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GENERALIZED DEHN-SOMMERVILLE RELATIONS FOR POLYTOPES, SPHERES AND EULERIAN PARTIALLY ORDERED SETS

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ABSTRACT

In this paper we generalize the Dehn-Sommerville equations to the cases of completely balanced spheres and Eulerian poset complexes. For such complexes, we consider an extension of the usual notion of f-vector, counting the number of faces having a prescribed label set. In the case of poset complexes, this is equivalent to counting the number of chains having elements of prescribed ranks. In each case we determine the affine span of the extended f-vectors of the class of objects. The result for Eulerian posets also gives the affine span of the extended f-vectors of convex polytopes.

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ABSTRACT

In this paper we generalize the Dehn-Sommerville equations to the cases of completely balanced spheres and Eulerian poset complexes. For such complexes, we consider an extension of the usual notion of f-vector, counting the number of faces having a prescribed label set. In the case of poset complexes, this is equivalent to counting the number of chains having elements of prescribed ranks. In each case we determine the affine span of the extended f-vectors of the class of objects. The result for Eulerian posets also gives the affine span of the extended f-vectors of convex polytopes. For d-polytopes, the dimension of this affine span is given by the $d^{\underline{th}}$ Fibonacci number.

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1. Introduction

This paper generalizes the Dehn-Sommerville equations for simplicial spheres to related classes of objects. The underlying motivation is to understand the combinatorial structure of arbitrary polytopes, that is, polytopes that are not necessarily simplicial. Towards this end we determine the affine span of the extended f-vectors of d-polytopes.

A polytope is the convex hull of finitely many points in R^d. We will generally consider a polytope of affine dimension d to be a subset of R^d; this is referred to as a d-polytope. A face of a polytope is the intersection of a supporting hyperplane with the polytope. For the most part we identify a polytope P with the abstract cell complex (or lattice) realized by the boundary of P, and write a face of P as the set of vertices of P it contains. That is, we will shorten $F = \mathrm{conv}\{v_0,\dots,v_k\}$ to $F = \{v_0,\dots,v_k\}$ when there is no risk of confusion. By convention, the empty set is considered a (-1)-dimensional face, and the polytope itself is a d-dimensional face; these faces will be called improper faces of P.

A polytope P is called simplicial if each of its faces, except possibly P itself, is a simplex (the convex hull of affinely independent points). We will write $\mathcal{P}^d$ (respectively, $\mathcal{P}^d_s$) for the set of all (respectively, all simplicial) d-polytopes.

The number of i-dimensional faces (or i-faces) of a polytope P is written f_i, and $f(P) = (f_0,f_1,\dots,f_{d-1})$ is called the f-vector of P. The set of f-vectors of all (simplicial) polytopes is written $f(\mathcal{P}^d)$ $(f(\mathcal{P}^d_s))$. A certain transformation on the f-vectors of simplicial polytopes has arisen in a number of different contexts, and will play an important part here. For a d-polytope P define the h-vector $h(P) = (h_0,h_1,\dots,h_d)$ by

$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}$ (here we use the convention $f_{-1} = 1$). This relation can be inverted to give $f_j = \sum_{i=0}^{j+1} \binom{d-i}{d-j-1} h_i$. The f- and h-vector are defined in exactly the same way for simplicial complexes.

The f-vector of any d-polytope P satisfies Euler's relation: $f_0 - f_1 + f_2 - \ldots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d$. This is the only (affine) linear relation satisfied by all f-vectors ([14], Chapter 8); i.e., the dimension of the affine hull of the f-vectors of d-polytopes, denoted $\dim \operatorname{aff} f(\mathcal{P}^d)$, is $d-1$. The only other general result is the Upper Bound Theorem: given integers n, d with $n \geq d+1 \geq 3$, among all d-polytopes with n vertices there exists one, called the cyclic polytope, that maximizes the number of faces of every dimension ([20], [24]). Euler's formula and the Upper Bound Theorem are not sufficient conditions for a vector of positive integers to be the f-vectors of a polytope. A characterization of $f(\mathcal{P}^d)$ is known only for $d = 3$:

$$f(\mathcal{P}^3) = \{f_0, f_0+f_2-2, f_2) \mid 4 \leq f_0 \leq 2f_2-4 \text{ and } 4 \leq f_2 \leq 2f_0-4\}.$$

For $d = 4$ the projections of $f(\mathcal{P}^4)$ into two dimensions have been characterized (see [14],[2],[3]).

When we restrict consideration to simplicial polytopes, a complete characterization of the f-vectors is known. This was conjectured by McMullen [19] in 1971 and proved in 1980 by Billera and Lee (sufficiency [7],[8]) and Stanley (necessity [27]). To state the theorem, we need the following definition. For h and i positive integers there is a unique decomposition of h, called the <u>i-canonical representation</u>, as

$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \ldots + \binom{n_j}{j}$$

such that $n_i > n_{i-1} > \ldots > n_j \geq j \geq 1$. The <u>i-th pseudo-power</u> of h is then $h^{\langle i \rangle} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \ldots + \binom{n_j+1}{j+1}$. Recall that there is a one-to-one correspondence between the f-vector and h-vector of a polytope. Write $h(P_s^d)$ for the set of h-vectors of simplicial d-polytopes.

<u>Theorem 1.1</u>. For an integer $h = (h_0, h_1, \ldots, h_d)$, $h \in h(P_s^d)$ if and only if the following conditions hold:

(i) $h_i = h_{d-i}$, $0 \leq i \leq [\frac{1}{2} d]$;

(ii) $h_{i+1} \geq h_i$, $0 \leq i \leq [\frac{1}{2} d] - 1$;

(iii) $h_0 = 1$ and $h_{i+1} - h_i \leq (h_i - h_{i-1})^{\langle i \rangle}$, $1 \leq i \leq [\frac{1}{2} d] - 1$.

The conditions in (i) are equivalent to the relations

$$f_k = \sum_{j=k}^{d-1} (-1)^{d-1-j} \binom{j+1}{k+1} f_j, \quad 1 \leq k \leq d-2.$$

These are the Dehn-Sommerville equations, named for Dehn, who conjectured their existence in 1905, and Sommerville, who first discovered and proved them in 1927 [23]. The equations did not become well-known until Klee reproved them in a more general context in 1963 [15].

In our discussions of generalizations of the Dehn-Sommerville equations, we will need to use several operations on convex polytopes. We briefly describe them here and indicate their effect on the h-vector. Note first that for any d-polytope Q, $h_d(Q) = 1$ because of Euler's formula.

If Q is a d-polytope, the <u>pyramid</u> on Q, PQ, is the convex (d+1)-polytope formed by taking the convex hull of Q with a point not in the affine span of Q. Sommerville [23] showed that $h(PQ) = (h(Q),1)$. Note that PQ is not simplicial unless Q is itself a simplex since Q is a facet of PQ. In this case PQ is again a simplex, and we get by induction that $h(T_0^d) = (1,1,\ldots,1)$, where T_0^d is the d-simplex.

