THESIS

A Study of the Plow-Bottom and Its Action on the Furrow-Slice

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Date Due

A STUDY OF THE PLOW-BOTTOM AND ITS ACTION ON THE FURROW-SLICE

A THESIS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL

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By

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INTRODUCTION.

The most ancient records show that from a very remote period man has used the plow, in one form or another, to assist him in stimulating the earth to bring forth a more bountiful harvest. As has been the case in many other lines of endeavor; theory has trailed far behind observation and experience, in developing this implement. In fact, as far as can be ascertained it was not until the last half of the eighteenth century that any serious attempt was made to develop a plow-bottom from a theoretical standpoint and even then the productions of Jefferson,* Lambruschini, Small, Rahm and others, cannot be considered as thoroughly grounded upon well developed theories; rather their works should be looked upon as hypotheses. Experience in the field generally proved that the machines designed by these men were not all that could be desired. For example: it is reported** that when Lambruschini's helicodial mouldboard was taken into the field for trial the driver of the draft animals immediately observed that the force required to move this plow was too great for the results obtained. To be sure,

*See the table.XII.

***Giornale Agrario Toscano*, 1832, vol. 3, *D'un nuovo orechio da Cottii Lambruschini*.

geometrically exact mouldboards furnished the basis in many instances for more perfect developments, but the results obtained by empirical plow designers who worked in the field, were so far superior to the results obtained by the men who worked in the laboratory, that the theorists were soon completely outstripped and even held up to ridicule by the men who developed their machines in the hard school of experience; until at the present time we find special types of plow-bottoms designed to meet certain field conditions, but no well developed theory is available to serve as a guide in this work.

This thesis is an attempt to begin a fundamental analysis of the plow-bottom and its work in the hope that some light may be thrown upon the theory of this humble but perplexing machine, and other attempts stimulated to delve further into the secrets which are still to be revealed regarding the theory of this important implement. Empirical methods have given the world plow-bottoms which work well. It is still to be hoped that scientific investigation can refine and further perfect, supplement as it were, the productions of experience.

The work undertaken by the writer can be naturally divided into three parts, viz.; a study of the forms of plow-bottoms, an attempt to analyse the motion of the soil parti-

Fig. 1

FORMS OF THE PLOW-BOTTOM.

A study of modern, American-manufactured plow-bottoms reveals the fact that a large number of these are so constructed that their surfaces contain sets of straight lines, (each set consisting of an infinite number of straight lines), so related that an equation or equations satisfied by the coordinates of points on the surface can be found.

Fig. 1 represents a bottom with two sets of straight lines. The few lines shown in the picture indicate that through every point of the surface two straight lines can be drawn which lie wholly on the surface until they pass off the edges of the bottom. These straight lines furnish the basis for the proof that such a surface is a portion of an hyperboloid of one sheet, (for the form of this surface see Figs. 8 to 12), whose equation can be developed and studied with mathematical exactness. The method of developing this equation will be given later, but at present we are mainly interested in the fact that there is a class of plow-bottoms on whose surfaces lie sets of straight lines, and, further, that one equation can be developed which will approximately represent the working surface of such a bottom.

Further study shows that the surfaces of other plow-bottoms contain sets of straight lines but that one equation

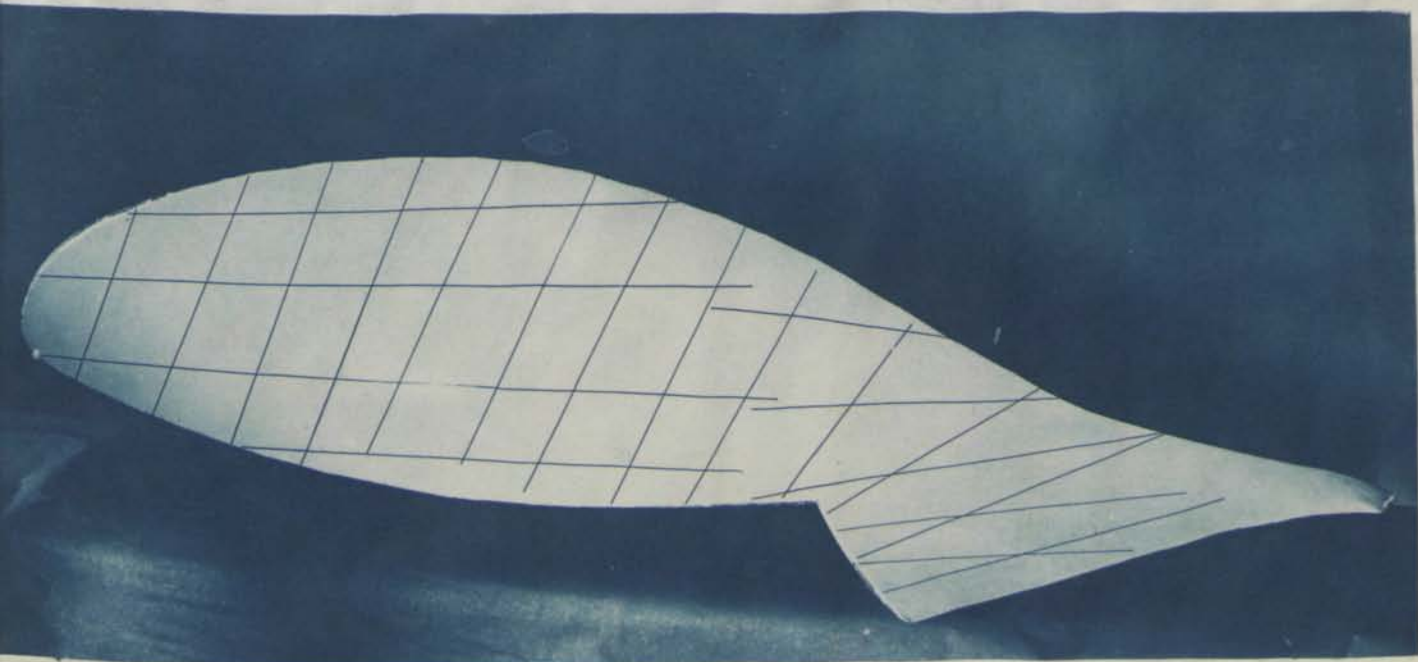

Fig. 2

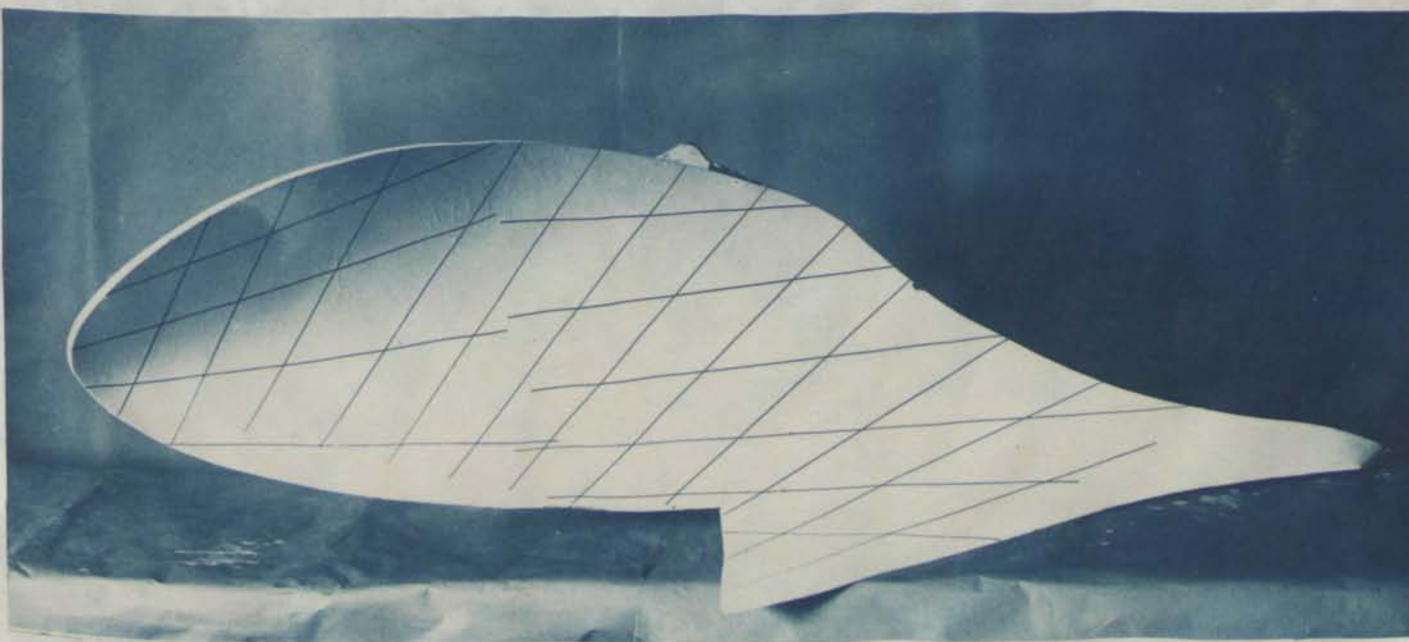

Fig. 3

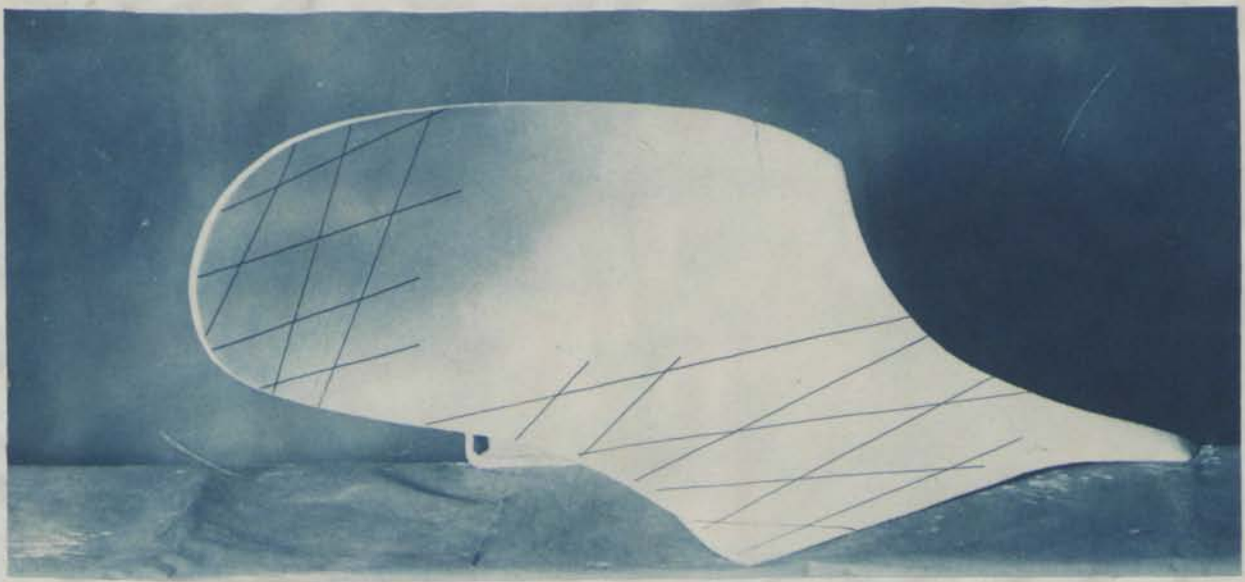

Fig. 4

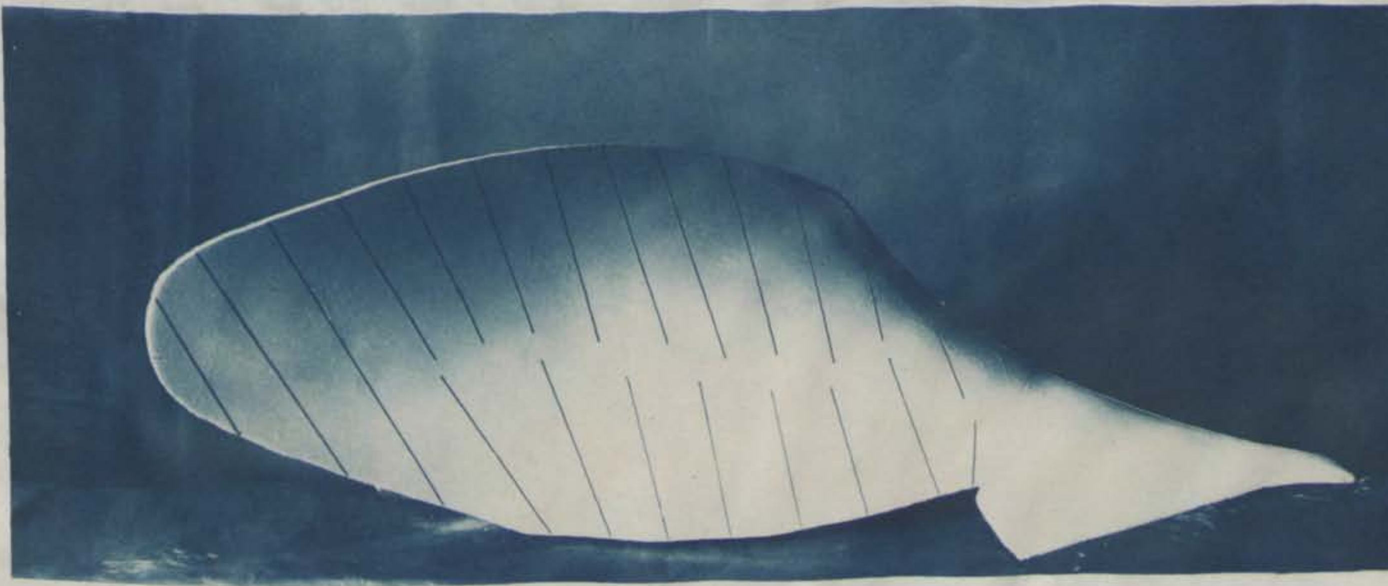

Fig. 5

will not completely describe such a surface. In Fig. 2 a bottom is shown whose surface is composed of a portion of each of two surfaces. Fig. 3 shows a similar bottom but in this case the two surfaces merge into each other further back upon the mouldboard.

In Fig. 4 a class of bottoms is represented whose entire surfaces do not contain an infinite set of straight lines. It is true that the share and back end of the mouldboard exhibit the same characteristics that the first two classes have shown, but the lines do not continue to the fore part of the mouldboard.

Fig. 5 shows a plow-bottom with a convex surface which has two sets of straight lines.

The American-manufactured plow-bottoms studied can thus be divided into three general classes, viz.; a portion of one quadric surface; a portion of each of two quadric surfaces; and non-quadric surfaces. Nearly all forged bottoms belong to classes one and two, with the majority falling into class two; while most of the cast bottoms belong to class three. It should be noted, however, that some recently designed cast bottoms depart from the general characteristics of class three and show clearly the two quadric surfaces of class two. The lines running in the general direction, front to rear, marked L, Fig. 1, will be called longitudinal lines; and those running

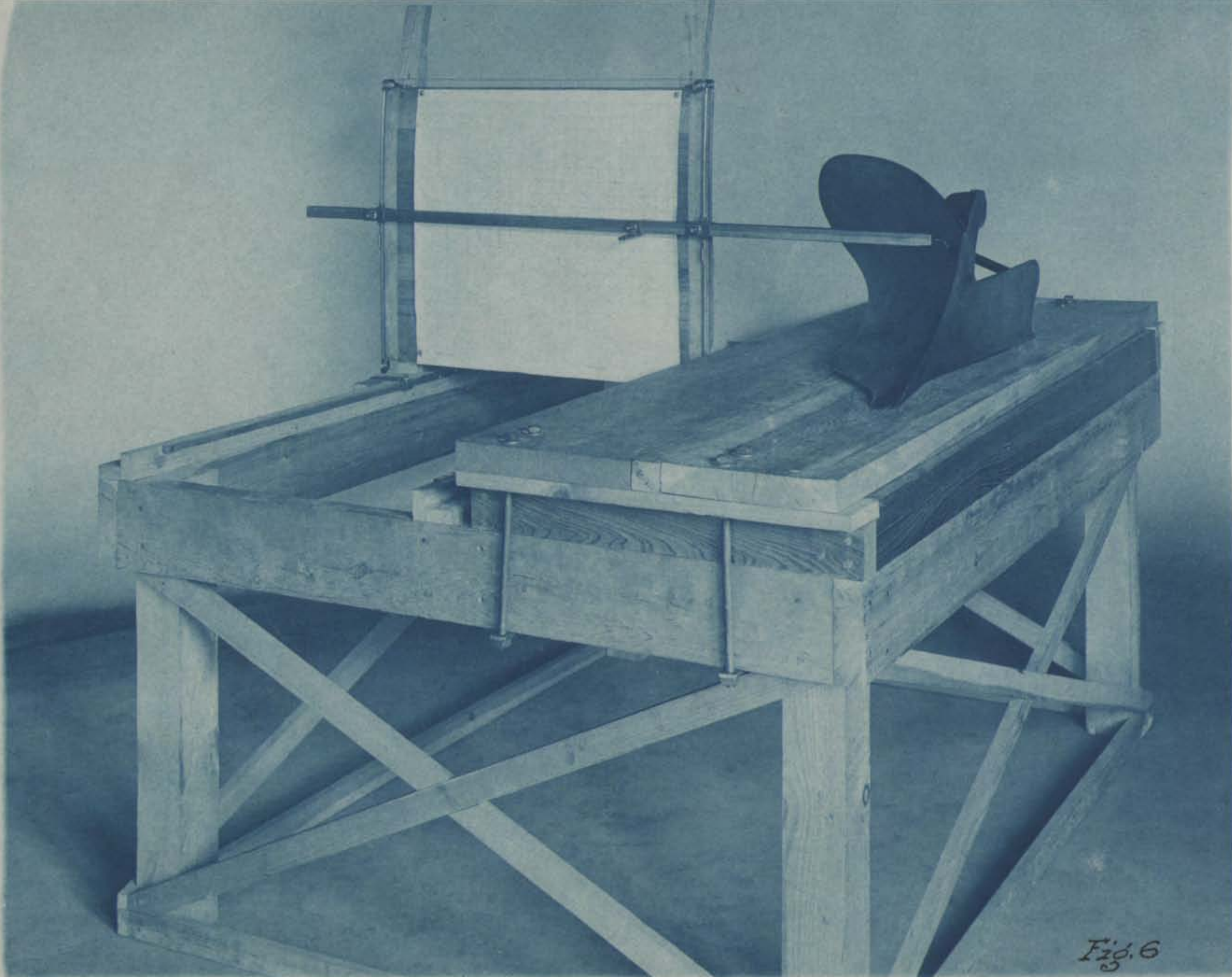

Fig. 6

in the general direction, top to bottom, marked T, Fig. 1, will be called transverse lines.

For the purpose of studying the forms of the various surfaces under consideration a machine, illustrated in Fig. 6, was designed and built by means of which the space coordinates of any desired point could be measured.* By means of slots and a system of pulleys attached to the drafting board, the cross bar can be kept horizontal and be moved both laterally and vertically, while the drafting board is attached to a frame which can be moved backward and forward upon guides so marked that the board in all positions will be squarely across the guides. When a plow-bottom is properly placed upon the platform the x, y and z coordinates of any point upon the surface can thus be recorded upon coordinate paper fastened upon the drafting board.

*Similar machines are described in the Report of the New York State Agricultural Society for 1867, vol. 1, page 426; and *Le Ricerche sperimentali di Meccanica Agraria*, by F. Giordano, page 110.

