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RANDOM USC FUNCTIONS, MAX-STABLE PROCESSES AND CONTINUOUS CHOICE

by

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Abstract

The theory of Random Utility maximization for a finite set of alternatives is generalized to alternatives which are elements of a compact separable metric space T. We model the random utility of these alternatives ranging over a continuum as a random process $\{Y_t,\, t \in T\}$ with upper semi–continuous (usc) sample paths. The alternatives which achieve the maximum utility levels constitute a *random closed, compact set* M. We specialize to a model where the random utility is a max–stable process with a.s. usc paths. Further path properties of these processes are derived and explicit formulae are calculated for the hitting and containment functionals of M. The hitting functional corresponds to the choice probabilities.

1 Introduction

We describe a general approach to the modeling of probabilistic choice from a set of alternatives whose cardinality need not be finite. We assume that the set of alternatives T, is a compact separable metric space. In concrete examples, T is usually the unit interval, the unit square or the unit circle. We postulate that the preferences of an individual over the range of alternatives are represented by a real–valued utility function. Individuals are assumed to adhere to utility maximization as the criterion for selecting a particular alternative. The randomness in the utility function is assumed since even if the choice process is deterministic for a particular individual, the analyst is in general not cognizant of its precise specification. (For further discussions on this and on models of choice from *finite* sets of alternatives, see McFadden(1981)). To ensure that the choice problem is well–defined, we assume that the random utility function has upper semicontinuous (usc) realizations implying that the maximum level of utility is achieved by at least one alternative in the space T.

The study of this problem is motivated by choices in sets which are *not* finite. A potential area of application includes the issue of choice of retail store location where the choice set may be modeled as a compact subset of $\mathbf{R}^2$. Lerman(1985) contains a discussion on issues related to continuous choice sets in the context of choice for spatial alternatives. Alternatively, a media manager selecting a time–slot in a media vehicle may be viewed as facing a continuum of alternatives. The brand and quantity purchase decision of a consumer may be modeled as choice from a compact subset of $\mathbf{R}^n$ where the n-vector $\mathbf{x}$ has as its i^{th} component the quantity of brand i which is purchased. Note that here one would be implicitly assuming that the brand and quantity purchase decision are simultaneously executed.

The choice of mode of transportation may be dependent on the time of travel, which can be viewed as alternatives which are elements of a closed interval in $\mathbf{R}_+$. In the area of transportation demand forecasting, Ben–Akiva et.al.(1984) provide a comprehensive exposition for spatial choice models based on the Continuous Logit model, which was derived by taking limits of Logit choice probabilities for finite sets of alternatives as the size of these sets tended to infinity (cf. also Ben–Akiva and Watanada(1981)). McFadden(1976) initially derived the Continuous Logit model by defining an ' independence from irrelevant alternatives ' (IIA) principle for non–finite sets in terms of an absolute continuity condition on the choice probability measures.

Section 2 contains the description of the general model with utility functions as random elements of the space of upper semicontinuous functions on T denoted by $US(T)$. The topological preliminaries concerning $US(T)$ and $\mathcal{F}(T)$, the space of closed subsets of T, are discussed. We then show that the set of alternatives which achieve the maximum utility level, M, is a random element of $\mathcal{F}(T)$ and deduce that the choice probabilities correspond to the hitting functional of M (cf. Matheron(1975), Salinetti and Wets(1986b)). At this stage the model is still very general, and in Section 3 we specialize to a model where the random utility function is a max–stable process (cf. de Haan(1984)) on T with a.s. usc sample paths. The use of max–stable processes in modeling continuous choice was ingeniously proposed by Cosslett(1988) who selected a specific parametrization of a stationary moving–maximum process (cf. Balkema and de Haan(1988)) with a.s. continuous paths to represent the random utility functions. We develop characterizations of max–stable processes with usc and continuous sample paths. Subsequently in section 4, by suitably projecting the underlying Poisson random measure onto subspaces, we compute the functionals of M which correspond to the choice probabilities. Then explicit characterizations are provided for max–stable processes which result in M being a.s. singleton. Finally, by invoking results on measurable selections we provide insight into how one can develop tractable choice models in this framework and present some illustrative examples.

2 Preliminaries on US(T) and $\mathcal{F}$(T).

The space of non–negative upper semicontinuous (usc) functions on T denoted $US(T)$, is a convenient setting for considering utility maximization with a continuous range of alternatives. This is because a usc function on a compact set achieves its maximum. Recall we assume that T is a compact Polish space. The σ–algebra of Borel subsets is denoted by $\mathcal{B}(T)$.

A standard topology for $US(T)$ is the *sup–vague* topology (cf. Vervaat(1988)) which has basis sets of the form

$$\{f \in US(T) \ : \ \bigvee_{t \in K} f(t) < x\}$$

and

$$\{f \in US(T) \ : \ \bigvee_{t \in G} f(t) > x\}$$

where $K \in \mathcal{F}(T)$, the closed subsets of T, and $G \in \mathcal{G}(T)$, the open subsets of T. We denote by $\mathcal{B}(US)$ the usual Borel σ–algebra on $US(T)$, i.e. the σ–algebra generated by open sets. Henceforth we will use the abbreviation :

$$f^{\vee}(B) := \bigvee_{t \in B} f(t), \ B \in \mathcal{B}(T).$$

If $(\Omega, \mathcal{A}, \mathbf{P})$ is a complete probability space, we say that the map

$$\xi \ : \ \Omega \rightarrow \ US(T)$$

is a random usc function if it is a random element of $(US(T), \mathcal{B}(US))$. This means

$$\xi^{-1}(\mathcal{B}(US)) \subseteq \mathcal{A}.$$

To construct a random usc function proceed as follows. Suppose $Y = \{Y_t, t \in T\}$ is a separable stochastic process with values in $[0, \infty)$ and that for almost all $\omega \in \Omega$ we have $Y_t(\omega)$ is an usc function of t. Modify Y so that all paths are separable and in $US(T)$. This change on a ω–set of measure 0 produces a new version which we also call Y. This Y is a random element of $US(T)$ (cf. Salinetti & Wets(1986a), Theorems 6.1, 6.2, and Vervaat(1988), Theorem 7.2). To check this we need to verify

$$Y^{-1}\{f \in US(T) \; : \; f^{\vee}(K) < x\} \;\; \in \;\; \mathcal{A} \tag{1}$$

$$Y^{-1}\{f \in US(T) \; : \; f^{\vee}(G) > x\} \;\; \in \;\; \mathcal{A}. \tag{2}$$

for $x \in \mathbf{R}_+$, $K \in \mathcal{F}(T)$, $G \in \mathcal{G}(T)$. If D is a separant for the separable process Y then (Billingsley(1986) p. 550 ff., Ash & Gardner(1975), Kendall(1974)) (2) above becomes

$$\{\omega : Y^{\vee}(G, \omega) > x\} = \{\omega : Y^{\vee}(G \cap D, \omega) > x\} \in \mathcal{A}$$

since D is countable. For (1) let $G_n \in \mathcal{G}(T), \forall n$, and $G_n \supset K$, $G_n \downarrow K$. By upper semi–continuity of paths

$$Y^{\vee}(G_n) \downarrow Y^{\vee}(K)$$

and so

$$\begin{aligned} \{\omega : Y^{\vee}(K, \omega) < x\} &= \bigcup_n \{\omega : Y^{\vee}(G_n, \omega) < x\} \\ &= \bigcup_n \{\omega : Y^{\vee}(G_n \cap D, \omega) < x\} \in \mathcal{A}\}. \end{aligned}$$

This argument also shows that $Y^{\vee}(K)$, and $Y^{\vee}(G)$ are random variables.

The following fact will be essential for using measurable selection theorems in the last section. If $Y(\omega) = \{Y(t, \omega), t \in T\}$ is a random element of $US(T)$, then Y is measurable, i.e.

$$(t, \omega) \mapsto Y(t, \omega)$$

is measurable

$$\mathcal{B}(T) \times \mathcal{A} \mapsto \mathcal{B}(\mathbf{R}_+)$$

(cf. O'Brien, Torfs, Vervaat(1988), Salinetti & Wets(1986a)).

