# How heavy are the tails of a stationary HARCH(k) process? A study of the moments *†

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**Abstract**

*Probabilistic properties of HARCH(k) processes, as special stochastic volatility models, are investigated. We present necessary and sufficient conditions for existence of a stationary version of a HARCH(k) process with finite $(2m)$th moments, $m \geq 1$. Our approach is based on the general Markov chain techniques of (Meyn and Tweedie, 1990). The conditions are explicit in the case of second moments, and also in the case of 4th moments of the HARCH(2) process. We also deduce explicit* necessary *and explicit* sufficient *conditions for higher order moments of general HARCH(k) models. We start by studying the HARCH(2) process (in which case our results are the most explicit) and then generalize the results to a general HARCH(k) process.*

## 1 Introduction

In (Müller et al., 1995), the Heterogeneous Auto-Regressive Conditional Heteroskedastic process (HARCH) has been introduced. This HARCH process has been developed as an improvement of traditional ARCH-type models in order to describe the behavior of financial time series, including some newly detected properties such as the long memory of volatility and the asymmetry between volatilities with different degrees of resolution in time.

The HARCH process is useful in practice only if its basic properties are known. In section 2, we give some motivation for using HARCH and determining its stationarity and moment conditions.

The HARCH process is a random recursion, whose equations are presented in section 3. In order to understand how heavy are the tails of a stationary HARCH(k) process, we study its moments. In section 4, we start by presenting in more details how a necessary condition for the existence of the

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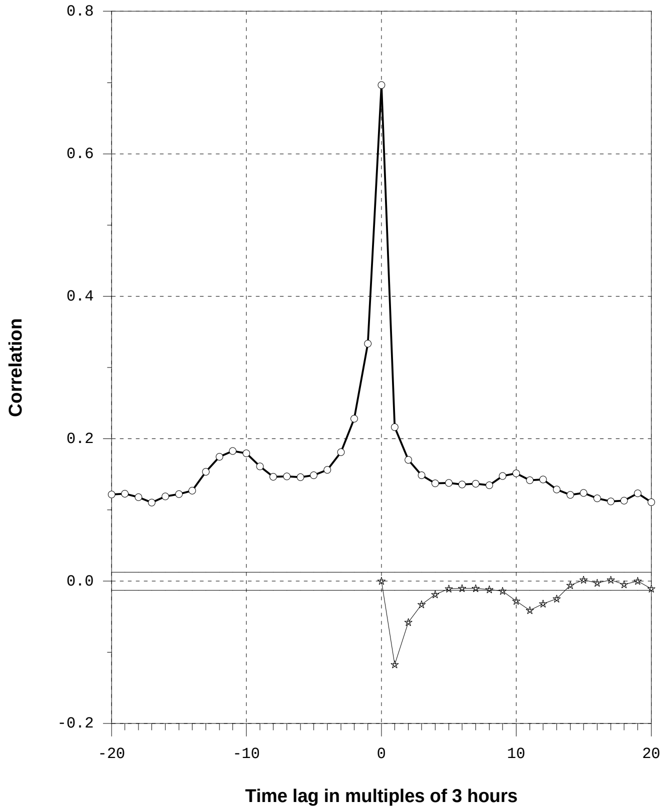



Lead-lag correlation of fine and coarse volatilities of a USD/DEM time series with a half-hourly grid in $\vartheta$-time, as defined in (Dacorogna et al., 1993). The fine volatility is defined as the mean absolute half-hourly price change within 3 hours ($\vartheta$-time); the coarse volatility is the absolute price change over a whole 3 hour interval ($\vartheta$-time). The thin curve indicates the asymmetry: the difference between correlations at positive and corresponding negative lags. Sampling period: 8 years, from 1 Jan 1987 00:00 to 1 Jan 1995 00:00 (GMT). The confidence limits represent the 95% confidence interval of a Gaussian random walk.

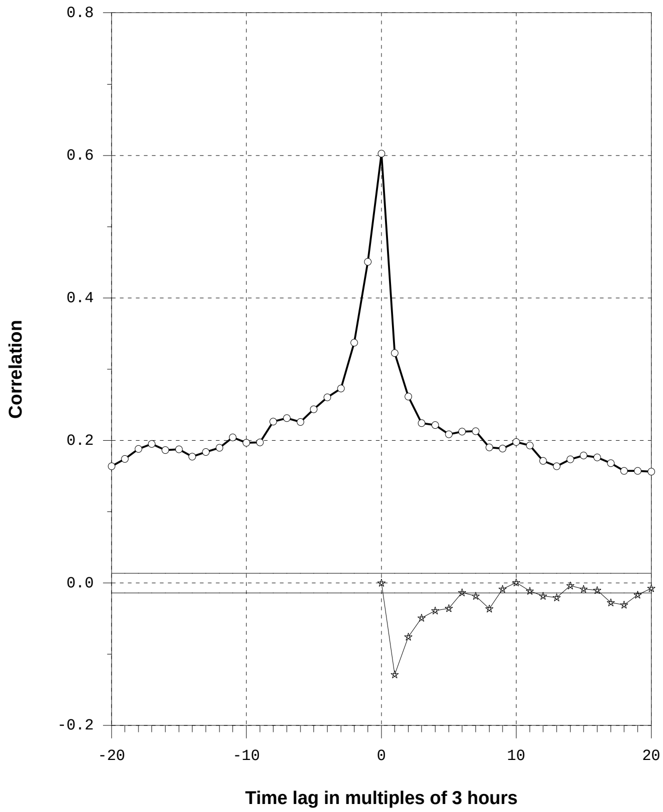
Figure 1: Asymmetric lead-lag correlation of fine and coarse volatilities of a half-hourly USD/DEM series

2nd moment can be derived and to show using a Markov chain approach that the necessary condition presented in (Müller et al., 1995) is also sufficient for the existence of a stationary HARCH(k) process with a finite second moment. This is done first for the HARCH(2) process and then generalized to HARCH(k). As a rule, we first analyze the HARCH(2) process, and then consider the general HARCH(k) process. The reason for this is two-fold. First of all, the results are often most explicit for the HARCH(2) model. Second, the arguments and main ideas are the easiest to follow in this case.

With the above in mind, we next present the derivation of an explicit necessary and sufficient condition for the existence of the 4th moment of a HARCH(2) model. The following step is to derive a general theorem giving necessary and sufficient conditions for the existence of a stationary HARCH(2) process with finite $(2m)$th moments. We are then able to derive an explicit necessary condition for a finite $(2m)$th moment of an HARCH(2) and derive an explicit sufficient condition for the existence of the $(2m)$th moment.

The second part of the paper contains a generalization to HARCH(k) models. As mentioned above, we start by proving an explicit necessary and sufficient condition for a finite 2nd moment. We further give a general theorem for stationarity with finite moments, where we prove that the condition is both necessary and sufficient. The condition is in terms of existence of a nonnegative solution of a certain system of linear equations. For higher order moments of HARCH(k) models we give an explicit necessary condition and an explicit sufficient condition for a finite $(2m)$th moment.

## 2 Motivation




Lead-lag correlation of fine and coarse volatilities of a synthetic HARCH time series, fitted to a USD/DEM series on a half-hourly time grid in $\vartheta$-time. The fine volatility is defined as the mean absolute half-hourly price change in an interval of 3 hours; the coarse volatility is the absolute price change over the whole 3-hour interval. The thin curve indicates the asymmetry: the difference between correlations at positive and corresponding negative lags. Sampling period: 7 years. The confidence limits represent the 95% confidence interval of a Gaussian random walk.

Figure 2: Asymmetric lead-lag correlation of fine and coarse volatilities of a HARCH series

The HARCH process has been developed for describing the behavior of financial time series, with price quotes from the foreign exchange market being the best-studied example; see (Müller et al., 1995). Financial time series exhibit clusters of high and low volatility, i. e. autoregressive conditional heteroskedasticity.

Additional facts have been found in recent empirical research of high-frequency data in finance: (1) a long memory in the volatility (positive autocorrelation of absolute price changes declining slower than exponentially) and (2) asymmetry between volatilities observed with different time resolutions. The HARCH process precisely reproduces these empirical facts.

The asymmetry between different volatilities deserves some special attention here as it is a quite newly detected effect. Figure 1 shows that the lead-lag correlation between a high-resolution volatility and another volatility measured with lower resolution in time is asymmetric: low-resolution (coarse) volatility predicts high-resolution (fine) volatility better than the other way around. In Figure 1, the case of the US Dollar expressed in German Marks (USD/DEM) is shown, but the effect is found also for all other foreign exchange rates we studied.

Figure 2 shows the case of a time series synthetically generated from a HARCH process whose 8 parameters were fitted on a long USD/DEM sample. This synthetic data exhibits the same asymmetric behavior of lead-lag correlation as the empirical USD/DEM data in Figure 1, whereas the other ARCH-type processes lead to symmetric lead-lag correlograms. More details on this study can be found in (Müller et al., 1995).

The correct reproduction of many empirical properties makes HARCH an attractive choice for modeling the behavior of financial time series. In order to use a HARCH model in practice, however,

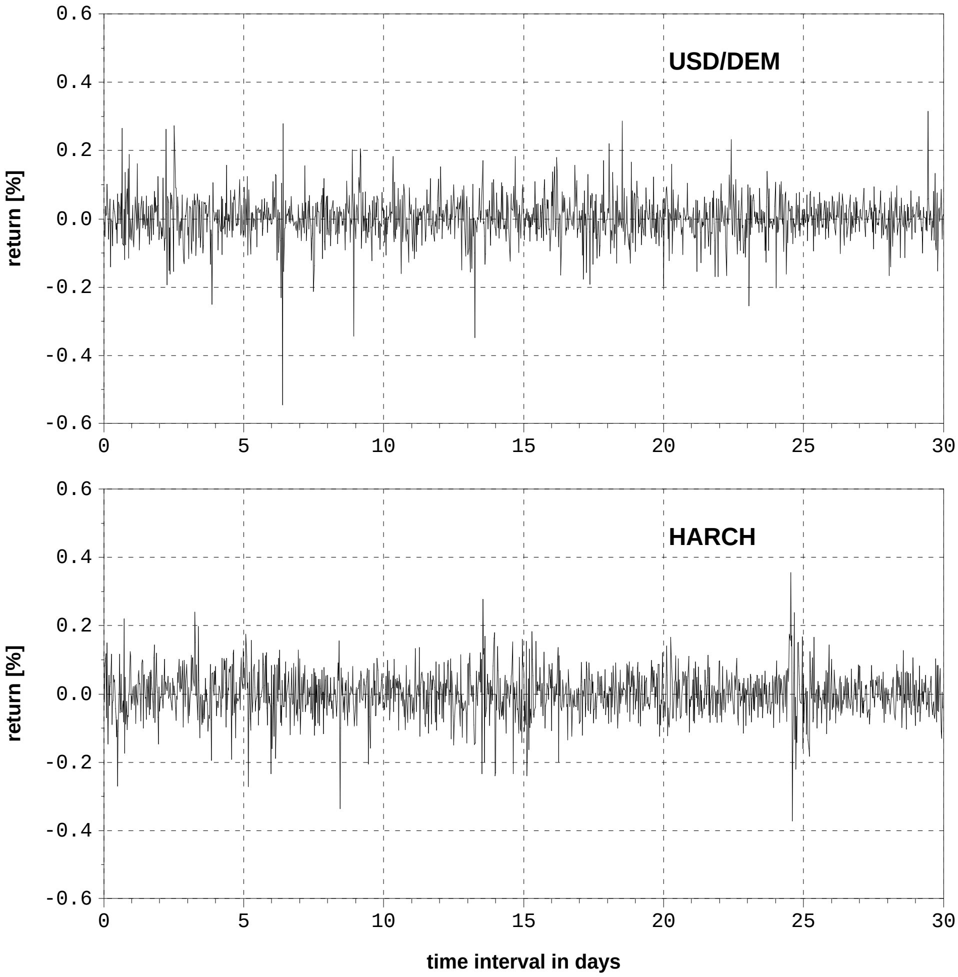



Figure 3: A comparison between 1 month of 30 minutes (in business time scale) returns of the USD/DEM foreign exchange rate and a Monte-Carlo realization of a HARCH process fitted to the same rate over 30 minute time intervals.

some basic theoretical properties of HARCH should be known, among these certainly the conditions for the stationarity and the existence of finite higher moments of the unconditional distribution function are crucial.