For a d-polytope Q, the <u>bipyramid</u> over Q, BQ, is defined to be the convex (d+1)-polytope formed by taking the convex hull of Q with a line segment which meets Q in a relative interior point of each. For example, the bipyramid over an interval is a square and the bipyramid over a square is an octahedron. For the h-vectors, we have $h(BQ) = (h(Q),0) + (0,h(Q))$. Note that if Q is simplicial, then so is BQ. For example, since the h-vector of the interval is $(1,1)$, that of the square is $(1,2,1) = (1,1,0) + (0,1,1)$, and that of the octahedron is $(1,3,3,1) = (1,2,1,0) + (0,1,2,1)$.

To define the final operation, let Q be a simplicial d-polytope and F a proper face of Q. If H is a ((d-1)-dimensional) hyperplane containing Q in one of its closed half spaces, then a point $x \notin H$ is said to be <u>beneath</u> H if it is on the same side of H as Q, and <u>beyond</u> H otherwise. Now let $F_1,\ldots,F_k$ $(k \geq 1)$ be all the facets ((d-1)-dimensional faces) of Q which contain F. Let x be a point which is beyond the hyperplanes generated by these F_i's and beneath the hyperplanes generated by any other facets. (A point $x \notin Q$ sufficiently close to the centroid of F will do.) Define the <u>stellar subdivision</u> of the face F in Q, st(F,Q), to be the (simplicial) d-polytope which is the convex hull of $Q \cup \{x\}$. (See [12] where this operation is described for nonsimplicial Q as well.)

To describe st(F,Q) combinatorially, let Δ be the boundary complex of Q, and let σ be the set of vertices of F. Then the boundary complex of st(F,Q) is the complex $st(\sigma,\Delta)$, the stellar subdivision of simplex σ in Δ, where

$$st(\sigma,\Delta) = \{\Delta\backslash\sigma\} \cup \overline{x}\cdot\partial\sigma\cdot\ell k_{\Delta}\sigma.$$

Here $\Delta\backslash\sigma = \{\tau \in \Delta \mid \tau \not\supseteq \sigma\}$, $\ell k_\Delta \sigma$ is the link of σ in Δ, defined by

$$\ell k_{\Delta}\sigma = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset, \ \tau \cup \sigma \in \Delta\},$$

$\partial\sigma$ is the complex of proper subsets of σ, $\overline{x}$ denotes the complex consisting of $\{x\}$ and $\emptyset$ and $\cdot$ denotes the join of simplicial complexes.

Stellar subdivision can be used to construct a set of simplicial polytopes whose f-vectors span the Dehn-Sommerville space and, from these, a set of polytopes whose f-vectors span the Euler hyperplane. If T_0^d is the d-simplex and F^k is a k-face of T_0^d, define $T_k^d = st(F^k, T_0^d)$. The f-vectors of the polytopes T_k^d, $0 \leq k \leq [d/2]$, span the subspace determined by the Dehn-Sommerville equations. Their h-vectors are given by

$$h(T_k^d) = (1,2,3,\ldots,k,k+1,k+1,\ldots,k+1,k,\ldots,3,2,1).$$

Now for $0 \leq r \leq d-2$, define $T_k^{d,r}$ to be the r-fold pyramid over the (d-r)-polytope T_k^{d-r}. These polytopes constitute all the d-polytopes with d+2 vertices; in fact, the T_k^d are all the simplicial d-polytopes with d+2 vertices. Their f-vectors span the Euler hyperplane. For details see [14],[6].

In this paper we will develop similar results for the extended f-vectors of completely balanced spheres and Eulerian poset complexes.

2. Labeled simplicial complexes

Let Δ be a simplicial complex with vertex set V. A labeling of Δ by labels $0,1,\ldots,r$ is a partition of $V = V_0 \cup V_1 \cup \ldots \cup V_r$; the vertices in V_i are said to have label i. If each maximal simplex of Δ has precisely one vertex with each label, then Δ is said to be completely balanced; in this case Δ is a pure simplicial r-complex, that is, a simplicial complex in which each maximal face has $r+1$ vertices.

An interesting class of completely balanced complexes arises in the study of ranked posets. Let P be a poset, having a least element $\hat{0}$ and a greatest element $\hat{1}$. We form a simplicial complex $\Delta(P)$ having vertex set $P\setminus\{\hat{0},\hat{1}\}$ by defining $\{x_1,x_2,\ldots,x_k\}$ to be a simplex in $\Delta(P)$ if (after reordering, if necessary) $x_1 < x_2 < \ldots < x_k$. That is, the simplices of $\Delta(P)$ are the chains of $P\setminus\{\hat{0},\hat{1}\}$. $\Delta(P)$ is called the order complex of P or, more generally, a poset complex.

P is said to be ranked if for each $x \in P$, $x \neq \hat{0}$, all maximal chains $\hat{0} < x_0 < \ldots < x_k = x$ in P have the same length $k+1$. We then call k the rank of x, written $r(x)$, and make the convention $r(\hat{0}) = -1$. (Note that this rank function corresponds to the usual rank function shifted down by 1; it corresponds to the usual rank in the poset $P\setminus\{\hat{0}\}$.) A labeling of the vertices of $\Delta(P)$ with the ranks of the corresponding elements in P makes $\Delta(P)$ completely balanced, since every maximal chain contains exactly one element of each rank.

An especially interesting special case of ranked poset complexes occurs when P is the lattice of faces of a convex d-polytope Q. In this case, $\Delta(P)$ is the barycentric subdivision of the polytope Q and is itself a convex polytope [12]. Each vertex of $\Delta(P)$ corresponds to a face of Q, its label being the dimension of that face. The label set here is $\{0,1,\ldots,d-1\}$, which we will denote by the symbol $\langle d\rangle$.

We extend the notion of f-vector and h-vector to labeled simplicial complexes as follows. Suppose Δ is labeled by $0,1,\ldots,r$. For each $z = (z_0,z_1,\ldots,z_r) \in \mathbb{N}^{r+1}$, define $f_z = |\Delta_z|$ where

$$\Delta_z = \{\sigma \in \Delta: |\sigma \cap V_i| = z_i \quad \text{for} \quad 0 \le i \le r\}.$$

Note that the number of j-simplices in Δ is

$$f_j(\Delta) = \sum_{\substack{z \\ |z|=j+1}} f_z(\Delta),$$

where for $z \in \mathbb{N}^{r+1}$, $|z| = \sum_{i=0}^{r} z_i$. Finally, define

$$h_z(\Delta) = \sum_{w\le z} (-1)^{|z|-|w|} f_w(\Delta).$$

When there is at most one vertex of any label in a simplex, we have $f_z(\Delta) = 0$ unless each $z_i \le 1$. Write $S = \text{supp } z = \{i: z_i > 0\}$ and define $\Delta_S = \Delta_z$ and $f_S(\Delta) = f_z(\Delta)$. Then

$$f_j(\Delta) = \sum_{\substack{S\subseteq \langle r+1\rangle \\ |S|=j+1}} f_S(\Delta),$$

and if the labeling makes Δ a completely balanced complex of dimension $r = d-1$, then

$$h_S(\Delta) = \sum_{T\subseteq S} (-1)^{|S|-|T|} f_T(\Delta).$$

It is straightforward to verify that

$$h_i(\Delta) = \sum_{\substack{S \subseteq \langle d \rangle \\ |S| = i}} h_S(\Delta).$$

Thus we will refer to $(f_S;\ S \subseteq \langle d \rangle)$ and $(h_S;\ S \subseteq \langle d \rangle)$ as the <u>extended</u> f-vector and h-vector. Note that by Möbius inversion [13], we can write $f_S(\Delta) = \sum_{T \subseteq S} h_T(\Delta)$.