This is checked as follows : Let $\{\{G_i^{(n)}, i \leq k_n\}, n \geq 1\}$ be a nested sequence of open coverings of T and suppose diam$(G_i^{(n)}) \leq 1/n$. If we define for $t \in T$

$$Y^{(n)}(t) = \bigvee_{i : t \in G_i^{(n)}} Y^{\vee}(G_i^{(n)})$$

then upper semi–continuity implies for $t \in T$

$$Y^{(n)}(t) \downarrow Y(t).$$

To prove Y is measurable it is enough to prove $Y^{(n)}$ measurable. For $x \in \mathbf{R}_+$

$$\{(t, \omega) : Y^{(n)}(t, \omega) > x\} = \bigcup_{1 \leq i \leq k_n} G_i^{(n)} \times \{\omega : Y^{\vee}(G_i^{(n)}, \omega) > x\} \in \mathcal{B}(T) \times \mathcal{A}$$

since $Y^{\vee}(G_i^{(n)})$ is a random variable whenever Y is a random element of $US(T)$.

Recall $\mathcal{F} = \mathcal{F}(T)$ is the class of closed subsets of T. We may give $\mathcal{F}$ a topology by declaring the following collection as sub–basis sets of the topology :

$$\{F \in \mathcal{F} \ : \ F \cap K = \phi\}, \ \{F \in \mathcal{F} \ : \ F \cap G \neq \phi\}$$

for $K \in \mathcal{F}(T)$, $G \in \mathcal{G}(T)$. Since T is compact, Polish, $\mathcal{F}(T)$ coincides with $\mathcal{K}(T)$, the space of compact subsets of T. Then the *hit–miss* topology defined above on $\mathcal{F}(T)$ is the same as the topology generated by the Hausdorff metric on $\mathcal{K}(T)$ (cf. Vervaat(1988)). Let $\mathcal{B}(\mathcal{F})$ be the Borel σ–algebra generated by the open subsets of $\mathcal{F}$. A random element of $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$ is a *random closed set*(RACS) (cf. Matheron(1975), Vervaat(1988)) .

If $f \in US(T)$, then

$$F := \{t \in T \ : \ f(t) = f^{\vee}(T)\} \ \in \ \mathcal{F}$$

For if $t_n \in F$ and $t_n \to t_0$ then since $f(t_n) = f^{\vee}(T)$ we have by upper semicontinuity

$$f^{\vee}(T) = \limsup_{n\to\infty} f(t_n) \le f(t_0) \le f^{\vee}(T)$$

whence $t_0 \in F$, showing F is closed.

If $Y = \{Y_t,\, t \in T\}$ is a random element of $US(T)$, then define

$$M(\omega) = \{t \in T \ : \ Y_t(\omega) = Y^{\vee}(T, \omega)\}.$$

For each $\omega \in \Omega$, $M(\omega)$ is a closed subset of T, and in fact $M : \Omega \to \mathcal{F}$ is a random closed set. To verify this it is enough to show

$$\{\omega \ : \ M(\omega) \in \{F \in \mathcal{F} \ : \ F \cap K \neq \phi\}\} \in \mathcal{A} \tag{3}$$

for $K \in \mathcal{F}(T)$ (cf. Wagner(1977), Vervaat(1988, Theorem 11.9)). The set in (3) is

$$\{\omega \ : \ M(\omega) \cap K \neq \phi\} \ = \ \{\omega \ : \ \exists t \in K \text{ s.t. } Y_t(\omega) \ge Y^{\vee}(T, \omega)\}$$

and since the supremum of Y over K is achieved, the above is

$$\{\omega : Y^{\vee}(K, \omega) \ge Y^{\vee}(T, \omega)\} \in \mathcal{A}$$

since $Y^{\vee}(K)$ and $Y^{\vee}(T)$ are random variables.

We now summarize this discussion. We intend to model a random utility process corresponding to alternatives in a compact Polish space T, by a stochastic process $Y =$

$\{Y_t(\omega),\, t \in T,\, \omega \in \Omega\}$. We want Y to be a random element of $US(T)$ since functions in $US(T)$ achieve their maxima. If Y has all paths separable and in $US(T)$, then Y is a random element of $US(T)$ and enjoys the technical property of measurability. The set of alternatives

$$M(\omega) = \{t \in T \;:\; Y_t(\omega) = Y^{\vee}(T,\omega)\}$$

which provide the economic agent with maximum utility is a random element of $\mathcal{F}(T)$; i.e. a random closed set.

For a utility maximizing agent, the probability of selecting an alternative from a non–empty set $K \in \mathcal{F}(T)$ is specified by the Choquet capacity, or hitting functional of M :

$$\mathbf{P}[M \bigcap K \neq \phi] = \mathbf{P}[\text{ some alternatives in K maximize utility }]$$

As yet we have not yet specified any properties of the random utility process Y_t except that it be a random element of $US(T)$. In the interests of developing tractable continuous choice models, we specify in the next section that $\{Y_t,\, t \in T\}$ is a max–stable process.

3 Max–Stable Random Utility Processes : Specification and Path Properties

In this section we start from a Poisson process and specify a *max–stable* process as a functional of that Poisson process (cf. de Haan(1984)).

Let $(U, \mathcal{U}, \rho)$ be a complete probability space. Recall T is a compact, Polish space. Let $\{\Gamma_k, k \geq 1\}$ be the points of a homogeneous Poisson process with unit intensity so that

$$\Gamma_k = \sum_{i=1}^{k} E_i$$

where E_i is a sequence of iid unit exponentials. Suppose u_k is a sequence of iid U–valued rv's with distribution ρ, independent of Γ_k. Then $\{(u_k, \Gamma_k), k \geq 1\}$ are the points of N, a

Poisson process (PRM) on $U \times [0,\infty)$, with intensity measure $\mu(du,dx) = 1_U(u)\rho(du) \times 1_{[0,\infty)}(x)(dx)$ (Proposition 3.8, Resnick(1987)). Let $\{f_t, t \in T\}$ be a class of non–negative functions with domain U which are $L_1(\rho)$ (i.e. $\forall t \in T$, $\int_U f_t(u)\rho(du) < \infty$). Then

$$Y_t = \bigvee_{k\geq 1} \frac{f_t(u_k)}{\Gamma_k}$$

is a max–stable process with index set T (cf. de Haan(1984)). The finite–dimensional distributions of Y_t are :

$$\mathbf{P}\left[\bigcap_{i=1}^{n}\{Y_{t_i} \leq x^{(i)}\}\right] = \exp\left(-\int_U \bigvee_{i=1}^{n} \frac{f_{t_i}(u)}{x^{(i)}}\,\rho(du)\right). \tag{4}$$

for $t_i \in T$, $x^{(i)} \geq 0$, $i = 1, \ldots, n$. Cosslett(1988) and others specify their processes to have Gumbel marginals. This can be achieved in our framework by a trivial logarithmic transformation $Y_t \to \ln Y_t$. Cosslett(1988), de Haan and Balkema(1988) considered a special case of the max–stable model, namely the stationary max–moving average.

We may consider $\{f_t, t \in T\}$ as a stochastic process on $(U, \mathcal{U}, \rho)$. This process always has a separable version (Ash & Gardner(1975), Billingsley(1986)) which we also denote by $\{f_t, t \in T\}$. Thus, if the separant is D,

$$f^{\vee}(T) = f^{\vee}(T \cap D)$$

is measurable. Henceforth we assume that for all $u \in U$, $\{f_t(u),\ t \in T\}$ is separable. This assumption does not change the finite–dimensional distributions (4).

Two basic results are the following : Let $\{Y_t, t \in T\}$ be a separable max–stable process whose finite–dimensional distributions are given by (4) above. Then

(a) Y_t is a.s. finite in any measurable set $B \subseteq T$ iff

$$\int_U f^{\vee}(T \cap D, u)\rho(du) < \infty$$

(b) Assume $f^\vee(T) \in L_1(\rho)$, i.e. Y_t is a.s. finite on T. Then $\{Y_t, t \in T\}$ is stochastically continuous iff $\{f_t, t \in T\}$ is $L_1(\rho)$–continuous (i.e. as $t_n \to t$, $f_{t_n} \xrightarrow{L_1(\rho)} f_t$).