There is yet another motivation for studying the moment conditions. Financial risk analysis is based upon assumptions on the probability of extreme price movements. Extreme events in the tails of the distribution function are strongly related to the existence of the higher moments. If the $(2m)$th moment of a distribution is finite and the $(2(m+1))$th moment is not, the tail index of the distribution, if this distribution has a regularly varying tail, must lie between $2m$ and $2(m+1)$. The knowledge of the moment conditions might help to use HARCH in financial risk management.

## 3 The HARCH process

The HARCH(k) process equations as first presented in (Müller et al., 1995) are: specify $r_0, \ldots, r_{k-1}$ and

$$r_n \;=\; \sigma_n \, \varepsilon_n \;, \; n = 0, 1, 2, \ldots , \tag{3.1}$$

$$\sigma_n^2 \;=\; c_0 \,+\, \sum_{j=1}^{k} c_j \left(\sum_{i=1}^{j} r_{n-i}\right)^2 \;,$$

$$c_0 \,>\, 0\,,\;\; c_k \,>\, 0\,,\;\; c_j \,\geq\, 0 \text{ for } j \,=\, 1 \ldots k-1\,,$$

where $r_n$ is the return of a financial asset time series as defined in (Guillaume et al., 1994). The $\epsilon_n$'s are independent identically distributed (iid) symmetric random variables, whose distribution is assumed to have a non-zero absolutely continuous component, which has a positive density on a Borel set with a non-empty interior. Typical examples are a normal distribution $N(0,\sigma^2)$ or a Student-t distribution with zero expectation. Denote $a_m = \mathrm{E}(|\epsilon|^{2m})$, $m \geq 1$. For example, if the $\epsilon_n$'s have the standard normal distribution, then it is well known that $a_m = \prod_{j=1}^{m}(2j-1)$, $m \geq 1$. We will always assume that $\epsilon_n$ has a finite absolute moment of at least the same order as we want $r_n$ to have. (That is, if we are discussing, say, conditions for existence of the finite 6th moment of $r_n$, we will assume that $\mathrm{E}(|\epsilon_n|^6) < \infty$.) In (Müller et al., 1995) the moment conditions deduced in this paper were first announced.

In Figure 3, we present a realization of a HARCH process fitted to the returns of the USD/DEM foreign exchange rate measured over 30 minute time intervals. The figure displays 1 month of 30 minutes returns for the real FX rate (measured on the business time scale described in (Dacorogna et al., 1993)) and a Monte-Carlo realization of the fitted HARCH process. Both graphs present fat-tails and heteroskedasticity although not exactly at the same place since the realization of the HARCH process is drawn with a random number generator for the $\varepsilon_n$. The similarity in behavior between the two curves is striking and is confirmed by other statistical analysis see (Müller et al., 1995).

# 4 Condition for stationarity with a finite 2nd moment for HARCH(2)

## 4.1 Necessity

Let us first recall the way to get to the condition for the existence of a stationary distribution with a finite 2nd moment for a HARCH(2) process. The equation of such a process is:

$$\begin{aligned} r_n &= \sigma_n \varepsilon_n,\; n = 0,1,2,\ldots\;, \\ \sigma_n^2 &= c_0 \,+\, c_1 r_{n-1}^2 \,+\, c_2 (r_{n-1} + r_{n-2})^2\,. \end{aligned} \tag{4.1}$$

The scaled returns $\varepsilon_n$ satisfy the assumptions of the previous section. If $r_n$ is stationary with a finite 2nd moment, then

$$\mathrm{E}(r_n^2) \;=\; \mathrm{E}(r_{n-i}^2)\;,\quad i \,\geq\, 1\,. \tag{4.2}$$

Recall that $a_1 = \mathrm{E}(\epsilon_n)^2$. If we now use (4.1), we can write:

$$\mathrm{E}(r_n^2) \;=\; a_1\,\mathrm{E}(\sigma_n^2) \;=\; a_1 \Big(c_0 \,+\, (c_1 + c_2)\mathrm{E}(r_{n-1}^2) \,+\, c_2 \mathrm{E}(r_{n-2}^2) \,+\, 2c_2 \mathrm{E}(r_{n-1} r_{n-2})\Big)\,. \tag{4.3}$$

Since we know that $\varepsilon_n$ is iid with mean zero, the expectation of the cross product is zero and using (4.2) we obtain

$$\mathrm{E}(r_n^2) \;=\; \frac{c_0}{\frac{1}{a_1} - [(c_1 + c_2) + c_2]}\,. \tag{4.4}$$

The reason why we write the denominator this way will become clear later. A necessary condition for stationarity with a finite 2nd moment for a HARCH(2) process then becomes

$$(c_1 + c_2) + c_2 \; < \; \frac{1}{a_1} \tag{4.5}$$

which is equivalent to (3.6) in (Müller et al., 1995) when $k = 2$ (see condition (6.10) for the reason for writing the left hand side above in this form).

### 4.2 Sufficiency of the condition $(c_1 + c_2) + c_2 < 1/a_1$

The derivation of the expression of $\mathrm{E}(r_n^2)$ in (4.4) as a function of the coefficients $c_1$ and $c_2$ is not sufficient to prove the existence of a stationary version of the HARCH(2) process with finite second moments. In order to do this, we need to reformulate the problem in terms of a Markov chain. We do not follow here the path chosen by (Engle, 1982) and (Bollerslev, 1986) because of the cross term $r_{n-1}r_{n-2}$ which makes the matrix formulation of the problem difficult.

With the HARCH(2) process $(r_n)$ we can associate a two-dimensional Markov chain

$$X_n \;=\; (r_{n-1}, r_n),\;\; n \geq 1\,. \tag{4.6}$$

The various properties of $(r_n)$ can now readily be derived through standard results on Markov chains with state space $\mathbb{R}^2$. We base our analysis strongly on an excellent book by Mein and Tweedie (1990), where one can find many results that give sufficiency of various recurrence conditions and existence of stationary distributions with finite moments. The latter, in our language, means existence of a stationary solution to (4.1) with appropriate finite moments.

Let us summarize the main results. Let $\{X_n,\; n \geq 0\}$ be a Markov chain, with values in an Euclidian space. For such Markov chains the key notion is that of an *irreducible $T$-chain*. Because of the conditions on the density of the noise variables $\varepsilon_n$'s, the latter is equivalent to showing the follwing continuity property:

(C) The conditional distribution of $X_n$ given $X_{n-1} = y_k$ converges weakly to that of $X_n$ given $X_{n-1} = y$ if $y_k \to y$.

This result is part of Theorem 6.0.1 in (Meyn and Tweedie, 1990). See also the relevant definitions on irredicible $T$-chains (pages 87, 127) and the result that for a $T$-chain, every compact set is *petite* (page 121).

There are three levels of theorems in (Meyn and Tweedie, 1990) that will bring us to the desired result:

- **Level 1**: Suppose that there exists $V \geq 0$ with $V \not\equiv 0$ and a compact set $C$ such that

  $$V(X_n) - PV(X_n) \;\geq\; 0 \;\text{ for }\; X_n \;\notin\; C, \tag{4.7}$$

  where $P$ expresses the expectation value at the following step of the Markov chain, i.e. $PV(x) = \mathrm{E}(V(X_n)|X_{n-1} = x)$. Then there exists an invariant measure and the Markov chain is Harris recurrent.

  This is essentially Theorem 12.3.3 of (Meyn and Tweedie, 1990).

- **Level 2**: Suppose that there exists $V \geq 0$ and $V \not\equiv 0$ and a compact set C such that

$$V(X_n) - PV(X_n) \;\geq\; -b\, 1_C(X_n) \,+\, 1 \tag{4.8}$$

  where $1_C$ is the indicator function of the set C (1 if in C, 0 if outside), and b a finite positive number. Then there exists a unique stationary distribution $\pi$ and $(X_n)$ is positive Harris recurrent.

  This is essentially Theorem 11.3.4 or 13.0.1(iv) of (Meyn and Tweedie, 1990). Level 2 gives a sufficient condition for the stationarity of the Markov chain.

- **Level 3**: Suppose that $X_n$ is positive Harris recurrent and there exist non-negative measurable functions $f,\ V$ so that

$$V(X_n) - PV(X_n) \;\geq\; f(X_n) \,-\, b \tag{4.9}$$

  with a finite $b \;\geq\; 0$. Then the $\pi$-expectation of $f$, $\pi f \;=\; \int f(x)\, \pi(dx)$, is finite, specifically: $\pi f \;\leq\; b < \infty$.

  This is essentially Theorem 14.3.7 of (Meyn and Tweedie, 1990). Level 3 gives a sufficient condition for the existence of the appropriate moments of the stationary distribution.

If all three levels hold, we have established a sufficient condition for stationarity (level 2) and existence of moments (level 3).

To prove the ergodicity and existence of moments of HARCH(2), or, indeed, for any HARCH(k), we thus need to prove the weak continuity (C) of the distribution of $X_n$ given $X_{n-1} = y$ in $y$, and then to find a condition on the coefficients of the model so that levels 2 and 3 hold. The main idea behind the above approach is to prove that the Markov chain tends to drfift back to a neighbourhood of its initial position from any position far out.

### 4.2.1 Continuity condition (C) on X

We consider a Markov chain $(X_n)$ as defined in (4.6). To prove that it is a T-chain, we need to examine the continuity condition (C). Because of (4.1) and (4.6) every new state is described by analytic functions of the previous state and a new random variable. Thus the weak continuity of the Markov chain holds by the continuous mapping theorem (see (Billingsley, 1968), Theorem 5.1 page 30).

### 4.2.2 Drift from the tails to the center

We now turn to show that our Markov chain eventually tends to drift back to the center if the coefficients $c_1, c_2$ satisfy condition (4.5), $c_1 + 2c_2 < 1/a_1$. We start by choosing a number $0 < \alpha < 1$ such that

(i) $\alpha \;>\; c_2\, a_1$,

(ii) $c_1 + c_2 + \alpha/a_1 < 1/a_1$.

We then define a function $V$ as:

$$V(x,y) \;\equiv\; \alpha x^2 \,+\, y^2 \,+\, 2c_2 a_1 xy\,, \tag{4.10}$$

where, because of condition (i) above, $V \geq 0$. Indeed the discriminant satisfies

$$c_2^2 a_1^2 - \alpha < c_2 a_1 - \alpha < 0 . \tag{4.11}$$

By inserting (4.10) and (4.1),

$$V(r_{n-1}, r_n) - PV(r_{n-1}, r_n) = \alpha r_{n-1}^2 + 2c_2 a_1 r_{n-1} r_n + r_n^2 -$$
$$- E\Big[\alpha r_n^2 + 2c_2 a_1 r_n \varepsilon_{n+1} (c_0 + c_1 r_n^2 + c_2 (r_n + r_{n-1})^2)^{1/2} +$$
$$+ \varepsilon_{n+1}^2 (c_0 + c_1 r_n^2 + c_2 (r_n + r_{n-1})^2)\Big] . \tag{4.12}$$

Using $\mathrm{E}(\varepsilon_n) = 0$ and $\mathrm{E}(\varepsilon_n^2) = a_1$, we have

$$V(r_{n-1}, r_n) - PV(r_{n-1}, r_n) = (\alpha - c_2 a_1) r_{n-1}^2 + (1 - \alpha - (c_1 + c_2) a_1) r_n^2 - c_0\, a_1 . \tag{4.13}$$

The two coefficients $\alpha - c_2 a_1$ and $1 - \alpha - (c_1 + c_2) a_1$ are positive, and the last expression can be made as large as we wish if $(r_{n-1}, r_n)$ is outside of a compact set.

Now, under condition (4.5), we have found a function $V$ that satisfies the requirements of level 2 and also 3 with $f(x_1, x_2) = \theta(x_1^2 + x_2^2)$, with $0 < \theta < (\alpha - c_2 a_1) \wedge (1 - \alpha - (c_1 + c_2))$. Thus the stationarity and the existence of the second moment are proven if (4.5) holds.

