SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853

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EMPTY SIMPLICES IN EUCLIDEAN SPACE

By

Imre Bárány† & Zoltán Füredi†

†This work was finished when both authors were on leave from the Mathematical Institute of the Hungarian Academy of Science, 1364 Budapest, P.O.B. 127, HUNGARY.

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By Imre Bárány† and Zoltán Füredi†
OR, Cornell University
Ithaca, NY 14853

RUTCOR, Rutgers University
New Brunswick, NJ 08903

Abstract. Let $P = \{p_1, p_2, \ldots, p_n\}$ be an independent point-set in $\mathbb{R}^d$ (i.e., there are no $d+1$ on a hyperplane). A simplex determined by $d+1$ different points of P is called empty if it contains no point of P in its interior. Denote the number of empty simplices in P by $f_d(P)$. Katchalski and Meir pointed out that $f_d(P) \geq \binom{n-1}{d}$. Here a random construction P_n is given with $f_d(P_n) < K(d)\binom{n}{d}$, where $K(d)$ is a constant depending only on d. Several related questions are investigated.

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1. INTRODUCTION

We call a set P of n points $(n \geq d+1)$ in the d-dimensional Euclidean space $\mathbb{R}^d$ <u>independent</u> if P contains no $d+1$ on a hyperplane. We call a simplex determined by $d+1$ different points of P <u>empty</u> if the simplex contains no point of P in its interior and denote the number of empty simplices of P by $f_d(P)$, or briefly $f(P)$.

Katchalski and Meir [KM] asked the following question: Given an independent set P of n points in $\mathbb{R}^d$, what can one say about the values of $f(P)$? If P consists of the vertices of a convex polytope, then clearly $f(P) = \binom{n}{d+1}$. So the interesting question is to find a lower bound for $f(P)$. Define

$$f_d(n) = \min\{f(P) \colon |P| = n, \quad P \subset \mathbb{R}^d \quad \text{independent}\}.$$

They proved that there exists a constant $K > 0$ such that for all $n \geq 3$,

$$(1) \qquad \binom{n-1}{2} \leq f_2(n) \leq Kn^2,$$

and in general, for every independent $P \subset \mathbb{R}^d$, $|P| = n$

$$(2) \qquad \binom{n-1}{d} \leq f_d(P).$$

(The case $d = 1$ has no importance, obviously $f_1(P) = n-1$.) The aim of this paper is to give bounds for $f_d(n)$ and to consider several related questions.

Our paper is organized as follows. In section 2 we state the upper bound for $f_d(n)$. Section 3 contains the results about the number of empty k-gons in the plane. In section 4 we deal with a related question: how many points are needed to pin the interiors of the empty simplices? Finally sections 5-12 contain the proofs.

A preliminary version of this work was presented in the 2nd Austrian Geometric Conference in Salzburg, 1985.

2. RANDOM CONSTRUCTIONS

Theorem 2.1. Let $A \subset R^d$ be a convex, bounded set with nonempty interior. Choose the points $p_1,\dots,p_n$ randomly and independently from A with uniform distribution. Then we have for the expected value of $f(P)$

$$E(\#\text{ empty simplices in } P) \leqq K\binom{n}{d}.$$

Here K is very large:

$$K = 2^{\binom{d}{2}} d!\, d^{d^2} \pi^{\frac{d-1}{2}} [\Gamma(\tfrac{d}{2}+1)]^{-1} \Big(\prod_{i=1}^{d-1} \Gamma(\tfrac{i}{2}+1)\Big)^2 < (2d)^{2d^2}$$

but independent of the shape of A! It is very likely that this value can be decreased, e.g., when A is a ball we can prove $K < d^{d^2}$.

Corollary 2.2. $f_d(n) < d^{d^2}\binom{n}{d}$.

The example of Katchalski and Meir gives in (1) that $K < 200$. Corollary 2.2 yields $K \leq 16$. The following random construction gives a much better upper bound. Let $I_1, I_2, \dots, I_n$ be parallel unit intervals on the plane, $I_i = \{(x,y): x = i,\ 0 \leq y \leq 1\}$. Choose the point p_i randomly from I_i with uniform distribution. Let $P_n = \{p_1,\dots,p_n\}$. Then

Theorem 2.3. $E(f_2(P_n)) = 2n^2 + O(n \log n)$.

