Nonlinear Generalizations of Matrix Diagonal Dominance with Application to Gauss-Seidel Iterations

by

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Abstract

A new class of nonlinear mappings is introduced which contains, in the linear case, the strictly and irreducibly diagonally dominant matrices as well as other classes of matrices introduced by Duffin and Walter. We then extend some of the properties of the above mentioned matrices to these weakly Ω-diagonally dominant functions, and point out their connection to the M- and P- functions studied by Rheinboldt, and More' and Rheinboldt, respectively. Finally, new convergence theorems for the nonlinear Jacobi and Gauss-Seidel iterations are presented.

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I. Introduction.

For a nonlinear mapping $F:D \subset R^n \to R^n$, we analyze the convergence of two iterative schemes for finding a solution of the system of equations $Fx = 0$ by considering a generalization of the concept of strictly and irreducibly diagonally dominant matrices to nonlinear mappings.

If F has component functions $f_1, \ldots, f_n$, we consider the (underrelaxed) Gauss-Seidel iteration: Solve

$$f_i(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, x_i, x_{i+1}^k, \ldots, x_n^k) = 0 \tag{1.1}$$

for x_i, and set

$$x_i^{k+1} = (1-\omega)x_i^k + \omega\, x_i, \quad i = 1, \ldots, n, \quad k = 0, 1, \ldots, \tag{1.2}$$

and the (underrelaxed) Jacobi iteration: Solve

$$f_i(x_1^k, \ldots, x_{i-1}^k, x_i, x_{i+1}^k, \ldots, x_n^k) = 0 \tag{1.3}$$

for x_i, and set

$$x_i^{k+1} = (1-\omega)x_i^k + \omega\, x_i, \quad i = 1, \ldots, n, \quad k = 0, 1, \ldots, \tag{1.4}$$

where $\omega \in (0,1]$ is a given relaxation parameter.

Take, for example, the Gauss-Seidel method. For affine mappings $Fx = Ax - b$ where A is some $n \times n$ matrix and b is a vector in R^n, it is well known (Varga [19]) that the

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following conditions guarantee the existence of a unique solution x^* of $Fx = 0$ and that the Gauss-Seidel iterates with $\omega \in (0,1]$ converge to x^* for any starting vector x^0:

1. A is symmetric and positive definite.
2. A is an M-matrix.
3. A is strictly diagonally dominant.
4. A is irreducibly diagonally dominant.

The first two of these conditions have been extended to nonlinear systems, and appropriate convergence results have been given. To be specific, Schechter [19] proved global convergence for the nonlinear Gauss-Seidel method for a certain generalization of the first condition, and Elkin [8], using a weaker generalization, extended Schechter's results. Concerning the second condition, Rheinboldt [16], following an unpublished suggestion of J. M. Ortega, investigated an extension of the M-matrix concept and proved a global convergence result for the (underrelaxed) nonlinear Jacobi and Gauss-Seidel processes. These M-functions, and the corresponding global convergence theorem, brought together a number of apparently separate results of Bers [2], Ortega and Rheinboldt [13], and Porsching [14].

In Section 2 we present the mentioned generalizations of the strictly and irreducibly diagonally dominant matrices — the weakly Ω - diagonally dominant functions - and give sufficient conditions for a (Gateaux) differentiable function to belong to this class of mappings. At this stage we point out that certain classes of matrices considered by Walter [20], and Bramble and Hubbard [3], as well as the strictly and irreducibly diagonally dominant matrices are covered by our definitions.

Section 3 contains some of the basic properties of weakly Ω - diagonally dominant functions - and their subfunctions - that are necessary in the next two sections. As a by-product of these properties we obtain a necessary condition for a differentiable function to be weakly Ω - diagonally dominant.

It is then shown in Section 4 how our results are related to the M- and P-functions studied in Rheinboldt [16] and More' and Rheinboldt [11], respectively. These results show that many of the functions arising from nonlinear network problems and discretizations of partial differential equations are, in fact, weakly Ω - diagonally dominant. Finally, Section 5 contains convergence results for the Jacobi and Gauss-Seidel iterations and an application of these results to finding nonnegative solutions of two-point boundary value problems.