# 5 Existence of the $4$th moment of a HARCH(2)

## 5.1 A necessary condition

Since $r_n = \sigma_n \varepsilon_n$, we deduce that:

$$\mathrm{E}(r_n^4) = \mathrm{E}(\sigma_n^4)\, \mathrm{E}(\varepsilon_n^4) = a_2\, \mathrm{E}(\sigma_n^4). \tag{5.1}$$

Therefore,

$$\mathrm{E}(\sigma_n^4) = \frac{1}{a_2}\, \mathrm{E}(r_n^4). \tag{5.2}$$

With the help of (4.1), we obtain:

$$\begin{aligned} \mathrm{E}(\sigma_n^4) &= \mathrm{E}([c_0 + c_1 r_{n-1}^2 + c_2 (r_{n-1} + r_{n-2})^2]^2) \\ &= c_0^2 + 2c_0 (c_1 + c_2) \mathrm{E}(r_{n-1}^2) + (c_1 + c_2)^2 \mathrm{E}(r_{n-1}^4) + 2c_0 c_2 \mathrm{E}(r_{n-2}^2) + \\ &\quad + 2c_2 (c_1 + 3c_2) \mathrm{E}(r_{n-1}^2 r_{n-2}^2) + c_2^2 \mathrm{E}(r_{n-2}^4) , \end{aligned} \tag{5.3}$$

where we have used

$$\mathrm{E}(r_{n-1} r_{n-2}) = \mathrm{E}(r_{n-1}^3 r_{n-2}) = \mathrm{E}(r_{n-1} r_{n-2}^3) = 0, \tag{5.4}$$

deduced from the iid nature of $\varepsilon_n$'s and their symmetric distribution. For ease of notation, we define:

$$L = \mathrm{E}(r_n^2), \quad M_0 = \mathrm{E}(r_n^4) \quad \text{and} \quad M_2 = \mathrm{E}(r_{n-1}^2 r_{n-2}^2), \tag{5.5}$$

which do not depend on $n$ if the process is stationary. Rewrite (5.3) as

$$\frac{1}{a_2} M_0 = c_0^2 + 2c_0(c_1 + 2c_2) L + (c_2^2 + (c_1 + c_2)^2) M_0 + 2c_2(c_1 + 3c_2) M_2$$

and compute, in a similar way, an expression for $M_2$:

$$\begin{aligned} M_2 &= \mathrm{E}(r_n^2 r_{n-1}^2) = \mathrm{E}(\sigma_n^2 r_{n-1}^2) E(\varepsilon_n^2) = a_1 \mathrm{E}(\sigma_n^2 r_{n-1}^2) \\ &= a_1 \mathrm{E}(r_{n-1}^2 [c_0 + c_1 r_{n-1}^2 + c_2 (r_{n-1} + r_{n-2})^2]) \\ &= a_1 \Big( c_0 L + (c_1 + c_2) M_0 + c_2 M_2 \Big). \end{aligned} \tag{5.6}$$

We are now left with a system of two equations with two variables $M_0$ and $M_2$ (the quantity $L$ is known from the stationarity condition (4.4)):

$$\left\{ \begin{array}{lcl} [1 - a_2 (c_2^2 + (c_1 + c_2)^2)] M_0 - 2a_2c_2(c_1 + 3c_2) M_2 &=& a_2c_0^2 + 2a_2c_0(c_1 + 2c_2) L \\ -(c_1 + c_2) M_0 + (\frac{1}{a_1} - c_2) M_2 &=& c_0 L. \end{array} \right. , \tag{5.7}$$

This linear system can be solved by standard methods, yielding

$$M_0 = \frac{a_2c_0^2(1 - a - 1c_2) + 2a_2c_0(c_1 + 2c_2 + c_2^2) L}{(1 - a_1c_2) - a_2[c_2^2 + (c_1 + c_2)^2] - a_1a_2c_2(c_1^2 + 6c_1c_2 + 4c_2^2)}. \tag{5.8}$$

Because of the condition (4.5), the numerator is positive and hence a necessary condition for the existence of a stationary HARCH(2) process with finite 4th moment becomes

$$1 - a_2[c_2^2 + (c_1 + c_2)^2] - a_1c_2[1 + a_2(c_1^2 + 6c_1c_2 + 4c_2^2)] > 0. \tag{5.9}$$

This condition is more stringent than (4.5). The solution for $M_2$ is:

$$M_2 = \frac{a_1a_2c_0^2(c_1 + c_2)}{D} + \frac{2a_1a_2c_0(c_1 + c_2)(c_1 + 2c_2 + a_1c_2^2)}{(1 - c_2) D} L + \frac{a_1c_0}{1 - c_2} L, \tag{5.10}$$

where $D$ is the denominator of the right hand side of (5.8).

An easy check of this condition is to compare it to the condition for an ARCH(1) process. We know that HARCH(2) becomes ARCH(1) if $c_2 = 0$. On page 992 of (Engle, 1982), the condition for the existence of the $2m$th moment is given. Assume that the $\epsilon_n$'s have the standard normal distribution. In that case, translated into our notation that condition becomes:

$$c_1^m \prod_{j=1}^{m} (2j - 1) < 1. \tag{5.11}$$

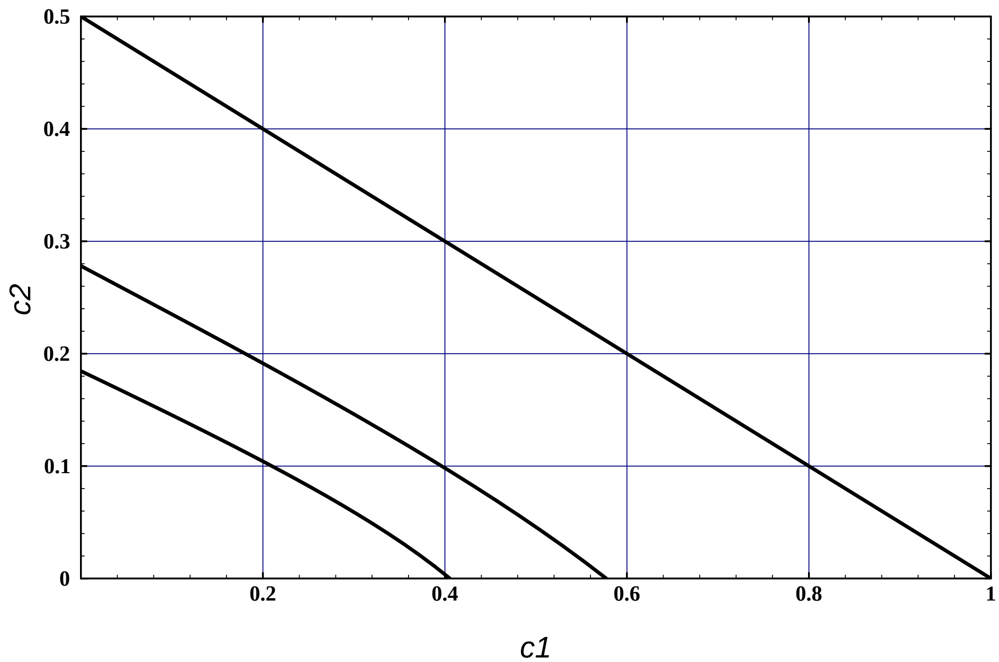



Figure 4: Moment conditions for HARCH(2) processes. The straight line on the right represents the boundary for the 2nd moment condition. The middle curve is the boundary for the existence of a finite 4th moment. The left curve represents the boundary for the existence condition of the 6th moment.

In the case of interest, the 4th moment, we get

$$3c_1^2 \; < \; 1 \quad \Rightarrow \quad c_1^2 \; < \; \frac{1}{3} \, ,$$

the same condition as in (5.9) if $c_2 = 0$ (and $a_2 = 3$).

The fact that (5.9) is also sufficient for the existence of the 4th moment will follow from the general theory for HARCH(2) models to which we now turn.

# 6 $(2m)$th moment of HARCH(2)

## 6.1 The system of equations

To get a necessary condition for the existence of a stationary distribution of the HARCH(2) process with a finite $(2m)$th moment, $m \geq 2$, we denote

$$\begin{aligned} M_0 &= \mathrm{E}(r_n^{2m}) \, , \\ M_i &= \mathrm{E}(r_n^{2(m-i)} r_{n-1}^{2i}) \, , \quad i = 1, \, ... \, , m-1 \end{aligned} \tag{6.1}$$

under stationarity conditions, in which case there is no dependence on $n$. If the $(2m)$th moment is finite, then all the moments of smaller orders are finite, $a_m < \infty$, and $M_0, M_1, \ldots , M_{m-1}$ above are all finite. Observe that

$$\mathrm{E}(r_n^{2m}) \;=\; \mathrm{E}(\sigma_n^{2m})\,\mathrm{E}(\varepsilon_n^{2m}) \,, \tag{6.2}$$

and so using (6.2) and $a_m = \mathrm{E}(\varepsilon_n^{2m})$ we have that

$$\begin{aligned}
M_0 \;=\; \mathrm{E}(r_n^{2m}) \;&=\; a_m\,\mathrm{E}\,[\,(c_0 + c_1 r_{n-1}^2 + c_2(r_{n-1}+r_{n-2})^2)^m\,] \\
&=\; a_m\,\mathrm{E}\,[\,(c_0 + (c_1+c_2) r_{n-1}^2 + c_2 r_{n-2}^2 + 2c_2 r_{n-1} r_{n-2})^m\,] \\
&=\; a_m \sum_{j=1}^{m} \binom{m}{j} c_0^j\,\mathrm{E}\,[\,((c_1+c_2) r_{n-1}^2 + c_2 r_{n-2}^2 + 2c_2 r_{n-1} r_{n-2})^{m-j}\,] + \\
&\quad + a_m\,\mathrm{E}\,[\,((c_1+c_2) r_{n-1}^2 + c_2 r_{n-2}^2 + 2c_2 r_{n-1} r_{n-2})^m\,] \\
&\equiv\; a_m \vartheta_{0,m} \;+\; a_m\,\mathrm{E}\,[\,((c_1+c_2) r_{n-1}^2 + c_2 r_{n-2}^2 + 2c_2 r_{n-1} r_{n-2})^m\,]\,,
\end{aligned} \tag{6.3}$$

where $\vartheta_{0,m}$ involves only moments of order less than $2m$. Moreover, we have:

$$\begin{aligned}
&\mathrm{E}[\,((c_1+c_2) r_{n-1}^2 + c_2 r_{n-2}^2 + 2c_2 r_{n-1} r_{n-2})^m\,] \;= \\
&= \sum_{j=0}^{[m/2]} \binom{m}{2j} (2c_2)^{2j}\,\mathrm{E}\,[\,(r_{n-1}^{2j} r_{n-2}^{2j})\,\left((c_1+c_2) r_{n-1}^2 + c_2 r_{n-2}^2\right)^{m-2j}\,] \\
&= \sum_{j=0}^{[m/2]} \binom{m}{2j} (2c_2)^{2j}\,\mathrm{E}\,[\,(r_{n-1}^{2j} r_{n-2}^{2j}) \sum_{i=0}^{m-2j} \binom{m-2j}{i} (c_1+c_2)^i\, c_2^{m-2j-i}\, r_{n-1}^{2i}\, r_{n-2}^{2(m-2j-i)}\,] \\
&= \sum_{j=0}^{[m/2]} \sum_{i=0}^{m-2j} \frac{m!}{(2j)!\,i!\,(m-2j-i)!}\, 2^{2j}\, (c_1+c_2)^i\, c_2^{m-i}\, \mathrm{E}(r_{n-1}^{2(i+j)}\, r_{n-2}^{2(m-j-i)}) \\
&= \sum_{j=0}^{[m/2]} \sum_{i=0}^{m-2j} \frac{m!}{(2j)!\,i!\,(m-2j-i)!}\, 2^{2j}\, (c_1+c_2)^i\, c_2^{m-i}\, M_{m-(i+j)}\,,
\end{aligned} \tag{6.4}$$

where $M_m \equiv M_0$, and $[\cdot]$ is the entier function. By replacing (6.4) in (6.3), we obtain:

$$M_0 \;=\; a_m \vartheta_{0,m} \;+\; a_m \sum_{j=0}^{[m/2]} \sum_{i=0}^{m-2j} \frac{m!}{(2j)!\,i!\,(m-2j-i)!}\, 2^{2j}\, (c_1+c_2)^i\, c_2^{m-i}\, M_{m-(i+j)}\,. \tag{6.5}$$