On the other hand we have

Theorem 2.4. Let $P \subset \mathbb{R}^2$ be an independent point-set with $|P| = n$.

Then

$$n^2 - O(n \log n) \leq f_2(P).$$

We have to remark here that G. Purdy [Pu] announced $f_2(n) = O(n^2)$ without proof. H. Harborth [Ha] pointed out that $f_2(n) = n^2 - 5n + 7$ for $n = 3,4,5,6,7,8,9$ but not for $n = 10$ because $f_2(10) = 58$.

3. EMPTY POLYGONS ON THE PLANE

More than 50 years ago Erdös and Szekeres [ES] proved that for every integer $k \geq 3$ there exists an integer $n(k)$ with the following property: If $P \subset R^2$, $|P| \geq n(k)$ and P is independent, then there exists a subset $A \subset P$ such that $|A| = k$ and conv A is a convex k-gon.

We call a k-subset A of P empty if conv A contains no point of P in its interior. Erdös [Er] asked whether the following sharpening of the Erdös-Szekeres theorem is true. Is there an $N(k)$ such that if $|P| \geq N(k)$, $P \subset R^2$ independent, then there exists an empty k-gon with vertex set $A \subset P$. He pointed out that $N(4) = 5$ $(= n(4))$ and [Ha] proved that $N(5) = 10$ (while $n(5) = 9$). A proof of the existence of $N(k)$ was presented at a combinatorial conference in 1978 but it turned out to be wrong. This is no wonder because Horton [Ho] proved that $N(7)$ does not exist. The question about the existence of $N(6)$ is still open; a recent example of Fabella and O'Rourke [FO] shows twenty-two independent points in the plane without an empty hexagon.

Example 3.1. (Horton [Ho]). This is a squashed version of the well-known van der Corput sequence. We will define by induction a pointset $Q(n)$ where n is a power of 2. In $Q(n)$ each point has positive integer coodinates and the set of the first coordinates is just $\{1,2,\dots,n\}$. To start with let $Q(1) = \{(1,1)\}$ and $Q(2) = \{(1,1),(2,2)\}$. When $Q(n)$ is defined, set

$$Q(2n) = \{(2x-1,y): (x,y) \in Q(n)\} \cup \{(2x,y+d_n): (x,y) \in Q(n)\}$$

where d_n is a large number, e.g., $d_n = 3^n$ will do.

Now denote by $f^k(P)$ the number of empty k-gons in P and let $f^k(n) = \min\{f^k(P): P \subset R^2 \text{ independent}, |P| = n\}$. So $f^3(n)$ is just $f_2(n)$ defined in the previous section. Though $f^k(P)$ can be as large as $\binom{n}{k}$, Example 3.1 shows the following estimations.

Theorem 3.2. When n is a power of 2, then

(3) $f^3(n) \leqq 2n^2$

(4) $f^4(n) \leqq 3n^2$

(5) $f^5(n) \leqq 2n^2$

(6) $f^6(n) \leqq \frac{1}{2} n^2$

(7) $f^k(n) = 0$ for $k \geqq 7$.

We remark that the random example of Theorem 2.3 gives a quadratic upper bound on $f^k(n)$, too. The only lower bounds we can prove are

Theorem 3.3.

(8) $f^4(n) \geqq \frac{1}{4} n^2 - O(n)$, $f^5(n) \geqq \lfloor \frac{n}{10} \rfloor$.

The second inequality here is implied by $N(5) = 10$.

4. THE COVERING NUMBER OF SIMPLICES

Let P be an independent set of points in R^d. We say that $Q \subset R^d$ is a cover of the simplices of P if for every $(d+1)$-tuple $\{p_1,\dots,p_{d+1}\} \subset P$ there exists a $q \in Q$ with $q \in \text{int conv}\{p_1,\dots,p_{d+1}\}$. Denote by $g(P)$ the minimum cardinality of a cover and let $g_d(n) = \max\{g(P): P \subset R^d, |P| = n\}$. Katchalsky and Meir [KM] proved that $g_2(n) = 2n-5$ and $g_3(n) \leqq (n-1)^2$. Actually they proved

$$g_2(P) = 2|P| - (\#\text{ vertices of conv } P) - 2.$$

Though such an exact result seems to be elusive in higher dimensions, we can determine the asymptotic value of $g_d(n)$.