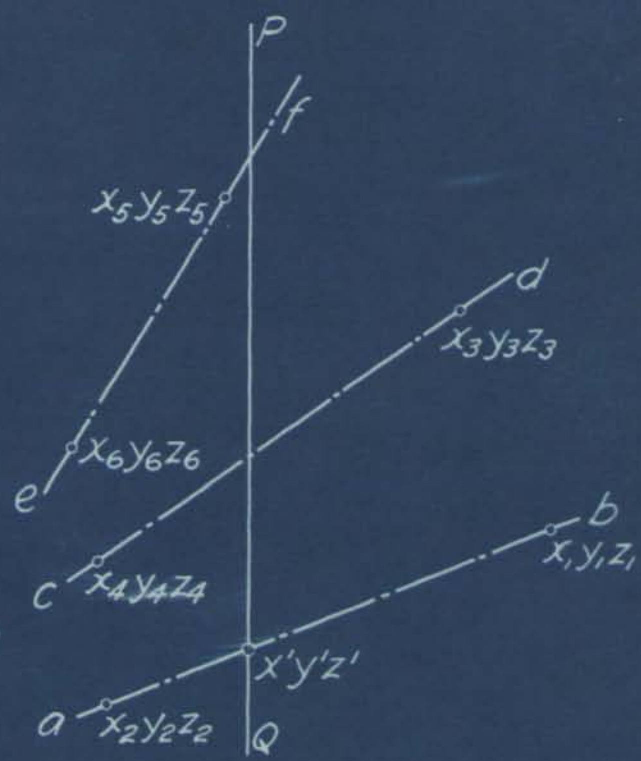

Fig. 7

Development of the Equation.

From a mathematical standpoint the surface shown in Fig. 1 presents the problem of finding the equation of a surface given two sets of straight line generators. This can be done if the equations of any three lines in the same set are known. Select three lines ab, cd, and ef, Fig. 7. Let x_1, y_1, z_1 and x_2, y_2, z_2 be the coordinates of two points upon line ab; x_3, y_3, z_3 and x_4, y_4, z_4 of two points upon line cd; and x_5, y_5, z_5 and x_6, y_6, z_6 of two points upon line ef.

The equations of the lines ab, cd, and ef, are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}; \qquad (1)$$

$$\frac{x - x_3}{x_4 - x_3} = \frac{y - y_3}{y_4 - y_3} = \frac{z - z_3}{z_4 - z_3}; \qquad (2)$$

and

$$\frac{x - x_5}{x_6 - x_5} = \frac{y - y_5}{y_6 - y_5} = \frac{z - z_5}{z_6 - z_5}. \qquad (3)$$

From (2) the following equation for a plane perpendicular to the XY-plane and containing the line cd is obtained:

$$u_4 \equiv (x - x_3)(y_4 - y_3) - (y - y_3)(x_4 - x_3) = 0. \qquad (4)$$

Similarly from (2) the equation of a plane perpendicular to the YZ-plane and containing the line cd is

$$u_5 \equiv (y - y_3)(z_4 - z_3) - (z - z_3)(y_4 - y_3) = 0. \qquad (5)$$

From (3), the equation of a plane perpendicular to the XY-plane and containing the line ef is

$$u_6 \equiv (x - x_5)(y_6 - y_5) - (y - y_5)(x_6 - x_5) = 0. \quad (6)$$

Similarly, from (3) the equation of a plane perpendicular to the YZ-plane and containing the line ef is

$$u_7 \equiv (y - y_5)(z_6 - z_5) - (z - z_5)(y_6 - y_5) = 0. \quad (7)$$

Consider

$$u_4 = Au_5 \quad (8)$$

where A is a constant. This is the equation of a plane which contains the intersection of planes (4) and (5); hence it contains the line cd.

Similarly

$$u_6 = Bu_7, \quad (9)$$

where B is a constant, is the equation of a plane which contains the line ef.

If A and B have such values that the point x', y', z', is on (8), (9) and (1) the line of intersection of (8) and (9) meets (1) and is a generator; (see Fig. 7). Hence,

$$A = \frac{(x' - x_3)(y_4 - y_3) - (y' - y_3)(x_4 - x_3)}{(y' - y_3)(z_4 - z_3) - (z' - z_3)(y_4 - y_3)}; \quad (10)$$

$$B = \frac{(x' - x_5)(y_6 - y_5) - (y' - y_5)(x_6 - x_5)}{(y' - y_5)(z_6 - z_5) - (z' - z_5)(y_6 - y_5)}; \quad (11)$$

and

$$\frac{x' - x_1}{x_2 - x_1} = \frac{y' - y_1}{y_2 - y_1} = \frac{z' - z_1}{z_2 - z_1} = K; \tag{12}$$

where K is a constant.

From equations (12)

$$x' = K(x_2 - x_1) + x_1 \tag{13}$$

$$y' = K(y_2 - y_1) + y_1 \tag{14}$$

$$z' = K(z_2 - z_1) + z_1 \tag{15}$$

From equations (10), (13), (14) and (15)

$$A = \frac{([K(x_2 - x_1) + x_1 - x_3](y_4 - y_3)) - ([K(y_2 - y_1) + y_1 - y_3](x_4 - x_3))}{([K(y_2 - y_1) + y_1 - y_3](z_4 - z_3)) - ([K(z_2 - z_1) + z_1 - z_3](y_4 - y_3))}; \qquad (16)$$

and from equation (8)

$$A = \frac{(x - x_3)(y_4 - y_3) - (y - y_3)(x_4 - x_3)}{(y - y_3)(z_4 - z_3) - (z - z_3)(y_4 - y_3)}. \qquad (17)$$

From equations (11), (13), (14) and (15)

$$B = \frac{([K(x_2 - x_1) + x_1 - x_5](y_6 - y_5)) - ([K(y_2 - y_1) + y_1 - y_5](x_6 - x_5))}{([K(y_2 - y_1) + y_1 - y_5](z_6 - z_5)) - ([K(z_2 - z_1) + z_1 - z_5](y_6 - y_5))}; \qquad (18)$$

and from equation (9)

$$B = \frac{(x - x_5)(y_6 - y_5) - (y - y_5)(x_6 - x_5)}{(y - y_5)(z_6 - z_5) - (z - z_5)(y_6 - y_5)}. \qquad (19)$$

Eliminating A, B and K from (16), (17), (18) and (19) we have the equation of a surface through the lines ab, cd and ef. The equations are left in this form because numerical substitutions are more easily made at this point than would be the case if the indicated operations were first performed with the symbols.* The general form of the equation resulting from the previous operations is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy$$
$$+ 2lx + 2my + 2nz + d = 0, \qquad (20)$$

To reduce equation (20) to its simplest form the axes must be translated and rotated.

Translation of Axes.‡

The origin of equation (20) is translated to the center by putting

$$x = x' + x_0, \qquad y = y' + y_0, \qquad z = z' + z_0; \qquad (21)$$

the values of x_0, y_0 and z_0 being obtained from the following:

$$ax_0 + hy_0 + gz_0 + l = 0 \qquad (22)$$

$$hx_0 + by_0 + fz_0 + m = 0 \qquad (23)$$

$$gx_0 + fy_0 + cz_0 + n = 0. \qquad (24)$$

These substitutions give, (after dropping the accents from x', y' and z'), an equation of the following form:

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy + G = 0; \qquad (25)$$

where $$G = lx_0 + my_0 + nz_0 + d. \qquad (25a)$$

*A numerical problem is developed by this method upon pages 14 to 19.

‡*Analytic Geometry of Space,* Snyder and Sisam, page 77.

Rotation of Axes.*

Equation (25) can be further reduced by a rotation of the axes. This is accomplished by means of a cubic equation

$$k^3 - (a + b + c)k^2 + (ab + ac + bc - f^2 - g^2 - h^2)k - D = 0; \tag{26}$$

where
$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}. \tag{26a}$$

Let the roots of (26) be k_1, k_2, and k_3. The desired equation, after translating and rotating the axes is

$$k_1x^2 + k_2y^2 + k_3z^2 + \frac{\Delta}{k_1k_2k_3} = 0;\ddagger \tag{27}$$

where
$$\Delta = DG. \tag{27a}$$

The direction cosines λ, μ, ν, of the angles which the new X-axis makes with the original axes are obtained from the following:

$$(a - k_1)\lambda + h\mu + g\nu = 0 \tag{28}$$

$$h\lambda + (b - k_1)\mu + f\nu = 0 \tag{29}$$

$$g\lambda + f\mu + (c - k_1)\nu = 0 \tag{30}$$

$$\lambda^2 + \mu^2 + \nu^2 = 1. \tag{31}$$

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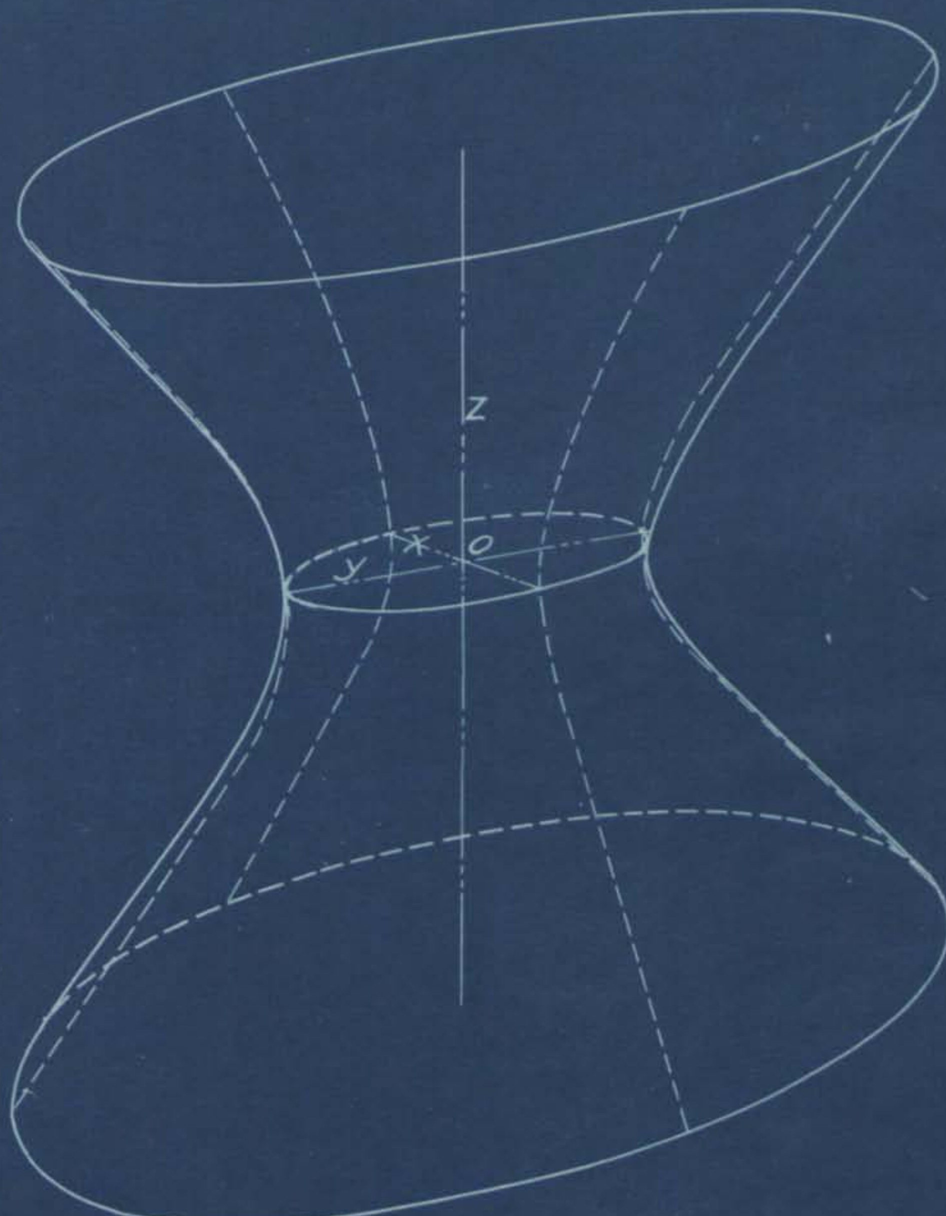
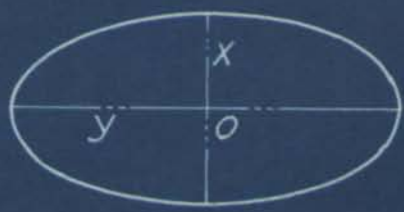
*Analytic Geometry of Space, Snyder and Sisam, page 79.

‡Analytic Geometry of Space, Snyder and Sisam, page 86.



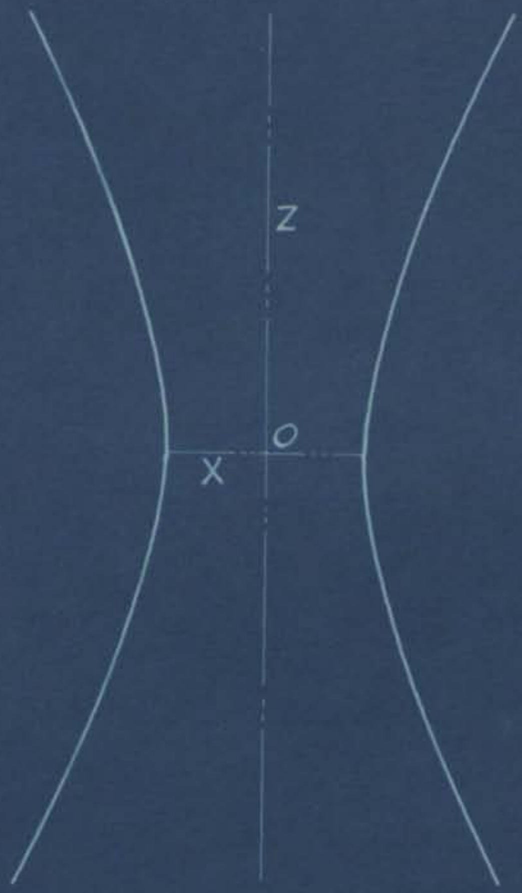
Skeleton. Hyperboloid of One Sheet

Fig. 8



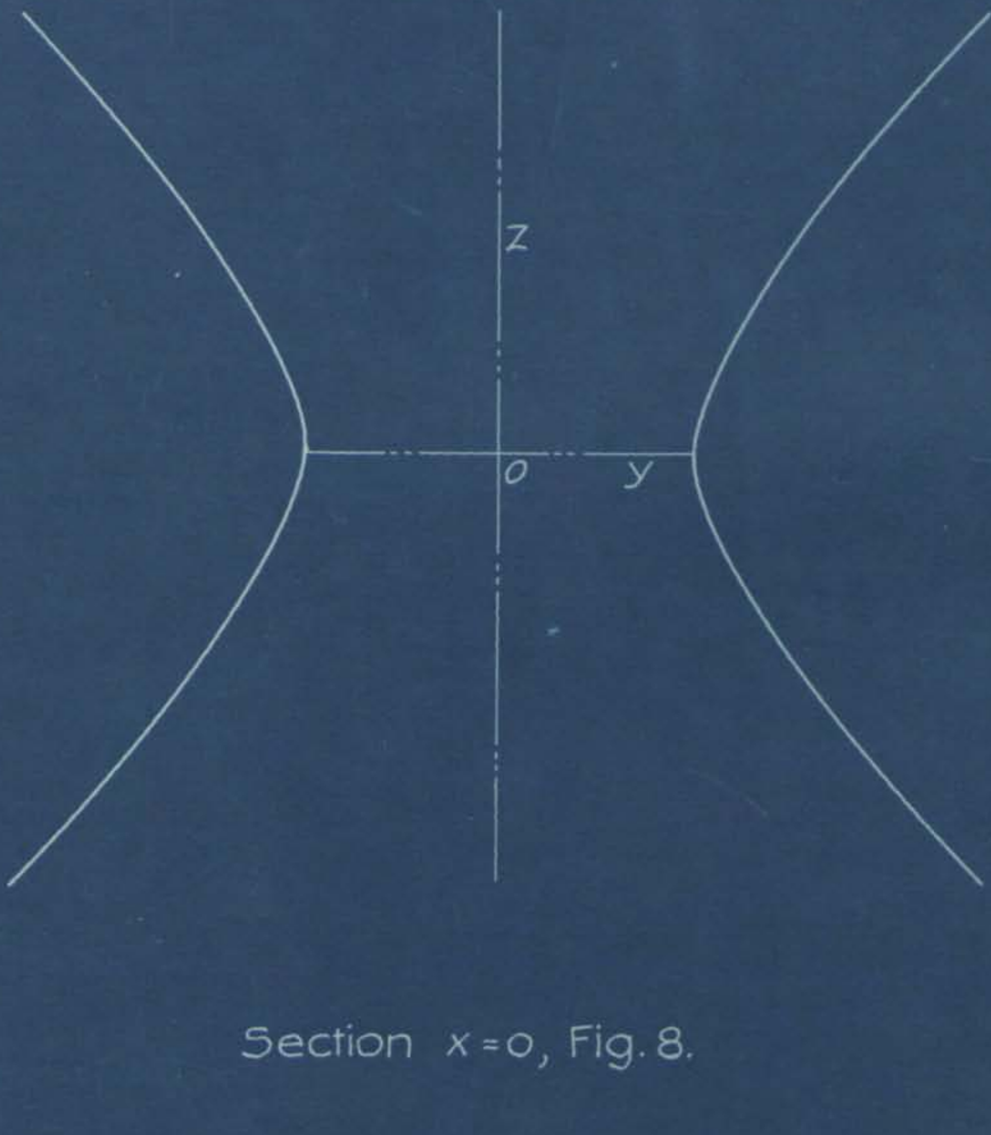
Section z=o, Fig. 8.