Furthermore, for every $i = 1, \ldots, m-1$,

$$M_i \;=\; \mathrm{E}(r_n^{2(m-i)} r_{n-1}^{2i}) \;=\; \mathrm{E}(\varepsilon_n^{2(m-i)})\,\mathrm{E}(r_{n-1}^{2i}\sigma_n^{2(m-i)})$$

$$= \; a_{m-i} \, \mathrm{E} \, [ \, r_{n-1}^{2i} (c_0 + (c_1 + c_2) r_{n-1}^2 + c_2 r_{n-2}^2 + 2 c_2 r_{n-1} r_{n-2})^{m-i} \, ]$$

$$= \; a_{m-i} \sum_{j=1}^{m-i} \binom{m-i}{j} c_0^j \, \mathrm{E} \, [ \, r_{n-1}^{2i} ((c_1 + c_2) r_{n-1}^2 + c_2 r_{n-2}^2 + 2 c_2 r_{n-1} r_{n-2})^{m-i-j} \, ]$$

$$+ \, a_{m-i} \, \mathrm{E} \, [ \, r_{n-1}^{2i} ((c_1 + c_2) r_{n-1}^2 + c_2 r_{n-2}^2 + 2 c_2 r_{n-1} r_{n-2})^{m-i} \, ]$$

$$= \; a_{m-i} \, \vartheta_{i,m} \; + \; a_{m-i} \, \mathrm{E} \, [ \, r_{n-1}^{2i} ((c_1 + c_2) r_{n-1}^2 + c_2 r_{n-2}^2 + 2 c_2 r_{n-1} r_{n-2})^{m-i} \, ] \, , \qquad (6.6)$$

where $\vartheta_{i,m}$ involves only moments of orders less than $2m$. Now,

$$\mathrm{E} \, [ \, r_{n-1}^{2i} ((c_1 + c_2) r_{n-1}^2 + c_2 r_{n-2}^2 + 2 c_2 r_{n-1} r_{n-2})^{m-i} \, ] \; =$$

$$= \sum_{j=0}^{[\frac{m-i}{2}]} (2c_2)^{2j} \binom{m-i}{2j} \mathrm{E} \, [ \, r_{n-1}^{2(i+j)} r_{n-2}^{2j} ((c_1 + c_2) r_{n-1}^2 + c_2 r_{n-2}^2)^{m-i-2j} ] \; =$$

$$= \sum_{j=0}^{[\frac{m-i}{2}]} (2c_2)^{2j} \binom{m-i}{2j} \sum_{d=0}^{m-i-2j} \binom{m-i-2j}{d} (c_1 + c_2)^d \, c_2^{m-i-2j-d} \, \cdot$$

$$\cdot \, \mathrm{E} \, [ \, r_{n-1}^{2(i+j)} r_{n-2}^{2j} r_{n-1}^{2d} r_{n-2}^{2(m-i-2j-d)} \, ]$$

$$= \sum_{j=0}^{[\frac{m-i}{2}]} \sum_{d=0}^{m-i-2j} \frac{(m-i)!}{(2j)! \, d! \, (m-i-2j-d)!} \, 2^{2j} \, (c_1 + c_2)^d \, c_2^{m-i-d} \, M_{m-(i+j+d)} \, . \qquad (6.7)$$

Therefore,

$$M_i \; = \; a_{m-i} \, \vartheta_{i,m} \; + \; a_{m-i} \sum_{j=0}^{[\frac{m-i}{2}]} \sum_{d=0}^{m-i-2j} \frac{(m-i)!}{(2j)! \, d! \, (m-i-2j-d)!} \, \cdot$$

$$\cdot \, 2^{2j} \, (c_1 + c_2)^d \, c_2^{m-i-d} \, M_{m-(i+j+d)} \, , \;\; i \; = \; 1, \, \dots \, , \; m-1 \, . \qquad (6.8)$$

Equations (6.5) and (6.8) form a system of $m$ equations with $m$ unknowns $(M_0, \; M_1, \; \dots \, , \; M_{m-1})$. Let us call this system $\varrho_m$.

## 6.2 The main theorem: necessary and sufficient moment conditions for HARCH(2)

The following theorem presents a general result on the existence of a stationary version of the HARCH(2) process with a finite $(2m)$th moment.

**Theorem 1** *The HARCH(2) process has a stationary version with a finite (2m)th moment, if and only if, for every* $j = 1, \ldots, m$*, the system* $\varrho_j$ *has a positive solution, with* $\vartheta_{i,j}, j = 2, \ldots, m, i = 0, \ldots, j-1$ *determined by solving* $\varrho_1, \ldots, \varrho_{j-1}, j = 2, \ldots, m$*. If this is the case, then* $\varrho_m$ *has a unique solution given by (6.1).*

**Proof** The proof is by induction on $m$. For $m = 1$, we have already proved this statement. Assume now that this statement is also true for $m \geq 1$, and let us prove it for $m+1$.

Assume first that the the process is stationary, and $(2(m+1))$th moment is finite. Then the $(2m)$th moment is finite as well, and so by the assumption of the induction, $\varrho_1, \ldots, \varrho_m$ all have a unique (positive) solution given by (6.1), and we use these solutions to compute $\vartheta_{i,m+1}$ for $i = 0, 1, \ldots, m$. Since $M_0, M_1, \ldots, M_m$, defined in (6.1), obviously yield a positive solution to $\varrho_{m+1}$, we only need to prove that this solution is unique.

Suppose, first of all, that there is a nonnegative solution $\underline{y}$ to $\varrho_{m+1}$, such that $y_i < M_i$ for at least one $i = 0, \ldots, m$. We now proceed as follows: take a nonstationary HARCH(2) with the same coefficients, beginning with $r_0 = r_1 = 0$. Observe that all the moments of the type $\mathrm{E}(r_n^{2j} r_{n-1}^{2k})$, $0 \leq j+k \leq m$, are for $n = 1$ less or equal to their corresponding stationary values (this is just the positivity of the moments), and

$$\mathrm{E}(r_n^{2(m+1-k)} r_{n-1}^{2k}) \quad \leq \quad y_k, \quad \text{with } k = 0, 1, \ldots, m \tag{6.9}$$

(once again this is just the nonnegativity of $\underline{y}$). Since $\varrho_1, \ldots, \varrho_{k+1}$ are systems of equations with nonnegative coefficients, we conclude that the above is true for all $n \geq 1$.

Since we have assumed that the HARCH(2) process $(r_n)$ has a stationary version, it follows immediately that our Markov chain $(r_{n-1}, r_n)$ has a stationary distribution. Moreover, we have proved that the existence of a finite 2nd moment implies (4.5), which yields that the Markov chain $(r_{n-1}, r_n)$ is positive Harris recurrent. Therefore by Theorem 13.0.1 of (Meyn and Tweedie, 1990) we conclude that $(r_{n-1}, r_n)$ converges weakly to its stationary version. Therefore, so do the products $r_n^{2(m+1-k)} r_{n-1}^{2k}$, $k = 0, 1, \ldots, m$. Hence, by Fatou's lemma, each $M_k$ does not exceed the lowest subsequential limit of $E(r_n^{2(m+1-k)} r_{n-1}^{2k})$, and so $M_k \leq y_k$, $k = 0, 1, \ldots, m$, which contradicts our assumption on $\underline{y}$.

Therefore $\underline{m}$, given by (6.1), is the smallest nonnegative solution of $\varrho_{m+1}$ and, if $\varrho_{m+1}$ has another nonnegative solution $\underline{y}$, we must have $y_i \geq M_i$ for all $i = 0, 1, \ldots, m$, and $y_i > M_i$ for at least one $i$. Thus for any $\alpha > 0$, $\underline{y}_\alpha = (1+\alpha)\underline{m} - \alpha \underline{y}$ is yet another solution to $\varrho_{m+1}$. However, if $\alpha$ is small enough, we have $\underline{y}_\alpha \geq 0$, and some components of $\underline{y}_\alpha$ will be less then those of $\underline{m}$. This contradicts the already established fact that $\underline{m}$ is the smallest nonnegative solution to $\varrho_{m+1}$. Therefore, $\underline{m}$ is the only nonnegative solution of $\varrho_{m+1}$.

Suppose now that $\varrho_{m+1}$ has another, not necessary nonnegative, solution $\underline{y}$. Consider $\underline{y}_\alpha$ above. For all $|\alpha|$ small enough, $\underline{y}_\alpha \geq 0$, and for no $\alpha \neq 0$ it is equal to $\underline{m}$. Therefore, $\underline{m}$ is the *only* solution of $\varrho_{m+1}$.

In the opposite direction, assume that the system $\varrho_j$ has a positive solution for all $j = 1, \ldots, m+1$. By the assumption of the induction, we know that the model has a stationary distribution with finite $(2m)$th moment, and, as above, is positive Harris recurrent. Once again, set $r_0 = r_1 = 0$, and observe that all the moments of the type $\mathrm{E}(r_n^{2i} r_{n-1}^{2k})$, $0 \leq i+k \leq m+1$, do not exceed the corresponding solution of the systems $\varrho_1, \ldots, \varrho_{k+1}$ for $n = 1$, and so, by the nonnegativity of the coefficients of these systems of equations, for each $n \geq 1$. By Fatou's lemma, we conclude that the stationary version of $\mathrm{E}(r_n^{2(m+1)})$ does not exceed the corresponding solution of $\varrho_{m+1}$, and so is finite.

This completes the proof of Theorem 1. □

### 6.3 An explicit necessary and an explicit sufficient condition

We now move to establish some more explicit necessary conditions for the existence of the $(2m)$th moment of the HARCH(2) models. First of all, it follows from (6.5) ($i = j = 0$, $i = m$, $j = 0$) that:

$$M_0 > a_m \, (c_2^m \; + \; (c_1 + c_2)^m) M_0,$$

implying that

$$(c_1 + c_2)^m \; + \; c_2^m \; < \frac{1}{a_m} \tag{6.10}$$

is a necessary condition for the existence of a finite $(2m)$th moment, $m \; = \; 1, \; 2, \; \ldots$ . We can get a stricter necessary condition. It follows from (6.8) with $j = 0$, $d = m - i$ that:

$$M_i \quad > \quad a_{m-i} \, (c_1 + c_2)^{m-i} \, M_0, \qquad i \; = \; 1, \; \ldots \, , \; m - 1 \, . \tag{6.11}$$

We rewrite (6.5) in the form

$$M_0 \quad = \quad a_m \vartheta_{0,m} \; + \; a_m \sum_{d=0}^{m} \; M_d \; \cdot$$

$$\cdot \sum_{j=0}^{\text{Min}[d,(m-d)]} \; 2^{2j} \, (c_1 + c_2)^{m-d-j} \, c_2^{d+j} \, \frac{m!}{(2j)! \, (m-d-j)!(d-j)!} \, , \tag{6.12}$$

and substituting (6.11) into (6.12) we obtain:

$$M_0 \quad > \quad a_m M_0 [(c_1 + c_2)^m + c_2^m] +$$

$$+ \, a_m \sum_{d=1}^{m-1} M_d \sum_{j=0}^{\text{Min}[d,(m-d)]} \; 2^{2j} \, (c_1 + c_2)^{m-d-j} \, c_2^{d+j} \, \frac{m!}{(2j)! \, (m-d-j)!(d-j)!}$$

$$> \, a_m M_0 \, [ \, ((c_1 + c_2)^m + c_2^m) \; + \; \sum_{d=1}^{m-1} a_{m-d} \, (c_1 + c_2)^{m-d} \; \cdot$$

$$\cdot \sum_{j=0}^{\text{Min}[d,(m-d)]} \; 2^{2j} \, (c_1 + c_2)^{m-d-j} \, c_2^{d+j} \, \frac{m!}{(2j)! \, (m-d-j)!(d-j)!} \, ] \, . \tag{6.13}$$