Theorem 4.1.

$$g_d(n) = \begin{cases} 2\binom{n}{d/2} + O(n^{d/2-1}) & \text{if } d \text{ is even} \\ \binom{n}{\lceil d/2\rceil} + O(n^{\lfloor d/2\rfloor}) & \text{if } d \text{ is odd} \end{cases}$$

holds for any fixed d when $n \to \infty$.

Corollary 4.2. $g_3(n) = \binom{n}{2} + O(n)$.

The constructions and proofs will be given in section 11.

The high value of $g_d(n)$ is a bit surprising (at least for the authors), because it was proved in [BF] and [Bá] that there exists a positive constant $c(d)$ $(c(2) = 1/27,\ c(d) > d^{-d})$ with the following property. For any pointset $P \subset R^d$, $|P| = n$ there exists a point contained in at least $c(d)\binom{n}{d+1}$ simplices of P.

5. THE DISTRIBUTION OF VOLUMES OF RANDOM SIMPLICES

Consider a bounded convex set $A \subset R^d$ wih $\mathrm{Vol}(A) > 0$. Choose randomly and independently the points $p_1, \dots, p_{d+1}$ from A with uniform distribution.

Lemma 5.1. There exists a $C = C(d) > 0$ such that for every $0 < v < 1$, $h > 0$

$$\mathrm{Prob}(v < \mathrm{Vol}(p_1, \dots, p_{d+1})/\mathrm{Vol}(A) < v+h) < Ch$$

where $\mathrm{Vol}(p_1, \dots, p_{d+1})$ is a shorthand for $\mathrm{Vol}(\mathrm{conv}\{p_1, \dots, p_{d+1}\})$.

Proof. A theorem of Fritz John [Jo] says that there exist two concentrical and homothetic ellipsoids E_1 and E_2 with $E_1 \subset A \subset E_2$ and $E_2 \subset dE_1$. As an affine transformation does not change the value of $\mathrm{Vol}(p_1, \dots, p_{d+1})/\mathrm{Vol}(A)$ we may assume that E_1 and E_2 are balls of radius r_1 and r_2 and $r_2 \leq dr_1$. Define w_d to be the volume of the d-dimensional unit ball, i.e., $w_d = \pi^{d/2}(\Gamma(\frac{d}{2}+1))^{-1}$. Let $0 < t < t+a$ and denote the Euclidean distance between $\mathrm{aff}(p_1, \dots, p_i)$ and p_{i+1} by D_i. Then

$$\mathrm{Prob}(t < D_i < t+a) \leqq \frac{w_{i-1} r_2^{i-1}}{\mathrm{Vol}(A)} (w_{d+1-i}(t+a)^{d+1-i} - w_{d+1-i} t^{d+1-i})$$

holds for every $i = 1, \dots, d$; the right hand side is the volume of the difference of two cylinders. Hence we have

$$\mathrm{Prob}(t < D_i < t+a) \leqq \frac{a}{r_2}\left(\frac{t}{r_2}\right)^{d-i} \frac{(d+1-i)w_{d+1-i}w_{i-1}}{w_d} \frac{w_d r_2^d}{\mathrm{vol}(A)} + O\left(\left(\frac{a}{r_2}\right)^2\right)$$

$$< \frac{a}{r_2}\left(\frac{t}{r_2}\right)^{d-i} 2^d d^{d+1}\left(1 + O\left(\frac{a}{r_2}\right)\right).$$

The choice of p_i and p_j is independent so we have

(9) $$\mathrm{Prob}(t_i < D_i < t_i+a \text{ holds for } i = 1,\ldots,d)$$

$$\leqq \left(\frac{a}{r_2}\right)^d \left(\frac{t_1}{r_2}\right)^{d-1} \left(\frac{t_2}{r_2}\right)^{d-2} \cdot \ldots \cdot \left(\frac{t_{d-1}}{r_2}\right) 2^{d^2} d^{d^2+d}\left(1 + O\left(\frac{a}{r_2}\right)\right).$$

Now $\mathrm{Vol}(p_1,\ldots,p_{d+1}) = (d!)^{-1}D_1 \cdot D_2 \cdot \ldots \cdot D_d$. Hence (9) yields