To check (a) note that $Y^\vee(B) \leq Y^\vee(T)$ and that by separability and (4)

$$\begin{aligned} -\log \mathbf{P}[Y^\vee(T) \leq x] &= -\log \mathbf{P}[Y^\vee(T \cap D) \leq x] \\ &= x^{-1} \int_U f^\vee(T \cap D, u)\rho(du), \; x > 0 \end{aligned}$$

and the result is immediate (cf. de Haan and Pickands(1986)). The result in (b) is Lemma 2 of de Haan(1984).

For the purpose of modeling the random utilities of the alternatives in T as a max–stable process, it follows from the discussion in section 2 that we would like the utility process to have usc realizations. The next theorem characterizes a.s. finite max–stable processes with usc paths.

It is convenient to define

$$f^*(u) = f^\vee(T, u) = f^\vee(T \cap D, u), \; u \in U$$

and recall that $\{f_t(u), \; t \in T\}$ is separable with separant D, for all u.

Theorem 3.1 *Suppose $f^* \in L_1(\rho)$. If for ρ–a.a. $u \in U$*

$$t \mapsto f_t(u)$$

is usc then $\{Y_t, t \in T\}$ is separable with separant D and for a.a. ω, $Y_.(\omega) \in US(T)$. Conversely, if $Y = \{Y_t, t \in T\}$ has a.a. paths usc then Y is separable with separant D and for ρ–a.a. u

$$t \mapsto f_t(u)$$

is usc.

Proof: Let $N = \sum_k \varepsilon_{(u_k, \Gamma_k)}$ be the Poisson process with points $\{(u_k, \Gamma_k),\ k \geq 1\}$ and mean measure $1_U(u)\rho(du)1_{[0,\infty)}(x)dx$. Then for any $\delta > 0$,

$$\begin{aligned}
\mathrm{E}[N(\{(u,x) : \frac{f^*(u)}{x} > \delta\})] &= \int \left(\int_{\{(u,x): f^*(u)/x > \delta\}} 1_U(u)\rho(du) \right) 1_{[0,\infty)}(x)dx \\
&= (1/\delta) \int_U f^*(u)\rho(du) \ < \ \infty
\end{aligned}$$

This implies $\mathrm{E}[\#k : \frac{f^*(u_k)}{\Gamma_k} > \delta] < \infty$ and consequently for all n

$$\Omega_n := \{\omega : \sum_k 1_{[\frac{f^*(u_k)}{\Gamma_k} > n^{-1}]}(\omega) < \infty\}$$

satisfies $\mathbf{P}[\Omega_n] = 1$. Define

$$M_n(\omega) = \sup\{k \ : \ \frac{f^*(u_k)}{\Gamma_k} \ > \ n^{-1}\}$$

so that on Ω_n, $M_n(\omega) < \infty$.

Now we proceed with the proof of the theorem.

(Sufficiency) For ρ–a.a. $u \in U$, suppose $f_t(u)$ is usc in t. Let

$$U_1 = \{u \in U \ : \ f_\cdot(u) \in US(T)\}$$

so that $\rho(U_1) = 1$. Define

$$\Omega_* = \{\omega \ : \ u_k(\omega) \in U_1,\ \forall k\}$$

so that

$$\begin{aligned}
\mathbf{P}[\Omega_*^c] &= \rho\left(\bigcup_k [u_k \notin U_1] \right) \\
&\leq \sum_k \rho(U_1^c) = 0
\end{aligned}$$

and

$$\mathbf{P}[\Omega_*] = 1.$$

We show that for $\omega \in (\cap_n \Omega_n) \cap \Omega_*$, $Y_.(\omega)$ is usc. Pick $t_0 \in T$ and consider two cases.

Case 1 : If $Y_{t_0}(\omega) > 0$, then there exists n_0 such that $1/n_0 < Y_{t_0}(\omega)$ and since

$$\omega \in \left(\bigcap_n \Omega_n\right) \bigcap \Omega_* \subseteq \Omega_{n_0}$$

we have

$$M_{n_0}(\omega) < \infty.$$

Thus

$$\begin{aligned}
\limsup_{t \to t_0} Y_t(\omega) &= \limsup_{t \to t_0} \bigvee_{k \geq 1} \frac{f_t(u_k(\omega))}{\Gamma_k(\omega)} \\
&= \limsup_{t \to t_0} \left[\left(\bigvee_{k=1}^{M_{n_0}(\omega)} \frac{f_t(u_k(\omega))}{\Gamma_k(\omega)} \right) \bigvee \left(\bigvee_{k > M_{n_0}(\omega)} \frac{f_t(u_k(\omega))}{\Gamma_k(\omega)} \right) \right] \\
&\leq \limsup_{t \to t_0} \left[\left(\bigvee_{k=1}^{M_{n_0}(\omega)} \frac{f_t(u_k(\omega))}{\Gamma_k(\omega)} \right) \bigvee \left(\bigvee_{k > M_{n_0}(\omega)} \frac{f^*(u_k(\omega))}{\Gamma_k(\omega)} \right) \right] \\
&\leq \limsup_{t \to t_0} \left(\bigvee_{k=1}^{M_{n_0}(\omega)} \frac{f_t(u_k(\omega))}{\Gamma_k(\omega)} \right) \bigvee n_0^{-1}.
\end{aligned}$$

Since $u_k(\omega) \in U_1$, $f_t(u_k(\omega)) \in US(T)$ for $k = 1, \ldots, M_{n_0}(\omega)$, whence $\bigvee_{k=1}^{M_{n_0}(\omega)} \frac{f_t(u_k(\omega))}{\Gamma_k(\omega)} \in US(T)$. Therefore the previous expression is bounded above by

$$\begin{aligned}
\left(\bigvee_{k=1}^{M_{n_0}(\omega)} \frac{f_{t_0}(u_k(\omega))}{\Gamma_k(\omega)} \right) \bigvee n_0^{-1} &\leq \left(\bigvee_{k=1}^{\infty} \frac{f_{t_0}(u_k(\omega))}{\Gamma_k(\omega)} \right) \bigvee n_0^{-1} \\
&= Y_{t_0}(\omega) \bigvee n_0^{-1} = Y_{t_0}(\omega)
\end{aligned}$$

since n_0 was chosen to satisfy $n_0^{-1} < Y_{t_0}(\omega)$.

Case 2 : If $\omega \in (\bigcap_n \Omega_n) \bigcap \Omega_*$ and $Y_{t_0}(\omega) = 0$, then for any n

$$\limsup_{t \to t_0} Y_t(\omega) = \limsup_{t \to t_0} \left[\left(\bigvee_{k=1}^{M_n(\omega)} \frac{f_t(u_k(\omega))}{\Gamma_k(\omega)} \right) \bigvee \left(\bigvee_{k > M_n(\omega)} \frac{f_t(u_k(\omega))}{\Gamma_k(\omega)} \right) \right]$$

$$
\begin{aligned}
&\leq \limsup_{t\to t_0}\left[\left(\bigvee_{k=1}^{M_n(\omega)}\frac{f_t(u_k(\omega))}{\Gamma_k(\omega)}\right)\bigvee\left(\bigvee_{k>M_n(\omega)}\frac{f^*(u_k(\omega))}{\Gamma_k(\omega)}\right)\right]\\
&\leq \limsup_{t\to t_0}\left(\bigvee_{k=1}^{M_n(\omega)}\frac{f_t(u_k(\omega))}{\Gamma_k(\omega)}\right)\bigvee n^{-1}\\
&\leq Y_{t_0}(\omega)\bigvee n^{-1}=n^{-1}
\end{aligned}
$$

and since n is arbitrary

$$\limsup_{t\to t_0} Y_t(\omega)=Y_{t_0}(\omega)=0.$$

For either Case 1 or Case 2, we have shown that for $\omega\in(\bigcap_n\Omega_n)\bigcap\Omega_*$ and any $t_0\in T$

$$\limsup_{t\to t_0} Y_t(\omega)\leq Y_{t_0}(\omega),$$

whence $t\mapsto Y_t(\omega)$ is usc.