Fig.9



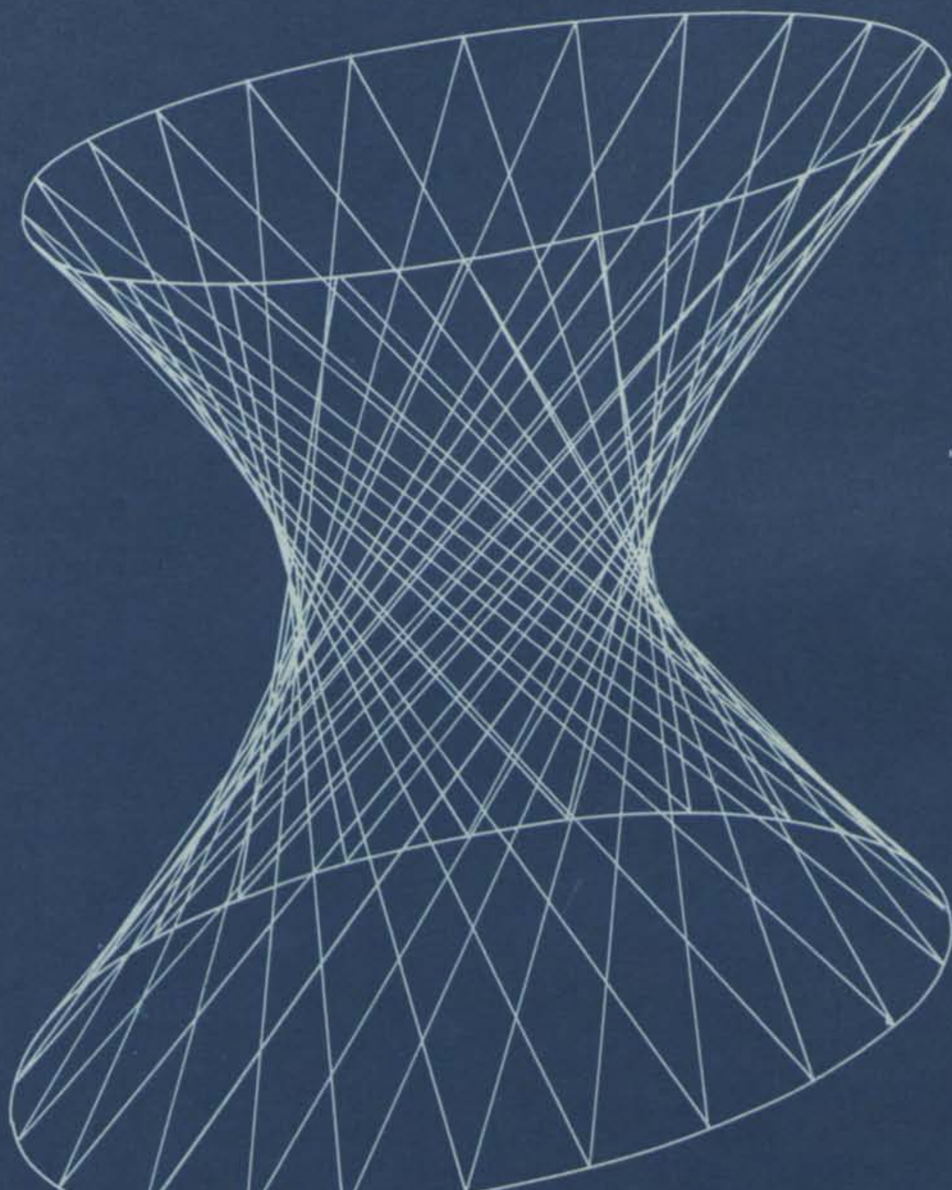
Section y=o, Fig. 8.

Fig. 10



Section $x = 0$, Fig. 8.

Fig. 11



Hyperboloid of One Sheet, showing Lines upon the Surface

Fig. 12

Similarly, the direction cosines of the angles which the Y- and Z-axes make, (after rotation), with the original axes are found by substituting k_2 and k_3 respectively for k_1 in equations (28), (29), (30) and (31).

When equation (27) was developed from the surface of a plow-bottom having two sets of straight line generators it had the following general form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1^* \qquad (32)$$

This is the equation of an hyperboloid of one sheet, a vase-shaped figure, the skeleton of a section of which is shown in Fig. 8. When $z = 0$ equation (32) becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the cross-section through the plane $z = 0$, Fig. 9, is an ellipse. When $y = 0$ the equation becomes $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$, and the section through the plane $y = 0$, Fig. 10 is an hyperbola. Similarly, when $x = 0$, $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, Fig. 11. Fig. 12 indicates the two sets of straight line generators which lie on the surface of an hyperboloid of one sheet.‡

*The constants a, b and c of this equation do not necessarily have the same numerical value as in previous equations.

‡The method for obtaining the equations of any line on the surface is given in *Analytic Geometry of Space*, Snyder and Sisam, page 93.

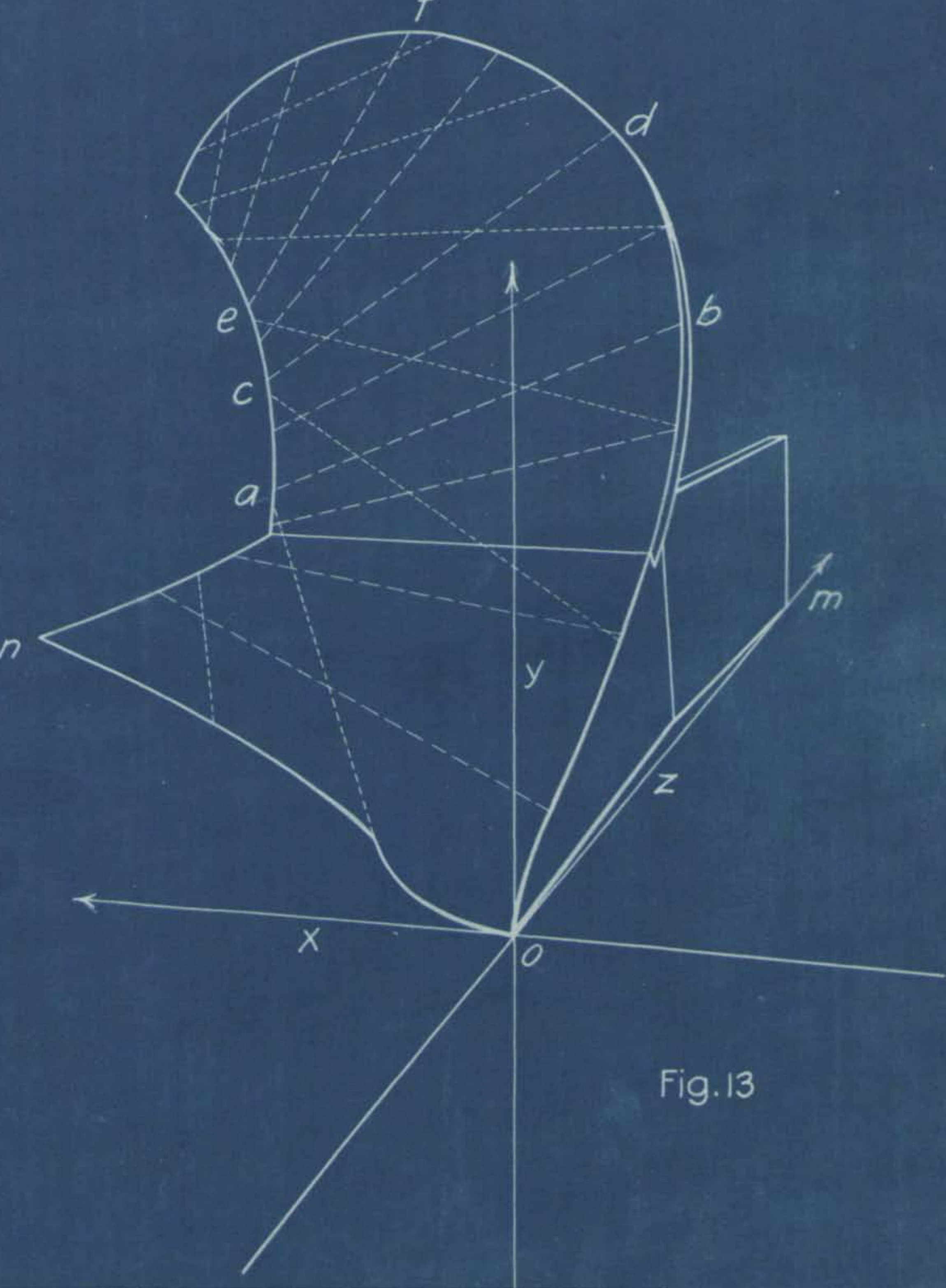


Fig. 13

Application of the Development to a Problem.

In order to develop the equation which will describe the surface of a plow-bottom it is necessary to obtain the data called for in equations (16), (17), (18) and (19). This application of the development will be carried through for the bottom represented in Fig. 1, which bottom was placed upon the machine shown in Fig. 6 so that the origin of co-ordinates came at O, Fig. 13. The plane $y = 0$ contains the points O, m and n; and the plane $x = 0$ contains the points O and m and is perpendicular to the plane $y = 0$. The plane $z = 0$ is perpendicular to both the planes $y = 0$ and $x = 0$. The axes are considered to be positive in the directions indicated by the arrow heads, Fig. 13. Three transverse lines ab, cd and ef, Fig. 13, were selected and the following data obtained:

Table I.*

$x_1 = 2.84$	$x_2 = 7.42$	$x_3 = 4.42$
$y_1 = 5.7$	$y_2 = 3.78$	$y_3 = 8.74$
$z_1 = 16$	$z_2 = 19$	$z_3 = 20$
$x_4 = 8.54$	$x_5 = 9.7$	$x_6 = 12.58$
$y_4 = 6.43$	$y_5 = 10.88$	$y_6 = 7.65$
$z_4 = 23$	$z_5 = 26$	$z_6 = 28$

*All measurements are reported in inches.

When the above values are substituted in the equations already developed;

From (16),

$$A = \frac{-\ 2.28K + 13.83}{K - 15.7}; \tag{33}$$

From (17),

$$A = \frac{-\ x - 1.78y + 20}{1.299y + z - 31.35}. \tag{34}$$

From (33) and (34)

$$K = \frac{15.7x + 10.04y - 13.83z + 11.95}{x - 1.177y - 2.28z + 51.45}. \tag{35}$$

From (18)

$$B = \frac{-\ 1.584K + 6.335}{K - 7.29}; \tag{36}$$

From (19)

$$B = \frac{-\ x - .892y + 19.4}{.619y + z - 32.74}. \tag{37}$$

From (36) and (37)

$$K = \frac{7.29x + 2.58y - 6.335z + 65.85}{x + .088y - 1.54z + 32.46}. \tag{38}$$

Eliminating K from equations (35) and (38) the following equation for the surface of the plow-bottom is obtained:

$$3.9x^2 + y^2 + 3.45z^2 - 7.53yz - 7.28xz + 6.79xy$$

$$+ 87.1x + 120.75y - 75.05z + 227.25 = 0. \qquad (39)$$

Table II is compiled for purposes of checking the values computed from equation (39) with those obtained by measuring.

Table II.*

z measured	y measured	x measured	x computed from eq. (39)	Difference
10	2	2.9	2.27	.63
15	6	1.53	1.56	- .03
15	4	3.58	3.77	- .19
15	2	6.9	6.32	.58
20	10	3.72	3.8	- .08
20	8	4.73	4.76	- .03
20	4	7.83	7.94	- .13
25	12	8.22	8.12	.1
25	9	9.07	9.2	- .13
25	6	10.43	10.46	- .03
30	10	14	13.86	.14
32	9	16.5	16.1	.4

To find the geometric center substitute the coefficients

*All measurements are given in inches.

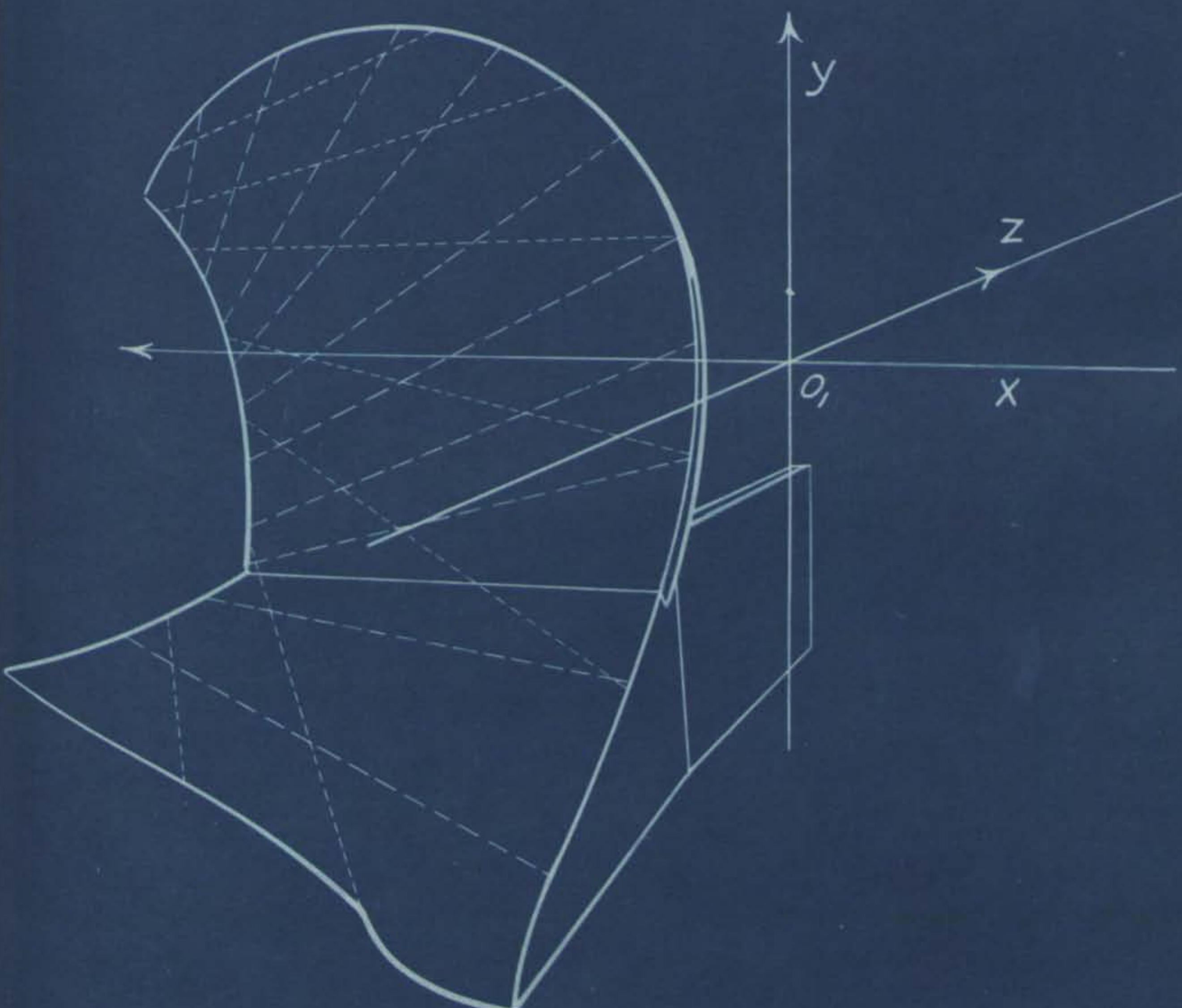


Fig. 14

$$10.27x^2 + .128y^2 - 2.05z^2 = 57.3$$

or

$$\frac{x^2}{(2.36)^2} + \frac{y^2}{(21.2)^2} - \frac{z^2}{(5.29)^2} = 1. \qquad (42)$$

The direction cosines of the angles which the axes make after rotation with the original axes are obtained by making the proper substitutions in equations (28), (29), (30), and (31).

For the X-axis

$$\lambda = \mp .6136$$

$$\mu = \mp .48$$

$$\nu = \pm .627.$$

For the Y-axis

$$\lambda = \pm .7515$$

$$\mu = \mp .1437$$

$$\nu = \pm .6445.$$

For the Z-axis

$$\lambda = \mp .1415$$

$$\mu = \pm .828$$

$$\nu = \pm .5425.$$

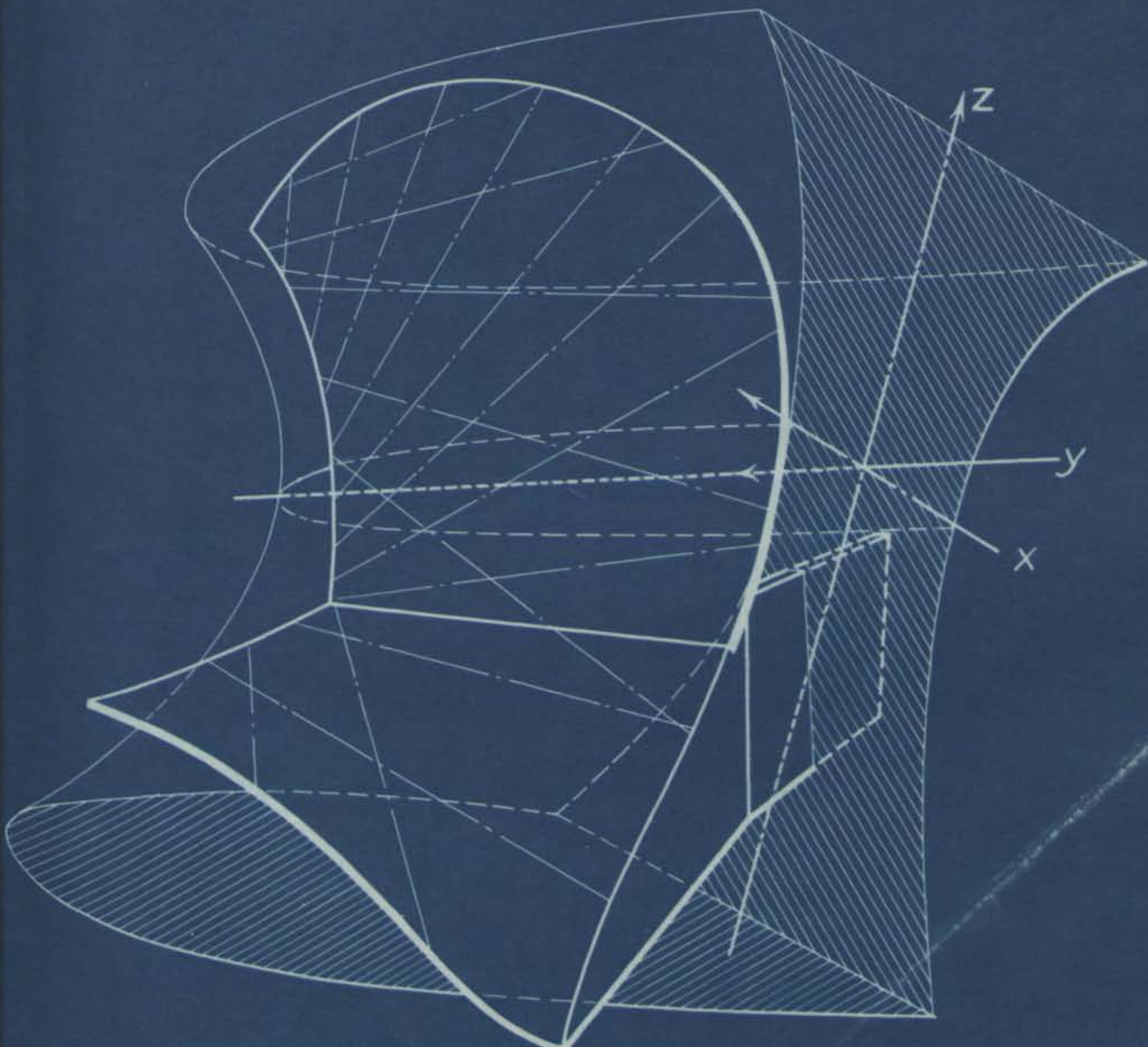


Fig. 15

Fig. 15 shows the axes after translation and rotation and the portion of the hyperboloid of one sheet which is a close approximation to the surface of this plow-bottom.

Surfaces One Portion from each of Two Quadric Surfaces.

By the use of the method which has just been employed to develop the equation of the surface of the plow-bottom shown in Fig. 1, two equations can be developed which will approximately represent the surface of the bottom shown in Fig. 2. Taking the origin, as at O, Fig. 13, the following data were obtained from the share and the front portion of the mouldboard:

Table III.*

$x_1 = 3.92$	$x_2 = 7.4$	$x_3 = 1.73$
$y_1 = .8$	$y_2 = .75$	$y_3 = 2.67$
$z_1 = 8$	$z_2 = 12$	$z_3 = 12$
$x_4 = 6.78$	$x_5 = 2.36$	$x_6 = 5.87$
$y_4 = 1.75$	$y_5 = 4.05$	$y_6 = 2.7$
$z_4 = 16$	$z_5 = 15$	$z_6 = 17.$

*All measurements are given in inches.

$$.25x^2 + 2.34y^2 + .46z^2 - 3.25yz - .77xz + 2.66xy + 6.88x + 32.3y - 5.81z - 4.4 = 0 \quad (43)$$

Table IV.*

z measured	y measured	x measured	x computed from eq (43)	Difference
10	1	4.75	4.75	0
10	2	1.75	1.54	.21
15	1	8.37	9.00	− .63
15	2	5.47	5.64	− .17
15	3	3.77	3.82	− .05
15	4	1.1	1.3	− .2

From the remaining surface of the mouldboard the following data were obtained:

Table V.*

$x_1 = 8.67$	$x_2 = 4.96$	$x_3 = 11.73$
$y_1 = 4.95$	$y_2 = 8.64$	$y_3 = 4.81$
$z_1 = 24$	$z_2 = 22$	$z_3 = 29$

*All measurements are given in inches.