(10) $$\mathrm{Prob}(v < \mathrm{Vol}(p_1,\ldots,p_{d+1})/\mathrm{Vol}(A) < v+h)$$

$$\leqq \int_{x_1=0}^{2} \cdots \int_{x_d=0}^{2} x_1^{d-1} x_2^{d-2} \ldots x_{d-1} 2^{d^2} d^{d^2+d} dx_1 dx_2 \ldots dx_d$$

where the integration is taken for $(x_1,\ldots,x_d)$ with $v \cdot \mathrm{Vol}(A) < r_2^d x_1 \ldots x_d (d!)^{-1} < (v+h)\mathrm{Vol}(A)$. Because $0 \leq x_d - d! v r_2^{-d} \cdot \mathrm{Vol}\, A/(x_1 \ldots x_{d-1}) \leq hd!(\mathrm{Vol}\, A/r_2^d)/(x_1 \ldots x_{d-1})$ we have $\int dx_d = hd!(\mathrm{Vol}\, A/r_2^d)/(x_1 \ldots x_{d-1})$. Hence the right-hand-side of (10) equals

$$\underbrace{(2^{d^2} d^{d^2+d}) d! \frac{\mathrm{Vol}\, A}{r_2^d}}_{C_0} h \int_{0 \leq x_1 \leq 2} \cdots \int_{0 \leq x_{d-1} \leq 2} x_1^{d-2} \ldots x_{d-2}^{1}\, dx_1 \ldots dx_{d-1}$$

$$= 2^{\binom{d}{2}}/(d-1)! \cdot C_0 h < (2d)^{2d^2} h.$$

6. PROOF OF THEOREM 2.1

For given $p_1,\dots,p_{d+1}$ choose the points $p_{d+2},\dots,p_n$ randomly. Define $\mu(v) = \mathrm{Prob}(\mathrm{Vol}(p_1,\dots,p_{d+1}) < v)$. Obviously we have

$$\mathrm{Prob}(p_1,\dots,p_{d+1} \text{ is empty}) = \int_{0\le v\le 1} (1-v)^{n-d-1}\, d\mu(v)$$

$$\le \int_{0\le v\le 1} (1-v)^{n-d-1}\, C\, dv = C/(n-d).$$

Hence

$$E(f(P)) \le \binom{n}{d+1} \frac{C}{n-d} = \frac{C}{d+1} \binom{n}{d}.$$

7. PROOF OF THEOREM 2.3

Consider the points $A = (i,x)$, $B = (i+a,y)$, and $C = (i+k,z)$ where $k = a+b \ge 3$. Let $m = |y-x+(a/k)(z-x)|$, i.e., the distance between B and $I_{i+a} \cap [AC]$. Choose randomly a point p_j on I_j, $(i < j < i+k,$ $j \ne i+a)$. Then

Prob(ABC is an empty triangle)

$$= (1 - \tfrac{m}{a})(1 - 2\,\tfrac{m}{a})\dots(1-(a-1)\tfrac{m}{a})(1-(b-1)\tfrac{m}{b})\dots(1 - \tfrac{m}{b})$$

$$\le \exp[- \tfrac{m}{a} - 2\tfrac{m}{a} - \dots - (a-1)\tfrac{m}{a} - (b-1)\tfrac{m}{b} - \dots - 2\,\tfrac{m}{b} - \tfrac{m}{b}]$$

$$= \exp(-\tbinom{a}{2}\tfrac{m}{a} - \tbinom{b}{2}\tfrac{m}{b}) = \exp(-(k-2)m/2).$$

Now choose the points p_i $(1 \le i \le n)$ randomly. We obtain

$$\mathrm{Prob}(p_i p_{i+a} p_{i+k} \text{ is empty}) \le \int_{0<x<1} \int_{0<y<1} \int_{0<z<1} \exp(-(k-2)m/2)\,dx\,dy\,dz$$

$$\le 2 \int_{0\le m\le 1/2} \exp(-(k-2)m/2)\,dm \le 4/(k-2).$$

Hence we have

$$E(f(P)) \le n-1 + \sum_{1\le i\le n} \sum_{3\le k\le n-i} \sum_{1<a<k} 4/(k-2)$$