Conversely define

$$\Omega_{USC}:=\{\omega\ :\ t\mapsto Y_t(\omega)\text{ is usc}\}$$

and therefore we have $\mathbf{P}[\Omega_{USC}]=1$. If $\int_U f^*(u)\rho(du)=0$ then $f^*(u)=0$ for ρ–a.e. u and for all $t\in T$, $f_t(u)=0$ for ρ–a.e. u. So for ρ–a.a. u : $f_t(u)$ is continuous in t. Hence suppose henceforth that $\int_U f^*(u)\rho(du)>0$. Then there exists $c>0$ such that

$$\int_U (f^*(u)-c)^+\,\rho(du)>0.$$

Write (cf. Balkema and de Haan(1988))

$$Y_t=\left(\bigvee_{k\,:\,\Gamma_k\leq c}\frac{f_t(u_k)}{\Gamma_k}\right)\bigvee\left(\bigvee_{k\,:\,\Gamma_k>c}\frac{f_t(u_k)}{\Gamma_k}\right)=Y_t^{'}\bigvee Y_t^{''}$$

so that $Y_t^{'}$ is independent of $Y_t^{''}$ by the complete randomness of the underlying Poisson process.

Define $E^{'} := \{\omega : N(U \times [0,c], \omega) = 1\}$ and we have

$$\begin{aligned}\mathbf{P}[E^{'}] &= \mathbf{P}[N(U \times [0,c]) = 1] \\ &= \mathbf{E}N(U \times [0,c]) \exp\{-\mathbf{E}N(U \times [0,c])\} = ce^{-c} > 0.\end{aligned}$$

Define $E^{''}$ as the event

$$\begin{aligned}E^{''} &= \left[\bigvee_{t \in T} Y_t^{''} \leq 1\right] \\ &= \{\omega \ : \ N\{(u,x) \ : \ x > c, \ \frac{f^*(u)}{x} \ > \ 1\} \ = \ 0\} \\ &= \{\omega \ : \ \bigvee_{k:\Gamma_k > c} \frac{f^*(u_k)}{\Gamma_k} \ \leq \ 1\}\end{aligned}$$

and we have

$$\begin{aligned}\mathbf{P}[E^{''}] &= \mathbf{P}[N(\{(u,x) \ : \ x > c, \frac{f^*(u)}{x} \ > \ 1\} \ = \ 0] \\ &= \exp\left(-\int_{\{(u,x):x>c, \ \frac{f^*(u)}{x}>1\}} \rho(du)dx\right) \\ &= \exp\left(-\int_U (f^*(u) - c)^+\right) \rho(du) \ > \ 0.\end{aligned}$$

Again from the complete randomness of N, $E^{'}$ and $E^{''}$ are independent so that $\mathbf{P}[E^{'} \bigcap E^{''}] > 0$. Note that if $\omega \in E^{'}$ then

$$Y_t^{'}(\omega) \ = \ \frac{f_t(u_1(\omega))}{\Gamma_1(\omega)}.$$

For $\omega \in E^{'} \bigcap E^{''}$

$$Y_t(\omega) = Y_t^{'}(\omega) \bigvee Y_t^{''}(\omega) = \frac{f_t(u_1(\omega))}{\Gamma_1(\omega)} \bigvee Y_t^{''}(\omega)$$

whence

$$\begin{aligned}Y_t(\omega) \bigvee 1 &= \frac{f_t(u_1(\omega))}{\Gamma_1(\omega)} \bigvee Y_t^{''}(\omega) \bigvee 1 \\ &= \frac{f_t(u_1(\omega))}{\Gamma_1(\omega)} \bigvee 1.\end{aligned}$$

and therefore

$$\Gamma_1(\omega)(Y_t(w)\bigvee 1) = f_t(u_1(\omega))\bigvee \Gamma_1(\omega).$$

For $\omega \in E' \bigcap E'' \bigcap \Omega_{USC}$ we have that $Y_t(\omega)$ is an usc function of t and from the preceding equation conclude $f_t(u_1(\omega)) \vee \Gamma_1(\omega)$ is usc in t. This implies

$$\mathbf{P}[\{f_t(u_1)\bigvee \Gamma_1 \notin US(T)\} \cap \{E' \cap E''\}] = 0$$

Consequently,

$$\mathbf{P}[f_t(u_1)\bigvee \Gamma_1 \notin US(T) \mid E' \cap E''] = 0$$

Conditional on $E' \bigcap E''$, (u_1, Γ_1) has distribution $\rho(du)c^{-1}dx$ on $U \times [0, c]$. From this and Fubini's theorem we get

$$\begin{aligned}\mathbf{P}[f_t(u_1)\bigvee \Gamma_1 \notin US(T) \mid E' \cap E''] &= c^{-1}\int_{[0,c]}\left(\int_{\{u\in U: f_t(u)\bigvee x \notin US(T)\}} \rho(du)\right) dx \\ &= c^{-1}\int_{[0,c]} \rho\{u \in U \ : \ f_t(u)\bigvee x \notin US(T)\} dx = 0.\end{aligned}$$

We conclude that for Lebesgue a.a. $x \in [0, c]$,

$$\rho(\{u \in U \ : \ f_t(u)\bigvee x \notin US(T)\}) = 0$$

Now pick a sequence $x_n \downarrow 0$, such that $\rho\{u \in U \ : \ f_t(u) \vee x_n \notin US(T)\} = 0$. Then the sets

$$A_{x_n} := \{u \in U : f_t(u)\bigvee x_n \notin US(T)\}$$

satisfy

$$A_{x_n} \uparrow A_0 =: \{u \in U : f_t(u) \notin US(T)\}.$$

From monotone convergence $\rho(A_{x_n}) \uparrow \rho(A_0)$ whence $\rho(A_0) = 0$ and we have our required result.

It remains to prove that if Y has a.s. usc paths, then Y is a separable random function. Set

$$\Omega_{sep} := \{\omega \ : \ t \mapsto f_t(u_k(\omega)) \text{ is separable, } \forall k \geq 1\}.$$

Since $\{f_t,\, t \in T\}$ is assumed separable

$$\mathbf{P}[\Omega_{sep}] \ = \ 1.$$

We show for $\omega \in (\bigcap_n \Omega_n) \bigcap \Omega_{sep} \bigcap \Omega_{USC}$ that $\{Y_t(\omega) \ : \ t \in T\}$ is a separable function with separant D.

Suppose initially that $Y_{t_0}(\omega) > 0$ and let n_0 be an integer satisfying $1/n_0 < Y_{t_0}(\omega)$. Then

$$Y_{t_0}(\omega) \ = \ \bigvee_{k=1}^{M_{n_0}(\omega)} \frac{f_{t_0}(u_k(\omega))}{\Gamma_k(\omega)}.$$

Suppose for $1 \leq j_0 \leq M_{n_0}(\omega)$

$$\bigvee_{k=1}^{M_{n_0}(\omega)} \frac{f_{t_0}(u_k(\omega))}{\Gamma_k(\omega)} \ = \ \frac{f_{t_0}(u_{j_0}(\omega))}{\Gamma_{j_0}(\omega)}.$$

Since $f_t(u_{j_0})$ is separable, there exist $t_n \in D$, $t_n \to t_0$ such that

$$Y_{t_n}(\omega) \ \geq \ \frac{f_{t_n}(u_{j_0}(\omega))}{\Gamma_{j_0}(\omega)} \to \frac{f_{t_0}(u_{j_0}(\omega))}{\Gamma_{j_0}(\omega)}.$$

Therefore

$$\liminf_{n\to\infty} Y_{t_n}(\omega) \geq \frac{f_{t_0}(u_{j_0}(\omega))}{\Gamma_{j_0}(\omega)} = Y_{t_0}(\omega).$$

Also, by upper semi–continuity,

$$\limsup_{n\to\infty} Y_{t_n}(\omega) \leq Y_{t_0}(\omega),$$

whence

$$Y_{t_n}(\omega) \to Y_{t_0}(\omega).$$

If on the other hand $Y_{t_0}(\omega) = 0$ then by upper semi–continuity

$$\limsup_{n\to\infty} Y_{t_n}(\omega) \leq Y_{t_0}(\omega) = 0$$

so that again

$$Y_{t_n}(\omega) \to Y_{t_0}(\omega).$$

This demonstrates separability.