$x_4 = 9.08$ $\quad x_5 = 13.62$ $\quad x_6 = 12.24$

$y_4 = 9$ $\quad y_5 = 6.23$ $\quad y_6 = 11.89$

$z_4 = 27$ $\quad z_5 = 33$ $\quad z_6 = 31.$

$$1.07x^2 - 1.07y^2 + z^2 - 3.99yz - 1.5xz + 16.37xy$$
$$+ 60.55x + 125.3y - 48.5z + 109.5 = 0. \quad (44)$$

Table VI.*

z measured	y measured	x measured	x computed from eq (44)	Difference
20	2	8.85	8.68	.17
20	4	6.67	6.78	- .11
20	6	4.9	4.95	- .05
20	8	3.4	3.5	- .1
25	3	10.6	10.5	.1
25	5	9.3	9.2	.1
25	7	8.23	8.12	.11
25	9	7.4	7.34	.06
25	11	6.82	6.77	.05
30	5	12.2	12.1	.1
30	7	11.7	11.63	.07
30	9	11.38	11.35	.03

*All measurements given in inches.

Table VI, (Cont).

z measured	y measured	x measured	x computed from eq (44)	Difference
30	11	11.3	11.24	.06
30	13	11.4	11.3	.1
35	5	14.65	14.53	.12
35	7	14.72	14.66	.06
35	9	15	14.93	.07
35	11	15.45	15.32	.13
35	13	16.1	15.85	.25
40	8	17.57	17.62	− .05
40	10	18.5	18.52	− .02

From a study of Tables II, IV and VI it is evident that the share cannot be as accurately described by mathematical equations as the mouldboard. However the differences even upon the share are not very great. It must be remembered that these surfaces have been developed empirically; experience and an extensive knowledge of the conditions to be met have been the chief guides. Yet, this implement produced in the school of experience has a surface approximately mathematically exact in form! Further, the surfaces of cast bottoms, which because of the difficulty of manufacture are not changed unless necessity demands, consist in some cases approximately of a portion from each of two quadric surfaces. It will be shown later in discussing the history of the plow that the surfaces of the Holbrook bottoms were designed to be portions of hyperboloids of one sheet. In the Utica Plow Trials these machines received many first awards and much commendation from the judges for the excellence of their work. In addition to this, Mr. J. J. Washburn, Vice President of the Wiard Plow Co., Batavia, New York, who knew Mr. Holbrook and was present at the Utica Plow Trials, testifies: "That the Holbrook plows did as good work as any that it has ever been his pleasure to witness." Thus there is considerable evidence, based upon field experience which indicates that a portion of a hyperboloid of one sheet is the proper form for the surface of a plow-bottom. As far as is known this hypothesis awaits definite proof.

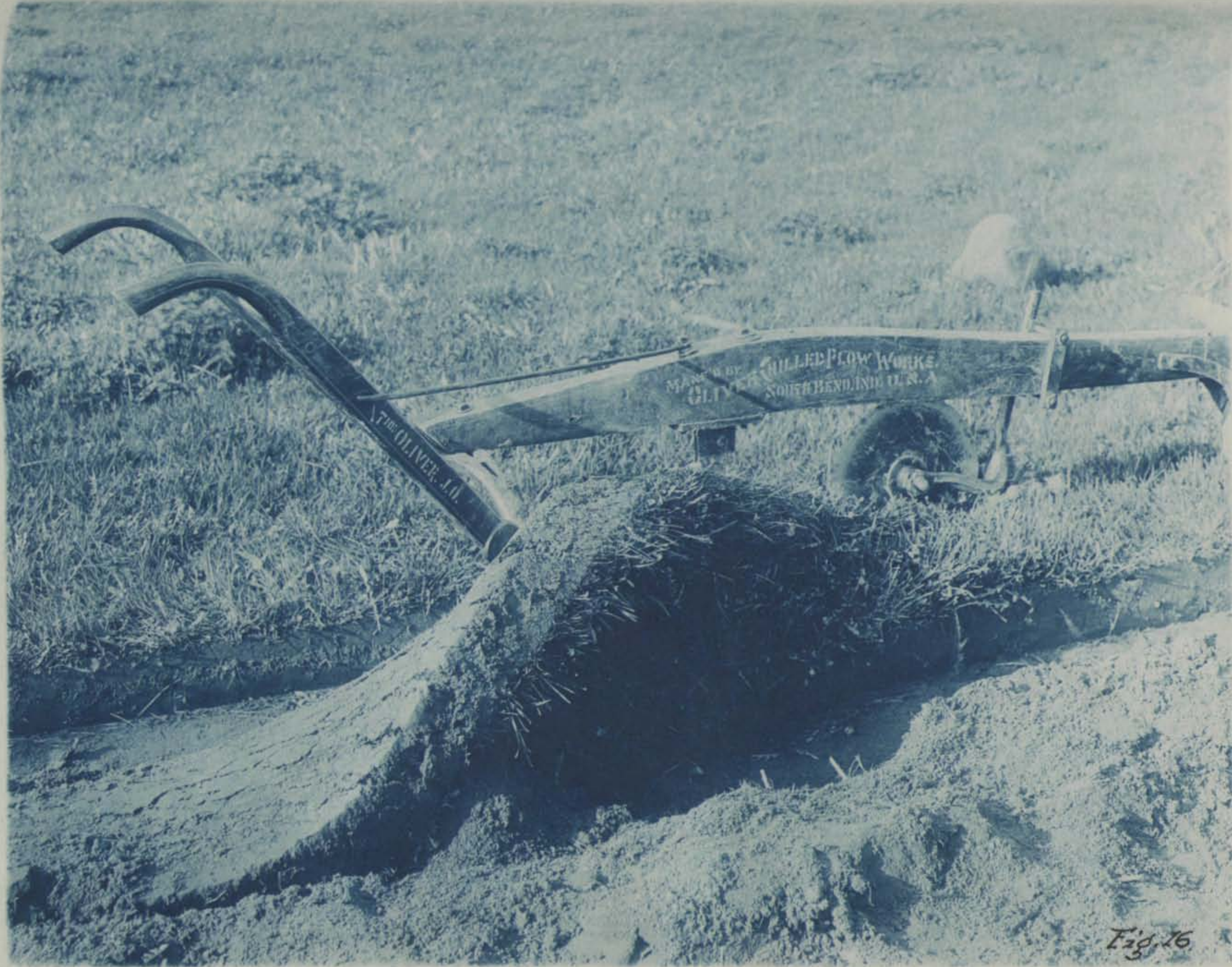


Fig. 26

MOTION OF THE SOIL PARTICLES IN PLOWING.

For the purpose of studying the motion of the soil particles in plowing, the work was limited to sod ground available in the vicinity of Ithaca, New York. From observations on a sod plow at work in the field, (see Fig. 16), the following general facts regarding the furrow-slice were noted:

The lower outside* edge of the furrow-slice did not appear to be either stretched or compressed.

The upper outside edge of the furrow-slice appeared to be compressed.

The inside of the furrow-slice was stretched and the lower edge more than the upper edge.

As the furrow-slice passed over the mouldboard the cracks which had formed on the inside in traveling over the share and the front portion of the mouldboard, closed up as the soil passed over the rear of the plow-bottom, indicating a point of maximum stretching.

The above considerations made it evident that a more detailed study of the behavior of the furrow-slice was desirable. For this purpose rows of pins, (the pins were driven in the ground to the estimated depth of plowing), were set

*The portion of the furrow-slice immediately adjacent to the furrow is called the outside.


Fig. 17

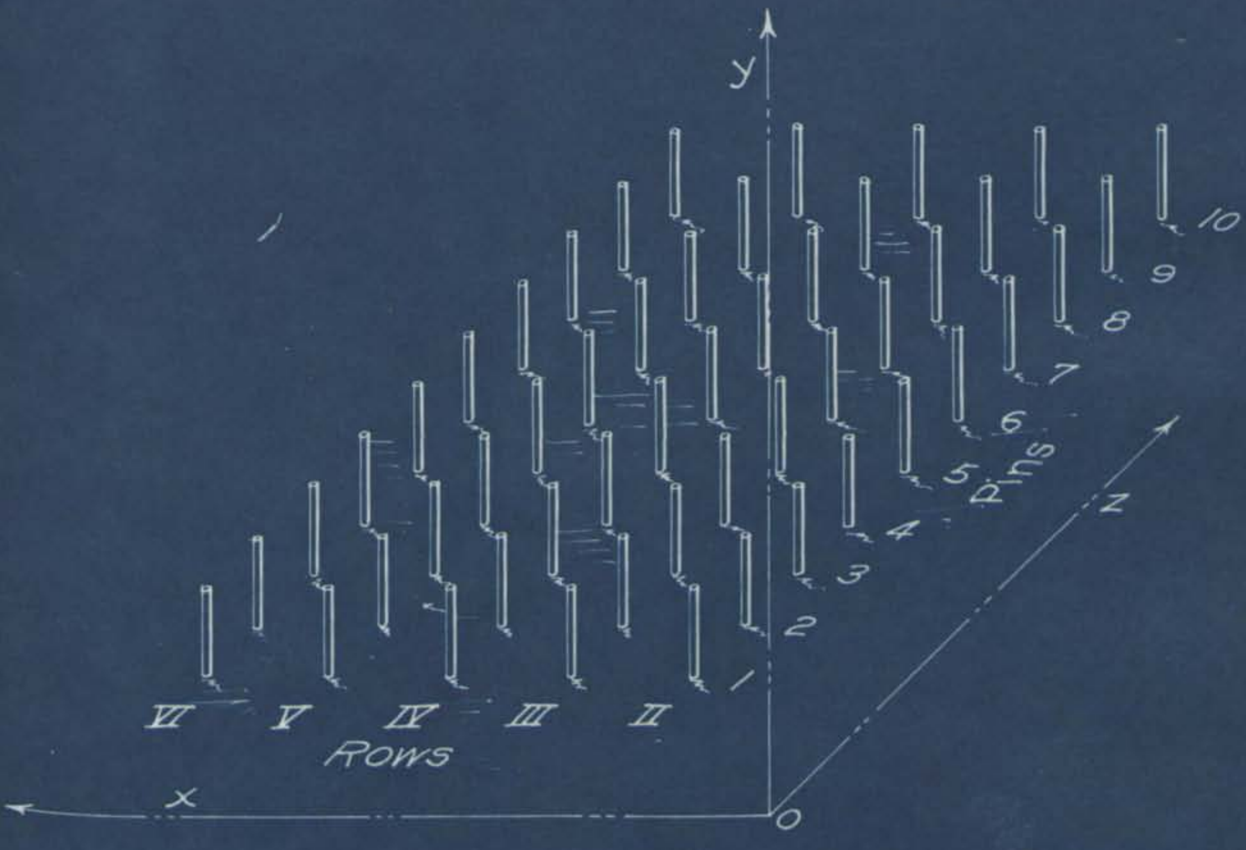


Fig. 18

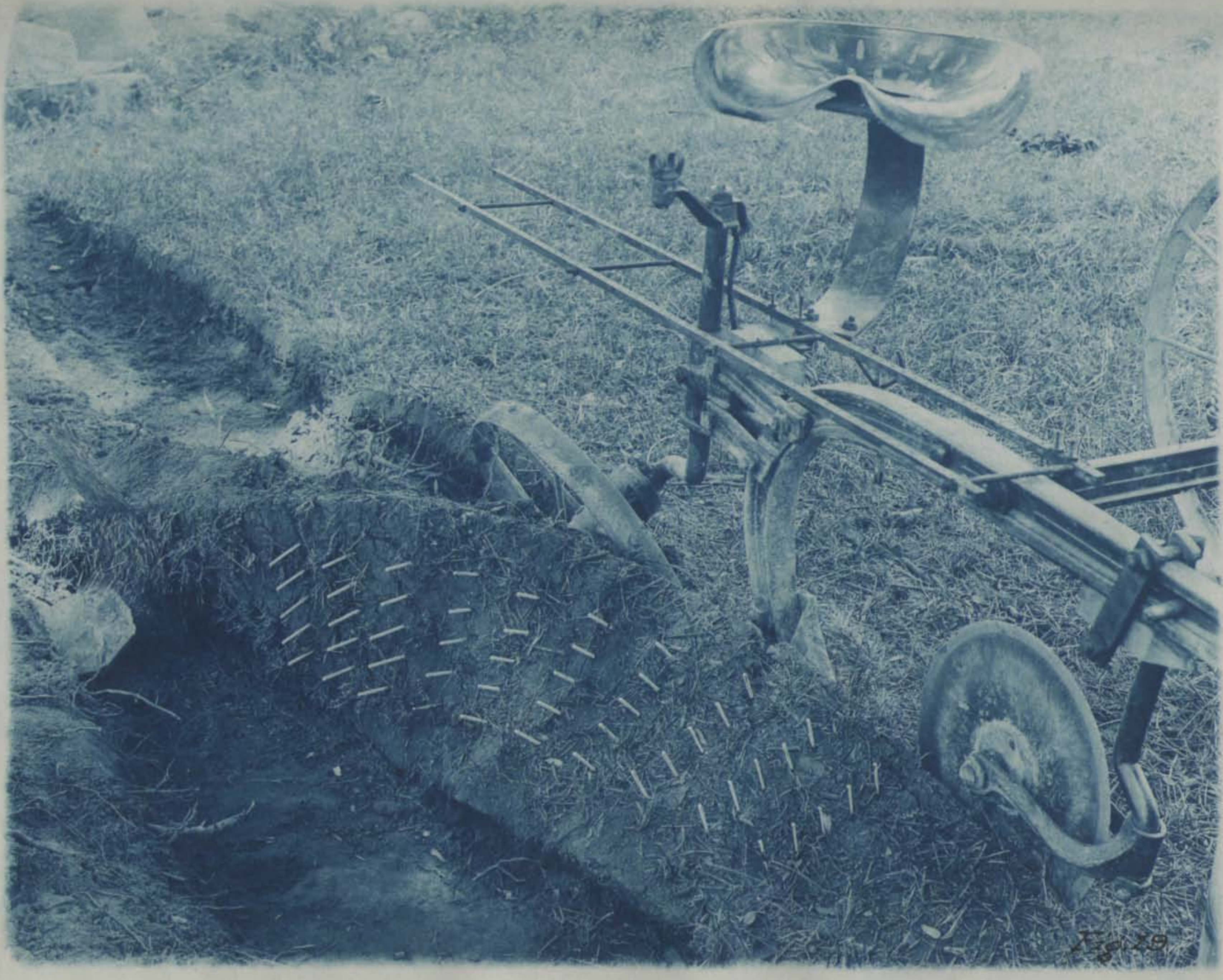


Fig. 29

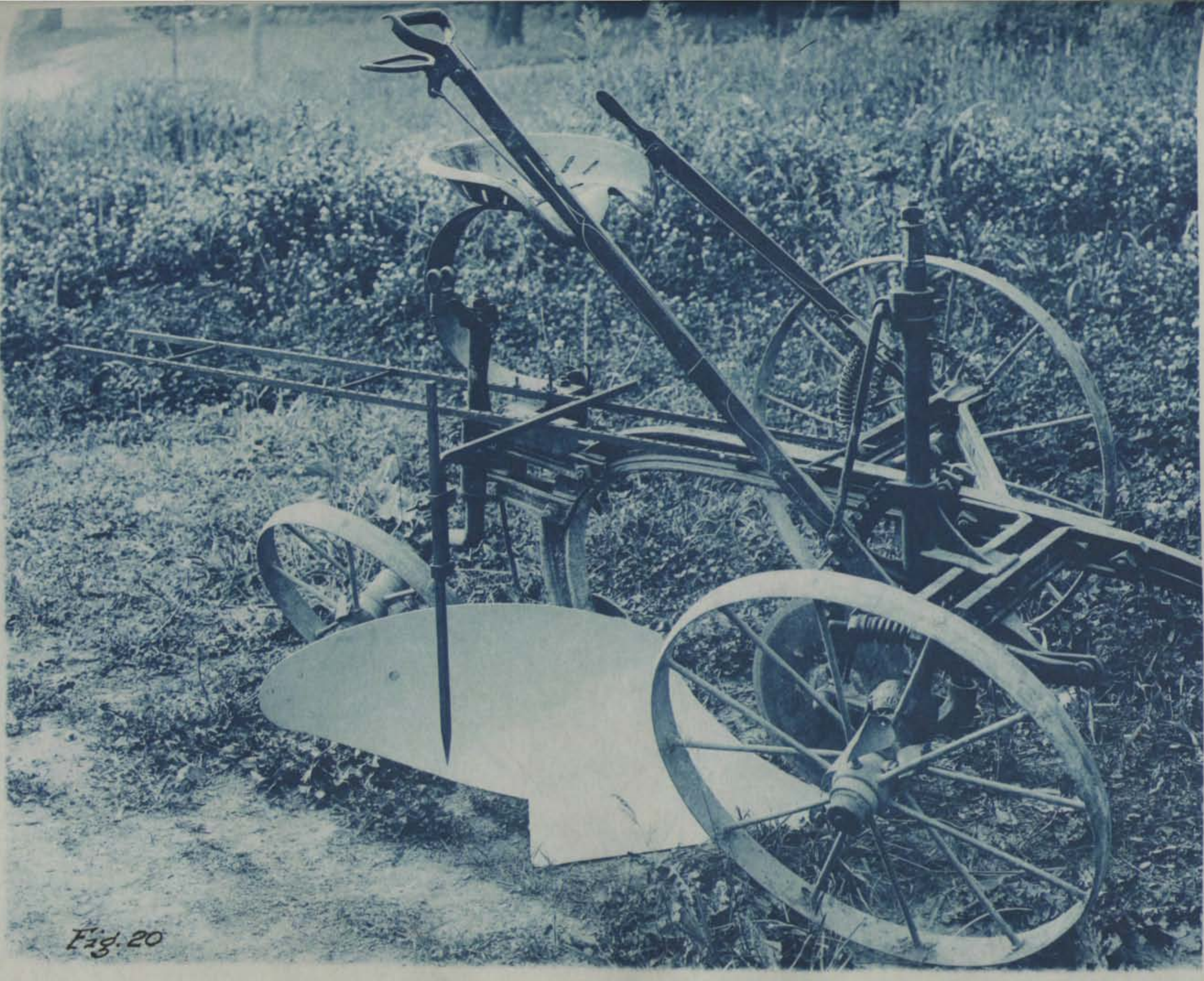


Fig. 20

in the unplowed ground as shown in Fig. 17. The longitudinal rows are parallel to the line of motion of the plow; which is also parallel to the Z-axis, Fig. 13, and the transverse rows perpendicular to this same line of motion. The longitudinal rows are numbered from II to VI, (Row I was omitted because the colter upset them), and the pins in each row numbered from 1 to 10; as shown in Fig. 18. When the part of the furrow-slice in which the pins were set was upon the mouldboard it took the form shown in Fig. 19. In order to obtain the x, y and z coordinates of points in the furrow-slice upon the mouldboard the apparatus shown in Fig. 20 was used. In this apparatus the axes have the same relation to the plow-bottom as those shown in Fig. 13. This more detailed study of the furrow-slice upon the mouldboard revealed the following:

The length of row II, pin 1 to pin 10, on top of the furrow-slice was greater than the length before the soil had passed upon the mouldboard indicating that this portion of the furrow-slice had been stretched.