$$= n-1 + \sum_{3\le k\le n} (n-k+1) \frac{4(k-1)}{k-2} = n-1 + \sum_{3\le k\le n} (n-k+1)4/(k-2)$$

$$+ 4 \sum_{3\le k\le n} (n-k+1) = O(n \log n) + 2n^2.$$

8. A LEMMA ON GRAPHS

Lemma 8.1. Let G be a graph on the vertices $\{1,2,\dots,n\}$. Suppose that there exist no four vertices $i < j < k < \ell$ such that (i,k), (i,ℓ), and $(j,\ell) \in E(G)$. Then

$$|E(G)| \le 3n\lceil \log_2 n\rceil. \tag{11}$$

Proof: Let $E(G) = E(G_1) \cup \dots E(G_i) \cup \dots$ where $1 \le i \le \lceil \log_2 n\rceil$ and $E(G_i) = \{(u,v)\colon 1 \le u \le v \le n,\ 2^{i-1} \le v-u < 2^i,\ (u,v) \in E(G)\}$. Split $E(G_i)$ into three parts U, D and T:

$$U = \{(u,v)\colon (u,v) \in E(G_i) \text{ and } \exists\, w \text{ such that } u < w < v \text{ and } (w,v) \in E(G_i)\}$$

$$D = \{(u,v)\colon (u,v) \in E(G_i) \text{ and } \exists\, w \text{ such that } u < w < v \text{ and } (u,w) \in E(G_i)\}$$

and $T = E(G_i) - U - D$.

Clearly $U \cap D = \phi$, U, D and T do not contain a circuit. Hence their cardinality is at most $n-1$.

We note that (11) can be improved to $\lceil n \log_2 n \rceil$, and there exists a graph G^n with $|E(G)| \geq n(\log_2 n-2)$ which fulfills the constraints of Lemma 8.1.

9. PROOF OF THEOREM 2.4

Consider the points $p_1,\dots,p_n \in \mathbb{R}^2$ and an arbitrary line $e \subset \mathbb{R}^2$. Let q_i be the projection of p_i on e. We can choose e such that $q_i \neq q_j$. We can suppose that q_i lays between q_{i-1} and q_{i+1} (eventually reordering the indeces.)

Let G_u and G_d be two graphs on vertices $\{q_1,\dots,q_n\}$ such that

$$E(G_u) = \{q_i q_j\text{: every } p_k \text{ for } i < k < j \text{ is below the } [p_i p_j] \text{ and only (at most) one } p_i p_k p_j \text{ triangle is empty}\}$$

$$E(G_d) = \{(q_i q_j)\text{: every } p_k \text{ for } i < k < j \text{ is above the } [p_i p_j] \text{ and only (at most) one of the triangles } p_i p_k p_j \text{ is empty}\}.$$

It is easy to see that G_u and G_d fulfills the constraints of Lemma 8.1. Indeed, suppose on contrary $(q_i q_k), (q_i q_\ell), (q_j q_\ell) \in E(G_u)$. Then one can find an j', $i < j' \leq j$ and a k', $k \leq k' < \ell$ such that the triangles $p_i p_j, p_\ell$ and $p_i p_k, p_\ell$ are empty, contradicting $p_i p_\ell \in E(G_u)$. Hence

$$\begin{aligned} f(P) &= \sum_{1 \leq i < j \leq n} \#(\text{empty triangles with vertices } p_i p_k p_j,\ i < k < j) \\ &\geq 2\binom{n}{2} - |E(G_u)| - |E(G_d)| \\ &= n^2 - O(n \log n). \end{aligned}$$

10. PROOF OF 3.2

Let P be a pointset on the plane, consider $u_1, u_2 \in P$ with $u_1 = (x_1, y_1)$, $u_2 = (x_2, y_2)$. We say that the line segment $[u_1, u_2]$ connecting u_1 and u_2 is empty from below if the interior of the "infinite triangle" with vertices $u_1, u_2, (\frac{x_1+x_2}{2}, -\infty)$ contains no point of P. Emptiness from above is defined analogously. Denote by $h_2^-(P)$ and $h_2^+(P)$, respectively the number of segments in P empty from below and above.