Remark (1) : If we assume $\{Y_t\}$ is stochastically continuous or equivalently that $\{f_t\}$ is $L_1(\rho)$–continuous, then any countable set may serve as the separant. By mimicking the construction of say Neveu(1965, p.92) or Ash and Gardner(1975), we observe that if $\{f_t\}$ is $L_1(\rho)$–continuous and $t \mapsto f_t(u)$ is ρ–a.e. usc, then there is a version of $\{f_t\}$, call it $\{f_t^{\#}\}$ which is $L_1(\rho)$ continuous, ρ–a.e. usc and separable. Note that if the functions $\{f_t\}$ are ρ–a.e. continuous in t and $f^{\vee}(T,\cdot) \in L_1(\rho)$ then $\{f_t\}$ is $L_1(\rho)$–continuous and it follows that $\{Y_t\}$ is stochastically continuous. To see this note that for any $t_n \to t$, $f_{t_n}(\cdot) \to f_t(\cdot)$ ρ–a.e. and from dominated convergence we get $f_{t_n} \stackrel{L_1(\rho)}{\to} f_t$.

Remark (2) : Theorem 3.1 and the discussion in section 2 show how to construct a max–stable process which is a random element of $US(T)$.

The same methods allow one to give a criterion for sample path continuity. See Balkema and de Haan(1988) for the max–moving average case. Continue to suppose $\{Y_t,\, t \in T\}$ is max–stable with spectral functions $\{f_t,\, t \in T\}$ and that $\{f_t\}$ is separable with separant D.

Theorem 3.2 $Y = \{Y_t, t \in T\}$ ***is almost surely continuous iff***

(i) $f^* = f^{\vee}(T) = f^{\vee}(T \cap D) \in L_1(\rho)$.

(ii) *for ρ–a.a. $u \in U$, $t \mapsto f_t(u)$ is continuous.*

Proof : (Sufficiency) Given (i) and (ii) we get from Theorem 3.1 that $Y_{\cdot}(\omega)$ is usc for a.a. ω. To check that paths are also lower semi–continuous (lsc) and hence continuous, observe that since arbitrary maxima of lsc functions are lsc, we have for any $t_0 \in T$

$$\begin{aligned}\liminf_{t \to t_0} Y_t(\omega) &= \liminf_{t \to t_0} \bigvee_{k \geq 1} \frac{f_t(u_k(\omega))}{\Gamma_k(\omega)} \\ &\geq \bigvee_{k \geq 1} \frac{f_{t_0}(u_k(\omega))}{\Gamma_k(\omega)} = Y_{t_0}(\omega)\end{aligned}$$

for $\omega \in \{\omega : f_{\cdot}(u_k(\omega))$ is continuous on $T, \forall k \geq 1\}$ i.e. for a.a. ω. Thus for a.a. ω, $t \mapsto Y_t(\omega)$ is both usc and lsc.

Necessity : Almost sure continuity of paths implies a.a. paths are finite whence (i) follows from (3.1). The proof of (ii) is very similar to the comparable part of Theorem 3.1.

The motivation behind considering max–stable random utility processes Y which are random elements of $US(T)$, is twofold. First, it ensures that there exists an alternative which achieves the maximum level of utility and secondly it allows utilities to vary discontinuously over T.

We note from the discussion in section 2 that $\{Y_t, t \in T\}$, a separable max–stable process with a.s. usc sample paths on a complete probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is a measurable stochastic process, i.e.

$$(\omega, t) \mapsto Y_t(\omega) \text{ is } \mathcal{A} \times \mathcal{B}(T) \text{ measurable.}$$

Similarly, by considering the spectral functions $f = \{f_t, t \in T\}$ as a separable stochastic process on the probability space $(U, \mathcal{U}, \rho)$ with usc realizations for ρ–a.a. $u \in U$, we get

$$(u, t) \mapsto f_t(u) \text{ is } \mathcal{U} \times \mathcal{B}(T) \text{ measurable.}$$

We modify the paths of $\{f_t, t \in T\}$ on the ρ–null set U_1^c (see Theorem 3.1) so that $t \mapsto f_t(u)$ is usc *for all* $u \in U$, and it is clear that this modification does not affect the finite–dimensional distributions (4) of $\{Y_t, t \in T\}$. This modification on a ρ–null set simplifies matters related to the computations of the choice probabilities.

4 The Choice Probabilities

Consider a separable max–stable random utility process $Y = \{Y_t, t \in T\}$ with a.s. usc sample paths. Then from the discussion in section 2 it follows that Y is a random element of $US(T)$. This implies that $M = \{ t : Y_t = Y^{\vee}(T) \}$ is a random element of $\mathcal{F}(T)$. In cases where M consists a.s. of a single element, it is natural to imagine that the alternative chosen is the one with maximum utility. In this case, the probability that an alternative is chosen from a closed set K is

$$\mathbf{P}[\text{ choose an alternative in K }] = \mathbf{P}[M \subseteq K].$$

In cases where $M(\omega)$ is not a.s. singleton, the situation for the analyst is complicated by the fact that the rule " pick the alternative with maximum utility " does not uniquely specify an alternative. This creates an identification problem with respect to the sets containing the utility maximizing alternatives. The ambiguity that results from this may be used to develop models representing flexible preferences (cf. Kreps(1979)). Eventually we will concentrate on understanding characteristics of max–stable processes which result in unambiguous choice probabilities stemming from $M(\omega)$ being a.s. singleton.

We first specify the distribution of the random set M by giving the hitting and containment functionals. $(U, \mathcal{U}, \rho)$ is a complete probability space, and we are given the functions $\{f_t(u), t \in T\}$ such that for each $u \in U$, $f_{.}(u) \in US(T)$. Then from the discussion in Section

2,

$$(u,t) \mapsto f_t(u) \text{ is } \mathcal{U} \times \mathcal{B}(T) \text{ measurable.}$$

and for each $u \in \mathcal{U}$, the set

$$M_f(u) = \{t \in T \;:\; f_t(u) \;=\; f^*(u)\}$$

is closed, where we recall that we set $\bigvee_{t \in T} f_t(\cdot) = f^*(\cdot)$. Consequently from the analogous discussion in Section 2, the closed set–valued map

$$M_f : U \to \mathcal{F}(T)$$

is a random element of $(\mathcal{F}(T), \mathcal{B}(\mathcal{F}(T)))$ with probability space $(U, \mathcal{U}, \rho)$; i.e. $M_f^{-1}(\mathcal{B}(\mathcal{F}(T)) \in \mathcal{U}$.