The length of row II, pin 1 to pin 10, was greater upon the bottom of the furrow-slice than its length before the soil passed upon the mouldboard.

The length of row VI, pin 1 to pin 10, on top of the furrow-slice was less than its length before the soil passed upon the mouldboard, indicating that this portion of the

Fig. 21

furrow-slice had been compressed.

The length of row VI, pin 1 to pin 10, on the bottom of the furrow-slice was greater than its length before the soil passed upon the mouldboard.

The lengths of rows IV and V, pin 1 to pin 10, on top of the furrow-slice were approximately the same as their lengths before the soil passed upon the plow-bottom, indicating neither compression nor stretching.

The lengths of rows IV and V, pin 1 to pin 10, on the bottom of the furrow-slice was greater than their lengths before the soil had passed upon the plow-bottom.

The z distances of pins 10 on top of the furrow-slice were approximately the same for each row but less than the distance which the plow had moved forward.

The z distances of pins 10 on the bottom of the furrow-slice were approximately the same for each row and equal to the distance which the plow had moved forward. (The coordinates of the pins at the bottom of the furrow-slice were measured by cutting away a portion of the soil but leaving the pins in place).

These observations reveal firstly; that when a cross-section of the furrow-slice is considered, Fig. 21, the portion marked A is compressed in plowing and the portion marked B is stretched, while the soil in the position of line 1j is

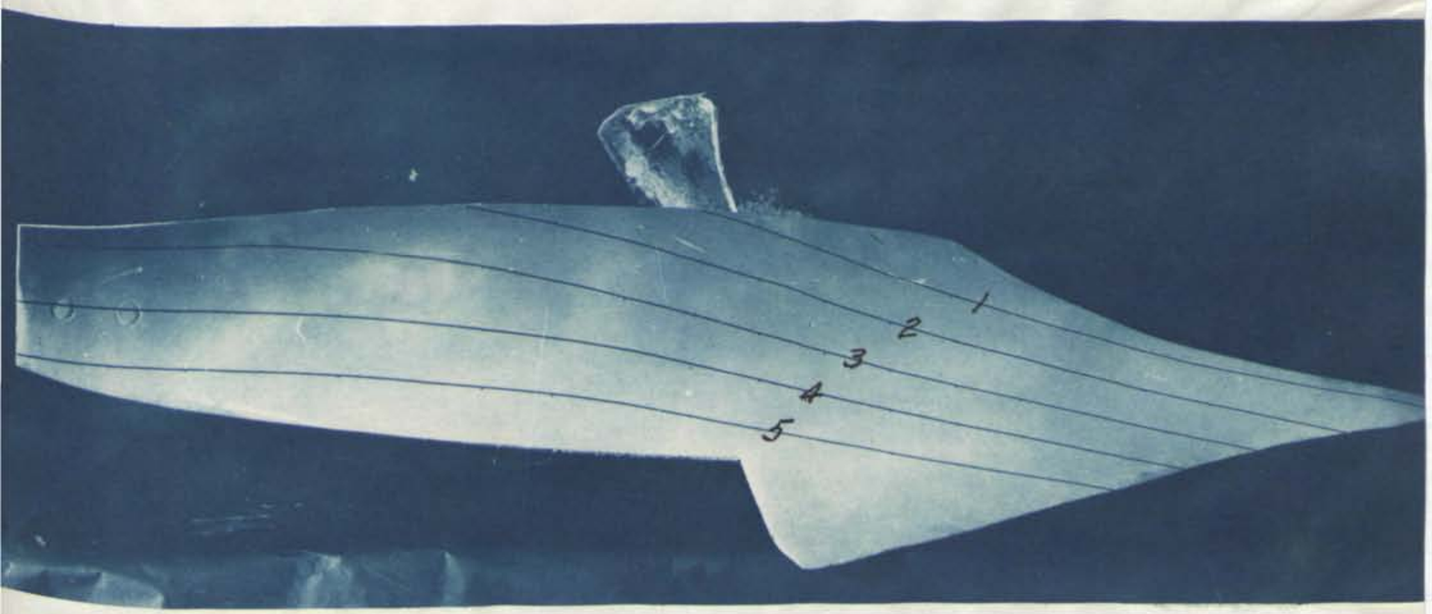


Fig. 22

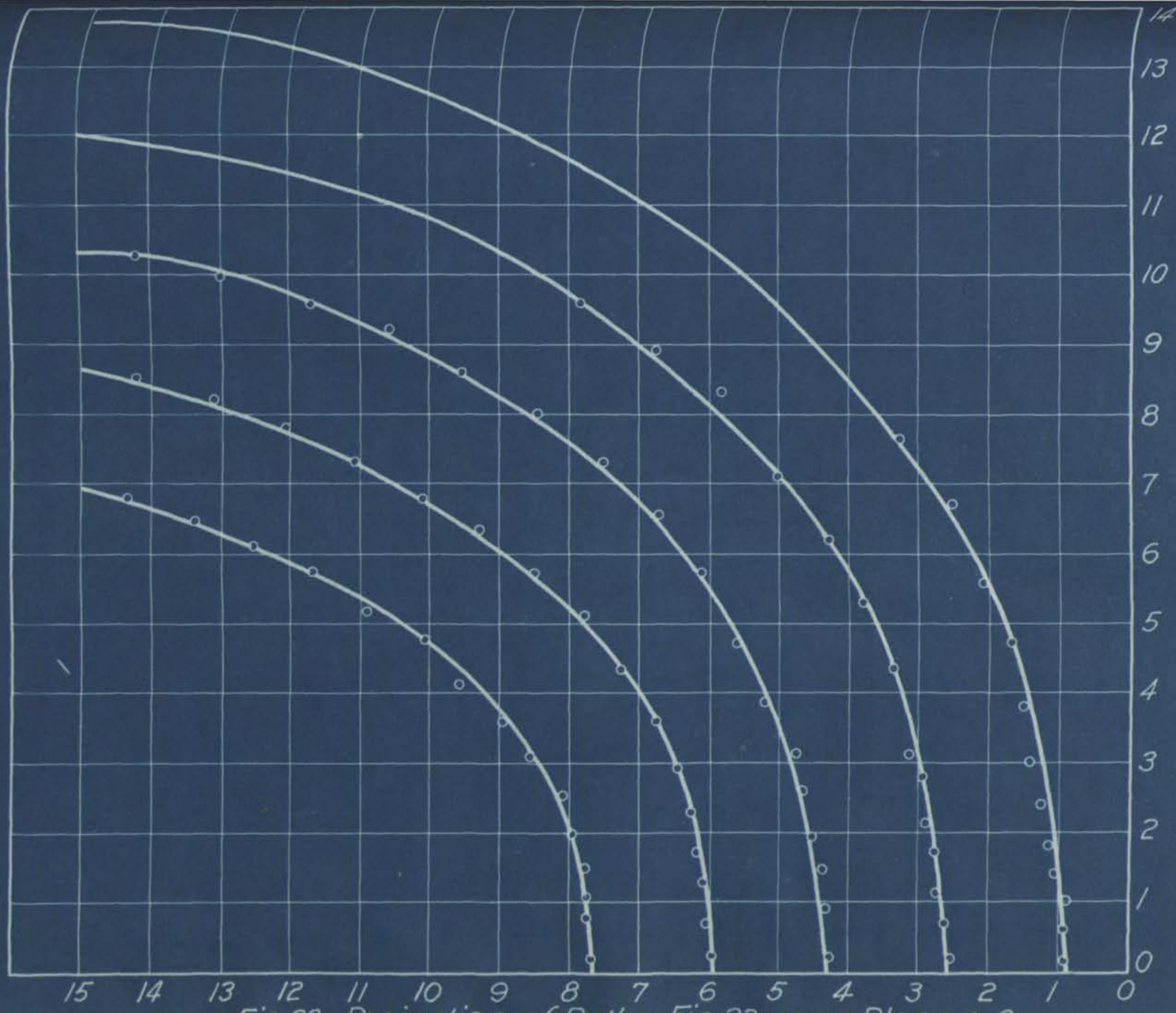


Fig.23. Projection of Paths, Fig.22 upon Plane z=0

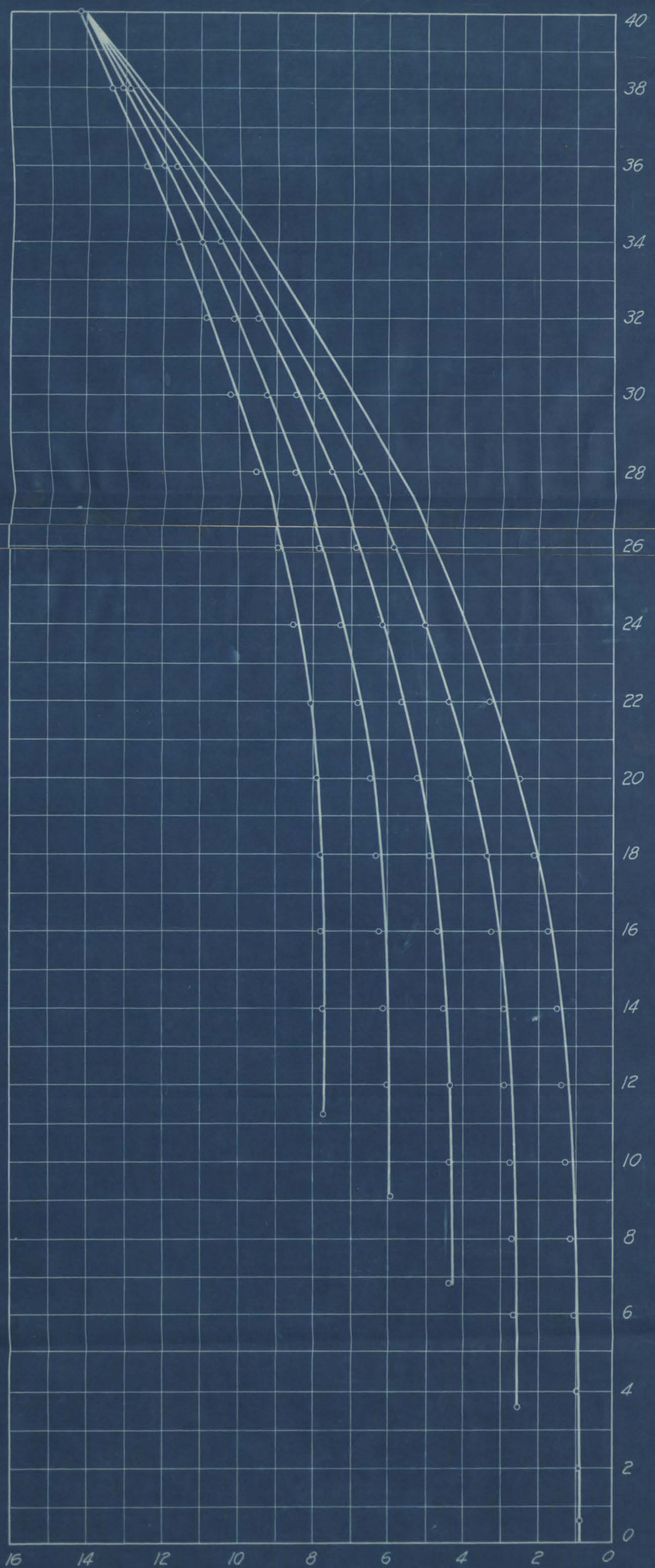


Fig. 24. Projection of the Paths Shown in Fig. 22 upon the Plane y=0

neither compressed nor stretched; and secondly that there is a definite relation between the z coordinate of a soil particle and the distance the plow has moved forward. This relation is developed on pages 28 to 32.

The next step was to analyse in detail the motion of the soil particles. This study was limited to the soil particles upon the bottom of the furrow-slice but the methods developed are applicable to other portions of the furrow-slice. The paths of the soil particles upon the bottom of the furrow-slice can be very accurately traced from the scratches which they make upon the mouldboard. Fig. 22 shows the paths of five soil particles. Taking the axes as shown in Fig. 13, a projection of these paths upon the plane $z = 0$ showed a very uniform set of curves. Each of these curves, shown in Fig. 23, can be very accurately described by equations of the general form

$$ax^2 + by^2 + lx + my + d = 0, \tag{45}$$

When these same paths are projected upon the plane $y = 0$ a set of curves resulted, Fig. 24, each of which could be very accurately described by equations having the following general form:

$$ax^2 + bz^2 + lxz + mx + nz + d = 0. \tag{46}$$

From equation (45) $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$, the velocity and acceleration respectively of a soil particle in the y direction can be found if $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ are known. The values of $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ can be found from equation (46) if $\frac{dz}{dt}$ and $\frac{d^2z}{dt^2}$ are known. Thus to analyse the velocity and acceleration of any soil particle whose path upon the surface of the plow-bottom is known, an equation must be found between z and time, (t). This was accomplished by comparing the z coordinates of the bottom ends of the pins with the distance which the plow had moved forward. The distance which the plow moved forward is designated by s, so that

$$s = vt, \tag{47}$$

where v = velocity of the plow, and t = time.

By the use of the apparatus illustrated in Fig. 20 the data given in Table VII were obtained for the soil particles upon the bottom of the furrow-slice whose paths are shown in Fig. 22. These data are typical of twelve sets of observations.

Table VII.*

Row II

z	s	z - s
$16\frac{1}{2}$	$15\frac{3}{4}$	$\frac{3}{4}$
$20\frac{1}{2}$	$19\frac{3}{4}$	$\frac{3}{4}$
24	$23\frac{3}{4}$	$\frac{1}{4}$
$27\frac{3}{4}$	$27\frac{3}{4}$	0
$32\frac{1}{4}$	$31\frac{3}{4}$	$\frac{1}{2}$
$35\frac{5}{8}$	$35\frac{3}{4}$	$-\frac{1}{8}$
$39\frac{3}{8}$	$39\frac{3}{4}$	$-\frac{3}{8}$

Row III

z	s	z - s
16	$15\frac{3}{4}$	$\frac{1}{4}$
$20\frac{3}{8}$	$19\frac{3}{4}$	$\frac{5}{8}$
$23\frac{7}{8}$	$23\frac{3}{4}$	$\frac{1}{8}$
$27\frac{5}{8}$	$27\frac{3}{4}$	$-\frac{1}{8}$
$32\frac{1}{8}$	$31\frac{3}{4}$	$\frac{3}{8}$
$35\frac{3}{4}$	$35\frac{3}{4}$	0
$39\frac{1}{2}$	$39\frac{3}{4}$	$-\frac{1}{4}$

Row IV

z	s	z - s
$15\frac{7}{8}$	$15\frac{3}{4}$	$\frac{1}{8}$
$19\frac{7}{8}$	$19\frac{3}{4}$	$\frac{1}{8}$
$24\frac{1}{4}$	$23\frac{3}{4}$	$\frac{1}{2}$
$27\frac{1}{2}$	$27\frac{3}{4}$	$-\frac{1}{4}$
32	$31\frac{3}{4}$	$\frac{1}{4}$
$35\frac{7}{8}$	$35\frac{3}{4}$	$\frac{1}{8}$
$39\frac{5}{8}$	$39\frac{3}{4}$	$-\frac{1}{8}$

Row V

z	s	z - s
$15\frac{3}{4}$	$15\frac{3}{4}$	0
$20\frac{1}{4}$	$19\frac{3}{4}$	$\frac{1}{2}$
$23\frac{3}{4}$	$23\frac{3}{4}$	0
$27\frac{1}{4}$	$27\frac{3}{4}$	$-\frac{1}{2}$
$31\frac{7}{8}$	$31\frac{3}{4}$	$\frac{1}{8}$
$35\frac{1}{4}$	$35\frac{3}{4}$	$-\frac{1}{2}$
$39\frac{1}{4}$	$39\frac{3}{4}$	$-\frac{1}{2}$

*All measurements are given in inches.

Unfortunately the soil available in the vicinity of Ithaca, New York was not well adapted for taking observations of the kind reported in Table VII. This soil is not uniform in texture, contains many stones, cracks much more readily than it stretches and the surface is not as level as could be desired for this work. At times it was difficult to drive the pins straight into the ground. The data of Table VII show, however, a distinct tendency for the difference between z and s to reach a maximum value and then decrease again to zero; and also a slight tendency for this maximum difference to decrease from Row I to Row V. When the work was commenced it was hoped that sufficiently accurate data could be obtained from which a law between z and s could be developed but on account of the difficulties already explained this was impossible; consequently, in order to develop a method for future work, a set of conditions were assumed which agreed qualitatively with the observed facts. It should always be kept in mind that this was done simply as an hypothesis whose exactness should be thoroughly tested upon the soil better adapted to this work. The conditions assumed for the relations between z and s are as follows:

That, for each path, when $z = 40$, $s = 40$.

That there was no stretching or compression in the outside bottom edge of the furrow-slice up to the point $z =$

40.

That the maximum difference, $z - s$, for path I was 1.05."

That the maximum difference, $z - s$, for each path decreased uniformly across the furrow-slice. Thus for Row I, $x = .85$", the maximum $z - s = 1.05$", and when $x = 13.6$" the width of the furrow-slice, the maximum $z - s = 0$; so when $x = 7.5$" the maximum $z - s$ for Row V is .45".

That the stretching in each row took place uniformly up to the maximum point and then decreased uniformly until it was zero when $z = s = 40$.

That the maximum stretching occured midway between the point where the soil particle passed upon the plow-bottom and the point $s = 40$. Thus for path I where the soil particle passed upon the mouldboard at the point $s = .6$;

$$40 - .6 = 39.4"$$

$$39.4 \div 2 = 19.7"$$

$$19.7 + .6 = 20.3"$$

For path I the point of maximum stretching was at $s = 20.3$".

The computations below show that for path V, where the soil particle passed upon the share at the point $s = 11.6$, the point of maximum stretching occurs at $s = 25.8$".

$$40" - 11.6" = 28.4"$$

$$28.4" \div 2 = 14.2"$$

$$14.2'' + 11.6'' = 25.8''$$

The following is the simplest form of a function which meets the requirements imposed by the above conditions and, when the constants are determined, will describe the relations between z and s for a soil particle on the bottom of the furrow-slice as it passes over the surface of the plow-bottom.

$$z - s = a(s^2 + bs + c)^2 \tag{48}$$

From equations (47) and (48)

$$z - vt = a[(vt)^2 + bvt + c]^2; \tag{49}$$

From (49) $\frac{dz}{dt}$ and $\frac{d^2z}{dt^2}$, the velocity and acceleration respectively of a soil particle in the z direction, can be obtained.