2. Definitions and Preliminary Results. We denote by R^n the real n - dimensional linear space of column vectors $x = (x_1,\ldots,x_n)^T$ and by $L(R^n)$ the linear space of real matrices $A = (a_{ij})$ of order n. The ℓ_∞ norm $||x||_\infty = \max\{|x_i|: i=1,\ldots,n\}$ in R^n and the corresponding induced operator norm in $L(R^n)$,

$$||A||_\infty = \max\{\sum_{j=1}^{n} |a_{ij}| \;:\; i = 1,\ldots,n\},$$

will be used frequently. In addition, we use the coordinatewise partial orderings on R^n and $L(R^n)$; that is, if x,y in R^n then $x \geq y$ ($x>y$) if and only if $x_i \geq y_i$ ($x_i > y_i$) for $i=1,\ldots,n$, and similarly for $L(R^n)$.

A rectangle Q in $L(R^n)$ is the Cartesian product of n intervals, each of which may be either open, closed, or semi-open. In particular, any of these intervals may be unbounded, and thus, a rectangle may be all of R^n. The line segment $[x,y]$ is the set $\{ z \in R^n: z = ty + (1-t)x$ for some $t \in [0,1] \}$, and the set $\{1,\dots,n\}$ will always be denoted by N. Finally, the vector $e \in R^n$ is defined by $e_i = 1$ for each $i \in N$, while e^j is the usual j-th unit vector.

We now recall the definitions of certain classes of matrices that will play a role in this article.

<u>Definition 2.1</u> a) A in $L(R^n)$ is (strictly) diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \qquad (>)$$

for each $i \in N$, where for $n=1$ the sum on the right is defined to be zero.

b) A in $L(R^n)$ is irreducibly diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$$

for each $i \in N$, where for at least one $i \in N$ strict inequality holds, and if for every i,j in N, there is a sequence of non-zero elements of A of the form $a_{i,i_1}, a_{i_1,i_2},\dots, a_{i_r,j}$.

c) A in $L(R^n)$ is an M- matrix if $a_{ij} \leq 0$ for $i \neq j$ in N, and $A^{-1} \geq 0$.

In order to introduce our generalization of a strictly diagonally dominant matrix, we will have to look at this class of matrices from a somewhat different point of view than is usual. The following result will indicate the way.

<u>Lemma 2.2</u> Let $v \in R^n$; then

a) $|v_k| > \sum_{j \neq k} |v_j|$ for some $k \in N$

if and only if for any $x \in R^n$

b) $\sum_{j=1}^{n} v_j x_j = 0$, $x \neq 0$, implies that $|x_k| < ||x||_\infty$

<u>Proof</u>. Assume that a) holds, and that

$$\sum_{j=1}^{n} v_j x_j = 0, \; x \neq 0.$$

Then $v_k x_k = -\sum_{j \neq k} v_j x_j$, and $|v_k||x_k| \leq \sum_{j \neq k} |v_j| \; ||x||_\infty$, from which b) follows.

If b) holds but $|v_k| \leq \sum_{j \neq k} |v_j|$, then $|v_k| = \alpha \sum_{j \neq k} |v_j|$ where $\alpha \leq 1$. Define $x \in R^n$ by $x_k = \operatorname{sgn} v_k$, $x_j = -\alpha \operatorname{sgn} v_j$, $j \neq k$; then $||x||_\infty = 1 = |x_k|$ and $\sum_{j=1}^{n} v_j x_j = 0$. This contradicts b) since $x \neq 0$. Hence, a) must hold.

If $A \in L(R^n)$, and for some $k \in N$, $v_j = a_{kj}$, $j = 1,\dots,n$, then a) is equivalent to assuming "strict diagonal dominance on the kth row". Condition b) can be generalized to the nonlinear case.

<u>Definition 2.3</u> a) A functional $f: D \subset R^n \to R^1$ is strictly diagonally dominant on D with respect to the kth variable if for every $x \neq y$ in D,

$$f(x) = f(y), \text{ implies that } |x_k - y_k| < \|x-y\|_\infty .$$

b) A function $F: D \subset R^n \to R^n$ is strictly diagonally dominant on D if for each $k \in N$, the kth component function of F, f_k, is strictly diagonally dominant with respect to the kth variable.

From Lemma 2.2 we obtain immedietely:

<u>Theorem 2.4</u> Let $A \in L(R^n)$. Then A is a strictly diagonally dominant matrix if and only if the induced mapping $Fx = Ax$ is a strictly diagonally dominant function on R^n.