A necessary condition for existence of a finite $(2m)$th moment then is

$$(c_1 + c_2)^m \; + \; c_2^m \; + \; \sum_{d=1}^{m-1} a_{m-d} \; \cdot \sum_{j=0}^{\text{Min}[d,(m-d)]} \; 2^{2j} \, (c_1 + c_2)^{2(m-d)-j} \, c_2^{d+j} \, \frac{m!}{(2j)! \, (m-d-j)!(d-j)!}$$

$$< \frac{1}{a_m}, \quad m = 1, 2, \dots . \tag{6.14}$$

Now we switch to computing a more explicit sufficient condition for the existence of a $(2m)$th moment of the HARCH(2) models. We claim that

$$(c_1 + 4c_2)^m < \frac{1}{a_m} \tag{6.15}$$

is such a sufficient condition. Indeed, define

$$V(x, y) = \alpha x^{2m} + y^{2m}, \quad 0 < \alpha < 1 . \tag{6.16}$$

It is enough to prove that, under (6.15), with a suitable choice of $\alpha$ we have

$$V(x, y) - PV(x, y) \geq \theta(x^{2m} + y^{2m}), \quad \text{with } \theta > 0 \tag{6.17}$$

outside of a compact set, as in subsection 4.2. Observe that

$$V(x, y) - PV(x, y) = (\alpha x^{2m} + y^{2m}) - (\alpha y^{2m} + a_m(c_0 + c_1 y^2 + c_2(x+y)^2)^m) =$$

$$= \alpha x^{2m} + (1-\alpha) y^{2m} - a_m(c_1 y^2 + c_2(x+y)^2)^m - \gamma(x, y) , \tag{6.18}$$

where $\gamma(x, y)$ is a polynomial of a lower order. It is, therefore, enough to prove that there is an $\alpha$ such that for all $x, y$ not both equal to zero,

$$\alpha x^{2m} + (1-\alpha) y^{2m} - a_m(c_1 y^2 + c_2(x+y)^2)^m > \theta(x^{2m} + y^{2m}), \quad \text{with } \theta > 0 . \tag{6.19}$$

By the homogeneity of the terms in the inequality (6.19), it suffices to show that for all $(x, y) \neq (0, 0)$,

$$\alpha x^{2m} + (1-\alpha) y^{2m} > a_m(c_1 y^2 + c_2(x+y)^2)^m . \tag{6.20}$$

We are finally ready to specify $\alpha$. Let

$$\alpha = \frac{1 - a_m^{1/m} c_1}{2} . \tag{6.21}$$

We have, by convexity of the function $f(t) = t^m$,

$$\alpha x^{2m} + (1-\alpha) y^{2m} \geq (\alpha x^2 + (1-\alpha) y^2)^m , \tag{6.22}$$

and so inequality (6.20) will follow once we prove that

$$\alpha x^2 + (1-\alpha) y^2 > a_m^{1/m} (c_1 y^2 + c_2(x+y)^2) . \tag{6.23}$$

We have

$$\alpha x^2 + (1-\alpha) y^2 - a_m^{1/m} (c_1 y^2 + c_2(x+y)^2) =$$

$$= \; x^2 \left(\alpha - a_m^{1/m} c_2\right) \; - \; 2 a_m^{1/m} c_2 \, xy \; + \; \left(1 - \alpha - (c_1 + c_2) a_m^{1/m}\right) y^2 \, . \qquad (6.24)$$

Using (6.15) and (6.21), we can write

$$\alpha \; - \; a_m^{1/m} \, c_2 \; = \; \frac{1 - a_m^{1/m} \, (c_1 + 2c_2)}{2} \; > \; 0 \, . \qquad (6.25)$$

Moreover,

$$(1 - \alpha) \; - \; (c_1 + c_2) \, a_m^{1/m} \; = \; \frac{1 + a_m^{1/m} \, c_1}{2} \; - \; a_m^{1/m} \, (c_1 + c_2) \; =$$

$$= \; \frac{1 - a_m^{1/m} \, (c_1 + 2c_2)}{2} \; = \; \alpha \; - \; a_m^{1/m} \, c_2 \; > \; 0 \, . \qquad (6.26)$$

Finally, by (6.25) and (6.26),

$$(\alpha - a_m^{1/m} c_2) \left((1 - \alpha) - (c_1 + c_2) \, a_m^{1/m}\right) \; - \; a_m^{2/m} \, c_2^2 \; =$$

$$= \; \left(\frac{1 - a_m^{1/m} \, (c_1 + 2c_2)}{2}\right)^2 \; - \; a_m^{2/m} \, c_2^2 \; > \; 0$$

because, by (6.15),

$$\frac{1 - a_m^{1/m} \, (c_1 + 4c_2)}{2} \; > \; 0 \, .$$

This proves the inequality (6.23), and so the condition (6.15) is a sufficient condition for the existence of a $(2m)$th moment.

## 7 Stationarity condition with finite 2nd moment for HARCH(k)

We now discuss the more difficult case: conditions for stationarity and finite moments of the general HARCH(k) process, and we start with the easiest and most explicit part: finiteness of the second moment. The following condition has been shown in (Müller et al., 1995) (relation (3.6)) to be necessary for existence of a finite second moment:

$$\sum_{j=1}^{k} j \; c_j \; < \; \frac{1}{a_1} \, . \qquad (7.1)$$

We will prove now that this condition is also sufficient.

Let $\alpha_1 = 1$. A simple inductive argument establishes that we can choose $\alpha_2, \; ... \; , \; \alpha_k$ in such a way that

$$a_1 (S_i \; + \; ... \; + \; S_k) \; < \; \alpha_i \; < \; \alpha_{i-1} \; - \; a_1 S_{i-1}, \quad i \; = \; 2, \; ... \; , \; k \, , \qquad (7.2)$$

where $S_i = c_i + ... + c_k$ with $i = 1, ..., k$.

Indeed, a condition equivalent to (7.1) is

$$\sum_{j=1}^{k} S_j \ < \ \frac{1}{a_1}, \tag{7.3}$$

and so $a_1(S_2 + ... + S_k) < 1 - a_1 S_1$, implying that there is a number strictly between the two, which we declare to be $\alpha_2$. Assuming that we have chosen $\alpha_1, \ ... \ , \ \alpha_i$, $i < k$, we have by (7.2),

$$a_1(S_{i+1} \ + \ ... \ + \ S_k) \ < \ \alpha_i \ - \ a_1 S_i \ , \tag{7.4}$$

allowing us to choose $\alpha_{i+1}$ in the interior of the above interval. Having chosen $\alpha_1, ..., \alpha_k$ as above, we define a function $V : \mathbb{R}^k \longrightarrow \mathbb{R}$ by

$$V(x_1, ..., x_k) \ = \ \sum_{i=1}^{k} \alpha_i x_i^2 \ + \ 2a_1 \sum_{i=1}^{k} \sum_{j=i+1}^{k} \beta_j x_i x_j, \tag{7.5}$$

where

$$\beta_j \ = \ S_j \ + \ ... \ + \ S_k, \quad j \ = \ 2, \ ... \ , \ k \ . \tag{7.6}$$

We claim that $V(x_1, ..., x_k) \geq 0$. To this end we need to show that the matrix

$$A \ = \ \begin{pmatrix} \alpha_1 & a_1\beta_2 & a_1\beta_3 & ..... & a_1\beta_k \\ a_1\beta_2 & \alpha_2 & a_1\beta_3 & ..... & a_1\beta_k \\ . & & & & \\ . & & & & \\ . & & & & \\ a_1\beta_k & a_1\beta_k & a_1\beta_k & ..... & \alpha_k \end{pmatrix} \tag{7.7}$$

is non-negatively definite. For this, it suffices to note that $A$ is the covariance matrix of the following Gaussian random vector

$$X_i \ = \ a_1^{1/2} \sum_{j=i}^{k} S_j^{1/2} G_j \ + \ (\alpha_i - a_1(S_i + ... + S_k))^{1/2} U_i, \quad i \ = \ 1, \ ... \ , \ k, \tag{7.8}$$

where $G_1, ..., G_k, \ U_1, ..., U_k$ are iid $N(0,1)$ random variables.

Now, we turn to the quantity needed for the different levels described in section 4.2.2:

$$V(r_{n-k+1}, ..., r_n) - PV(r_{n-k+1}, ..., r_n) \ =$$

$$\sum_{i=1}^{k} \alpha_i \, r_{n+1-i}^2 \ + \ 2a_1 \sum_{i=1}^{k} \sum_{j=i+1}^{k} \beta_j \, r_{n+1-i} \, r_{n+1-j} \ -$$

$$-E\left[\sum_{i=2}^{k}\alpha_i\, r_{n+2-i}^2 \;+\; \alpha_1\, \varepsilon_{n+1}^2\,(c_0+\sum_{i=1}^{k}c_i\,(\sum_{j=1}^{i}r_{n+1-j})^2\,)+\right.$$

$$+\,2a_1\sum_{i=2}^{k}\sum_{j=i+1}^{k}\beta_j\, r_{n+2-i}\, r_{n+2-j}\,+$$

$$\left.+\,2\varepsilon_{n+1}\,(c_0+\sum_{i=1}^{k}c_i\,(\sum_{m=1}^{i}r_{n+1-m})^2\,)^{1/2}\sum_{j=2}^{k}\beta_j r_{n+2-j}\right]. \tag{7.9}$$

Using similar arguments as in the case $k=2$, we can simplify the above as

$$V(r_{n-k+1},...,r_n) - PV(r_{n-k+1},...,r_n) \;=$$

$$\sum_{i=1}^{k}\alpha_i\, r_{n+1-i}^2 \;+\; 2a_1\sum_{i=1}^{k}\sum_{j=i+1}^{k}\beta_j\, r_{n+1-i}\, r_{n+1-j}\;-$$

$$-\sum_{i=2}^{k}\alpha_i\, r_{n+2-i}^2 \;-\; a_1(c_0\;-\;\sum_{i=1}^{k}c_i\,(\sum_{j=1}^{i}r_{n+1-j})^2)\;-$$

$$-\,2a_1\sum_{i=2}^{k}\sum_{j=i+1}^{k}\beta_j\, r_{n+2-i}\, r_{n+2-j}\,. \tag{7.10}$$

Setting $\alpha_{k+1}=\beta_{k+1}=0$, we can rewrite the expression in terms of the $S_i$ and we get

$$V(r_{n-k+1},...,r_n) - PV(r_{n-k+1},...,r_n) \;=$$

$$\sum_{i=1}^{k}(\alpha_i-\alpha_{i+1}-a_1S_i)\, r_{n+1-i}^2 \;-\; 2a_1\sum_{i=1}^{k}\sum_{j=i+1}^{k}(\beta_j-\beta_{j+1}-S_j)\, r_{n+1-i}\, r_{n+1-j}\;-\;c_0\,. \tag{7.11}$$

By (7.6), we conclude that

$$V(r_{n-k+1},...,r_n) - PV(r_{n-k+1},...,r_n) \;=\; \sum_{i=1}^{k}(\alpha_i\,-\,\alpha_{i+1}\,-\,a_1S_i)\, r_{n+1-i}^2\;-\;c_0\,. \tag{7.12}$$

Let

$$\vartheta_i \;=\; \alpha_i\;-\;\alpha_{i+1}\;-\;a_1S_i,\quad i\;=\;1,\;...\;,\;k\,. \tag{7.13}$$

It follows from (7.2) that $\vartheta_i>0$, $i=1,...,k$, and so

$$\Theta \;=\; \min_{i=1,..,k}\;\vartheta_i>0\,. \tag{7.14}$$

Letting

$$f(x_1, ..., x_k) \;=\; x_1^2 \;+\; ... \;+\; x_k^2 \tag{7.15}$$

and denoting by $B_a$ ($a > 0$) the sphere centered at 0 with radius $a$, we conclude from (7.12) that

$$V(r_{n-k+1}, ..., r_n) - PV(r_{n-k+1}, ..., r_n) \;\geq\; \Theta f(x_1, ..., x_k)/2 \;-\; c_0 1_{B(2c_0/\Theta)}(x_1,\; ...\;,\; x_k) \tag{7.16}$$

for all $(x_1,\; ...\;,\; x_k)$. Therefore, all levels of Section 4.2 are satisfied for this Markov chain. That is, the stationary HARCH(k) process has, under the condition (7.1), finite second moments.