From equation (46), by differentiation, we have

$$(2ax + lz + m)\frac{dx}{dt} + (2bz + lx + n)\frac{dz}{dt} = 0; \tag{50}$$

and

$$(2ax + lz + m)\frac{d^2x}{dt^2} + (2a\frac{dx}{dt} + l\frac{dz}{dt})\frac{dx}{dt}$$

$$+ (2bz + lx + n)\frac{d^2z}{dt^2} + (2b\frac{dz}{dt} + l\frac{dx}{dt})\frac{dz}{dt} = 0. \tag{51}$$

Similarly from equation (45) we find

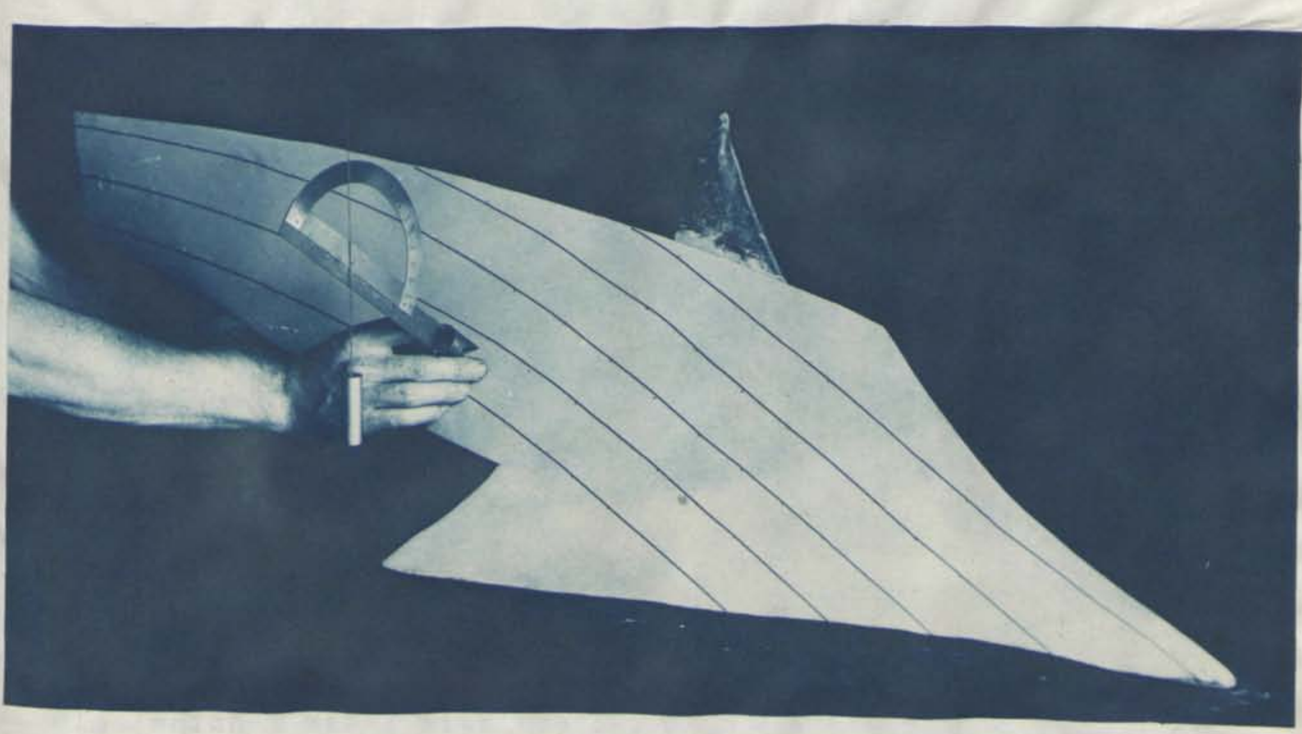


Fig. 25

$$(2ax + l)\frac{dx}{dt} + (2by + m)\frac{dy}{dt} = 0; \quad (52)$$

and

$$(2ax + l)\frac{d^2x}{dt^2} + 2a(\frac{dx}{dt})^2 + (2by + m)\frac{d^2y}{dt^2} + 2b(\frac{dy}{dt})^2 = 0. \quad (53)$$

From equations (50), (51), (52) and (53) the velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$ and the accelerations $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, of a soil particle on the bottom of the furrow-slice can be obtained when $\frac{dz}{dt}$ and $\frac{d^2z}{dt^2}$ are known.

In this problem however, we are interested in the accelerations in the directions of the normal to the surface, designated by N; the tangent to the soil path, T; and the perpendicular to the plane formed by the normal and the tangent, R.

We can find λ_1, μ_1, ν_1, the direction cosines of the angles which N makes with the X-, Y-, and Z-axis, in either of the following ways:

If (20), (the equation of the surface of the plow-bottom), is known we have by differentiation

$$\frac{\lambda_1}{ax_0 + by_0 + gz_0 + l} = \frac{\mu_1}{hx_0 + by_0 + fz_0 + m}$$

$$= \frac{\nu_1}{gx_0 + fy_0 + cz_0 + n} = \quad (54)$$

$$\frac{1}{\sqrt{(ax_0 + by_0 + gz_0 + l)^2 + (hx_0 + by_0 + fz_0 + m)^2 + (gx_0 + fy_0 + cz_0 + n)^2}};$$

or if the paths of the soil particles are known but the equation of the surface is unknown, the angle Ny can be measured by means of a protractor and plumb-bob, as shown in Fig. 25. The direction cosines λ_1 and ν_1 can then be computed from the following:

$$(\lambda_1)^2 + (\mu_1)^2 + (\nu_1)^2 = 1 \tag{55}$$

$$\lambda_1\frac{dx}{dt} + \mu_1\frac{dy}{dt} + \nu_1\frac{dz}{dt} = 0; \tag{56}$$

where the values for $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ can be obtained from (49), (50) and (52).

The direction cosines of T, (λ_2, μ_2, ν_2), are proportional to $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$. Hence

$$\frac{\lambda_2}{\frac{dx}{dt}} = \frac{\mu_2}{\frac{dy}{dt}} = \frac{\nu_2}{\frac{dz}{dt}} = \frac{1}{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}} \tag{57}$$

The direction cosines of R, (λ_3, μ_3, ν_3), can be computed from the following:*

$$(\lambda_3)^2 + (\mu_3)^2 + (\nu_3)^2 = 1 \tag{58}$$

*_Analytic Geometry of Space_, Snyder and Sisam, page 40.

$$\frac{\lambda_1}{\mu_3\nu_2 - \nu_3\mu_2} = \frac{\lambda_2}{\mu_3\nu_1 - \nu_3\mu_1} = \frac{\lambda_3}{\mu_1\nu_2 - \nu_1\mu_2} = \pm 1 \qquad (59)$$

The components in the directions N, T and R of the forces acting on a soil element of mass M, moving with the component accelerations $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$ are

$$F_N = M(\lambda_1\frac{d^2x}{dt^2} + \mu_1\frac{d^2y}{dt^2} + \nu_1\frac{d^2z}{dt^2}) \qquad (60)$$

$$F_T = M(\lambda_2\frac{d^2x}{dt^2} + \mu_2\frac{d^2y}{dt^2} + \nu_2\frac{d^2z}{dt^2}) \qquad (61)$$

$$F_R = M(\lambda_3\frac{d^2x}{dt^2} + \mu_3\frac{d^2y}{dt^2} + \nu_3\frac{d^2z}{dt^2}) \qquad (62)$$

Evaluating the Constants in Equations (48), (46) and (45).

The methods of evaluating the constants in equations (48), (46) and (45) for a given soil path will now be considered. For this purpose path V Fig. 22 will be taken. The general form of equation (48) is

$$z - s = a(s^2 + bs + c)^2. \qquad (48)$$

From the assumptions that have already been made, pages 28 to 32 the following data for this curve are obtained:

s	z
11.6	11.6
25.8	26.25
40	40.

Substituting the above values for s and z in equation (48) three equations are obtained from which it is found that

$$a = .00001114$$
$$b = -51.6$$
$$c = 464$$

giving

$$z - s = .00001114(s^2 - 51.6s + 464)^2 \tag{63}$$

To determine the values of the constants in

$$ax^2 + bz^2 + lxz + mx + nz + d = 0, \tag{46}$$

the origin is moved to, $x = 7.65$, $z = 11.6$. For this point as origin an equation of the following form describes the curve,

$$a(x')^2 + b(z')^2 + l_1x'z' + m_1x' = 0. \tag{64}$$

Taking $a = 1$ only three constants b, l_1, and m_1 remain to be evaluated. From the trace of path V on the surface of the plow-bottom the following data were obtained:

x'	z'
1	13.55
3	20.05
6	27.15

Substituting these values for x' and z' in equation (64) gives three equations from which

$$b = -.019$$

$$l_1 = -.453$$

$$m_1 = 8.63$$

$$(x')^2 - .019(z')^2 - .453x'z' + 8.63x' = 0. \qquad (65)$$

Translating the axes back to the original origin,

$$x' = x - 7.65$$

$$z' = z - 11.6$$

gives

$$x^2 - .019z^2 - .453xz - 1.45x + 3.91z - 49.92 = 0 \qquad (66)$$

To determine the values of the constants in

$$ax^2 + by^2 + lx + my + d = 0, \qquad (45)$$

the origin is moved to, $x = 7.65$, $y = .2$. This changes the form of the equation to

$$a(x')^2 + b(y')^2 + l_1x' + m_1y' = 0. \qquad (67)$$

Taking $a = 1$ three constants remain to be evaluated. From the trace of path V upon the surface of the plow-bottom,

x'	y'
1	3.1
4	5.45
7	6.68

Substituting these values of x' and y' in equation (67) gives

$$a = 1$$

$$b = 4.29$$

$$l_1 = -\ 30.85$$

$$m_1 = -\ 3.67$$

$$(x')^2 + 4.29(y')^2 - 30.85x' - 3.67y' = 0 \qquad (68)$$

The axes are translated back to the original origin by substituting

$$x = x' - 7.65$$

$$y = y' - .2$$

in equation (68), which gives

$$x^2 + 4.29y^2 - 46.15x - 5.39y + 295.45 = 0. \qquad (69)$$

Numerical Example.

The surface of a plow-bottom is represented by the equation

$$.54x^2 - 1.52y^2 + 1.12z^2 - 3.69yz - 1.62xz + 2.04xy$$
$$+ 53.63x + 114.90y - 46.4z + 49.4 = 0$$

The motion of a soil particle which passes upon this bottom at the point $x = 6.9$, $y = .2$, $z = 9.5$ is described by the following equations:

$$z = .00001622(s^2 - 45.5s + 342)^2 + s \tag{70}$$

$$x^2 - .119z^2 - 1.126xz + 20.78x + 10.03z - 201.63 = 0 \tag{71}$$

$$x^2 + 1.8y^2 - 42.41x - 1.5y + 245.25 = 0 \tag{72}$$

$$s = vt \tag{47}$$

From equations (70), (71), (72) and (47) the following are obtained:

Table VIII.*

s	z	x	y
18	18.4	7.55	3.6
27	27.4	11.5	8.25
36	36.	19.5	11.

$$z = .00001622(v^2t^2 - 45.5vt + 342)^2 + vt \tag{73}$$

$$\frac{dz}{dt} = .00003244[(v^2t^2 - 45.5vt + 342)(2v^2t - 45.5v)] + v \tag{74}$$

*All measurements are given in inches.

$$\frac{d^2 z}{dt^2} = .00003244[(v^2 t^2 - 45.5vt + 342)(2v^2) + (2v^2 t - 45.5v)^2] \tag{75}$$

$$\frac{dx}{dt} = \frac{(.238z + 1.126x - 10.03)\frac{dz}{dt}}{2x - 1.126z + 20.78} \tag{76}$$

$$\frac{d^2 x}{dt^2} = \frac{(.238z + 1.126x - 10.03)\frac{d^2 z}{dt^2} - 2(\frac{dx}{dt})^2 + .238(\frac{dz}{dt})^2 + 2.252(\frac{dx}{dt})(\frac{dz}{dt})}{2x - 1.126z + 20.78} \tag{77}$$

$$\frac{dy}{dt} = \frac{(-2x + 42.41)\frac{dx}{dt}}{3.6y - 1.5} \tag{78}$$

$$\frac{d^2 y}{dt^2} = \frac{(-2x + 42.41)\frac{d^2 x}{dt^2} - 2(\frac{dx}{dt})^2 - 3.6(\frac{dy}{dt})^2}{3.6y - 1.5} \tag{79}$$

The plow moved forward with a velocity of 36 inches per second giving

$$s = 36t \tag{80}$$

From equations (74), (75), (76), (77), (78), (79) and (80) the values listed in Table IX are computed.

Table IX.

s	t in sec.	$\frac{dx}{dt}$	$\frac{d^2x}{dt^2}$	$\frac{dy}{dt}$	$\frac{d^2y}{dt^2}$	$\frac{dz}{dt}$	$\frac{d^2z}{dt^2}$
18	$\frac{1}{2}$	7.09	53.6	16.9	28.4	37.7	−9.07
27	$\frac{3}{4}$	25.15	47.75	17.32	−50	34.44	−10.21
36	1	38.4	41.6	3.44	−74.8	36.	29.52

By making the proper substitutions from (86), (74), (76) and (78) in equations (54), (57), (58) and (59) the values of the direction cosines for the normals (N) the tangents to the path (T) and the perpendiculars to the planes formed by the normals and tangents (R), for three points are computed and listed in Table X.

Table X.

$x = 7.55 \qquad y = 3.6 \qquad z = 18.4$

$\cos N_x = .549$	$\cos T_x = .169$	$\cos R_x = .817$
$\cos N_y = .716$	$\cos T_y = .4025$	$\cos R_y = -.564$
$\cos N_z = -.429$	$\cos T_z = .9$	$\cos R_z = .0977$

Table X (cont).

$x = 11.5 \quad y = 8.25 \quad z = 27.4$

$\cos N_x = .728$	$\cos T_x = .546$	$\cos R_x = .4145$
$\cos N_y = .229$	$\cos T_y = .376$	$\cos R_y = -.897$
$\cos N_z = -.646$	$\cos T_z = .749$	$\cos R_z = .149$

$x = 19.5 \quad y = 11 \quad z = 36$

$\cos N_x = .698$	$\cos T_x = .728$	$\cos R_x = .102$
$\cos N_y = -.215$	$\cos T_y = .065$	$\cos R_y = -.975$
$\cos N_z = -.683$	$\cos T_z = .683$	$\cos R_z = .2$

For the purpose of computing the forces a block of soil 2 inches wide, 1 inch long, and $\frac{1}{2}$ of an inch thick is taken. The mass of this soil is

$$M = \frac{(2\cdot 1\cdot .5)62.5e}{1728\cdot 32.2\cdot 12} = \frac{.0362\ e}{32.2\cdot 12} \tag{81}$$

e = density.

By the proper substitutions from Tables IX and X into equations (60), (61) and (62) the forces necessary to produce the accelerations are computed and listed in Table XI.

Table XI

$x = 7.55$	$y = 3.6$	$z = 18.4$
$F_N = .00503e$	$F_T = .00116e$	$F_R = .00252e$

Table XI (cont).

$x = 11.5$	$y = 8.25$	$z = 27.9$
$F_N = .00281e$	$F_T = .000248e$	$F_R = .00592e$
$x = 19.5$	$y = 11$	$z = 36$
$F_N = .00234e$	$F_T = .00428e$	$F_R = .00778e$

A soil particle in passing over the surface of the plow-bottom will be acted upon by the following:

A force from the surface of the bottom acting in the direction of the normal.

Gravity.

Pressure from the weight of the soil above the particle.

Friction between the particle and the surface.

Shearing, stretching or compression on each of the remaining five sides of the particle due to its contact with other soil particles.

The force which produces the movement of a soil particle in any direction will be the resultant of the components of the above listed forces which act in the direction of the movement.

The preceeding analysis of the motion which certain soil particles have in the operation of plowing has not been developed from as refined methods nor as uniform data in all cases as could be desired but the results obtained furnish abundant evidence that the problem here attempted is by no means hopeless. The study should be continued upon a tough sod which would stretch more uniformly and some apparatus which would remove the necessity of certain soil particles remaining in line with each other substituted for the pins.

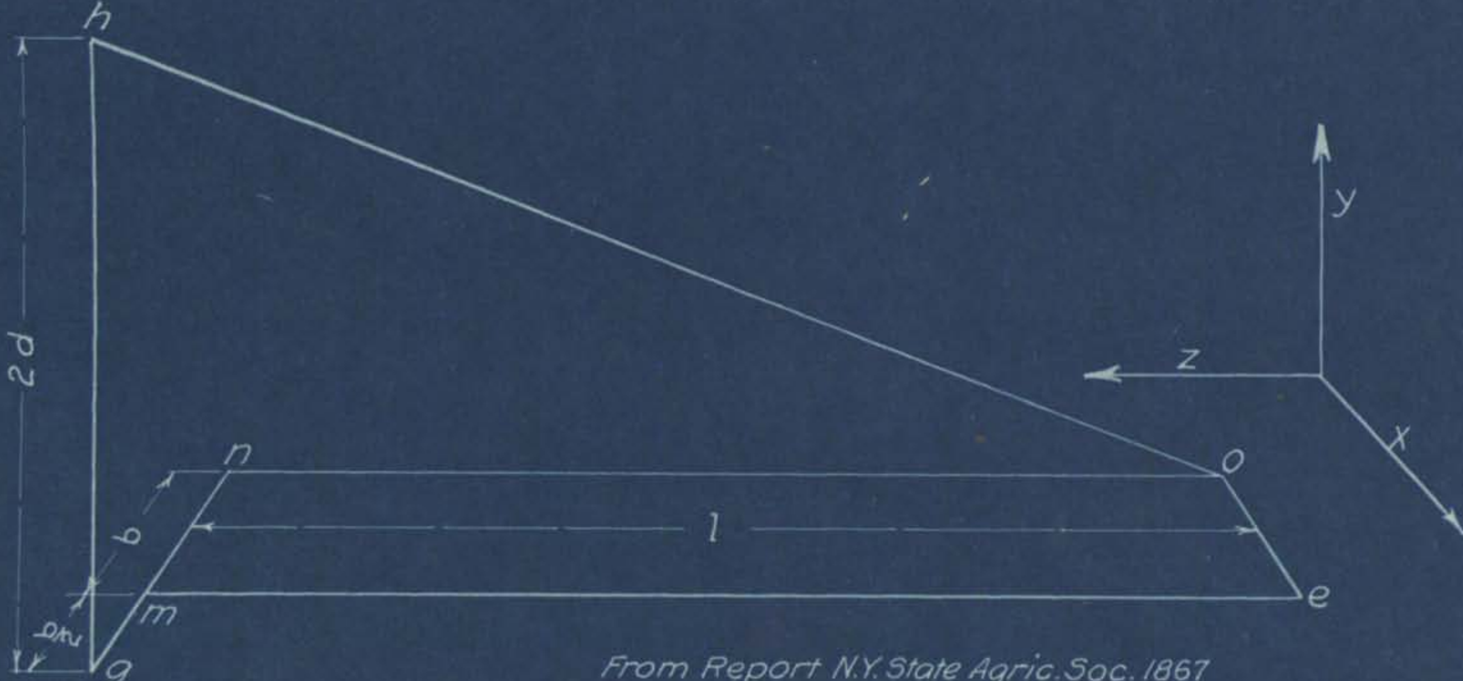


Fig. 26

HISTORY.

The Annual Report of the New York State Agricultural Society for 1867 contains an excellent treatise giving the geometrical construction of the surfaces of many historical plow-bottoms but no attempt has been made in this report to classify these surfaces upon the basis of their mathematical forms. Using the above mentioned work as a basis the author has attempted to work out the mathematical forms of the most important of these historical surfaces with a view to making fundamental comparisons with present day plow-bottoms.

Jefferson's Plow-bottom.