We next prove several results that give sufficient conditions for a function to be strictly diagonally dominant in terms of its derivative. The notion of differentiability to be used is that of the well-known Gateaux derivative. Briefly: $F: D \subset R^n \to R^m$ is <u>G-differentiable</u> at an interior point $x \in D$ if there is an m by n matrix A such that for any $h \in R^n$,

$$\lim_{t \to 0} \frac{1}{|t|} \| F(x+t\,h) - F(x) - t\,Ah \| = 0.$$

It is clear that there is only one such A, denoted by $F'(x)$, namely, the Jacobian matrix $(\partial_j f_i(x))$ where $\partial_j f_i(x) \equiv \frac{\partial f_i(x)}{\partial x_j}$. For a summary of the properties of G-differentiable functions, see Ortega and Rheinboldt [12].

<u>Theorem 2.5</u> Let $F \cdot D \subset R^n \to R^n$ be G-differentiable on the convex set $D_0 \subset D$, and assume that $F'(x)$ is a strictly diagonally dominant matrix for each x in D_0. Then F is a strictly diagonally dominant function on D_0.

<u>Proof</u>. Let $k \in N$ be given, and assume that $f_k(x) = f_k(y)$ for some $x \neq y$ in D_0. Then $\psi(t) = f_k(x+t(y-x))$ is differentiable on $[0,1]$, and $\psi(0) = \psi(1)$. By Rolle's theorem, there is a $t_0 \in (0,1)$ such that

$$\psi'(t_0) = \sum_{j=1}^{n} \partial_j f_k(x+t_0(y-x))(y_j-x_j) = 0.$$

The conclusion now follows from Lemma 2.2 with

$$v_j = \partial_j f_k(x+t_0(y-x)), \quad j = 1,\ldots,n.$$

Later we shall see that this result admits a certain converse. On the other hand, Theorem 2.5 does not account for the case where $F'(x)$ is not strictly diagonally dominant at all points. The next result will point out how this theorem can be extended to cover this case.

<u>Theorem 2.6</u> Let $f: D \subset R^n \to R^1$ be G-differentiable on the convex set $D_0 \subset D$, and assume that for some fixed $k \in N$,

$$(2.1) \qquad |\partial_k f(x)| \geq \sum_{j \neq k} |\partial_j f(x)|$$

for each $x \in D_0$, $\partial_k f(x)$ does not change sign on D_0, and f is not constant on any line segment $[x,y]$ for which $|x_k-y_k| = ||x-y||_\infty > 0$. Then f is strictly diagonally dominant on D_0 with respect to the kth variable.

<u>Proof</u>. If for some $x \neq y$ in D_0, $f(x) = f(y)$ and $|x_k-y_k| = ||x-y||_\infty$, then $x_k = y_k$ and without loss of generality, we may suppose that $y_k-x_k>0$. Since $\partial_k f$ does not change sign on D_0, assume that $\partial_k f(x+t(y-x)) \geq 0$ for each $t \in [0,1]$. Then if $\psi(t) = f_k(x + t(y-x))$,

$$\psi'(t) \geq \sum_{j \neq k} |\partial_j f(x+t(y-x))| (||y-x||_\infty - |y_j - x_j|) \geq 0.$$

Since $\psi(1) = \psi(0)$, it follows that $\psi'(t) \equiv 0$ for $t \in [0,1]$ which contradicts the fact that f is not constant on the line segment $[x,y]$.

The preceding result extends the class of functions which Theorem 2.5 identifies as stictly diagonally dominant.

Example 2.7 Let $F:R^2 \to R^2$ be defined by

$$F(x_1,x_2) = \begin{bmatrix} x_1 - \sin\ x_2 \\ x_2^3 \end{bmatrix}$$

Direct computation shows that each component function of F satisfies the hypotheses of Theorem 2.6, and therefore, F is strictly diagonally dominant on R^2. Note, however, that $F'(x)$ is strictly diagonally dominant only if x_2 is not an even multiple of π.

Suppose now that $A \in L(R^n)$ is irreducibly diagonally dominant, but not strictly diagonally dominant; then $Fx = Ax$ is not a strictly diagonally dominant function. Since this type of matrix function arises frequently in practical situations, it is interesting to consider a corresponding extension of the diagonal dominance concept. We begin with an analog of Lemma 2.2.