In the remainder of this section we will derive a set of linear relations on the $f_Z(\Delta)$ which must hold when Δ triangulates a sphere or, more generally, is a homology sphere. These relations will hold, in particular, for barycentric subdivisions of convex polytopes. In the case of completely balanced complexes, these relations have a particularly nice form when expressed in terms of the h_S.

To derive the relations, we first recall the definition of the Stanley-Reisner ring of a simplicial complex [21], [24]. If Δ is a simplicial complex on vertex set $V = \{v_1,\ldots,v_n\}$, then define the <u>Stanley-Reisner ring</u> of Δ to be

$$A_\Delta = K[X_1,\ldots,X_n]/I_\Delta$$

where $K[X_1,\ldots,X_n]$ is the polynomial ring in n indeterminates over a field K (which we assume here to be the field of rational numbers) and I_Δ is the ideal generated by all monomials of the form $X_{i_1}X_{i_2}\cdots X_{i_k}$ where $i_1 < i_2 < \cdots < i_k$ and $\{v_{i_1},v_{i_2},\ldots,v_{i_k}\} \notin \Delta$.

If Δ is labeled by $0,1,\ldots,r$, we can define an $\mathbb{N}^{r+1}$ grading of A_Δ, that is, a K-vector space direct sum decomposition

$$A_\Delta = \sum_{z \in \mathbb{N}^{r+1}} A_z$$

where $A_z A_w \subseteq A_{z+w}$ and $A_0 = K$. If vertex v_j has label i, then let the degree of X_j be e_i, the i^{th} unit vector in $\mathbb{N}^{r+1}$. This defines an $\mathbb{N}^{r+1}$ grading on $K[X_1,\ldots,X_n]$, and since I_Δ is generated by monomials, it is a homogeneous ideal in this grading. Thus A_Δ inherits the grading from the polynomial ring; A_z is the subspace of A_Δ spanned by all monomials of degree z. We define the <u>Hilbert function</u> of A_Δ by $H(A,z) = \dim_K A_z$; the <u>Hilbert series</u> of A_Δ is then given by

$$P(A_\Delta,t) = \sum_{z \in \mathbb{N}^{r+1}} H(A_\Delta,z)t^z,$$

where $t^z = t_0^{z_0} t_1^{z_1} \cdots t_r^{z_r}$.

The following result of Stanley (stated in [26] only for balanced complexes but proved in general) gives the Hilbert function of A_Δ for a labeled complex Δ.

<u>Proposition 2.1</u>. If Δ is a simplicial complex labeled by $0,1,\ldots,r$, then for $w \in \mathbb{N}^{r+1}$,

$$H(A_\Delta,w) = \sum_{z \in \mathbb{N}^{r+1}} f_z(\Delta) \prod_{i=0}^{r} \binom{w_i - 1}{z_i - 1}.$$

We use the conventions that $\binom{n}{0} = 0$ if $n < 0$, $\binom{n}{-1} = 0$ if $n \neq -1$ and $\binom{-1}{-1} = 1$. When Δ is completely balanced, we have $f_z(\Delta) = 0$ unless $z_i \leq 1$ for all i; in this case, $\prod_{i=0}^{r} \binom{w_i - 1}{z_i - 1}$ is nonzero if and only if $z \leq w$ and, for all i, $z_i = 0$ if and only if $w_i = 0$. Thus the term $f_z(\Delta) \prod_{i=0}^{r} \binom{w_i - 1}{z_i - 1}$ is zero unless

$$z_i = \begin{cases} 1 & \text{if } w_i \geq 1 \\ 0 & \text{if } w_i = 0, \end{cases}$$

in which case it is equal to $f_z(\Delta)$. Thus $H(A_\Delta, w) = f_{\text{supp } w}(\Delta)$ for $w \in \mathbb{N}^{r+1}$ when Δ is completely balanced.

Now suppose Δ is a <u>homology</u> $(d-1)$-<u>sphere</u>, that is, a simplicial complex with the property that for each k-face $\sigma \in \Delta$, $\ell k_\Delta \sigma$ is a $(d-k-2)$-dimensional complex having the rational homology of a $(d-k-2)$-sphere, $-1 \leq k \leq d-1$. Let Δ_j be the set of j-simplices in Δ, $-1 \leq j \leq d-1$. If $\sigma \in \Delta_j$, $\sigma = \{v_{i_0}, v_{i_1}, \ldots, v_{i_j}\}$, let

$$A(\sigma) = K[X_{i_0}, X_{i_1}, \ldots, X_{i_j}],$$

and define

$$C_j = \sum_{\sigma \in \Delta_j} A(\sigma),$$

a direct sum of vector spaces over K.

If Δ is labeled by $0, 1, \ldots, r$, then A_Δ and each C_j has an $\mathbb{N}^{r+1}$ grading as discussed above. The following result is an application of an exact sequence stated without proof by Danilov [11], who referred to a similar result due to Kouchnirenko [16]. A proof, based on the proof in [16], can be found in [6].

<u>Theorem 2.2</u>. Let Δ be a homology $(d-1)$-sphere labeled by $0, 1, \ldots, r$. Then there exist homogeneous (in the $\mathbb{N}^{r+1}$ grading defined above) linear maps ∂_i so that the sequence of K-vector spaces

$$0 \to A_\Delta \xrightarrow{\partial_d} C_{d-1} \xrightarrow{\partial_{d-1}} C_{d-2} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \to 0$$

is exact.

The result actually states that the maps are homogeneous in the finest grading of these algebras by the semigroup of monomials in $X_1,\ldots,X_n$. To prove it, it is enough to show that for each such monomial $m = X_1^{a_1}\cdots X_n^{a_n}$, the sequence of m^{th} homogeneous components

$$0 \to A_m \to (C_{d-1})_m \to (C_{d-2})_m \to \cdots$$

is exact. The proof consists of relating the homology of this complex to the usual simplicial homology of the link in Δ of the simplex $\sigma = \{i: a_i > 0\}$. See [6] for details.