In 1788, Thomas Jefferson, while making a tour in Germany, developed what appears to be the one first recorded method for making the surface of the mouldboard geometrically exact in form.* He argued that the offices of the mouldboard were to receive the soil from the share and invert it with the least possible resistance. In order to do this Jefferson developed a surface which he considered best adapted for the work of plowing, but attention should be called to the fact that no evidence is offered to prove the assertion. Fig. 26 shows the framework for generating the Jefferson

*Report of the New York State Agricultural Society for 1867, vol. 1, page 403.

mouldboard in which lines *em* and *oh* are the directrices. To generate the surface a straight edge is laid upon *eo* and moved backward; the straight edge remaining parallel to the plane $z = 0$. Taking the point *o* as the origin, the equation of the surface is

$$3byz - 2dxz - 2bly + 2bdz = 0^{*} \qquad (82)$$

b = breadth of furrow

d = depth of furrow

l = length of mouldboard.

Rotating the XY-axes through $\tan^{-1} = 2d/3b$ the equation is

$$(9b^2 + 4d^2)y'z - 4bdlx' - 6b^2ly' + 2bd\sqrt{9b^2 + 4d^2}\, z = 0 \qquad (83)$$

Rotating the $Y'Z$-axes through $\tan^{-1} \tfrac{1}{2}\sqrt{2}$ the equation is

$$(9b^2 + 4d^2)[(y'')^2 - (z')^2] - 8bdlx'$$
$$+ 2(bd\sqrt{18b^2 + 8d^2} - 3b^2l\sqrt{2})[y'' + z'] = 0 \qquad (84)$$

Translating the axes to the points

$$y'' = y''' + y_0$$
$$z' = z'' + z_0$$

where y_0 has such a value that

$$2(9b^2 + 4d^2)y_0 + 2[bd\sqrt{18b^2 + 8d^2} - 3b^2l\sqrt{2}] = 0, \qquad (85)$$

---

*The method of developing the equation for this surface is given upon page 7 to 13.

and z_0 has such a value that

$$-2(9b^2 + 4d^2)z_0 + 2[bd\sqrt{18b^2 + 8d^2} - 3b^2 l\sqrt{2}] = 0, \tag{86}$$

gives

$$(9b^2 + 4d^2)[(y''')^2 - (z'')^2] - 8bdlx' + (y_0^2 - z_0^2)(9b^2 + 4d^2)$$
$$+ (y_0 + z_0)(2bd\sqrt{18b^2 + 8d^2} - 3b^2 l\sqrt{2}) = 0. \tag{87}$$

Letting the constant terms in (87) equal C gives

$$(9b^2 + 4d^2)[(y''')^2 - (z'')^2] - 8bdlx' + C = 0. \tag{88}$$

Translating the axes to the point $x' = x'' + x_0$ where x_0 has such a value that

$$-8bdlx_0 + C = 0$$

gives

$$(9b^2 + 4d^2)[(y''')^2 - (z'')^2] = 8bdlx''. \tag{89}$$

This is the equation of an hyperbolic paraboloid.*

Lambruschini's Plow-bottom.

In the Giornale Agrario Toscano, (1832, vol. VI) an Italian, Lambruschini, describes a method for generating the surface of a plow-bottom which he considered to be more efficient than the surface developed by the Jefferson method. Lambruschini proposed a helicoid generated as follows: Lay

*See *Analytic Geometry of Space,* Snyder and Sisam, p.73.

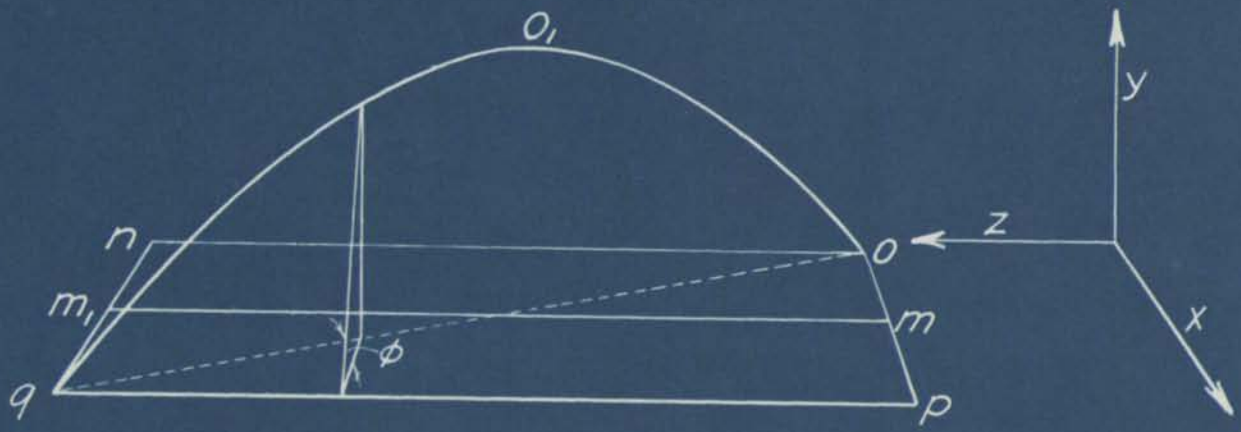


Fig. 27

out a rectangle *opqn*, Fig. 27, twice the desired width of the furrow and of an empirically determined length. Take the point *m* midway between points *o* and *p* and draw the line mm_1 parallel to *pq*. A straight edge laid upon *mo* and moved backward along the line mm_1 being kept parallel to the plane $z = 0$, and with an angular rotation proportional to the movement toward m_1, generates the surface of the Lambruschini bottom. The point of the straight edge which was at *o* will describe the helix oo_1q, Fig. 27. The equation of this surface is

$$\frac{y}{x} = \tan \theta,$$

where θ has uniformly increasing values as z increases.

Then $\theta = f(z)$, when $\theta = 90^\circ = \frac{\pi}{2}$ radians,

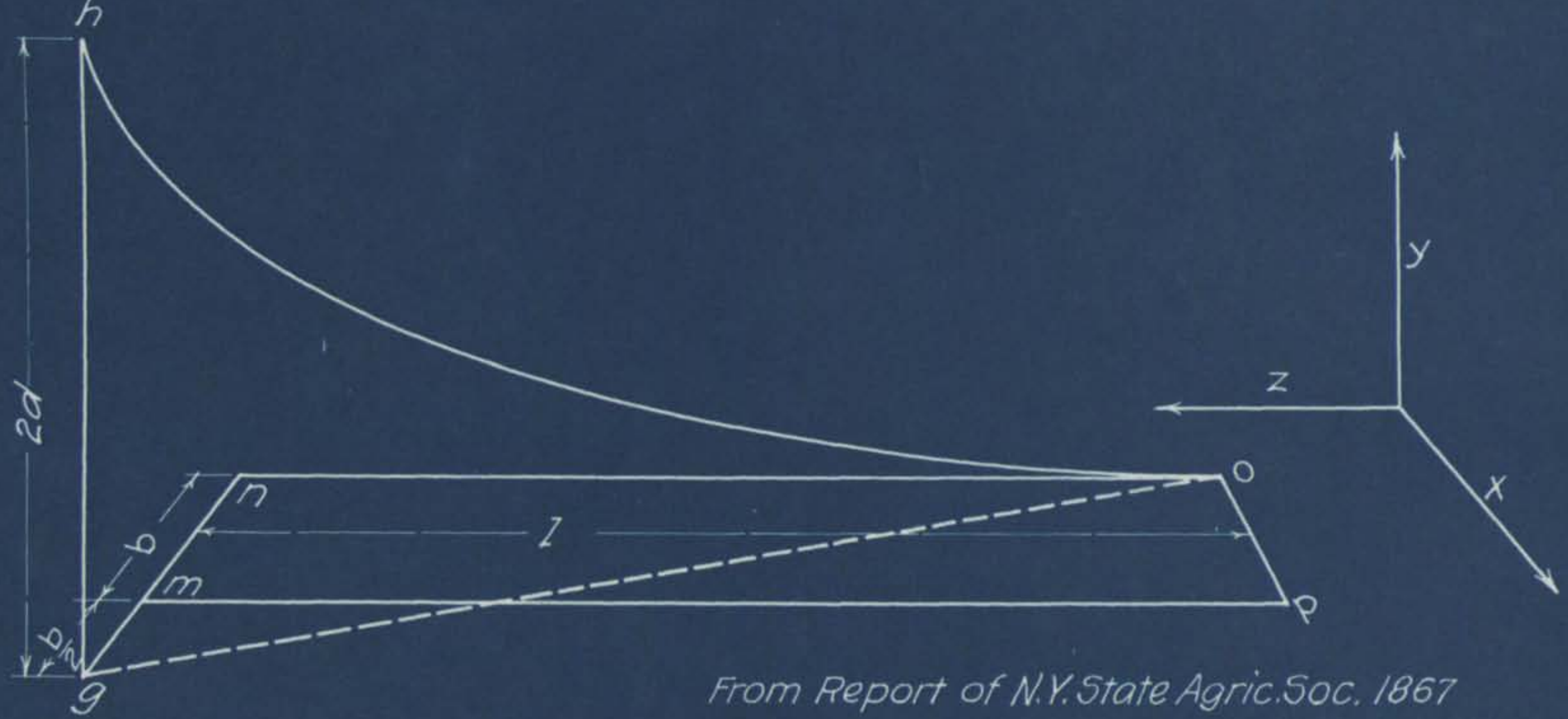
$$z = \frac{l}{2},$$

l = length of line mm_1.

$$\frac{\pi}{2} = c\frac{l}{2}$$

$$c = \frac{\pi}{l}.$$

Hence $$\frac{y}{x} = \tan\left(\frac{\pi}{l}z\right). \qquad (90)$$



From Report of N.Y. State Agric. Soc. 1867

Fig. 28

Fig. 29

Small's Plow-bottom.*

About 1760, a Scotchman, James Small, established a factory in Scotland for the manufacture of plows. The surface of Small's mouldboard is obtained by laying a straight edge upon *op*, Fig. 28 and moving it backward parallel to the plane $z = 0$, with the line *pm* and the curve *oh* as directrices. The equation of the curve, (a half catenary), is obtained by drawing a line *og*, Fig. 29, the length of line *og*, Fig. 28. At *o* erect a line oo_1 perpendicular to line *og* and equal in length to line *gh*, Fig. 28. Through point o_1, Fig. 29 draw a line o_1h parallel and equal to line *og*. With *h* and *o* as points of suspension describe a catenary with its lowest point at O. Taking the point O, Fig. 29 as origin, the equation of the catenary is

$$y = \frac{a}{2}(e^{21z/3ba} + e^{-21z/3ba}). \tag{91}$$

$$a = Og.$$

Transferring the origin to the point *o* gives

$$y = \frac{a}{2}(e^{21z/3ba} + e^{-21z/3ba}) - a \tag{92}$$

as the equation of the catenary *oh*, Fig. 28. The equations of line *pm*, Fig. 28 are

*The Annual Report of the New York State Agricultural Society for 1867, vol. I, page 415.

$$y = 0$$

$$x = b.$$

Any plane parallel to the plane $z = 0$ is given by $z = c$, and this plane cuts the line pm at the point

$$x_1 = b$$

$$y_1 = 0$$

$$z_1 = c.$$

It also cuts the catenary oh at the point

$$x_2 = \frac{3b}{2l}c$$

$$y_2 = f(c)$$

$$z_2 = c.$$

The equation of the line in the plane $z = c$ which cuts the line pm and the catenary oh, Fig. 28 is

$$\frac{x - b}{\frac{3b}{2l}c - b} = \frac{y - 0}{f(c) - 0} \tag{93}$$

or

$$(x - b)f(c) - y\left(\frac{3b}{2l}c - b\right) = 0 \tag{94}$$

As this line is always parallel to the plane $z = 0$ it follows that $c = z$ and

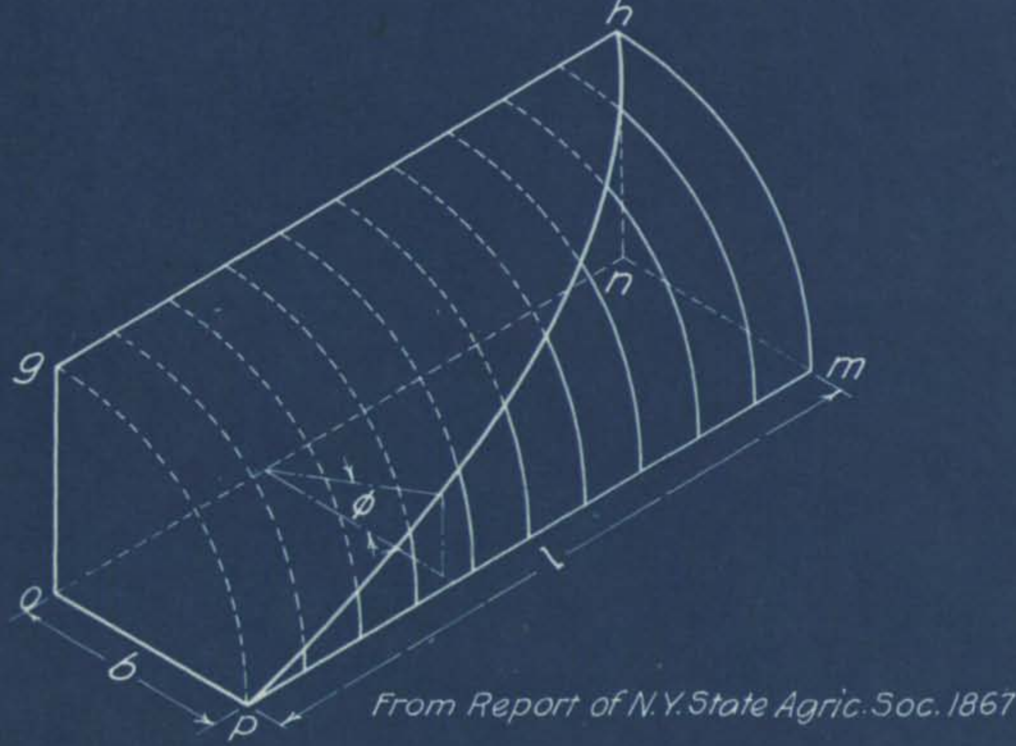


Fig.30

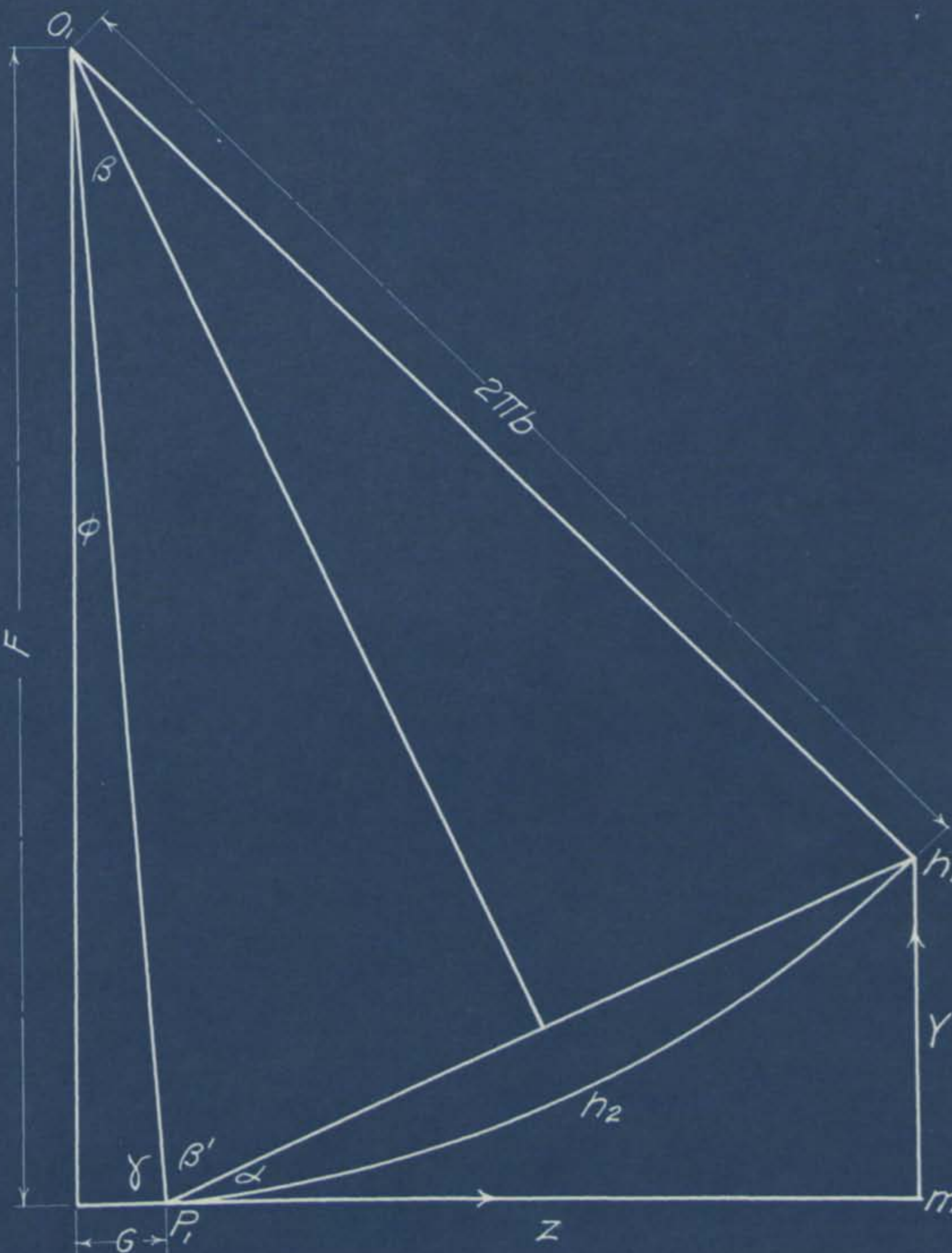
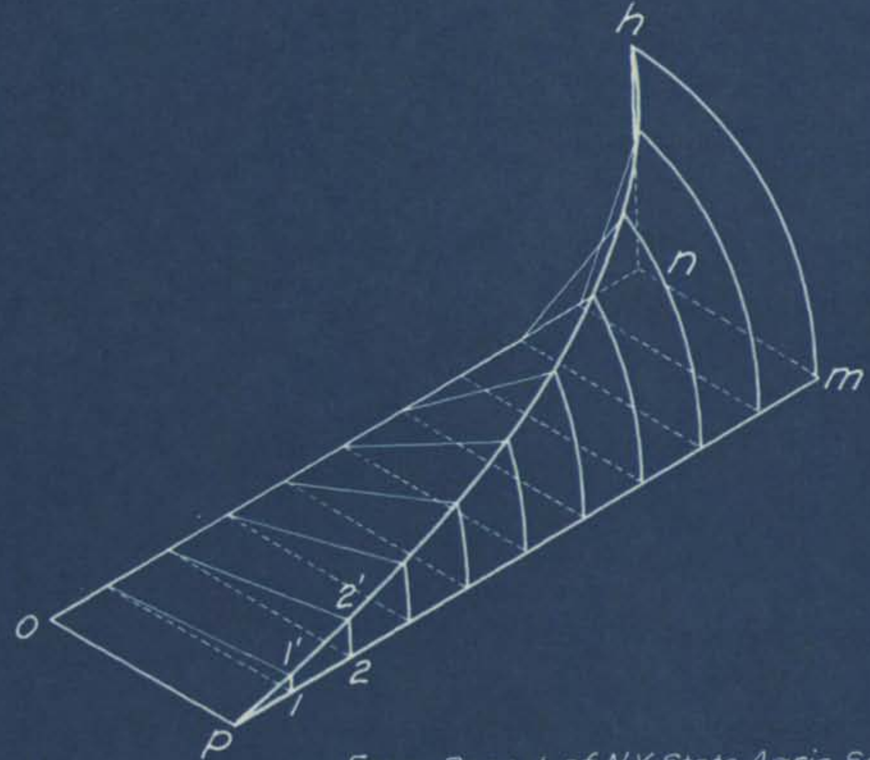


Fig. 31



From Report of N.Y. State Agric. Soc. 1867

Fig. 32

$$f(c) = f(z).$$

From equations (92) and (94) then,

$$(x - b)[\frac{a}{2}(e^{2lz/3ba} + e^{-2lz/3ba}) - a] - y(\frac{2b}{3l}c - b) = 0, \quad (95)$$

which is the equation of Small's mouldboard.