Consider $Q(2n)$ from Example 3.1. $Q(2n)$ splits in a natural way into two parts: $Q^+(n)$ and $Q^-(n)$ where $Q^+(n) = \{(2x, y+d_n): (x,y) \in Q(n)\}$ and $Q^-(n) = \{(2x-1, y); (x,y) \in Q(n)\}$. The next two statements are obvious.

(12) If $u_1, u_2 \in Q(2n)$ and $[u_1, u_2]$ is empty from below in $Q(2n)$ then either $u_1, u_2 \in Q^-(n)$ or $u_1 \in Q^-(n)$ and $u_2 \in Q^+(n)$ and $|x_1 - x_2| = 1$ or $u_1 \in Q^+(n)$ and $u_2 \in Q^-(n)$ and $|x_1 - x_2| = 1$.

(13) $$h_2^-(Q(2n)) = h_2^-(Q^-(n)) + 2n-1.$$

Using induction (13) implies that

(14) $$h_2^-(Q(n)) < 2n.$$

$Q(n)$ is centrally symmetric and so

(15) $$h_2^+(Q(n)) < 2n.$$

Now call a triple $(u_1, u_2, u_3) \in Q(n)$ empty from below all the three line segments $[u_1u_2], [u_1u_3], [u_2u_3]$ are empty from below and denote by $h_3^-(Q(n))$ the number of triples of $Q(n)$, that are empty from below. Clearly,

$$h_3^-(Q(2n)) = h_3^-(Q^-(n)) + n - 1$$

hence by induction

$$h_3^-(Q(n)) < n.$$

To prove (3),(4),...,(7) we can use induction and the facts established about h_2^+, h_2^-, h_3^+ and h_3^-. For instance, we can estimte $f^4(Q(2n))$ in the following way:

$$\begin{aligned} f^4(Q(2n)) &= f^4(Q^+(n)) + h_3^-(Q^+(n))n \\ &\quad + h_2^-(Q^+(n))h_2^+(Q^-(n)) + nh_3^+(Q^-(n)) + f^4(Q^-(n)) \\ &< 2f^4(Q(n)) + 6n^2, \end{aligned}$$

which shows that $f^4(Q(2n)) \leqq 12n^2$.

The proofs of (3), (5), (6) are similar.

11. PROOF OF 3.3

Consider an arbitrary n-element set P on the plane, and assume no three points of P are on a line.

Lemma 11.1. Suppose $u, v, a, b \in P$ and the segments $[u,v]$ and $[ab]$ intersect (in an interior point). Then there exist $a', b' \in P$ such that $uva'b'$ is an empty quadrilaterial with a diagonal $[uv]$.

Proof. Trivial: if the uva triangle is empty then take $a' = a$ if not let $a' \in P$ be the nearest to $[uv]$ point from the interior of the triangle uva.

Now define a graph G with vertex set P. A pair $\{u,v\} \subset P$ is an edge of G if $[uv]$ is not a diagonal of any convex empty quadrilateral of P. By the above Lemma G must be a planar graph hence the number of its edges is at most 3n–6. All other pairs are contained in an empty quadrilateral hence $f^4(P) \geq \frac{1}{2}(\binom{n}{2} - (3n-6))$.

12. Proof of 4.1.

First we give the upper bound. Our main tool is Radon's theorem [DGK] which we need in the following form.

Lemma 12.1. Let $x_1,\ldots,x_{d+1} \in R^d$ be the vertices of a simplex S and let L be a line not parallel to any one of the facets of S. Then there exists a line L' parallel to L such that $L' \cap S = [ab]$ and $a \in \operatorname{relint} F_a$ and $b \in \operatorname{relint} F_b$ with F_a and F_b disjoint faces of S.

Proof. Consider the projection of $x_1,\ldots,x_{d+1}$ onto the subspace orthogonal to L and apply Radon's theorem in that subspace.

We use the lemma in the following way. Pick a line L not parallel to any affine subspace spanned by at most d points of P. Choose $\epsilon > 0$ small enough and let v be a vector parallel to L and $\|v\| = \epsilon$. We define a covering system Q as follows:

$$Q = \{v + \frac{1}{t} \sum_{x \in X} x: t \leqq \frac{d+1}{2}, X \subset P, |X| = t\}$$

when d is odd, and

$$Q = \{\delta v + \frac{1}{t} \sum_{x \in X} x: \delta = \pm 1, t \leqq \frac{d}{2}, X \subset P, |X| = t\}.$$

when d is even.