If $K \in \mathcal{F}(T)$ then define

$$\begin{aligned}
K^{(>)} &= \{u \in U : \exists\, t_u \in K \text{ s.t. } f_{t_u}(u) > f_s(u),\ \forall s \in K^c\} \\
&= \{u \in U : f^{\vee}(K,u) > f_s(u), \forall s \in K^c\} \\
&= \{u \in U : M_f(u) \subseteq K\} = \{u \in U : M_f(u) \cap K^c = \phi\} \\
&= \{u \in U : M_f(u) \cap K^c \neq \phi\}^c \in \mathcal{U} \\
K^{(<)} &= \{u \in U : \exists\, s_u \in K^c \text{ s.t. } f_{s_u}(u) > f_t(u),\ \forall t \in K\} \\
&= \{u \in U : \exists\, s_u \in K^c \text{ s.t. } f_{s_u}(u) > f^{\vee}(K,u)\} \\
&= \{u \in U : M_f(u) \subseteq K^c\} = \{u \in U : M_f(u) \cap K = \phi\} \in \mathcal{U} \\
K^{(=)} &= \left(K^{(>)} \bigcup K^{(<)}\right)^c \\
&= \{u \in U : M_f(u) \cap K \neq \phi,\ M_f(u) \cap K^c \neq \phi\} \in \mathcal{U}.
\end{aligned}$$

The underlying Poisson process (PRM(μ)) on $U \times [0, \infty)$ of the max–stable process $\{Y_t\}$ is

$$N = \sum_j \varepsilon_{(u_j, \Gamma_j)}$$

where for $B \in \mathcal{U} \times \mathcal{B}[0,\infty)$,

$$\varepsilon_x(B) = \begin{cases} 1, & x \in B \\ 0, & x \notin B. \end{cases}$$

N has mean measure

$$\mu(du, d\Gamma) = 1_U \rho(du) \times 1_{[0,\infty)} d\Gamma$$

Now consider the Poisson processes

$$N_{K^{(>)}} = \sum_j \varepsilon_{(u_j,\Gamma_j)} 1_{\{u_j \in K^{(>)}\}} = N(\cdot \cap K^{(>)} \times [0,\infty))$$

$$N_{K^{(<)}} = \sum_j \varepsilon_{(u_j,\Gamma_j)} 1_{\{u_j \in K^{(<)}\}} = N(\cdot \cap K^{(<)} \times [0,\infty))$$

$$N_{K^{(=)}} = \sum_j \varepsilon_{(u_j,\Gamma_j)} 1_{\{u_j \in K^{(=)}\}} = N(\cdot \cap K^{(=)} \times [0,\infty))$$

Then by the complete randomness of N : $N_{K^{(<)}}$, $N_{K^{(=)}}$ and $N_{K^{(>)}}$ are mutually independent PRM's with mean measure

$$\mu_{K^{(>)}}(\cdot) = \mu(\cdot \cap K^{(>)} \times [0,\infty))$$

for $N_{K^{(>)}}$ and the mean measures of $N_{K^{(<)}}$ and $N_{K^{(=)}}$ are defined similarly. (Similar projections were employed in Resnick and Roy(1989) to derive choice probabilities for multivariate extremal random utility processes).

Define the random variables

$$\begin{aligned} X_{K^{(>)}} &= \bigvee_k \frac{f^*(u_k)}{\Gamma_k} 1_{\{u_k \in K^{(>)}\}}, \\ X_{K^{(<)}} &= \bigvee_k \frac{f^*(u_k)}{\Gamma_k} 1_{\{u_k \in K^{(<)}\}}, \\ X_{K^{(=)}} &= \bigvee_k \frac{f^*(u_k)}{\Gamma_k} 1_{\{u_k \in K^{(=)}\}}, \end{aligned}$$

Then $X_{K^{(>)}}$, $X_{K^{(<)}}$ and $X_{K^{(=)}}$ are independent random variables with distributions which are of Φ_1 extreme–value type :

$$\mathbf{P}[X_{K^{(>)}} \leq x] = \exp\left(-(1/x)\int_{K^{(>)}} f^*(u)\rho(du)\right), \; x > 0.$$

The distributions of $X_{K^{(=)}}$ and $X_{K^{(<)}}$ are similar, except that the domain of integration varies according to the underlying sets $K^{(<)}$ or $K^{(=)}$. Also define the random variable

$$X_{K^{(\geq)}} = X_{K^{(>)}} \vee X_{K^{(=)}}$$

which is also Φ_1 extreme–value distributed.

Theorem 4.1 *Suppose $\{Y_t, t \in T\}$ is a separable max–stable process with a.s. usc sample paths, and $f^* \in L_1(\rho)$. The random closed set M is defined as*

$$M := \{t : Y_t = Y^{\vee}(T)\}.$$

For an arbitrary $K \in \mathcal{F}(T)$,

- *The* containment functional*(cf. Eddy and Trader(1982)) is :*

$$\mathbf{P}[M \subseteq K] := \frac{\int_{K^{(>)}} f^*(u)\rho(du)}{\int_U f^*(u)\rho(du)}$$

- *The* hitting function or Choquet capacity *(Matheron(1975)) is :*

$$\begin{aligned} \mathbf{T}_M[K] &= \mathbf{P}[M \cap K \neq \phi] \\ &= \mathbf{P}[M \subseteq K] + \mathbf{P}[X_{K^{(=)}} > (X_{K^{(<)}} \vee X_{K^{(>)}})] \\ &= \frac{\int_{K^{(\geq)}} f^*(u)\rho(du)}{\int_U f^*(u)\rho(du)} \end{aligned}$$

Proof: The event that alternatives exclusively in some $K \in \mathcal{F}(T)$ achieve the maximum utility level corresponds to the event $[M \subseteq K]$ and has probability

$$\mathbf{P}[M \subseteq K] = \mathbf{P}[X_{K^{(>)}} > \left(X_{K^{(<)}} \bigvee X_{K^{(=)}}\right)]$$

$$= \int_0^\infty e^{\left(-(1/x)\int_{K(<)} f^*(u)\rho(du)\right)} e^{\left(-(1/x)\int_{K(=)} f^*(u)\rho(du)\right)} d\left[e^{\left(-(1/x)\int_{K(>)} f^*(u)\rho(du)\right)}\right]$$

$$= \frac{\int_{K(>)} f^*(u)\rho(du)}{\int_U f^*(u)\rho(du)}$$

The other probabilities are calculated similarly (cf. de Haan(1984), Resnick and Roy(1989)).

Now we note that for max–stable utility processes the maximum *value* of utility realized is independent of the alternative/s which actually attained this maximum utility level. For finite T, a similar result in the context of multivariate extremal processes is in Resnick and Roy(1989).

Corollary 4.1 *Assume the hypotheses of the previous theorem. Then $Y^\vee(T)$ and M are independent.*

Proof: We have that $Y^\vee(T)$ is Φ_1 extreme–value distributed with distribution function

$$\mathbf{P}[Y^\vee(T) \le z] = \exp\left(-(1/z)\int_U f^*(u)\rho(du)\right),\ z \ge 0.$$

Now

$$\begin{aligned}
\mathbf{P}[(Y^\vee(T) \le z) \cap (M \cap K \ne \phi)] &= \mathbf{P}[z \ge (X_{K(>)} \vee X_{K(=)}) \ge X_{K(<)}] \\
&= \frac{\int_{K(\ge)} f^*(u)\rho(du)}{\int_U f^*(u)\rho(du)} \exp\{-z^{-1}\int_U f^*(u)\rho(du)\} \\
&= \mathbf{P}[M \cap K \ne \phi]\mathbf{P}[Y^\vee(T) \le z].
\end{aligned}$$

This gives the desired independence.

We now discuss when $M(\omega)$ consists of a single element. We first review notation :

$$f^*(u) \quad := \quad \bigvee_{t \in T} f_t(u),$$

$$
\begin{aligned}
Y^*(\omega) &:= \bigvee_{t \in T} Y_t(\omega) = \bigvee_{k \geq 1} \frac{f^*(u_k(\omega))}{\Gamma_k(\omega)}, \\
M(\omega) &= \{t \in T : Y_t(\omega) = Y^\vee(T,\omega)\}, \\
M_f(u) &= \{t \in T : f_t(u) = f^*(u)\}.
\end{aligned}
$$

Theorem 4.2 *$\{Y_t, t \in T\}$ is a separable max–stable process with a.s. usc sample paths, and $f^*(\cdot) \in L_1(\rho)$. Then*

$$M(\omega) \text{ is a.s. singleton}$$

iff for ρ–a.a. $u \in U$

$$M_f(u) \text{ is singleton .}$$

Proof: $Y_\cdot(\omega) \in US(T)$ iff $f_\cdot(u) \in US(T)$ for ρ–a.a. u. Without loss of generality, by suitably modifying $\{f_t\}$ we assume that $t \mapsto f_t(u)$ is usc in t for *all* $u \in U$.