Now, since vector space dimension is additive over exact sequences, we get the relation in Hilbert functions

$$H(A_\Delta,z) = \sum_{j=-1}^{d-1} (-1)^{d-1-j} H(C_j,z)$$

for each $z \in \mathbb{N}^{r+1}$, which in turn gives the relation of Hilbert series

$$P(A_\Delta,t) = \sum_{j=-1}^{d-1} (-1)^{d-1-j} P(C_j,t).$$

By Proposition 2.1 and the discussion immediately following, we have

$$P(A_\Delta,t) = \sum_{w\in\mathbb{N}^{r+1}} \sum_{z\leq w} f_z(P) \prod_{i=0}^{r} \binom{w_i-1}{z_i-1} t^w.$$

On the other hand, C_j is the direct sum of the vector spaces $A(\sigma)$ for $\sigma \in \Delta_j$. Now if $z_i = |\sigma \cap V_i|$ for $0 \leq i \leq r$, then

$$P(A(\sigma),t) = 1 \Big/ \prod_{i=0}^{r} (1-t_i)^{z_i}.$$

Thus, since

$$P(C_j,t) = \sum_{\sigma\in\Delta_j} P(A(\sigma),t),$$

we get, by equating coefficients the two expressions for $P(A_\Delta,t)$, the following identities on the numbers $f_z(\Delta)$ for homology spheres. See [6] for details.

Theorem 2.3. Let Δ be a homology (d-1)-sphere with vertices labeled $\{0,1,\ldots,r\}$. Then for all $w \in \mathbb{N}^{r+1}$

$$\sum_{z\leq w} f_z(\Delta) \prod_{i=0}^{r} \binom{w_i-1}{z_i-1} = \sum_{z\in\mathbb{N}^{r+1}} (-1)^{d-|z|} f_z(\Delta) \prod_{i=0}^{r} \binom{w_i+z_i-1}{z_i-1}. \quad \square$$

By applying Theorem 2.3 to the case $r = 0$, we obtain the usual Dehn-Sommerville equations for homology (d-1)-spheres

$$f_k = \sum_{j=k}^{d-1} (-1)^{d-j-1} \binom{j+1}{k+1} f_j, \quad -1 \leq k \leq d-2.$$

Again, see [6] for details.

Corollary 2.4. If Δ is a completely balanced homology (d-1)-sphere, then for all $S \subseteq \{0,1,\ldots,d-1\}$,

$$f_S(\Delta) = \sum_{\substack{T \\ S\subseteq T\subseteq \langle d\rangle}} (-1)^{d-|T|} f_T(\Delta)$$

Equivalently, writing $\tilde{S} = \langle d\rangle\backslash S$, we have for all S,

$$h_S(\Delta) = h_{\tilde{S}}(\Delta).$$

This result has been noted in [9] and [29] for those completely balanced complexes arising from certain posets. Again, see [6] for details of proofs in this and other cases.

3. Generators for the h_S of spherical complexes

In this section we show that the relations given in Corollary 2.4 along with the trivial equation $h_\emptyset = 1$, are the only affine linear relations satisfied by the numbers $h_S(\Delta)$ for every completely belanced homology sphere Δ. To do this we describe a set of polytopes and partitions of their vertices that make them completely balanced.

Suppose in R^d we have d mutually orthogonal segments, $[v_i,w_i]$ $(0 \leq i \leq d-1)$, that intersect at a single point interior to each of the segments. The convex hull of these segments is a simplicial polytope, called the d-<u>crosspolytope</u> Q^d or just Q (see [14], section 4.3). Alternatively, Q is the polar to the d-cube, and its boundary complex can be viewed as the order complex of the rank d poset having 2 elements of each rank, any two elements of different rank being comparable.

For $0 \leq k \leq d-1$, the k-faces of Q are determined by sets F of $k+1$ of the points $\{v_i\} \cup \{w_i\}$ where no pair $\{v_i,w_i\}$ is in F. In particular, the facets consist of exactly one element from each pair $\{v_i,w_i\}$, $0 \leq i \leq d-1$. Thus, the sets $V_i = \{v_i,w_i\}$ $(0 \leq i \leq d-1)$ partition the vertices of Q, making Q a completely balanced simplicial complex. For $S \subseteq \{0,1,\ldots,d-1\}$, the elements of Q_S are the sets $\{y_i\colon i \in S\}$ where $y_i \in V_i = \{v_i,w_i\}$. There are clearly $2^{|S|}$ such sets, i.e., $f_S(Q) = 2^{|S|}$. It follows easily that $h_S(Q) = 1$ for all $S \subseteq \{0,1,\ldots,d-1\}$. This makes the crosspolytope the completely balanced analog of the simplex, which has $h_i = 1$ for all i, $0 \leq i \leq d$.

Suppose F is a j-face of Q labeled by some subset $X \subseteq \{0,1,\ldots,d-1\}$ $(|X| = j+1)$. Then $\ell k_Q F$ is a complex whose facets consist of exactly one element from each V_i, for $i \in \{0,1,\ldots,d-1\} \setminus X$. So $\ell k_Q F$ is itself a (d-j-1)-crosspolytope with vertices labeled by $\{0,1,\ldots,d-1\} \setminus X$.

We describe an operation that will produce from Q another completely balanced polytope with slightly different numbers h_S. To do this we need to define a stellar subdivision of a face in a simplicial complex.

Suppose K is a simplicial complex, and F is a j-face of K. Let v be a point not in the vertex set of K. Then the stellar subdivision of F in K is the simplicial complex whose maximal faces are either maximal faces of K that do not contain F or faces of the form $H \cup \{v\} \cup G$, where H is a maximal proper subface of F and G is a maximal face of $\ell k_K F$. We write this complex $st_K(F,v)$. If K is the boundary of a simplicial d-polytope then $st_K(F,v)$ can also be realized as a simplicial d-polytope [12].

Theorem 3.1. For any $X \subseteq \{0,1,\ldots,d-1\}$ there exists a simplicial d-polytope P^X and a labeling of the vertices of P^X such that

(i) P^X is a completely balanced complex, and

(ii) For each $S \subseteq \{0,1,2,\ldots,d-1\}$,

$$h_S(P^X) = \begin{cases} 1 & \text{if } S \cap X = \emptyset \text{ or } X \subseteq S \\ 2 & \text{else.} \end{cases}$$

Proof: Let Q be the d-crosspolytope with vertices labeled as above. If $|X| \leq 1$, then for any S, either $S \cap X = \emptyset$ or $X \subseteq S$, so $P^X = Q$

satisfies the conclusion of the theorem. So assume $|X| \geq 2$; without loss of generality, let $X = \{0,1,\ldots,r\}$ $(r \geq 1)$. Let $F = \{u_0,\ldots,u_r\}$ be a face of Q with u_i labeled i.

Now choose a new vertex t_0, label it 0, and form the polytope $Q_0 = st_Q(F,t_0)$. A facet of Q_0 is one of two types: it is either a facet of Q not containing F (in which case its vertices are labeled $\{0,1,\ldots,d-1\}$); or it is of the form $H \cup \{t_0\} \cup G$, where H is a facet of F and G is a facet of $\ell k_Q F$. In the latter case, $H = \{u_0,\ldots,u_r\} \setminus \{u_k\}$ for some k, and G is labeled $\{r+1,r+2,\ldots,d-1\}$, so $H \cup \{t_0\} \cup G$ is labeled $(\{0,\ldots,r\} \setminus \{k\}) \cup \{0\} \cup \{r+1,r+2,\ldots,d-1\}$ (considered as a multiset, i.e., two vertices will have label 0 if $k \neq 0$).