Stephen's Plow-bottom.*

About the same time Small brought out his mouldboard another Scotchman named Stephens developed a method for forming the surface of a mouldboard the general plan of which is shown in Fig. 30. The generator for this surface is a straight edge laid upon *op*, Fig. 30 and moved backward parallel to the plane $z = 0$ with the line *on* and the curve *ph* as directrices. Stephen designed his surface by taking a quarter cylinder *opmnhg* and laying out p_1m_1 Fig. 31, equal in length to *pm*, Fig. 30. Perpendicular to line p_1m_1 draw m_1h_1 equal to the length of arc *mh* Fig. 30. Through points *p*, h_1, Fig. 31 pass a circle of radius $2\pi b$. The plane figure $p_1m_1h_1h_2$, Fig. 31 is then laid upon the quarter cylinder, Fig. 30, so that p_1 falls upon *p*; m_1 upon *m*; and h_1 upon *h*. This will locate the curve *ph*, Fig. 30 leaving a figure as shown in Fig. 32. It will be observed in Fig. 32 that $\frac{y}{x} = \tan\theta$ where θ has gradu-

*The Report of the New York State Agricultural Society for 1867, vol. I, page 431.

ally increasing values from 0 at $z = 0$ to 90° at $z = l$. Further $\theta = \frac{\gamma}{b}$ radians where γ represents the lengths of arcs 11', 22', etc., then $\frac{y}{x} = \tan\left(\frac{\gamma}{b}\right)$. From Fig. 31, the equation of the circle with its center at O, taking p_1 as the origin is

$$(\gamma - F)^2 + (z + G)^2 = 4\pi^2 b^2 \tag{96}$$

$$\gamma = F + \sqrt{4\pi^2 b^2 - (z - G)^2} \tag{97}$$

In Fig. 31

$$\varphi + \gamma = 90°;$$

$$B + B' = 90°;$$

$$\alpha + B' + \gamma = 180°;$$

$$\varphi = \alpha - B;$$

$$F = 2\pi b \cos \varphi$$

$$= \frac{l\sqrt{4\pi^2 b^2 - \left(\frac{l^2}{4} + \frac{\pi^2 b^2}{16}\right)}}{2\sqrt{l^2 + \frac{\pi^2 b^2}{4}}} + \frac{\pi b}{4}; \tag{98}$$

$$G = 2\pi b \sin \varphi$$

$$= \frac{\pi b\sqrt{4\pi^2 b^2 - \left(\frac{l^2}{4} + \frac{\pi^2 b^2}{16}\right)}}{2\sqrt{l + \frac{\pi^2 b^2}{4}}} - \frac{l}{2\pi b}. \tag{99}$$

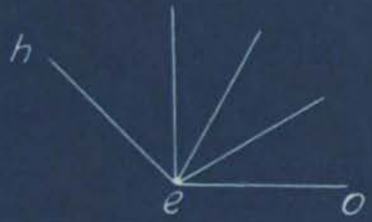
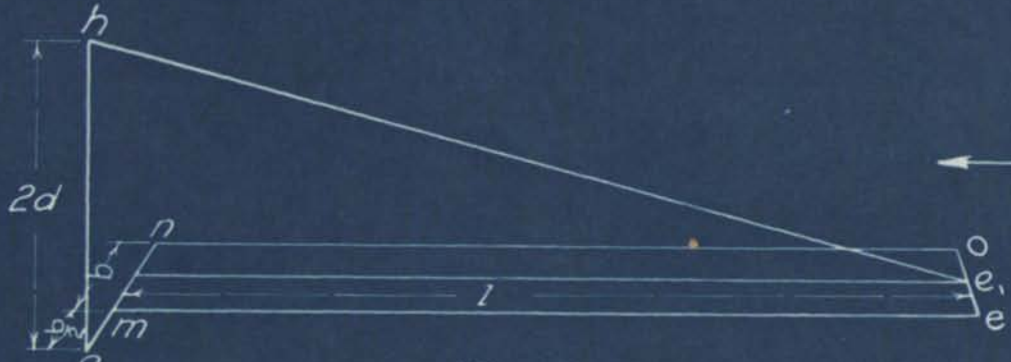


Fig. 34

Fig. 35

Fig. 36



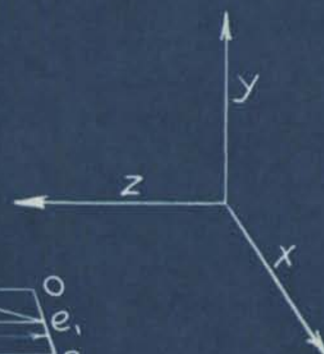


Fig. 33

From Report of N.Y. State Agric. Soc. 1867

Substituting the values for F from equation (98) and for G from equation (99) gives

$$\frac{y}{x} = \tan\,[f(z)], \tag{100}$$

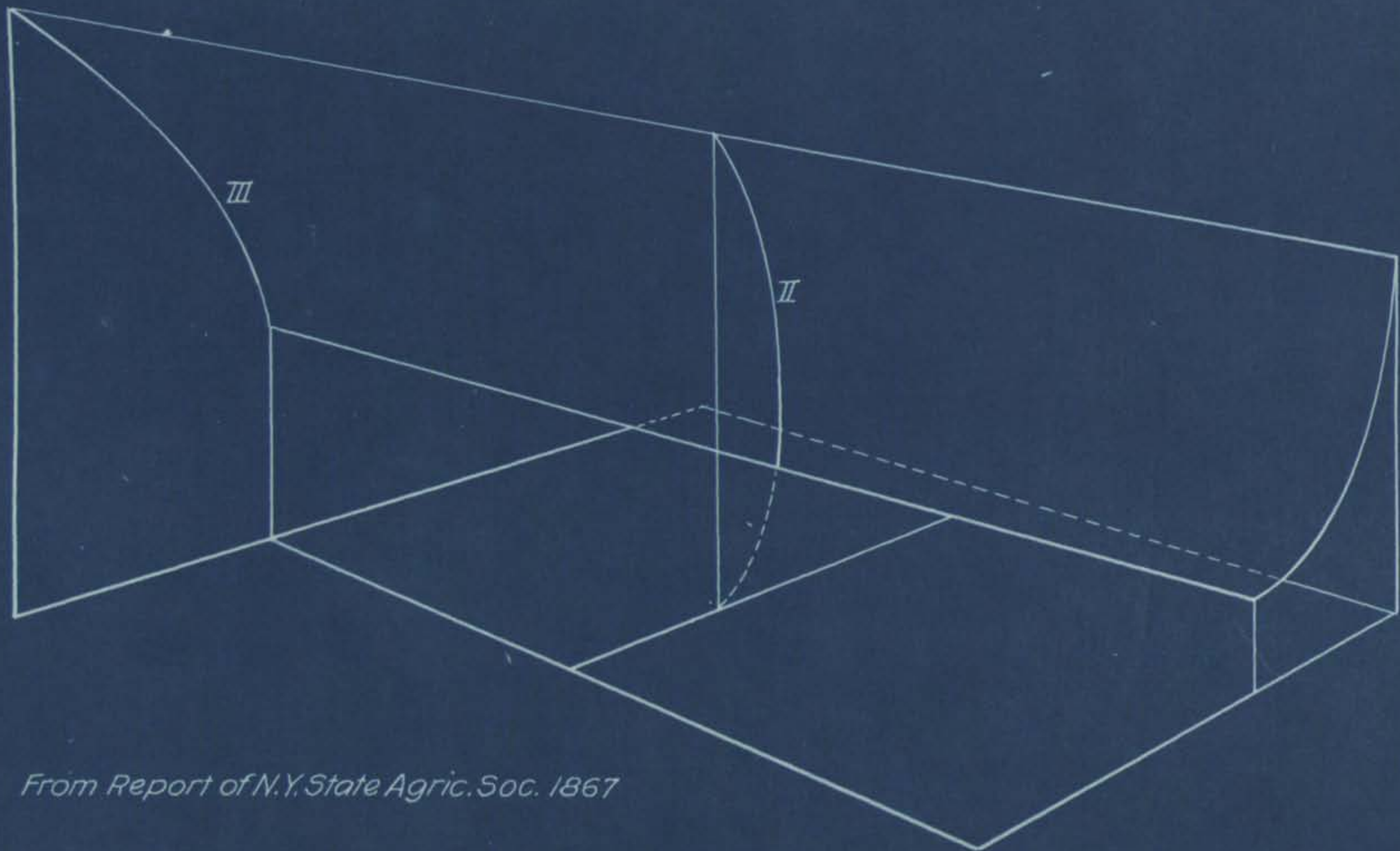
which is the equation of the surface.

Rahm's Plow-bottom.*

In 1846 Rev. W. L. Rahm, an Englishman, brought forward the theory that the lines of the mouldboard running in the longitudinal direction‡ should be straight but that the section of the mouldboard formed by any plane $z = c$, Fig. 33, should be a straight line or a curve according to the physical characteristics of the soil to be worked. Mr. Rahm agreed that for medium, mellow soils the surface of the mouldboard should be generated by laying a straight edge upon oe and moving it backwards parallel to the plane $z = 0$ with the lines e_1h and em as directrices. This surface will be a portion of an hyperbolic paraboloid, the same general type as the surface which Mr. Jefferson proposed. The orthogonal projection of the generator in various positions, upon the plane $z = 0$ will look as shown in Fig. 34. For stiff clay soils the lines, Fig. 35, are made concave, and for loose sandy soils Fig. 36

*The Report of the New York State Agricultural Society for 1867, vol. I, page 442.

‡See page .



From Report of N.Y. State Agric. Soc. 1867

Fig. 37

they are made convex. As no exact description was given regarding the shape of the curves, Fig. 35 and 36 it has not been possible to develop equations for the surfaces; however as it is known that these surfaces have straight lines in one direction and cannot be described by an equation of the second order, they are of the fourth order or higher.

Knox's Plow-bottom.*

In 1852 Samuel A. Knox of Worcester, Mass., applied for a patent upon the surface of a plow-bottom which was certainly unique. The skeleton of this surface is shown in Fig. 37. The segments of circles I, II, III are placed in parallel planes twelve inches apart so that a series of straight lines will cut the three circles. Circles I and III have equal diameters and the diameter of circle II is one-half that of circles I and III. As the equation of this surface is of the eighth order it will not be worked out in detail, but a development given showing how the equation could be obtained.

Let the equations of the three circles be‡

$$x^2 + y^2 = R^2$$

$$z = 0,$$

$$(x - a)^2 + (y - b)^2 = (\frac{R}{2})^2$$

*The report of the New York State Agricultural Society for 1867, vol. I, page 495.

‡This development is the work of Virgil Snyder, Professor of Mathematics, Cornell University.

$$z = k;$$

and
$$(x - c)^2 + (y - d)^2 = R^2$$
$$z = 2k.$$

Draw the line from a point $(x_1, y_1, 0)$ on the first circle to a point $(x_2, y_2, 2k)$ on the third. Its equations are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z}{2k},$$

from which

$$\frac{2k(x - x_1) + z(x_1 - c)}{z} = x_2 - c,$$

$$\frac{2k(y - y_1) + z(y_1 - d)}{z} = y_2 - d.$$

Since

$$(x_2 - c)^2 + (y_2 - d)^2 = R^2,$$

we have, after simplifying

$$4k^2[(x - x_1)^2 + (y - y_1)^2] + 4kz[(x - x_1)(x_1 - c)$$
$$+ (y - y_1)(y_1 - d)] + z^2[(x_1 - c)^2 + (y_1 - d)^2 - R^2] = 0. \quad (101)$$

This is the equation of a cone with vertex at $(x_1, y_1, 0)$ and passing through the third circle.

In the same way, find the equations of the line from $(x_1, y_1, 0)$ to (x_3, y_3, k) on the middle circle

$$\frac{x - x_1}{x_3 - x_1} = \frac{y - y_1}{y_3 - y_1} = \frac{z}{k},$$

$$\frac{k(x - x_1) + z(x_1 - a)}{z} = x_3 - a,$$

$$\frac{k(y - y_1) + z(y_1 - b)}{z} = y_3 - b.$$

Since

$$(x_3 - a)^2 + (y_3 - b)^2 = (\frac{R}{2})^2,$$

we have, after simplifying,

$$k^2[(x - x_1)^2 + (y - y_1)^2] + 2kz[(x - x_1)(x_1 - a)$$

$$+ (y_1 - b)(y - y_1)] + z^2[(x_1 - a)^2 + (y_1 - b)^2 - \frac{R^2}{4}] = 0 \quad (102)$$

When equations (101) and (102) are multiplied out, it will be seen that x_1^2, y_1^2 always enter in the form $x_1^2 + y_1^2 = R^2$. By substituting R^2 for $x_1^2 + y_1^2$ in each, the equations are of the form

$$Ax_1 + By_1 = C,$$

$$A'x_1 + B'y_1 = C'.$$

Solve these equations for x_1, y_1 and put their values in

$$x_1^2 + y_1^2 = R^2.$$

$$A = [4kz(x + c) - 2cz^2 - 8xk^2],$$

$$B = [4kz(y + d) - 2dz^2 - 8yk^2],$$

$$C = [4R^2k^2 - 4k^2(x^2 + y^2) - 4kxz - 4kzy - 4kR^2z + z^2(c^2 + d^2)]$$

$$A' = [2kz(x + a) - 2xk^2 - 2az^2],$$

$$B' = [2kz(y + b) - 2yk^2 - 2bz^2],$$

$$C' = [R^2k^2 + k^2(x^2 + y^2) - 4kz(ax + by - R^2) + z^2(a^2 + b^2 + \tfrac{3}{4}R^2)].$$

$$x_1 = \frac{B'C - BC'}{AB' - A'B},$$

$$y_1 = \frac{C'A - CA'}{AB' - A'B}$$

hence $$(B'C - BC')^2 + (C'A - CA')^2 = R^2(AB' - A'B)^2. \qquad (103)$$

Cylindrical Plow-bottoms.

In 1854 an American, Joshua Gibbs,* patented a plow-bottom the surface of which is a portion of a circular cylinder. Taking a point upon the axis of the cylinder as the origin, the equation of this surface is

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} - 1 = 0 \qquad (104)$$

In some foreign countries, notably Germany, the hyperbolic cylinder has been suggested as suitable for forming the surface of the mouldboard. In this connection it is interesting

*The Report of the New York State Agricultural Society for 1867, vol. I, page 502.

to note that any cylindrical surface can be described by an equation of the general form

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm 1 = 0^* \tag{105}$$

Mead's Plow-bottom.**

In 1863 Mr. Mead of New Haven, Conn., patented a plow-bottom the surface of which conformed exactly to a portion of a frustrum of a cone. The general equation of this surface is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \tag{106}$$

Holbrook's Plow-bottom.

The Report of the New York State Agricultural Society for 1867 contains a very complete report of the plow trials held at Utica, New York, in 1867, at which trials a line of plows designed by F. F. Holbrook of Boston, Mass., showed general superiority to all other makes. The following quotation gives a very good description of the Holbrook surfaces:

"We‡ were interested in the most minute details of these plows by Gov. Holbrook and the trials at Utica and subsequent-

*_Analytic Geometry of Space_, Snyder and Sisam, page 82.

**The Report of the New York State Agricultural Society for 1867, vol. I, page 505.

‡The Report of the New York State Agricultural Society for 1867, vol. I, page 586.

ly at Brattleboro, Vt., showed very clearly the influence of the warped surface which is generated by his method upon the texture of the soil. Gov. Holbrook is as yet unprotected by a patent on his method, and we are therefore most reluctantly compelled to withhold a description of it but we have no hesitation in saying that it is the best system for generating the true curves of the mouldboard which has been brought to our knowledge. This method is applicable to the most diversified forms of plows, to long or short, to broad or narrow, to high or low, no matter what the form may be, this method will impress a family likeness upon them all; There will be straight lines in each running from the front to the rear and from the sole to the upper parts of the share and mouldboard. None of these lines will be parallel to each other, nor will any of them be radii from a common center. The angle formed by any two of them will be unlike the angle formed by any other two; a change in the angle formed by any transverse lines will produce a corresponding change in the vertical lines, and there will always in every form of this plow, be a reciprocal relation between the transverse and vertical* lines. Plows made upon this plan may appear to the eye to be as widely different as it is possible to make them, and yet, on the application of the straight edge and protractor, it will be found that they

*It should be noted that the lines here called transverse are designated as longitudinal, Fig. 1; and the lines called vertical are designated as transverse, Fig. 1.

agree precisely in their fundamental character. The surface of the mouldboard is always such that the different parts of the furrow-slice will move over it with unequal velocities."

From the above description it is evident that the surfaces of the Holbrook plows are portions of an hyperboloid of one sheet, whose general equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Miscellaneous Plow-bottoms.

In addition to the surfaces already described there remain at least three which show unique characteristics but data were not available for developing the equations.

In 1818 Gideon Davis* of Maryland patented the surface of a plow-bottom which was obtained by using the segment of a circle as a generator and two segments of another circle as directrices. Somewhat later, 1834 James Jacobs,‡ an American, brought out a plow-bottom the surface of which was a combination of two mathematical surfaces each of which had sets of straight lines in two directions.

In 1839, Samuel Witherow of Gettysburg, Pa., and David Pierce of Philadelphia, Pa., brought out a plow-bottom whose

*The Report of the New York State Agricultural Society for 1867, vol. I, page 452.

‡The Report of the New York State Agricultural Society for 1867, vol. I, page 486.

surface was generated by the most ingenious use of the arc of a cycloid. A more detailed description of this plow can be found in the Report of the New York State Agricultural Society, 1867, vol. I, page 491.

TABLE XII. HISTORICAL PLOW-BOTTOMS

Date	Name	Generatrix	Directrixes	Equation of Surface
	Small	Straight Line	Straight Line & Catenary	
	Stephens	Straight Line	Straight Line and arc of Circle	$\frac{y}{x} = \tan[f(z)]$
1788	Jefferson	Straight Line	Straight Lines	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2nz$
1818	Davis	Arc of Circle	Arcs of Circle	
1832	Lambruschini	Straight Line	Straight Line and Helix	$\frac{y}{x} = \tan(az)$
1839	Witherow and Pierce	Arc of Cycloid	Arcs of Cycloid	
1840	Rham	Straight Line	Straight Lines	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2nz$
1840	Rham	Straight Line	Curves	
1852	Knox	Straight Line	Arcs of Circles	Ruled surface of eighth order
1854	Gibbs	Straight Line	Arcs of Circles	$\frac{x^2}{r^2} + \frac{y^2}{r^2} - 1 = 0$
1863	Mead	Straight Line	Arcs of Circles	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$
1867	Holbrook	Straight Line	Straight Lines	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
1834	Jacobs	A portion from each of 2 Surfaces; each surface having 2 sets of straight line generators.		

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