \\ a_1 \underset{\sim}{1} \cdots a_1 \underset{\sim}{1} & \cdots\cdots & a_z \underset{\sim}{1} \cdots a_z \underset{\sim}{1} & 0 \cdots 0 & \cdots\cdots & 0 \cdots 0 \end{array}\right)$$
$$\quad (\alpha_1 \text{ times}) \qquad\qquad (\alpha_z \text{ times}) \; (\alpha_{z+1} \text{ times}) \qquad\qquad (\alpha_n \text{ times})$$

where N_i is the incidence matrix of D_i, $\underset{\sim}{1}$ is a row vector of ones and a_i is a constant (integer ≥ 1).

The above design is connected and so to be variance balanced its C-matrix must be of the form $\lambda I - \frac{\lambda}{v+1} J$, λ is a constant. Thus the method is to form the C-matrix of $\hat{D}$, $\hat{C}$, and solve for the α's such that $\hat{C}$ is of the appropriate form.

$$\hat{C} = \hat{R} - \hat{N}\hat{K}^{-1}\hat{N}'$$

where

$$\hat{R} = \left(\begin{array}{cccc|c} \sum_{m=1}^{n} \alpha_m r_m & & & & \\ & \ddots & & & \\ & & & \sum_{m=1}^{n} \alpha_m r_m & \\ \hline & & & & \sum_{i=1}^{z} \alpha_i a_i b_i \end{array}\right) \qquad (2.1)$$

$$\hat{K}^{-1} = \begin{pmatrix} (k_1+a_1)^{-1} & & & & & \\ & \ddots & & & & \\ & & (k_z+a_z)^{-1} & & & \\ & & & k_{z+1}^{-1} & & \\ & & & & \ddots & \\ & & & & & k_n^{-1} \end{pmatrix} \tag{2.2}$$

$$\hat{N}\hat{K}^{-1}\hat{N}' = \left(\begin{array}{c|c} \sum_{i=1}^{z} \alpha_i (k_i+a_i)^{-1} N_i N_i + \sum_{\ell=z+1}^{n} \alpha_\ell k_\ell^{-1} N_\ell N_\ell^1 & \sum_{i=1}^{z} \alpha_i (k_i+a_i)^{-1} a_i N_i 1 \\ \hline \text{symmetric} & \sum_{i=1}^{z} \alpha_i a_i^2 (k_i+a_i)^{-1} b_i \end{array}\right) \tag{2.3}$$

2.1 - 2.3 yields $\hat{C}$.

For $\hat{C}$ to be of the appropriate form all its diagonal elements must be equal and all off-diagonal elements must also be equal. $\hat{C}$ has two types of diagonal and off-diagonal elements. The diagonal elements are:

1. $$\sum_{i=1}^{z} \alpha_i r_i \frac{(k_i+a_i-1)}{(k_i+a_i)} + \sum_{\ell=z+1}^{n} \alpha_\ell r_\ell \left(\frac{k_\ell-1}{k_\ell}\right)$$

2. $$\sum_{i=1}^{z} \alpha_i a_i b_i \left(\frac{k_i}{k_i+a_i}\right).$$

The off-diagonal elements are:

3. $$\sum_{i=1}^{z} \alpha_i (k_i+a_i)^{-1}\lambda_i + \sum_{\ell=z+1}^{n} \alpha_\ell k_\ell^{-1}\lambda_\ell .$$

4. $$\sum_{i=1}^{z} \alpha_i a_i (k_i+a_i)^{-1} r_i .$$

Variance balanceness requires 1. = 2. and 3. = 4., this gives the following solution for the α's:

$$\alpha_i = \Big(\sum_{\ell=z+1}^{n} \lambda_\ell\Big)(k_i+a_i); \quad \text{for all } i = 1,2,\cdots,z$$

and $$\alpha_\ell = \Big[\sum_{i=1}^{z} (a_i r_i - \lambda_i)\Big] k_\ell \quad ; \quad \text{for all } \ell = z+1,\ z+2,\cdots,n.$$

With the above α's $\hat{D}$ is variance balanced. Any multiple of the α's, such that the new α's are integers, is also a solution which makes $\hat{D}$ a variance balanced design. Thus we have constructed a variance balanced design with n different block sizes and two different replications. [Note that the design is binary if and only if $a_i = 1$ for all $i = 1,2,\cdots,z$ otherwise $\hat{D}$ is non-binary.]