We proceed to construct a sequence of polytopes inductively. The inductive hypothesis will be that for some $j \geq 0$ we have a simplicial d-polytope Q_j with the following properties: the vertices of Q_j are the vertices of Q plus $\{t_0,t_1,\ldots,t_j\}$ (t_i labeled i); and the facets of Q_j are

(1) facets of Q not containing F; or

(2) $\{x_0,x_1,\ldots,x_j\} \cup \{u_{j+1},u_{j+2},\ldots,u_r\} \cup G$, where for $0 \leq i \leq j$, $x_i \in \{u_i,t_i\}$, not every x_i is u_i, and G is a facet of $\ell k_Q F$; or

(3) $\{x_0,x_1,\ldots,x_{j-1}\} \cup \{u_j,t_j\} \cup (\{u_{j+1},u_{j+2},\ldots,u_r\} \setminus \{u_k\}) \cup G$, where for $0 \leq i \leq j-1$, $x_i \in \{u_i,t_i\}$, G is a facet of $\ell k_Q F$ and $j+1 \leq k \leq r$.

Note that type (1) and (2) facets are labeled by $\{0,1,\ldots,d-1\}$, but in a type (3) facet the label j occurs twice. The inductive hypothesis clearly holds for $j = 0$.

Suppose the hypothesis holds for some j, $0 \leq j \leq r$. Choose a new vertex t_{j+1}, label it $j+1$ and form the polytope $Q_{j+1} = st_{Q_j}(\{u_j,t_j\},t_{j+1})$. According to the definition of stellar subdivision we get facets of Q_{j+1}

in two ways. First there are the facets of Q_j that do not contain $\{u_j,v_j\}$. These are the type (1) and (2) facets of Q_j. They account for all of the type (1) facets of Q_{j+1}, and some of the type (2) facets of Q_{j+1} (those for which $x_{j+1} = u_{j+1}$). Secondly there are facets of the form $\{x_j\} \cup \{t_{j+1}\} \cup E$, where $x_j \in \{u_j,t_j\}$ and E is a facet of $\ell k_{Q_j}\{u_j,t_j\}$. Now from the description of the facets of Q_j we see that E must be of the form

$$\{x_0,x_1,\ldots,x_{j-1}\} \cup (\{u_{j+1},\ldots,u_r\}\backslash\{u_k\}) \cup G \quad (x_i,G,k \text{ as in (3)}).$$

When $k = j+1$ we get the rest of the type (2) facets of Q_{j+1} (those for which $x_{j+1} = t_{j+1}$). When $k > j+1$ we get the type (3) facets of Q_{j+1}. So Q_{j+1} satisfies the conditions of the inductive hypothesis.

Finally, we obtain from this sequence a simplicial d-polytope Q_r whose vertices are the vertices of Q plus $\{t_0,t_1,\ldots,t_r\}$ (t_i labeled i). We see that the construction of Q_r from Q_{r-1} produces no facets of type (3), because there is no $k > r$. So all the facets of Q_r are labeled $\{0,1,\ldots,d-1\}$, i.e., Q_r is completely balanced. Q_r is our desired polytope P^X. The facets of $P^X = Q_r$ are facets of Q not containing F and facets of the form $\{x_0,x_1,\ldots,x_r\} \cup G$, where for $0 \leq i \leq r$, $x_i \in \{u_i,t_i\}$, not every x_i is u_i, and G is a facet of $\ell k_Q F$.

Let Q' be the simplicial complex whose facets are $\{x_0,x_1,\ldots,x_r\} \cup G$, where for $0 \leq i \leq r$, $x_i \in \{u_i,t_i\}$, and G is a facet of $\ell k_Q F$. Since $\ell k_Q F$ is a crosspolytope, it is clear that Q' is a d-crosspolytope. Let $F' = \{u_0,u_1,\ldots,u_r\}$ (as a face of Q'). Then the facets of P^X can be thought of as the facets of Q not containing F and the facets of Q' not containing F'.

This description enables one to compute the extended f-vector and hence the extended h-vector of P^X. The details of the calculation can be found in [6; Theorem 6.1]. □

We note here that the sequence of stellar subdivisions constructed in the proof of Theorem 3.1 can be carried out in any completely balanced simplicial complex, defining the notion of a completely balanced stellar subdivision of any face in such a complex. See [6] for a discussion of this.

Let C^d be the set of all completely balanced homology (d-1)-spheres, and $(h_S(C^d))$ the set of the vectors $(h_S(\Delta))_{S\subseteq\langle d\rangle} \in \mathbb{N}^{2^d}$ for $\Delta \in C^d$.

Theorem 3.2. $\dim \operatorname{aff}(h_S(C^d)) = 2^{d-1} - 1$.

Proof: For all $\Delta \in C^d$ we have $h_\emptyset = 1$ and (by Corollary 2.4) $h_S(\Delta) = h_{\tilde{S}}(\Delta)$ for $S \subseteq \{0,1,\ldots,d-1\}$. These are clearly independent linear equations, and there are $2^{d-1} + 1$ of them, so $\dim \operatorname{aff} h_S(C^d) \leq 2^d - (2^{d-1}+1) = 2^{d-1} - 1$. The rest of the proof will consist in showing the other inequality by demonstrating 2^{d-1} affinely independent vectors in $h_S(C^d)$.

We first define a lexicographic order on the subsets of $\{0,1,\ldots,d-1\}$ as follows. If $S = \{s_1,\ldots,s_k\}$, $s_1 < s_2 < \ldots < s_k$, and $T = \{t_1,\ldots,t_{k'}\}$, $t_1 < t_2 < \ldots < t_{k'}$, then $S < T$ if $k < k'$ or if $k = k'$ and for some $j \leq k$, $s_j < t_j$, while for $i < j$, $s_i = t_i$. In this ordering $S < T$ if and only if $\tilde{S} > \tilde{T}$, so the complement of the nth subset is the (2^d-n+1)st subset.

Now define a $(2^{d-1}\times 2^{d-1})$ matrix A with columns indexed by the first 2^{d-1} subsets of $\{0,1,\ldots,d-1\}$ in increasing order, and the rows indexed by the last 2^{d-1} subsets of $\{0,1,\ldots,d-1\}$ arranged in decreasing order. If S is one of the first 2^{d-1} subsets and X is one of the last 2^{d-1} subsets, then the (X,S) entry in A is

$$a_{X,S} = 2 - h_S(P^X) = \begin{cases} 1 & \text{if } S \cap X = \emptyset \text{ or } X \subseteq S \\ 0 & \text{else,} \end{cases}$$

where P^X is given by Theorem 3.1. Note that for X and S within the range defined, $S < X$, so $X \not\subseteq S$.