# 8 Existence of the 4th moment of a general HARCH(k)

## 8.1 The system of equations

Assume first, that the 2nd moments are finite. That is, we assume that the condition (7.1) holds and denote,

$$\begin{aligned}
L &= \mathrm{E}(r_n^2), \\
M_0 &= \mathrm{E}(r_n^4), \\
M_i &= \mathrm{E}(r_n^2 r_{n-i}^2), \qquad i = 1,\; ...\;, k-1\,, \\
n_{ij} &= \mathrm{E}(r_n^2 r_{n-i} r_{n-j}), \quad i = 1,\; ...\;, k-2, \quad j = i+1,\; ...\;, k-1\,.
\end{aligned}$$

We have:

$$n_{ij} \;=\; \mathrm{E}(r_n^2 r_{n-i} r_{n-j}) \;=\; \mathrm{E}(\varepsilon_n^2)\; \mathrm{E}(\sigma_n^2 r_{n-i} r_{n-j})\,.$$

Recalling that $\mathrm{E}(\varepsilon_n^2) = a_1$, we can write the expression as:

$$\begin{aligned}
n_{ij} \;&=\; a_1\, \mathrm{E}(\sigma_n^2 r_{n-i} r_{n-j}) \;=\; a_1\, \mathrm{E}\,[\, r_{n-i} r_{n-j}\; (c_0 + \sum_{l=1}^{k} c_l\; (\sum_{d=1}^{l} r_{n-d})^2\,] \\
&=\; a_1 \sum_{l=1}^{k} c_l\, \mathrm{E}\,[\, r_{n-i} r_{n-j} (\sum_{d=1}^{l} r_{n-d})^2\,]\,.
\end{aligned} \tag{8.1}$$

We evaluate the expectation under the sum in (8.1) by examining different cases for the index $l$.

**<u>Case 1.</u>** $\;1 \;\leq\; l \;<\; i$.

$$\mathrm{E}\,[\, r_{n-i} r_{n-j} (\sum_{d=1}^{l} r_{n-d})^2\,] \;=\; \mathrm{E}\,[\, r_{n-i} r_{n-j} \sum_{d=1}^{l} r_{n-d}^2\,] \;+\; 2 \sum_{d_1=1}^{l} \sum_{d_2=d_1+1}^{l} \mathrm{E}\,[\, r_{n-i} r_{n-j} r_{n-d_1} r_{n-d_2}\,]$$

$$= \sum_{d=1}^{l} \mathrm{E}\,[\, r_{n-i} r_{n-j} r_{n-d}^2 \,] \;=\; \sum_{d=1}^{l} n_{i-d,j-d}\;, \tag{8.2}$$

since the expectation of a product of only odd powers of $r_n$ is zero.

**Case 2.** $\quad i \;\leq\; l \;<\; j.$
Here we have

$$\mathrm{E}\,[\, r_{n-i} r_{n-j} (\sum_{d=1}^{l} r_{n-d})^2 \,] \;=\; \sum_{d=1}^{i-1} \mathrm{E}\,[\, r_{n-d}^2 r_{n-i} r_{n-j} \,] \;+\; 2 \sum_{d_2=i+1}^{l} \mathrm{E}\,[\, r_{n-i}^2 r_{n-j} r_{n-d_2} \,]$$

$$= \sum_{d=1}^{i-1} n_{i-d,j-d} \;+\; 2 \sum_{d=i+1}^{l} n_{d-i,j-i}\;. \tag{8.3}$$

**Case 3.** $\quad j \;\leq\; l \;\leq\; k.$
Similarly to the above we have

$$\mathrm{E}\,[\, r_{n-i} r_{n-j} (\sum_{d=1}^{l} r_{n-d})^2 \,] \;=\; \sum_{d=1}^{i-1} \mathrm{E}\,[\, r_{n-d}^2 r_{n-i} r_{n-j} \,] \;+\; 2 \sum_{d=i+1}^{l} \mathrm{E}\,[\, r_{n-i}^2 r_{n-j} r_{n-d} \,] \;=$$

$$= \sum_{d=1}^{i-1} n_{i-d,j-d} \;+\; 2 \sum_{d=i+1}^{j-1} n_{d-i,j-i} \;+\; 2M_{j-i} \;+\; 2 \sum_{d=j+1}^{l} n_{j-i,d-i}\;. \tag{8.4}$$

Putting together all three cases and using (8.1), we obtain an equation for the $n_{ij}$'s:

$$\frac{1}{a_1}\, n_{ij} \;=\; \sum_{l=1}^{i-1} c_l \sum_{d=1}^{l} n_{i-d,j-d} \;+\; \sum_{l=i}^{j-1} c_l \left( \sum_{d=1}^{i-1} n_{i-d,j-d} + 2 \sum_{d=i+1}^{l} n_{d-i,j-i} \right) \;+$$

$$+ \sum_{l=j}^{k} c_l \left( \sum_{d=1}^{i-1} n_{i-d,j-d} + 2 \sum_{d=i+1}^{j-1} n_{d-i,j-i} + 2M_{j-i} + 2 \sum_{d=j+1}^{l} n_{j-i,d-i} \right)\;, \tag{8.5}$$

with $1 \leq i < j \leq k-1$.

We now turn to finding an equation for $M_0$:

$$M_0 \;=\; \mathrm{E}(r_n^4) \;=\; a_2\, \mathrm{E}(\sigma_n^4) \;=\; a_2\, \mathrm{E}\,[\, (c_0 + \sum_{l=1}^{k} c_l\, (\sum_{d=1}^{l} r_{n-d})^2)^2 \,]$$

$$= a_2 \left[\, c_0^2 \;+\; 2c_0 \sum_{l=1}^{k} c_l\, \mathrm{E}\,[\, (\sum_{d=1}^{l} r_{n-d})^2 \,] \;+\; \mathrm{E}\,[\, (\sum_{l=1}^{k} c_l\, (\sum_{d=1}^{l} r_{n-d})^2)^2 \,] \,\right]$$

$$= \; a_2 \left[ c_0^2 \; + \; 2c_0 \sum_{l=1}^{k} c_l \, l \, L \; + \; \mathrm{E} \, [ \, (\sum_{l=1}^{k} c_l \sum_{d=1}^{l} r_{n-d}^2 + 2 \sum_{l=1}^{k} c_l \sum_{d_1=1}^{l} \sum_{d_2=d_1+1}^{l} r_{n-d_1} r_{n-d_2})^2 \, ] \right] .$$

We continue to develop the right hand side of the equation by introducing more partial sums,

$$\frac{M_0}{a_2} \; = \; c_0^2 \; + \; 2c_0 \, L \sum_{l=1}^{k} l \, c_l \; + \; \mathrm{E} \, [ \, (\sum_{l=1}^{k} c_l \sum_{d=1}^{l} r_{n-d}^2)^2 \, ] +$$

$$+ \, 4 \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} c_{l_1} \, c_{l_2} \sum_{d_3=1}^{l_1} \sum_{d_1=1}^{l_2} \sum_{d_2=d_1+1}^{l_2} \mathrm{E}(r_{n-d_3}^2 r_{n-d_1} r_{n-d_2}) \; +$$

$$+ \, 4 \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} c_{l_1} \, c_{l_2} \sum_{d_1=1}^{l_1} \sum_{d_2=d_1+1}^{l_1} \sum_{d_3=1}^{l_2} \sum_{d_4=d_3+1}^{l_2} \mathrm{E}(r_{n-d_1} r_{n-d_2} r_{n-d_3} r_{n-d_4})$$

$$= \; c_0^2 \; + \; 2c_0 \, L \sum_{l=1}^{k} l \, c_l \; + \; \mathrm{E} \, [ \, (\sum_{d=1}^{k} r_{n-d}^2 \sum_{l=d}^{k} c_l)^2 \, ] +$$

$$+ \, 4 \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} c_{l_1} \, c_{l_2} \sum_{d_1=1}^{l_1} \sum_{d_2=d_1+1}^{l_2} \sum_{d_3=d_2+1}^{l_2} n_{d_2-d_1, d_3-d_1} \; +$$

$$+ \, 4 \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} c_{l_1} \, c_{l_2} \sum_{d_1=1}^{\mathrm{Min}[l_1,l_2]} \sum_{d_2=d_1+1}^{l_1} \sum_{d_3=d_1+1}^{l_2} \mathrm{E}(r_{n-d_1}^2 r_{n-d_2} r_{n-d_3}) \, .$$

Using now the definitions we set at the beginning of this section, we get

$$\frac{M_0}{a_2} \; = \; c_0^2 \; + \; 2c_0 \, L \sum_{l=1}^{k} l \, c_l \; + \; M_0 \sum_{d=1}^{k} (\sum_{l=d}^{k} c_l)^2 \; +$$

$$+ \, 2 \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} c_{l_1} \, c_{l_2} \sum_{d_1=1}^{l_1} \sum_{d_2=d_1+1}^{l_2} M_{d_2-d_1} \; +$$

$$+ \, 4 \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} c_{l_1} \, c_{l_2} \sum_{d_1=1}^{l_1} \sum_{d_2=d_1+1}^{l_2} \sum_{d_3=d_2+1}^{l_2} n_{d_2-d_1, d_3-d_1} \; +$$

$$+ \, 4 \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} c_{l_1} \, c_{l_2} \sum_{d_1=1}^{\mathrm{Min}[l_1,l_2]} \sum_{d_2=d_1+1}^{l_1} \sum_{d_3=d_2+1}^{l_2} n_{d_2-d_1, d_3-d_1} \; +$$

$$+ \, 4 \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} c_{l_1} \, c_{l_2} \sum_{d_1=1}^{\mathrm{Min}[l_1,l_2]} \sum_{d_2=d_1+1}^{l_2} \sum_{d_3=d_2+1}^{l_1} n_{d_2-d_1, d_3-d_1} \, . \qquad (8.6)$$

This is an equation for the variable $M_0$.

Finally, we study the equations for the variables $M_i$'s with the index $i = 1, \ldots, k-1$:

$$M_i = \mathrm{E}(r_n^2 r_{n-i}^2) = \mathrm{E}(\sigma_n^2 r_{n-i}^2)\,\mathrm{E}(\varepsilon_n^2) = a_1\,\mathrm{E}(\sigma_n^2 r_{n-i}^2)$$

$$= a_1\,\mathrm{E}\,[\,r_{n-i}^2\,(c_0 + \sum_{l=1}^{k} c_l\,(\sum_{d=1}^{l} r_{n-d})^2)\,] = a_1 c_0\,L + a_1 \sum_{l=1}^{k} c_l\,\mathrm{E}\,[\,r_{n-i}^2 (\sum_{d=1}^{l} r_{n-d})^2\,]$$

$$= a_1 c_0\,L + a_1 \sum_{l=1}^{i-1} c_l\,\mathrm{E}\,[\,r_{n-i}^2 \sum_{d=1}^{l} r_{n-d}^2\,] + a_1 \sum_{l=i}^{k} c_l\,\mathrm{E}\,[\,r_{n-i}^2 \sum_{d=1}^{l} r_{n-d}^2\,] +$$

$$+ 2a_1 \sum_{l=i}^{k} c_l\,\mathrm{E}\,[\,r_{n-i}^2 \sum_{d_1=1}^{l} r_{n-d_1} \sum_{d_2=d_1+1}^{l} r_{n-d_2}\,]\,.$$

Using again the definitions at the beginning of the section, we obtain the equations for $M_i$:

$$\frac{M_i}{a_1} = c_0\,L + \sum_{l=1}^{i-1} c_l \sum_{d=1}^{l} M_{i-d} + \sum_{l=i}^{k} c_l \left( M_0 + \sum_{d=1}^{i-1} M_{i-d} + \sum_{d=i+1}^{l} M_{d-i} \right) +$$

$$+ 2\sum_{l=i}^{k} c_l \sum_{d_1=i+1}^{l} \sum_{d_2=d_1+1}^{l} n_{d_1-i,d_2-i}\,. \tag{8.7}$$

Overall, equations (8.5), (8.6) and (8.7) give a system of $\frac{k^2-k+2}{2}$ equations with as many unknowns. We also mention that it follows from Theorem 2 below that existence of a positive solution to this system of equations, constitutes, together with (4.5), a necessary and sufficient condition for the existence of a stationary version of the HARCH(k) process with a finite 4th moment. The following two subsections give more explicit conditions.