Now we give a construction for the lower bound. Let $p(i) = (i, i^2, \ldots, i^d) \in R^d$, $i = 1, \ldots, n$ and set $P = \{p(i): i = 1, \ldots, n\}$. P is the set of vertices of the cyclic polytope [Mc,Gr]. We will use certain properties of the cyclic polytope without explanation. Consider first the case when d is odd. Define

$$\mathcal{F} = \{\{i_1, \ldots, i_{d+1}\} \subset \{1, \ldots, n\} \quad i_\alpha < i_{\alpha+1} \quad \text{for}$$
$$1 \leqq \alpha \leqq d \quad \text{and} \quad i_{2\beta} = i_{2\beta-1}+1 \quad \text{for} \quad 1 \leqq \beta \leqq \frac{d+1}{2}\}.$$

So the members of the family $\mathcal{F}$ are unions of sequences of $\{1,2,\ldots,n\}$ of even length. Clearly

$$|\mathcal{F}| = \binom{n}{\frac{d+1}{2}} - O\left(n^{\frac{d-1}{2}}\right).$$

We claim that the simplices $\text{conv}\{p(i): i \in F\}$, $F \in \mathcal{F}$ are pairwise disjoint. Let $F_1, F_2 \in \mathcal{F}$ with $F_1 = \{i_1, \ldots, i_{d+1}\}$, $F_2 = \{j_1, \ldots, j_{d+1}\}$ and let k be the minimal element of the symmetric difference $F_1 \Delta F_2$, $k \in F_1$, say. Clearly $k = i_{2\alpha-1}$, i.e., its order in F_1 is odd. Consider the hyperplane H passing through the vertices $\{p(i): i \in F_1 - \{k\}\}$. We claim that H separates conv F_1 and conv F_2. The equation of H is

$$H(x_1,x_2,\dots,x_d) = \det\begin{vmatrix} 1 & x_1 & & x_d \\ 1 & i_1 & & i_1^d \\ 1 & \vdots & & \\ 1 & i_{d+1} & \cdots & i_{d+1}^d \end{vmatrix} = 0$$

where the row corresponding to k is missing. Set $f(t) = H(t,t^2,\dots,t^d)$, this is a polynomial in t of degree d. Then $f(i_s) = 0$ for $i_s \neq k$, i.e., its roots are exactly $\{i_1,\dots,i_{d+1}\}\setminus\{k\}$. Let, say $f(k) > 0$. Then the sign of $f(t)$ is negative for every integer $t > k$ except for those with $t = i_s$. So $H(x) \geq 0$ for $x \in \{p(i): i \in F_1\}$ and $H(x) \leqq 0$ for $x \in \{p(i): i \in F_2\}$. Thus we obtained $|\mathcal{F}|$ pairwise disjoint simplices. To cover them requires at least that many points so $g_d(n) \geq |\mathcal{F}|$.

The case d is even is similar. We define

$$Q = \{p(i): i = 1,2,\dots,n-2\} \cup \{v,-v\}$$

where v is in general position with respect to $p(i)$ and $\|v\|$ is large enough. This means that each facet of $\Pi = \mathrm{conv}\{p(i): i = 1,\dots,n-2\}$ is visible from either v or $-v$. As it is well-known [Gr,Mc], Π has $\binom{n}{d/2} + O(n^{d/2-1})$ facets $F_1,\dots,F_s$. Now in the following set of simplices no two have a common interior point:

$$\{\mathrm{conv}(F_i \cup \{v\}): F_i \text{ is visible from } v\}$$

$$\cup\ \{\mathrm{conv}(F_i \cup \{v\}): F_i \text{ is visible from } -v\}$$

$$\cup\ \{\mathrm{conv}\{p(i_1),\dots,p(i_{d+1})\}: 1 \leqq i_1 < i_2 < \dots < i_d < i_{d+1} = n-2,$$

$$i_{2\beta} = i_{2\beta-1}+1 \text{ for } \beta = 1,\dots,d/2\}.$$

This set of simplices shows that the simplices of Q cannot be covered by less than $2\binom{n}{d/2} + O(n^{d/2-1})$ points. Details are left to the reader.

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