The matrix A is lower triangular with ones along the diagonal. To see this, let $1 \leq q < n \leq 2^{d-1}$, and let X be the (2^d-q+1)st set (the set indexing row q) and S the nth set. Then $\tilde{X}$ is the qth set, so $\tilde{X} < S$. This implies $S \not\subseteq \tilde{X}$, i.e., $S \cap X \neq \emptyset$. So $a_{X,S} = 0$. The diagonal elements of A are $a_{\tilde{S},S} = 1$. Thus, rank $A = 2^{d-1}$. This says that the polytopes P^X, as X ranges over the last 2^{d-1} subsets of $\{0,1,\ldots,d-1\}$, have affinely independent vectors $h_S \in \mathbb{N}^{2^{d-1}}$ (S ranges over the first 2^{d-1} subsets of $\{0,1,\ldots,d-1\}$). But then their complete h-vectors $(h_S(P^X))_{S \subseteq \langle d \rangle}$ must be affinely independent. Thus $\dim \operatorname{aff}(C^d) \geq 2^{d-1} - 1$; combined with the other inequality, this gives the desired result. □

Since the basis constructed for the proof of Theorem 3.2 actually consists of polytopes, we have the following.

Corollary 3.3. The dimension of the affine span of the extended h-vectors of completely balanced d-polytopes is $2^{d-1} - 1$.

In [26], a variety of other conditions on the extended h-vectors are shown to hold in the more general class of completely balanced Cohen-Macaulay complexes. Proposition 3.6 of [26] can be used to give a simpler proof of Corollary 2.4 for shellable completely balanced homology spheres, since every shelling of such a complex is reversible [17; Proposition 3.3.11].

4. Eulerian poset complexes

In this section we look at a special class of completely balanced complexes: the order complexes of Eulerian posets. A poset P is called Eulerian if for every $x < y$ in P, $\mu(x,y) = (-1)^{r(y)-r(x)}$. We shall see that this means that Euler's formula holds for intervals in the poset, and so the order complexes of these posets are Eulerian manifolds [15]. The face lattice of a convex polytope is always Eulerian [18],[22].

Theorem 4.1. Let P be an Eulerian poset of rank d, and $S \subseteq \{0,1,\ldots,d-1\}$. If $\{i,k\} \subseteq S \cup \{-1,d\}$, $i < k-1$, and S contains no j such that $i < j < k$, then

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup j}(P) = f_S(P)(1 - (-1)^{k-i-1}).$$

Proof: Let C be a chain in P with rank set S. Let x be the element of C with rank i ($\hat{0}$ if $i = -1$), and y the element with rank k ($\hat{1}$ if $i = d$). For $i \leq j \leq k$ write $f_j(x,y)$ for the number of rank j elements of P between x and y. Since P is Eulerian, $\mu(x,y) = (-1)^{k-i}$ and it follows from the fact that $\sum_{x \leq z \leq y} \mu(x,z) = 0$ (see, e.g., [18; Theorem 2]) that

$$(-1)^{k-i} = \mu(x,y) = -\sum_{j=i}^{k-1} (-1)^{j-i} f_j(x,y).$$

Then, since $f_i(x,y) = 1$ we get a form of the Euler equation:

$$1 - (-1)^{k-i-1} = \sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_j(x,y).$$

Summing over all S-chains C we get (here x and y depend on C)

$$f_S(P)(1-(-1)^{k-i-1}) = \sum_{C \text{ an } S\text{-chain}} \sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_j(x,y)$$

$$= \sum_{j=i+1}^{k-1} (-1)^{j-i-1} \sum_{C \text{ an } S\text{-chain}} f_j(x,y)$$

$$= \sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup j}(P). \quad \square$$

For face lattices of arbitrary polytopes, this theorem is the analog of the Dehn-Sommerville equations, which hold for the f-vectors of simplicial polytopes. The proof here follows Sommerville's original proof [23]. Note that taking $S = \emptyset$, $i = -1$, $k = d$, the equation given by Theorem 4.1 is Euler's formula.

We now analyze the dependencies among the variables f_S given by the equations of Theorem 4.1. For $d \geq 1$, let Ψ^d be the set of subsets $S \subseteq \{0,1,\ldots,d-2\}$ such that S contains no two consecutive integers.

<u>Proposition 4.2</u>. For all $T \subseteq \{0,1,\ldots,d-1\}$ there is a nontrivial linear relation expressing $f_T(P)$ in terms of $f_S(P)$, $S \in \Psi^d$, which holds for all Eulerian posets P of rank d. The cardinality of Ψ^d is c_d, the dth Fibonacci number $(c_d = c_{d-1} + c_{d-2};\ c_1 = 1,\ c_2 = 2)$.

<u>Proof</u>: Order the subsets of $\{0,1,\ldots,d-1\}$ as in the proof of Theorem 3.2. If $T \notin \Psi^d$ then for some k, $1 \leq k \leq d$, $\{k-1,k\} \subseteq T \cup \{d\}$. Let $S = T\setminus\{k-1\}$, and $i = \max\{j \in T \cup \{-1\}: j < k-1\}$. Then Theorem 4.1 for these S, i and k says

$$f_T(P) = \sum_{j=i+1}^{k-2} (-1)^{k-j} f_{S\cup j}(P) + f_S(P)(1 - (-1)^{k-i-1}).$$

All the subscripts appearing on the right-hand side of this equation are less than T in the lexicographic order. Repeating the process for any subscript not in Ψ^d we get the desired linear relation.

To compute $|\Psi^d|$, note that any element S of Ψ^d is one of two types: either $d-2 \notin S$ or $d-2 \in S$. In the first case $S \in \Psi^{d-1}$; in the second case $d-3 \notin S$, so $S\setminus\{d-2\} \in \Psi^{d-2}$. Thus $|\Psi^d| = |\Psi^{d-1}| + |\Psi^{d-2}|$; it is easy to see that $|\Psi^1| = 1$, $|\Psi^2| = 2$, so the proposition is proved. □

Adding the relation $f_\emptyset = 1$ we get that the dimension of the affine span of the extended f-vectors of Eulerian posets is at most c_d-1. In fact, this upper bound is the actual dimension, and its value gives us a hint as to the proof. We need to exhibit c_d affinely independent extended f-vectors; it turns out we can do this within the class of polytopes. Since $c_d = c_{d-1} + c_{d-2}$ we try to use bases for $\mathrm{aff}(f_S(\mathcal{P}^{d-1}))$ and $\mathrm{aff}(f_S(\mathcal{P}^{d-2}))$ to create a basis for $\mathrm{aff}(f_S(\mathcal{P}^d))$. We obtain d-polytopes by taking pyramids and bipyramids over (d-1)-polytopes. The faces of the pyramid PQ consist of the faces of Q, the polytope Q itself, pyramids over faces of Q, and the new vertex. The faces of the bipyramid BQ consist of the faces of Q (but not Q itself), two pyramids over each face of Q, and the two new vertices.

In what follows we will use the convention that the symbol P alone means the "0-dimensional" polytope, i.e., a single point (a pyramid over the empty polytope). An ordered string or word made up of the symbols B and P, and ending in P, stands for the polytope obtained by taking successive pyramids and bipyramids over the empty polytope in the order indicated by the word. Thus $P^2 = PP$ is an interval, P^3 is a triangle, BP^2 is a square, and B^2P^2 is an octahedron. Clearly, the dimension of the polytope is one less than the length of the word. For $d \geq 1$ let Ω^d be the set of d-polytopes named by words of length $d+1$ in P and B that end in P^2 and contain no two adjacent B's. We set $\Omega^0 = \{\emptyset, P\}$.