## 8.2 An explicit necessary condition for the existence of 4th moments of HARCH(k)

We start with showing that, if the 4th moment is finite, then we must have

$$n_{ij} \geq 0, \quad \text{with } i = 1, \ldots, k-2 \text{ and } j = i+1, \ldots, k-1\,. \tag{8.8}$$

To this end, observe that

$$P(\tau_n \geq 0, \tau_{n-1} \geq 0, \ldots, \tau_{n-k+1} \geq 0) \geq 2^{-k} > 0 \tag{8.9}$$

for every $n \geq k$.

Define a function

$$f(x_1, \ldots, x_k) = x_1^4 + x_2^4 + \ldots + x_k^4\,. \tag{8.10}$$

If the 4th moment of HARCH(k) is finite, we conclude from Theorem 14.3.3 of (Meyn and Tweedie, 1990) that the set $\mathcal{S}_f$ of f-regular points has probability 1 (under the steady state), and so by (8.9), it follows that there is an f-regular point $(x_1, \ldots, x_k)$ with $x_j \geq 0$, $j = 1, \ldots, k$. We then set

$$r_n = x_n , \quad n = 1, \ldots, k , \tag{8.11}$$

and we observe that

$$n_{ij}^{(n)} = \mathrm{E}(r_n^2 r_{n-i} r_{n-j}) \geq 0 , \quad i = 1, \cdots k-2, \; j = i+1, \cdots, k-1 \tag{8.12}$$

for $n = k$. We claim that (8.12) holds for all $n \geq k$. The proof is by induction on $n$. We have seen that (8.12) holds for $n = k$. Assume that it holds for all $k \leq l \leq n$ and let us prove it for $n+1$. As in equation 8.5, we have,

$$\frac{n_{ij}^{(n+1)}}{a_1} = \sum_{l=1}^{i-1} c_l \sum_{d=1}^{l} n_{i-d,j-d}^{(k_1(i,j,d))} + \sum_{l=i}^{j-1} c_l \left( \sum_{d=1}^{i-1} n_{i-d,j-d}^{(k_2(i,j,d))} + 2 \sum_{d=i+1}^{l} n_{d-i,j-i}^{(k_3(i,j,d))} \right) +$$

$$+ \sum_{l=j}^{k} c_l \left( \sum_{d=1}^{j-1} n_{i-d,j-d}^{(k_4(i,j,d))} + 2 \sum_{d=i+1}^{j-1} n_{d-i,j-i}^{(k_5(i,j,d))} + 2\mathrm{E}(r_{n+1-i}^2 r_{n+1-j}^2) + 2 \sum_{d=j+1}^{l} n_{i-d,j-d}^{(k_5(i,j,d))} \right) , \tag{8.13}$$

where

$$k_p(i,j,d) \leq n, \quad p = 1, \ldots, 5 . \tag{8.14}$$

Therefore, by the assumption of the induction, $n_{ij}^{(n+1)} \geq 0$, and (8.12) has been proved. Let

$$g(x_1, \ldots, x_k) = x_k^2 \, x_{k-i} \, x_{k-j}, \quad i = 1, \ldots, k-2, \; j = i+1, \ldots, k+1 . \tag{8.15}$$

Observe that

$$|g(x_1, \ldots, x_k)| \leq \max_{j \leq k} |x_j|^4 \leq f(x_1, \ldots, x_k) .$$

Therefore, by Theorem 14.3.3 of (Meyn and Tweedie, 1990), $\mathrm{E}(r_n^2 r_{n-i} r_{n-j})$ converges to its stationary version, and so it follows by (8.12) that this stationary expectation is nonnegative. This proves (8.8).

The discussion above allows us to get explicit necessary conditions for the existence of the 4th moment of the HARCH(k) models. First of all, it follows from (8.6) that

$$M_0 \geq a_2 [\, c_0^2 + 2c_0 \, M \sum_{l=1}^{k} l \, c_l + M_0 \sum_{d=1}^{k} (\sum_{l=d}^{k} c_l)^2 + 2 \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} c_{l_1} \, c_{l_2} \sum_{d_1=1}^{l_1} \sum_{d_2=d_1+1}^{l_2} M_{d_2-d_1} \,]$$

and so

$$M_0 \geq a_2 [\, c_0^2 + 2c_0 \, M \sum_{l=1}^{k} l \, c_l + M_0 \sum_{d=1}^{k} (\sum_{l=d}^{k} c_l)^2 + a_1 \sum_{l_1=1}^{k} \sum_{l_2=1}^{k} c_{l_1} \, c_{l_2} \sum_{j=1}^{l_2-1} \mathrm{Min}[l_1, (l_2 - j)] \, M_j \,]$$

$$\geq \; a_2 \, [\, c_0^2 \; + \; 2c_0 \, M \sum_{l=1}^{k} l \, c_l \; + \; M_0 \sum_{d=1}^{k} (\sum_{l=d}^{k} c_l)^2 \; + \; 2 \sum_{j=1}^{k-1} M_j \sum_{l=1}^{k-j} l(2k - 2l + 1 - j) \,] \, . \tag{8.16}$$

Moreover, it follows from (8.7) that

$$\frac{M_i}{a_1} \; \geq \; c_0 \, M \; + \sum_{l=1}^{i-1} c_l \, \sum_{d=1}^{l} M_{i-d} + $$

$$+ \sum_{l=i}^{k} c_l \, \left( \sum_{d=1}^{i-1} M_{i-d} + M_0 + \sum_{d=i+1}^{l} M_{d-i} \right) \, , \quad i \; = \; 1, \, \ldots \, , k - 1 \, . \tag{8.17}$$

We immediately see from (8.16) that

$$M_0 \; \geq \; a_2 \, [\, c_0^2 \; + \; 2c_0 \, M \sum_{l=1}^{k} l \, c_l \; + \; M_0 \sum_{d=1}^{k} (\sum_{l=d}^{k} c_l)^2 \,] \, ,$$

and so

$$\sum_{d=1}^{k} (\sum_{l=d}^{k} c_l)^2 \; < \; \frac{1}{a_2} \tag{8.18}$$

is a necessary condition for the existence of a finite 4th moment; but it is, clearly, insufficient. We can get a stricter necessary condition as follows. From (8.17), we know that for every $i \; = \; 1, \, \ldots \, , k - 1$,

$$M_i \; \geq \; M_0 \, a_1 \, \sum_{l=i}^{k} c_l \, . \tag{8.19}$$

Substituting (8.19) into (8.16), we obtain

$$M_0 \; \geq \; a_2 \, [\, c_0^2 \; + \; 2c_0 \, M \sum_{l=1}^{k} l \, c_l \; + \; M_0 \sum_{d=1}^{k} (\sum_{l=d}^{k} c_l)^2 \; + 2a_1 M_0 \sum_{j=1}^{k-1} \sum_{l=j}^{k} c_j \sum_{l=1}^{k-j} l(2k - 2l + 1 - j) \,] \, , \tag{8.20}$$

and so a necessary condition for the existence of a finite 4th moment is

$$\sum_{d=1}^{k} (\sum_{l=d}^{k} c_l)^2 \; + \; 2a_1 \sum_{j=1}^{k-1} \sum_{l=j}^{k} c_j \sum_{l=1}^{k-j} l(2k - 2l + 1 - j) \;\; < \;\; \frac{1}{a_2} \, . \tag{8.21}$$

## 8.3 A sufficient condition for the existence of the 4th moment of HARCH(k)

We now move to derive some explicit sufficient conditions for the existence of a finite 4th moment of a HARCH(k) model. Specifically, we claim that if

$$\left( \sum_{j=1}^{k} j^2 \, c_j \right)^2 \; < \; \frac{1}{a_2} \, , \tag{8.22}$$

then a finite 4th moment exists.

Indeed, if (8.22) holds, one can choose positive numbers $\alpha_1, \ldots, \alpha_{k-1}$ such that

$$\alpha_{k-1} \;>\; a_2(kc_k) \sum_{j=1}^{k} j^2\, c_j\,, \tag{8.23}$$

$$\alpha_{i-1} - \alpha_i \;>\; a_2 \left(\sum_{l=i}^{k} l\, c_l\right) \sum_{j=1}^{k} j^2\, c_j\,, \quad i \;=\; 2, \ldots, k-1\,, \tag{8.24}$$

$$1 - \alpha_1 \;>\; a_2 \left(\sum_{l=1}^{k} l\, c_l\right) \sum_{j=1}^{k} j^2\, c_j\,. \tag{8.25}$$

Using these numbers define a function $g$

$$g(x_0, x_1, \ldots, x_{k-1}) \;=\; \sum_{j=0}^{k-1} \alpha_j\, x_j^4\,, \quad \alpha_0 \;=\; 1\,. \tag{8.26}$$

As before, we need to show that the difference $g - Pg$ can be made as large as we wish, away from a compact set. We have:

$$g(r_n, r_{n-1}, \ldots, r_{n-k+1}) - Pg(r_n, r_{n-1}, \ldots, r_{n-k+1}) \;=$$

$$= \sum_{j=0}^{k-1} \alpha_j\, r_{n-j}^4 - \sum_{j=1}^{k-1} \alpha_j\, r_{n-j+1}^4 - a_2\,[\,c_0 + \sum_{l=1}^{k} c_l\, (\sum_{d=1}^{l} r_{n-d+1})^2\,]^2$$

$$= \sum_{j=0}^{k-1} \alpha_j\, r_{n-j}^4 - \sum_{j=1}^{k-1} \alpha_j\, r_{n-j+1}^4 - a_2\,[\sum_{l=1}^{k} c_l\, (\sum_{d=1}^{l} r_{n-d+1})^2\,]^2 - \Theta\,, \tag{8.27}$$

with $\Theta$ being a polynomial of order 2. Thus, it is enough to prove that, away from the origin,

$$\sum_{j=0}^{k-1} \alpha_j\, r_{n-j}^4 - \sum_{j=1}^{k-1} \alpha_j\, r_{n-j+1}^4 - a_2 \left(\sum_{l=1}^{k} c_l (\sum_{d=1}^{l} r_{n-d+1})^2\right)^2 \;>\; 0\,. \tag{8.28}$$

With the help of (8.23)-(8.25), we estimate the left hand side of (8.28) as

$$\geq \sum_{j=0}^{k-1} \alpha_j\, r_{n-j}^4 - \sum_{j=1}^{k-1} \alpha_j\, r_{n-j+1}^4 - a_2 \left(\sum_{l=1}^{k} l\, c_l \sum_{d=1}^{l} r_{n-d+1}^2\right)^2$$

$$= \sum_{j=0}^{k-1} \alpha_j\, r_{n-j}^4 - \sum_{j=1}^{k-1} \alpha_j\, r_{n-j+1}^4 - a_2 \left(\sum_{d=1}^{k} r_{n-d+1}^2 \sum_{l=d}^{k} l\, c_l\right)^2$$

$$\geq \sum_{j=0}^{k-1} \alpha_j \, r_{n-j}^4 \; - \; \sum_{j=1}^{k-1} \alpha_j \, r_{n-j+1}^4 \; - \; a_2 \, (\sum_{d=1}^{k} \sum_{l=d}^{k} l \, c_l \,) \sum_{d=1}^{k} r_{n-d+1}^4 \, \sum_{l=d}^{k} l \, c_l$$

$$= \sum_{j=0}^{k-1} \alpha_j \, r_{n-j}^4 \; - \; \sum_{j=1}^{k-1} \alpha_j \, r_{n-j+1}^4 \; - \; a_2 \, (\sum_{i=1}^{k} i^2 \, c_i \,) \, \sum_{j=0}^{k-1} r_{n-j}^4 \, \sum_{l=j+1}^{k} \, l \, c_l$$

$$= \sum_{j=0}^{k-2} r_{n-j}^4 \, \left( \alpha_j - \alpha_{j+1} - a_2 \, (\sum_{i=1}^{k} i^2 \, c_i \,) \, \sum_{l=j+1}^{k} \, l \, c_l \right) \, +$$

$$+ \; r_{n-k+1}^4 \, \left( \alpha_{k-1} - a_2 \, (k c_k) \sum_{i=1}^{k} i^2 \, c_i \right) \; > \; 0 \, .$$

This proves (8.28), and so (8.22) is a sufficient condition for the existence of a finite 4th moment for the HARCH(k) models.