(Sufficiency) Let

$$U_2 = \{u \in U : M_f(u) = \{t_u\}; \text{ i.e. } M_f(u) \text{ is singleton }\}$$

so that $\rho(U_2) = 1$. Then it follows that for any $K \in \mathcal{F}(T)$

$$K^{(=)} \bigcap U_2 = \phi$$

and hence $\rho(K^{(=)}) = 0$. Therefore from the formulae in Theorem 4.1

$$\mathbf{P}[X_{K^{(=)}} > (X_{K^{(<)}} \vee X_{K^{(>)}})] = 0.$$

and thus we conclude that for any $K \in \mathcal{F}(T)$ we have

$$\mathbf{P}[M \subseteq K] = \mathbf{T}_M[K] = \mathbf{P}[M \bigcap K \neq \phi],$$

i.e. the hitting function coincides with the containment functional, and by Eddy and Trader(1982)(Proposition 4.7), M is a.s. *singleton*.

(Necessity) Conversely we may suppose that $\int_U f^*(u)\rho(du) > 0$ and define

$$\Omega_P = [Y^* > 0] \cap \{\omega : M(\omega) \text{ is singleton }\}$$

so that $\mathbf{P}[\Omega_P] = 1$. Define E', E'' as in Theorem 3.1 so that E' and E'' are independent, with $\mathbf{P}[E' \cap E''] > 0$.

As before we have on $E' \cap E''$

$$Y_t(\omega) \bigvee 1 = \frac{f_t(u_1(\omega))}{\Gamma_1(\omega)} \bigvee 1.$$

so that on $\Omega_P \cap E' \cap E'' \cap [\bigvee_{t\in T} Y_t' > 1]$ we have $\{t \in T : f_t(u_1(\omega)) = f^*(u_1(\omega))\}$ is singleton. This follows from recalling that $t \mapsto f_t(u_1(\omega))$ is usc in t and hence $M_f(u_1(\omega))$ is non–empty. Therefore defining the event

$$\text{SING } = \{\omega : \{t \in T : f_t(u_1(\omega)) = f^*(u_1(\omega))\} \text{ is singleton }\}$$

we get

$$\mathbf{P}[(\text{ SING })^c \cap E' \cap E'' \cap [\bigvee_{t\in T} Y_t' > 1]] = 0.$$

Since on E' we have $\bigvee_{t\in T} Y_t' = \frac{f^*(u_1)}{\Gamma_1}$ we conclude

$$\mathbf{P}[\{\text{ SING }\}^c \cap \{f^*(u_1) > \Gamma_1\} \mid E' \cap E''] = 0.$$

Conditional on $E' = [N(U \times [0,c]) = 1] = [\Gamma_1 \le c < \Gamma_2]$ we have Γ_1 uniformly distributed on $[0,c]$ so we have

$$0 = c^{-1} \int_{[0,c]} \left(\int_{[\{t: f_t(u)=f^*(u)\} \textit{is not singleton }] \cap [f^*(u)>x]} \rho(du) \right) dx$$

whence for Lebesgue a.a. x

$$0 = \rho\left(\{u : \{t : f_t(u) = f^*(u)\} \textit{is not singleton }\} \cap \{u : f^*(u) > x\}\right)$$

and letting $x \downarrow 0$ through an appropriate sequence gives the desired result.

Remark (4) : Note that for Cosslett's(1988) parametrization of the spectral functions $\{f_t, t \in T\}$ of a stationary max–moving average, $M_f(u)$ is singleton $\forall\, u \in U$, and hence for his case M is a.s. singleton.

Given a max–stable random utility process with a.s. usc realizations, on the alternatives space T, we now find a measurable way of ***identifying*** the alternative/s which actually attain the maximum utility value.

From the discussion in the beginning of this section, we have that $(u,t) \mapsto f_t(u)$ is $\mathcal{U} \times \mathcal{B}(T)$ measurable, and for $M_f : U \to \mathcal{F}(T)$ we have

$$M_f^{-1}(\mathcal{B}(\mathcal{F}(T)) \subseteq \mathcal{U}.$$

Then from a classical result on measurable selections (cf. Wagner(1977), Theorem 4.1) there exists an $\mathcal{U}$–measurable function $h : U \to T$ such that $h^{-1}(\mathcal{B}(T)) \subseteq \mathcal{U}$ and

$$h(u) \in M_f(u).$$

Hence for any $u \in U$

$$f_{h(u)}(u) = f^*(u).$$

Suppose M is a.s. singleton, so that if

$$U_3 = \{u \in U : M_f(u) \text{ is singleton } \}$$

then $\rho(U_3) = 1$. For $u \in U_3$, there exists $t_u \in T$ such that $M_f(u) = \{t_u\}$. So if h is a measurable selection and $u \in U_3$ we have $h(u) = t_u$. This means all measurable selections agree on U_3, a set of ρ–measure 1. Conversely, if all measurable selections agree on a set U_4 of ρ–measure 1, $M_f(u)$ must be singleton for $u \in U_4$. We summarize :

Corollary 4.2 *If the hypotheses in Theorem 4.2 hold, then the following are equivalent :*

1. *M is a.s. singleton.*

2. *$M_f(u)$ is singleton for ρ–a.a. $u \in U$.*

3. *There exists a $\mathcal{U}$–measurable selection $h : U \to T$ such that*

$$h(u) \in M_f(u)$$

and $h(\cdot)$ is unique up to sets of ρ–measure 0. For any $K \in \mathcal{F}(T)$ we have

$$K^{(>)} = h^{-1}(K) = \{u : h(u) \in K\}$$

and

$$\mathbf{P}[M \subseteq K] = \frac{\int_{K^{(>)}} f^*(u)\rho(du)}{\int_U f^*(u)\rho(du)} = \frac{\int_{h^{-1}(K)} f_{h(u)}(u)\rho(du)}{\int_U f_{h(u)}(u)\rho(du)}. \tag{5}$$

5 Complements and Examples

5.1 Max–Stable Random Sup Measures and Max–Stable Processes

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space. and assume T is a compact, Polish space. A function $m : \mathcal{G}(T) \to \bar{\mathbf{R}}_+$ is called a *sup measure* if $m(\phi) = 0$ and for an arbitrary collection of open sets $(G_j)_{j \in J} \in \mathcal{G}(T)$

$$m(\bigcup_{j \in J} G_j) = \bigvee_{j \in J} m(G_j)$$

(cf. Vervaat(1988)). Denote the collection of sup–measures on T by $SM(T)$ and endow it with the sup–vague topology which has the following collection as subbasis sets : for $x \in \bar{\mathbf{R}}_+$

$$\{m \in SM(T) : m(K) < x\} \quad , \quad K \in \mathcal{F}(T),$$

$$\{m \in SM(T) : m(G) > x\} \quad , \quad G \in \mathcal{G}(T).$$

Let $\mathcal{B}(T)$ be the Borel σ–algebra generated by the open subsets of $SM(T)$. Then a measurable function $X : \Omega \to SM(T)$ is called a *random sup measure.*

For a sup measure m, define its *sup derivative* as the mapping $d^{\vee}m : T \to \bar{\mathbf{R}}_{+}$ where

$$d^{\vee}m(t) := \bigwedge_{G \ni t} m(G) = m(\{t\})$$

(cf. Vervaat(1988), O'Brien, Torfs, & Vervaat(1989)). Then it follows that $d^{\vee}m \in US(T)$ (cf. Vervaat(1988)). Assume $US(T)$ is topologized by the sup–vague topology as described in section 2.

Let X be a *max–stable* random sup measure. By this we mean that X is a random sup measure whose finite–dimensional distributions are max–stable. This implies there exists a collection of Lebesgue integrable functions $f(G, \cdot) : [0,1] \to \bar{\mathbf{R}}_{+}$ indexed by sets in $\mathcal{G}(T)$ such that for $G_i \in \mathcal{G}(T)$, $x^{(i)} > 0$, $i = 1, \ldots, n$:

$$\mathbf{P}[\bigcap_{i=1}^{n} \{X(G_i) \le x^{(i)}\}] = \exp\left(- \int_{[0,1]} \bigvee_{i=1}^{n} \frac{f(G_i, u)}{x^{(i)}} du \right)$$

(cf. Resnick(1987, Theorem 5.11)). The random sup derivative of X, $d^{\vee}X$ is a random element of $US(T)$ (cf. Vervaat(1988)). If $d^{\vee}X(t)$ is *non–degenerate* for all t, *then* by virtue of the max–stability of X, it follows that the sup derivative $\{d^{\vee}X(t), t \in T\}$ is a max–stable process.