First note that we restrict Ω^d to words ending in P^2 because $BP = P^2$. To calculate the cardinality of Ω^d, consider the two types of words in Ω^d: those beginning with P and those beginning with B. Words of the first type are of the form PQ, where Q is any word in Ω^{d-1}. Words of the second type must start with BP (since B^2 is not allowed) and thus are of the form BPQ, where Q is any word in Ω^{d-2}. So $|\Omega^d| = |\Omega^{d-1}| + |\Omega^{d-2}|$, i.e., the cardinality of Ω^d satisfies the Fibonacci recursion. Since $|\Omega^1| = |\{P^2\}| = 1$, $|\Omega^2| = |\{P^3, BP^2\}| = 2$, we get $|\Omega^d| = c_d$.

Proposition 4.3. For $d \geq 1$, the extended f-vectors of the c_d elements of Ω^d are affinely independent.

The proof that the extended f-vectors of elements of Ω^d are independent is difficult because the effect on the extended f-vector of taking a pyramid or bipyramid is not easily described. We have seen, however, that it is relatively easy to describe the faces of a pyramid or bipyramid in terms of the faces of the original polytope. In particular, all the faces of a polytope in Ω^d are in $(\cup_{i=0}^{d-1} \Omega^i)$. So we will define a vector which counts the number of faces of each combinatorial type in a given polytope $Q \in \Omega^d$, use the recursive construction of Ω^d to show these vectors independent, and prove that a nonsingular transformation takes these vectors to the extended f-vectors.

As it turns out, we will not need to consider the facets of $Q \in \Omega^d$. The remaining faces of these Q are indexed by $M^d = (\cup_{i=0}^{d-2} \Omega^i)$; it is easy to show by induction that $|M^d| = |\Omega^d| = c_d$. Associate with a polytope $Q \in \Omega^d$ the c_d-vector $(a_{Q,M})_{M \in M^d}$ where $a_{Q,M}$ is the number of faces of Q of combinatorial type M. We wish to show that the $c_d \times c_d$ matrix $A^d = (a_{QM})$ has full (row) rank. We do this by induction on d. However, in the inductive step we will need the fact that A^d is equivalent to the matrix of extended f-vectors, so we must work with two other matrices at the same time. We define K^d to be a submatrix of the matrix of extended f-vectors. K^d is a $c_d \times c_d$ matrix with rows indexed by Ω^d, columns indexed by Ψ^d and entries $k_{QS} = f_S(Q)$. Finally we define the transformation from A^d to K^d. Let T^d be the $c_d \times c_d$ matrix with rows indexed by M^d, columns indexed by Ψ^d, and block diagonal form as follows:

$$T^d = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & \bigcirc & \\ & & K^1 & & & \\ & \bigcirc & & K^2 & & \\ & & & & \ddots & \\ & & & & & K^{d-2} \end{pmatrix}.$$

Here the K^j block is located in the rows indexed by words in M^d of length $j+1$, and in the columns indexed by sets in Ψ^d with maximal element j. The first two columns are indexed by $\emptyset$ and $\{0\}$, respectively. Writing $T^d = (t_{MS})$ we have $t_{MS} = k_{M,S\setminus j} = f_{S\setminus j}(M)$ if M is of length $j+1$ and S has maximal element j, and $t_{MS} = 0$ otherwise.

Lemma 4.4. $A^dT^d = K^d$.

Proof: $(A^dT^d)_{QS}$ = (row Q of A^d)$\cdot$(column S of T^d). Let j be the maximal element of S; the product will only pick up components of row Q indexed by faces of length $j+1$. So

$$\begin{aligned}(A^dT^d)_{QS} &= \sum_{\substack{M\in M^d \\ \text{length}(M)=j+1}} a_{QM}\cdot k_{M,S\setminus j} \\ &= \sum_{\substack{M\in M^d \\ \text{length}(M)=j+1}} (\#\text{ of faces of } Q \text{ of type } M)(f_{S\setminus j}(M)) \\ &= \sum_{\substack{F \text{ face of } Q \\ \dim F = j}} f_{S\setminus j}(F) = f_S(Q) = k_{QS}. \quad \square\end{aligned}$$

Proof of Proposition 4.3: We wish to show by induction that rank A^d = rank T^d = rank $K^d = c_d$. For $d = 1,2$ it is clear:

$A^1 = T^1 = K^1 = (1)$; $A^2 = K^2 = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$, $T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So assume $d \geq 3$ and that for $r < d$, rank A^r = rank T^r = rank $K^r = c_r$.

We construct A^d out of A^{d-1} and A^{d-2}, and show that it is non-singular. First let $E = (e_{QM})$ be a $c_d \times c_d$ matrix of the form

$$E = \left(\begin{array}{c|c} A^{d-1} & * \\ \hline 0 & I \end{array} \right)$$

with rows indexed by Ω^d and columns indexed by M^d. Here for $Q \in \Omega^{d-1}$, row Q of A^{d-1} is located in row PQ of E, and for $M \in M^{d-1}$, column M of A^{d-1} is located in column M of E. For $Q \in \Omega^{d-1}$ and $M \in M^d \setminus M^{d-1}$ (i.e., M of length $d-1$), define $e_{PQ,M}$ = the number of faces of Q of combinatorial type M. I is a $c_{d-2} \times c_{d-2}$ identity matrix, so clearly rank $E = c_{d-1} + c_{d-2} = c_d$. We show that A^d can be obtained from E by elementary row and column operations. The last operations will be the column operations which will take the faces of Q to the faces of PQ. But the bottom rows of the matrix correspond to bipyramids, not pyramids. So we must put in the bottom rows vectors that will yield the face vector of a bipyramid when the pyramiding operation is applied.

We first do row operations on E to get a matrix G. Leave the first c_{d-1} rows (those whose indices start with P) alone. If $Q \in \Omega^{d-2}$ and Q does not begin with B, add row PBQ of E to row BPQ of E. If Q begins with B, then PBQ does not label a row of E. We mimic the above row operation by finding the face vector of the polytope BQ as a linear combination of the first c_{d-1} rows of E.

Lemma 4.5. Let v be the c_d-vector with v_M = the number of faces of BQ of type M $(M \in M^d)$. Then v is linearly dependent on the first c_{d-1} rows of E.