## 9 General moments of HARCH(k)

Let $m \geq 2$. As always, if the (2m)th moments are finite, there is a system of equations these moments must satisfy. Define, under the assumption of stationarity,

$$M_0 \; = \; \mathrm{E}(r_n^{2m}) \, , \tag{9.1}$$

$$n_{i_0, i_1, \ldots, i_{k-1}} \; = \; \mathrm{E}( \, r_n^{2 i_0} \, r_{n-1}^{i_1} \, \ldots \, r_{n-k+1}^{i_{k-1}} \,) \, , \tag{9.2}$$

$$1 \; \leq \; i_0 \; \leq \; m-1 \, , \quad 0 \; \leq \; i_j \; \leq \; 2m-2 \, , \quad j \; = \; 1, \, \ldots \, , k-1 \, ,$$

$$2 i_0 \; + \; i_1 \; + \; \ldots \; + \; i_{k-1} \; = \; 2m \, .$$

If the process has a finite (2m)th moment, then we have

$$M_0 \; = \; \mathrm{E}(r_n^{2m}) \; = \; \mathrm{E}(\varepsilon_n^{2m}) \, \mathrm{E}(\sigma_n^{2m}) \; = \; a_m \, \mathrm{E}(\sigma_n^{2m})$$

$$= \; a_m \, \mathrm{E} \, [ \, ( \, c_0 \; + \; \sum_{j=1}^{k} c_j \, (\sum_{d=1}^{j} r_{n-d})^2 \, )^m \, ] a_m \, \vartheta_{0,m} \; + \; a_m \, \mathrm{E} \, [ \sum_{j=1}^{k} c_j \, (\sum_{d=1}^{j} r_{n-d})^2 \, )^m \, ] \, , \tag{9.3}$$

where $\vartheta_{0,m}$ involves only moments of orders less than $2m$. We further have

$$\mathrm{E} \, [ \, ( \sum_{j=1}^{k} c_j \, (\sum_{d=1}^{j} r_{n-d})^2 \, )^m \, ] \; = \; \sum_{p_1, \, \cdots \, , p_k}^{m} \frac{m!}{p_1! \, \ldots \, p_k!} \, \prod_{i=1}^{k} c_i^{p_i} \, \mathrm{E} \, [ \, \prod_{j=1}^{k} (\sum_{d=1}^{j} r_{n-d} \,)^{2 p_j} \, ] \, . \tag{9.4}$$

with the following relations

$$0 \leq p_i \leq m , \quad \text{and} \sum_{i=1}^{k} p_i = m , \quad i = 1, \dots , k . \tag{9.5}$$

Concentrating on the expectation in (9.4), we further get

$$\mathrm{E}\,[\prod_{j=1}^{k} (\sum_{d=1}^{j} r_{n-d} )^{2p_j} ] =$$

$$= \sum_{l \in \mathcal{L}(p_1, \dots ,p_k)} \prod_{i=0}^{k-1} \binom{2(p_{k-i} + \dots + p_k) - (l_{k-i+1} + \dots + l_k)}{l_{k-i}} \cdot \mathrm{E}\,[\prod_{j=1}^{k} r_{n-j}^{l_j} ] , \tag{9.6}$$

where for a vector $\underline{p} = (p_1, \dots, p_k)$,

$$\mathcal{L}(\underline{p}) = \mathcal{L}(p_1, \dots, p_k)$$

$$= \{ \underline{l} = (l_1, \dots, l_k) \mid 0 \leq l_k \leq 2p_k, \;\; 0 \leq l_{k-1} + l_k \leq 2(p_{k-1} + p_k), \; \dots ,$$

$$0 \leq l_2 + \dots + l_k \leq 2(p_2 + \dots + p_k), l_1 + l_2 + \dots + l_k = 2(p_1 + p_2 + \dots + p_k) \} \tag{9.7}$$

Define a function $b \,:\, \mathbb{R}^k \longrightarrow \{1, \dots, k\}$ by

$$b(x_1, \dots, x_k) = \min\{j, x_j \neq 0\} . \tag{9.8}$$

Observe that for every $\underline{l} \in \mathcal{L}(\underline{p})$ we have

$$b(\underline{l}) \leq b(\underline{p}) . \tag{9.9}$$

Indeed, if $b(\underline{l}) > j$, then $l_1 = l_2 = \dots = l_j = 0$, and so we have

$$2(p_1 + \dots + p_k) = l_1 + \dots + l_k = l_{j+1} + \dots + l_k \leq 2(p_{j+1} + \dots + p_k) ,$$

implying that $p_1 = p_2 = \dots = p_j = 0$, and so $b(\underline{p}) > j$. It is clear that for every $\underline{l} \in \mathcal{L}(\underline{p})$ such that $l_{b(\underline{l})}$ is odd, we have

$$\mathrm{E}\,[\prod_{j=1}^{k} r_{n-j}^{l_j} ] = 0 .$$

Therefore, by (9.9), we have

$$\mathrm{E}\,[\prod_{j=1}^{k} (\sum_{d=1}^{j} r_{n-d} )^{2p_j} ] = \sum_{i=1}^{b(\underline{p})}$$

$$\left[ \sum_{\underline{l} \in \mathcal{L}(\underline{p})} \prod_{j=0}^{k-1} \binom{2(p_{k-j} + \ \dots \ + p_k) - (l_{k-j+1} + \ \dots \ + l_k)}{l_{k-j}} \cdot n_{l_1/2, l_2, \dots, l_k} + i\, M_0 \right] , \tag{9.10}$$

with the following condition on the indices of the second summation

$$b(\underline{l}) \ = \ i \ , \quad l_i \text{ even}, \quad l_i \neq 2m \ . \tag{9.11}$$

Substituting (9.10) into (9.4), we obtain

$$\mathrm{E}\,[\,(\sum_{j=1}^{k} c_j \,(\sum_{d=1}^{j} r_{n-d})^2)^m\,] \ = \ \sum_{i=1}^{k} \sum_{p_i, \dots, p_k} \frac{l!}{p_i! \dots p_k!} \prod_{j=i}^{k} c_j^{p_j} \cdot$$

$$\cdot \sum_{d=1}^{i} \left[ \sum_{\underline{l} \in \mathcal{L}(\underline{p})} \prod_{j=0}^{k-d} \binom{2(p_{k-j} + \ \dots \ + p_k) - (l_{k-j+1} + \ \dots \ + l_k)}{l_{k-j}} \cdot n_{l_d/2, l_{d+1}, l_k, 0, \dots, 0} + d\, M_0 \right] ,$$

where the indices $\underline{p}$ and $\underline{l}$ follow (9.5) and(9.11) respectively, with $p_1 = \dots = p_{i-1} = 0$. That is, we have:

$$\mathrm{E}\,[\,(\sum_{j=1}^{k} c_j \,(\sum_{d=1}^{j} r_{n-d})^2)^m\,] \ = \ M_0 \sum_{j=1}^{k} (\sum_{i=j}^{k} c_j)^m \ + \sum_{i=1}^{k} \sum_{p_i, \dots, p_k} \frac{l!}{p_i! \dots p_k!} \prod_{j=i}^{k} c_j^{p_j} \cdot$$

$$\cdot \sum_{d=1}^{i} \sum_{\underline{l} \in \mathcal{L}(\underline{p})} n_{l_d/2, l_{d+1}, \dots, l_k, 0, \dots, 0} \prod_{j=0}^{k-d} \binom{2(p_{k-j} + \ \dots \ + p_k) - (l_{k-j+1} + \ \dots \ + l_k)}{l_{k-j}} ,$$

where the subscript of $n$ above is of length $k$. Therefore, we obtain the following equation

$$M_0 \ = \ M_0 \, a_m \sum_{j=1}^{k} (\sum_{i=j}^{k} c_j)^m \ + \ a_m \sum_{i=1}^{k} \sum_{p_i, \dots, p_k} \frac{l!}{p_i! \dots p_k!} \prod_{j=i}^{k} c_j^{p_j} \cdot$$

$$\cdot \sum_{d=1}^{i} \sum_{\underline{l} \in \mathcal{L}(\underline{p})} n_{l_d/2, l_{d+1}, \dots, l_k, 0, \dots, 0} \prod_{j=0}^{k-d} \binom{2(p_{k-j} + \ \dots \ + p_k) - (l_{k-j+1} + \ \dots \ + l_k)}{l_{k-j}} \ + \ a_m \, \vartheta_{0,m} \ . \tag{9.12}$$

In a similar manner, it is possible to obtain a system of equations for $n_{i_0, i_1, \dots, i_{k-1}}$. This gives us a system of

$$\sum_{i=0}^{m-1} \binom{2i + k - 2}{k - 2}$$

linear equations with as many unknowns. We call this system $\varrho_m$.

Combining the ideas we used in analyzing the cases of $k = 2$ and $m = 1$ (Section 6.2), we obtain in the same way, the following theorem.

**Theorem 2** *A HARCH(k) model has a stationary version with a finite* $(2m)$*th moment if and only if, for every* $j = 1, ..., m$*, the system* $\varrho_j$ *has a positive solution. In this case,* $\varrho_k$ *has a unique solution, given by (9.1) and (9.2).*

### 9.1 An explicit necessary condition

Since by Theorem 2, we have:

$$n_{i_0, i_1, ..., i_k} \;\geq\; 0 \quad \text{for all } i_0, i_1, ..., i_k \,,$$

it follows from (9.12) that

$$M_0 \;>\; a_m \; M_0 \sum_{j=1}^{k} (\sum_{i=j}^{k} c_j)^m \,,$$

and so

$$\sum_{j=1}^{k} (\sum_{i=j}^{k} c_j)^m \;<\; \frac{1}{a_m} \tag{9.13}$$

is a necessary condition for the existence of a finite $(2m)$th moment.

### 9.2 An explicit sufficient condition

We claim that if

$$\left( \sum_{j=1}^{k} j^2 \, c_j \right)^m \;<\; \frac{1}{a_m} \,, \tag{9.14}$$

then the HARCH(k) model has a finite $(2m)$th moment.

The proof proceeds exactly like in the case $m = 2$ (Section 8.3). We take

$$g(x_0, x_1, ..., x_{k-1}) \;=\; \sum_{j=0}^{k-1} \alpha_j \, x_j^{2m}, \quad \alpha_0 \;=\; 1 \;, \tag{9.15}$$

and note that under (9.14) we may choose $\alpha_1, \; ... \,, \alpha_{k-1}$ in such a way that

$$\alpha_{k-1} \;>\; a_m (k c_k) \left( \sum_{j=1}^{k} j^2 \, c_j \right)^{m-1} \,, \tag{9.16}$$

$$\alpha_{i-1} - \alpha_i > a_m \left(\sum_{l=i}^{k} l\, c_l\right) \left(\sum_{j=1}^{k} j^2\, c_j\right)^{m-1}, \quad i = 2, \ldots, k-1, \tag{9.17}$$

$$1 - \alpha_1 > a_m \left(\sum_{l=1}^{k} l\, c_l\right) \left(\sum_{j=1}^{k} j^2\, c_j\right)^{m-1}. \tag{9.18}$$

Then proceed exactly as in Section 8.3.

## 10 Summary

We give in this paper a necessary and sufficient condition for the existence of moments of a general stationary HARCH(k) process using a Markov chain approach. Our interest in the tail behavior of this process leads us to study the convergence of the higher order moments of HARCH processes. We give a theorem that governs the stationarity with finite moments and we prove that the condition is both necessary and sufficient. The condition is in terms of the existence of non-negative solution of a system of linear equations. Unfortunately, the general condition we give is not explicit. For every HARCH(k) process one has explicit necessary and sufficient conditions for finiteness of the second moment. In the case of the HARCH(2) process, we can give the explicit necessary and sufficient condition for the existence of the 4th moment. In all other cases, we give explicit necessary conditions and explicit sufficient conditions which will allow one to study the tail behavior of HARCH processes.

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