Lemma 2.8 Let $v \in R^n$; then

a) $|v_k| \geq \sum_{j \neq k} |v_j|$ for some $k \in N$

if and only if for any $x \in R^n$,

b) $\sum_{j=1}^{n} v_j x_j = 0$, $x \neq 0$, implies that either $|x_k| < ||x||_\infty$, or $|x_k| = ||x||_\infty = |x_j|$ whenever $v_j \neq 0$.

<u>Proof</u>. Assume that a) holds and that $\sum_{j=1}^{n} v_j x_j = 0$ for some $x \neq 0$. If $|x_k| < ||x||_\infty$ there is nothing to prove; hence, suppose that $|x_k| = ||x||_\infty$. Then

$$v_k x_k = -\sum_{j\neq k} v_j x_j,$$

and

$$\sum_{j\neq k} |v_j| \; ||x||_\infty \leq |v_k| \; ||x||_\infty = |v_k||x_k| \leq \sum_{j\neq k} |v_j||x_j|.$$

Thus,

$$\sum_{j\neq k} |v_j|(||x||_\infty - |x_j|) \leq 0,$$

which shows that $|x_j| = ||x||_\infty$ whenever $v_j \neq 0$.

Conversely, if b) holds, but

(2.2) $\quad |v_k| < \sum_{j\neq k} |v_j|,$

then $|v_k| = \alpha \sum_{j\neq k} |v_j|$ where $\alpha < 1$. Define $x \in R^n$ by $x_k = \operatorname{sgn} v_k$, $x_j = -\alpha \operatorname{sgn} v_j$, $j \neq k$; then $||x||_\infty = 1 = |x_k|$, and $\sum_{j=1}^{n} v_j x_j = 0$. By b), $|x_j| = ||x||_\infty$ whenever $v_j \neq 0$, but since $|x_j| = \alpha < 1 = ||x||_\infty$, we have $v_j = 0$ for all $j \neq k$. This contradicts (2.2).

Lemma 2.8 is the clue to generalizing the notion of a diagonally dominant matrix; we only need to specify the non-linear counterpart of the condition $v_j \neq 0$ in b). To do this, we will use the well-known notion of a finite directed graph or network. For our purposes a <u>network</u> $\Omega = (N,\Lambda)$ consists of a set of n <u>nodes</u> $N = \{1,\ldots,n\}$, and a set $\Lambda \subset N \times N$ of (directed) <u>links</u> which contain no loops; that is, $(i,i) \notin \Lambda$ if $i \in N$. <u>A node i is connected to a node j</u> if there is a <u>directed path in Λ from i to j</u>; that is, a sequence of links of the form (i,i_1), (i_1,i_2), ..., (i_r,j).

Note that since we only consider directed paths, it will be possible for node i to be connected to node j without j being connected to i.

<u>Definition 2.9</u> A mapping $F: D \subset R^n \to R^n$ is diagonally dominant on D with respect to the family of networks $\{\Omega_x: x \in D\}$, if for every $x \in D$, the network $\Omega_x = (N, \Lambda_x)$ is such that $f_k(y) = f_k(x)$ for some $y \neq x$ in D and k in N, implies that either $|x_k - y_k| < ||x-y||_\infty$, or $|x_k-y_k| = ||x-y||_\infty = |x_j-y_j|$ whenever $(k,j) \in \Lambda_x$.

In some cases, the mapping F will be diagonally dominant with respect to a single term family of networks. For example, it follows directly from the above definition and Lemma 2.8 that $A \in L(R^n)$ is a diagonally dominant matrix if and only if the induced mapping $Fx=Ax$ is diagonally dominant on R^n with respect to the associated network Ω_A. Here $\Omega_A = (N,\Lambda_A)$ is defined by

$$\Lambda_A = \{(i,j) \in N \times N: \ i \neq j,\ a_{ij} \neq 0\}.$$

It should now be apparent how to generalize the notion of an irreducibly diagonally dominant matrix; instead we shall generalize a weaker concept which, in addition, contains the strictly diagonally dominant matrices and functions as a special case.