Proof: By looking at the proof that $A^d T^d = K^d$ we see that

$$v \left(\begin{array}{c|c} T^{d-1} & 0 \\ \hline 0 & K^{d-2} \end{array} \right) = (f_S(BQ)) \in \mathbb{N}^{c_d} ,$$

where the subscript S ranges over sets which have no two consecutive elements, but may contain $d-2$. Adjoining v to the bottom of the matrix consisting of the first c_{d-1} rows of E we get

$$\left(\begin{array}{c|c} A^{d-1} & * \\ \hline \multicolumn{2}{c}{v} \end{array} \right) \left(\begin{array}{c|c} T^{d-1} & 0 \\ \hline 0 & K^{d-2} \end{array} \right) = \left(\begin{array}{c|c} K^{d-1} & * \\ \hline \multicolumn{2}{c}{f_S(BQ)} \end{array} \right) .$$

By Proposition 4.2 there are at most c_{d-1} independent extended v-vectors of dimension $d-1$, so the rank of the matrix on the right-hand side is c_{d-1}. The matrix

$$\left(\begin{array}{c|c} T^{d-1} & 0 \\ \hline 0 & K^{d-2} \end{array} \right)$$

is invertible, so the left-hand matrix has rank c_{d-1}. Therefore v is a linear combination of the first c_{d-1} rows of E. $\square$

We can now finish defining the row operations on the matrix E. If $Q \in \Omega^{d-2}$ begins with B, we add to row BPQ of E the combination of the first c_{d-1} rows of E given by the lemma.

We now have the matrix $G = (g_{QM})$, whose first c_{d-1} rows are the same as those of E, and whose (BPQ,M)-entry is the number of faces of the polytope BQ of type M (or one more than this number if $M = Q$). Clearly, rank G = rank $E = c_d$.

Now we perform column operations on G to get $H = (h_{QM})$ and show $H = A^d$. The operations are indicated by the description of the faces of a pyramid. For M starting with B, leave column M alone. If $M = PN$, $N \in M^{d-1}$, add column N of G to column M of G to obtain column M of H. We now calculate the entries of H by considering four cases.

(i) Suppose $Q \in \Omega^{d-1}$, $M \in M^d$, and M starts with B. Then $h_{PQ,M} = g_{PQ,M} = e_{PQ,M}$ = the number of faces of Q of type M = the number of faces of PQ of type $M = a_{PQ,M}$, since no new bipyramid faces can be created by taking the pyramid over Q.

(ii) Suppose $Q \in \Omega^{d-1}$, $N \in M^{d-1}$. Then $h_{PQ,PN} = g_{PQ,N} + g_{PQ,PN} = e_{PQ,N} + e_{PQ,PN}$ = the number of faces of Q of type N + the number of faces of Q of type PN = the number of faces of PQ of type $PN = a_{PQ,PN}$.

(iii) Suppose $Q \in \Omega^{d-2}$, $M \in M^d$, and M starts with B. Then

$$\begin{aligned} h_{BPQ,M} = g_{BPQ,M} &= \text{the number of faces of } BQ \text{ of type } M + \chi(M = Q) \\ &= \text{the number of faces of } Q \text{ of type } M \\ &= \text{the number of faces of } PQ \text{ of type } M \\ &= \text{the number of faces of } BPQ \text{ of type } M \\ &= a_{BPQ,M}. \end{aligned}$$

Here $\chi(M = Q)$ is 1 if $M = Q$, and 0 otherwise. Again we are using the fact that no new bipyramid faces are created by taking pyramids and bipyramids.

(iv) Suppose $Q \in \Omega^{d-2}$, $N \in M^{d-1}$. Then

$h_{BPQ,PN} = g_{BPQ,N} + g_{BPQ,PN}$ = the number of faces of BQ of type N

+ the number of faces of BQ of type PN + $\chi(PN = Q)$.

If N itself is a pyramid over L, then

$h_{BPQ,PN}$ = 2 × the number of faces of Q of type L

+ 3 × the number of faces of Q of type N

+ the number of faces of Q of type PN

= 2 × the number of faces of PQ of type N

+ the number of faces of PQ of type PN

= the number of faces of BPQ of type PN

= $a_{BPQ,PN}$.

If N is, instead, a bipyramid, then

$h_{BPQ,PN}$ = 3 × the number of faces of Q of type N

+ the number of faces of Q of type PN

= 2 × the number of faces of PQ of type N

+ the number of faces of PQ of type PN

= $a_{BPQ,PN}$.

Thus $H = A^d$, and rank A^d = rank $H = c_d$.

To finish the proof we must compute the ranks of T^d and K^d, but fortunately these are easy. By the induction hypothesis K^r is nonsingular for $r < d$; so by the block diagonal construction of T^d, T^d is clearly nonsingular. Then K^d is the product of two nonsingular matrices, hence is nonsingular. So the proof of Proposition 4.3 is complete. □

Combining Propositions 4.2 and 4.3 gives us the following result.

Theorem 4.6. For $d \geq 1$,

$$\dim \operatorname{aff}\{(f_S(P))_{S\subseteq\langle d\rangle}: P \text{ is an Eulerian poset of rank } d\}$$

$$= \dim \operatorname{aff}\{(f_S(P))_{S\subseteq\langle d\rangle}: P \text{ is a } d\text{-polytope}\} = c_d - 1,$$

where c_d is the dth Fibonacci number. □

In particular, the extended f-vectors of Eulerian posets (or polytopes) are contained in a proper subspace of the affine span of the extended f-vectors of completely balanced homology spheres. In other words, the equations $h_S = h_{\tilde{S}}$ for Eulerian posets are dependent on the equations given by Theorem 4.1.

Already at dimension 4, the f-vectors of polytopes have not been characterized. The results of this section show that the extended f-vectors of 4-polytopes are determined linearly by the values of f_0, f_1, f_2 and $f_{\{0,2\}}$ (here we have dropped the set brackets on $f_{\{i\}}$, because it coincides with f_i in the original f-vector).

Conclusion

The results of this paper could all be viewed as generalizations of the Dehn-Sommerville equation for simplicial spheres. The equations $h_S = h_{\tilde{S}}$, which determine the affine span of the extended f-vectors of completely balanced spheres resemble the original Dehn-Sommerville equations most closely in form. It is the equations of Theorem 4.1,

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup j}(P) = f_S(P)(1 - (-1)^{k-i-1}),$$

however, whose proof follows Sommerville's original proof and which, up to affine span, describe the extended f-vectors of polytopes.

Part of the motivation for considering extended f-vectors is to derive information on the original f-vectors themselves. The linear equations obtained do not help with this problem. One attempt in this direction is the following conjecture [5]: If $f(P) = (f_0, f_1, \ldots, f_{d-1})$ is the f-vector of a d-polytope P, then for $0 \leq k \leq d-2$, $f_k \geq \sum_{j=k}^{d-1} (-1)^{d-1-j} \binom{j+1}{k+1} f_j$.

This conjecture is, in general, false. The inequality for $k = 0$ fails for the 5-polytope constructed by joining two 5-cubes along a common facet and performing stellar subdivisions on the (cubical) facets of the resulting polytope. On the other hand, the inequality holds for $k = d-2$ and, equivalently for $k = d-3$. The conjecture is true for polytopes of dimension ≤ 4; for simple polytopes (and, of course, simplicial polytopes, for which the relations are equalities); and for prisms on simplicial polytopes. If the inequalities hold for some polytope, then they hold for the pyramid and bipyramid over that polytope. For details see [5].

It would, of course, be of great interest to find characterizations of the f-vectors and extended f-vectors of polytopes.

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