Conversely suppose that $Y = \{Y_t, t \in T\}$ is a max–stable process which is a random element of $US(T)$. Then it follows that $Y^{\vee}(\cdot)$ is a max–stable random sup measure.

5.2 Choice Probability Densities

Let U be complete metric subspace of $\mathbf{R}$ and T is a compact subset of $\mathbf{R}$. Suppose $h(u)$ is monotone (say increasing) in u, implying that h is Lebesgue a.e. differentiable (cf. Hewitt

& Stromberg(1965)). Then for $K = [a, b] \subset T$

$$\mathbf{P}[M \subseteq [a,b]] = \frac{\int_{h^{-1}(a)}^{h^{-1}(b)} f_{h(u)}(u)\rho(du)}{\int_U f_{h(u)}(u)\rho(du)}.$$

Let ρ be Lebesgue measure. This implies for $[a, t] \subset T$ we have

$$\frac{d}{dt}\mathbf{P}[M \subseteq [a,t]] = \frac{f_t(h^{-1}(t))}{\int_U f_{h(u)}(u)du} \frac{dh^{-1}(t)}{dt}, \quad \text{Lebesgue a.e.}$$

In general, we obtain from the transformation theorem for integrals that the probability in (5) can be obtained by integrating over K, the density

$$\frac{f^*(h^{-1}(y))}{\int_U f^*(u)\rho(du)}$$

with respect to the measure $\rho \circ h^{-1}(dy)$.

5.3 Independence from Irrelevant Alternatives (IIA)

The IIA property for a compact, Polish choice set T is defined as follows : suppose T_1 is a compact subset of T. Then for any $K_i \in \mathcal{F}(T_1)$, $i = 1, 2$ IIA prescribes that

$$\frac{\mathbf{P}_T[M \subseteq K_1]}{\mathbf{P}_T[M \subseteq K_2]} = \frac{\mathbf{P}_{T_1}[M \subseteq K_1]}{\mathbf{P}_{T_1}[M \subseteq K_2]}.$$

(cf. McFadden(1976)), where $\mathbf{P}_T[\cdot]$ denotes the choice probability when the underlying choice set is T.

Then an inspection of equation (5) indicates that

$$\frac{\mathbf{P}_T[M \subseteq K_1]}{\mathbf{P}_T[M \subseteq K_2]} = \frac{\int_{K_1^{(>)}} f^{\vee}(T, u)\rho(du)}{\int_{K_2^{(>)}} f^{\vee}(T, u)\rho(du)}$$

is in general *not* equal to

$$\frac{\mathbf{P}_{T_1}[M \subseteq K_1]}{\mathbf{P}_{T_1}[M \subseteq K_2]} = \frac{\int_{K_1^{(>)}} f^{\vee}(T_1, u)\rho(du)}{\int_{K_2^{(>)}} f^{\vee}(T_1, u)\rho(du)}.$$

In situations where $U = T$ and the specification of f is such that the selection function $h(\cdot)$ satisfies

$$h(u) = u,$$

the choice probabilities *will satisfy* IIA.

5.4 Examples

Example (1) : Uniformly distributed random set M.

Suppose for any $K \in \mathcal{F}(T)$

$$\int_{K^{(>)}} f^*(u)\rho(du) = \rho(K),$$

and set $C = \int_U f^*(u)\rho(du)$. Then

$$\mathbf{P}[M \subseteq K] = \frac{\rho(K)}{C}, \ K \in \mathcal{F}(T).$$

For instance suppose $U = T = [0,1]^2$, $\|\cdot\|$ is Euclidean distance, $f_t(u) = e^{-\|\mathbf{t}-\mathbf{u}\|^2}$ and ρ is Lebesgue measure. Then $h(\mathbf{u}) = \mathbf{u}$, $K^{(>)} = K$ and $f^*(\mathbf{u}) = 1$, and for $\mathbf{t} = (t^{(1)}, t^{(2)}) \in T$

$$\mathbf{P}[M \subseteq [\mathbf{0}, \mathbf{t}]] = t^{(1)}t^{(2)}$$

i.e. M is uniformly distributed on $[0,1]^2$.

In general if ρ is Lebesgue measure on $T = U$ and $f_.(u)$ has the unique maximum 1 at u (for a.a. u), then $K^{(>)} = K$ and M is uniformly distributed.

Example (2) : Let $U = T = [0,1]$, $\theta \in \mathbf{R}$ is a constant and define

$$f_t(u) = \exp(-(1/2)[(u-\theta)^2 + (t-u)^2]).$$

ρ is Lebesgue measure. Then $h(u) = u$ and for $t \in [0,1]$,

$$\mathbf{P}[M \subseteq [0,t]] = \frac{\Phi(t-\theta) + \Phi(\theta) - 1}{\Phi(1-\theta) + \Phi(\theta) - 1},$$

where $\Phi(\cdot)$ denotes the standard Normal distribution function.

Example (3) : Suppose $U = T = [0,1]$ and ρ is Lebesgue measure. Define for $0 \leq \theta \leq \log 2$,

$$f_t(u) = \begin{cases} 1- \mid t-u \mid, & t \in [0, 1/2) \\ e^\theta(1- \mid t-u \mid), & t \in [1/2, 1] \end{cases}$$

Then

$$\begin{aligned} [q,r]^{(>)} &= [q,r],\ 0 \leq q < r \leq (1/e^\theta - 1/2) \\ [r,s]^{(>)} &= \phi,\ (1/e^\theta - 1/2) < r < s < 1/2 \\ \{1/2\}^{(>)} &= (1/e^\theta - 1/2, 1/2] \\ (s,t]^{(>)} &= [s,t],\ 1/2 < s < t \leq 1 \end{aligned}$$

and hence

$$\begin{aligned} \mathbf{P}[M \subseteq [s,t]] &= \frac{t-s}{C},\ 0 \leq s < t \leq (1/e^\theta - 1/2) \\ &= 0,\ (1/e^\theta - 1/2) < s < t < 1/2 \\ &= \frac{e^\theta(t-s)}{C},\ 1/2 < s < t \leq 1 \\ \mathbf{P}[M = \{1/2\}] &= \frac{e^\theta/2 - \frac{1}{2e^\theta}}{C} \end{aligned}$$

where

$$C = e^\theta + \frac{1}{2e^\theta} - 1/2.$$

6 Discussion

In this section we make some observations about the modeling framework analyzed above and discuss some research questions which arise in this context.

The choice of the spectral functions which are unimodal seem to be the obvious candidates for modeling purposes. Thought is being given to the basic issue of how to systematically select spectral functions. If one assumes that the spectral functions are functions of some underlying parameters θ in some parameter space Θ then issues related to the estimation of these parameters arise. This issue is left for future research. Cosslett(1988) does look into the estimation issue for a specific parametrization of the spectral functions.

The measurable selection notion is really an existence result, and not constructive. What it does provide though is insight into how one can construct a max–stable random utility process model, and then identify the relevant domains of integration in the formulae for the choice probabilities.

It may be possible to incorporate the model proposed in this essay in dynamic programming models where the action space is compact, Polish. For instance proceeding in a similar fashion as Rust(1988) (where the action space is assumed to be finite), the ' *social surplus function* ' (cf. McFadden(1981)) corresponding to the action space T, is just $\log[\int_U f^*(u)\rho(du)]$. Investigations into this problem are subjects of ongoing research. Also, it is often assumed in dynamic choice modeling that *exactly one* action maximizes utility (for instance see Manski(1988)). Hence it may be worth reiterating that in the discussion above, this situation has been completely characterized for max–stable random utility processes on compact Polish action spaces. Modeling the dynamic continuous choice problem is underway.

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