Part B

In this part we unionize three different types of block designs to form a variance balanced design. The three types are as follows:

1. D_a^{**} which is a BIBD $(v;b_a;r_a;k_a;\lambda_a)$ with every block augmented with two new treatments v+1 and v+2.

2. D_w^* which is a BIBD $(v;b_w;r_w;k_w;\lambda_w)$ with every block augmented with one new treatment either $v+1$ or $v+2$.

3. D_u which is a BIBD $(v;b_u;r_u;k_u;\lambda_u)$.

Now let us form the unionized design $\bar{D}$ as follows:

$$\bar{D} = (\alpha_1 D_1^*)\cup(\alpha_2 D_2^*)\cup\cdots\cup(\alpha_a D_a^{**})\cup(\alpha_{a+1} D_{a+1}^*)\cup\cdots\cup(\alpha_z D_z^*)\cup(\alpha_{z+1} D_{z+1})\cup\cdots\cup(\alpha_n D_n)$$

$\bar{D}$ has incidence matrix $\bar{N}$.

$$\bar{N} = \left(\underbrace{\begin{matrix} N_1\cdots N_1 \\ 1\cdots 1 \\ 0\cdots 0\end{matrix}}_{\alpha_1 \text{ times}} \;\vdots\cdots\vdots\; \underbrace{\begin{matrix} N_{a-1}\cdots N_{a-1} \\ 1\cdots 1 \\ 0\cdots 0\end{matrix}}_{\alpha_{a-1} \text{ times}} \;\vdots\; \underbrace{\begin{matrix} N_a\cdots N_a \\ 1\cdots 1 \\ 1\cdots 1\end{matrix}}_{\alpha_a \text{ times}} \;\vdots\; \underbrace{\begin{matrix} N_{a+1}\cdots N_{a+1} \\ 0\cdots 0 \\ 1\cdots 1\end{matrix}}_{\alpha_{a+1} \text{ times}} \;\vdots\cdots\vdots\; \underbrace{\begin{matrix} N_z\cdots N_z \\ 0\cdots 0 \\ 1\cdots 1\end{matrix}}_{\alpha_z \text{ times}} \;\vdots\; \underbrace{\begin{matrix} N_{z+1}\cdots N_{z+1} \\ 0\cdots 0 \\ 0\cdots 0\end{matrix}}_{\alpha_{z+1} \text{ times}} \;\vdots\cdots\vdots\; \underbrace{\begin{matrix} N_n\cdots N_n \\ 0\cdots 0 \\ 0\cdots 0\end{matrix}}_{\alpha_n \text{ times}}\right)$$

where N_i and 1 are the same as in part A.

The method of solution is the same as in part A. First we calculate $\bar{C}$, the C-matrix of $\bar{D}$, then equate all diagonal elements equal and all off-diagonal elements equal and solve for the α's. Thus we obtain the following set of solutions:

$$\alpha_i = \frac{1}{a-1}\left[\sum_{\ell=z+1}^{n} \lambda_\ell(v-1)\right]\frac{b_a-r_a}{r_i}(k_i+1) \;; \quad \text{for all } i = 1,2,\cdots,a-1$$

$$\alpha_a = \left[\sum_{\ell=z+1}^{n} \lambda_\ell(v-1)\right] k_a+2$$

$$\alpha_j = \frac{1}{z-(a+1)}\left[\sum_{\ell=z+1}^{n} \lambda_\ell(v-1)\right]\frac{b_a-r_a}{r_j}(k_j+1)\ ; \quad \text{for all } j = a+1,\ a+2,\cdots,z$$

$$\alpha_p = \frac{b_a-r_a}{n-(z+1)}\left[v - \frac{\sum_{i=1}^{a-1}(k_i-1)}{a-1} - \frac{\sum_{j=a+1}^{z}(k_j-1)}{z-(a+1)} + (k_a-1)\right]k_p\ ;$$

for all $p = z+1,\ z+2,\cdots,n$.

As in part A, any multiple of the above α's which yields integers is also a solution. Thus with the above α's we have a method for constructing a ternary ($n_{ij} = 0$, 1 or 2) variance balanced design with many different block sizes and three different replications.

Further extension of these methods is possible but the algebra becomes very difficult and the resulting design very large (i.e. many blocks).

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