<u>Definition 2.10</u> The mapping $F: D \subset R^n \to R^n$ is weakly Ω - diagonally dominant on D if for each $x \in D$ there is a network $\Omega_x = (N, \Lambda_x)$ such that

a) F is diagonally dominant with respect to the family of networks $\{\Omega_x : x \in D\}$, and

b) For each $x \in D$ there is a nonempty subset J_x of N such that for each $i \in J_x$, $f_i(y)=f_i(x)$ for $y \neq x$ in D implies that $|y_i - x_i| < ||y-x||_\infty$, and for each $i \notin J_x$, there is a path in Λ_x from i to some $j = j(i) \in J_x$.

An important special case of the above definition arises when the networks Ω_x and the sets J_x are independent of x in D.

<u>Definition 2.11</u> The mapping $F: D \subset R^n \to R^n$ is Ω-diagonally dominant on D if there is a network $\Omega = (N, \Lambda)$ such that

a) F is diagonally dominant with respect to the network $\Omega = (N, \Lambda)$, and

b) There is a nonempty subset J of N such that for each $i \in J$, f_i is strictly diagonally dominant with respect to the ith variable, and for each $i \notin J$, there is a path in Λ from i to some $j = j(i) \in J$.

Definition 2.10 and 2.11 seems to be new in this generality but for functions defined on all of R^n, many authors have considered special cases. In particular, for linear functions, Walter [20] considered matrices A satisfying a condition Z_2 which is equivalent to requiring that $I - |A|$ be Ω-diagonally dominant where $|A| = (|a_{ij}|)$, while Duffin [7] assumed that the matrix A had nonpositive off-diagonal elements and considered another special case of Definition 2.11. Their formulations are designated, in the theorem below, by c) and d), respectively. Of course, $A \in L(R^n)$ (weakly) Ω-diagonally dominant on R^n means that $Fx=Ax$ is (weakly) Ω-diagonally dominant on R^n with Ω_x being the associated network of A for each x in R^n.

Theorem 2.12 Let $A \in L(R^n)$ be given such that $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$ for each $i \in N$, and set $J = \{i \in N: |a_{ii}| > \sum_{j\neq i} |a_{ij}|\}$. Then the following conditions are equivalent.

a) A is weakly Ω-diagonally dominant.

b) A is Ω-diagonally dominant.

c) J is not empty, and for every non-empty subset L of N for which $L \cap J$ is empty, there is an $i \in L$ and a $j \notin L$ such that $a_{ij} \neq 0$.

d) J is not empty, and for each $i \notin J$ there is a sequence of nonzero elements of A of the form a_{i,i_1}, $a_{i_1,i_2}, \ldots, a_{i_r,j}$ where $j \in J$.

The equivalence of the first three conditions follows directly from the definitions and Lemmas 2.2 and 2.8, while the equivalence of c) and d) is a consequence of the next result which also shows that the connectivity assumptions of part b) of Definitions 2.10 and 2.11 could have been phrased in somewhat different terms.

<u>Lemma 2.13</u> Let $\Omega = (N,\Lambda)$ be a network and J a non-empty subset of N. Then for each $i \notin J$ there is a path from i to some $j = j(i) \in J$ if and only if for every non-empty subset L of N such that $L \cap J$ is empty, there is an (i,j) in Λ with $i \in L$ and $j \notin L$.

<u>Proof</u>. Assume first that for each $i \notin J$ there is a path from i to some $j = j(i) \in J$, and let L be a non-empty subset of N such that $L \cap J$ is empty. Choose $i_0 \in L$; then $i_0 \notin J$ and hence, there is a path $(i_0,i_1),\ldots,(i_{r-1},i_r)$ to some $i_r \in J$. Let p be the first integer such that $i_p \notin L$, and note that $1 \leq p \leq r$ since $i_0 \in L$ and $i_r \notin L$. Then $(i_{p-1},i_p) \in \Lambda$ with $i_{p-1} \in L$ and $i_p \notin L$.

Conversely, if $i_0 \notin J$, then $\{i_0\} \cap J$ is empty, and hence, i_0 is connected to some $j \neq i_0$. Thus, $L = \{j : i_0$ is connected to $j\}$ is not empty. If $L \cap J$ were empty, then by hypothesis there is a link (i,j) with $i \in L$ and $j \notin L$. Thus i_0 is not connected to node j. But this is absurd since i_0 is connected to i and $(i,j) \in \Lambda$. Consequently, $L \cap J$ is not empty which is what we wanted to show.

It is now easy to prove the next result.

<u>Theorem 2.14</u> Let $F: D \subset R^n \to R^n$ be continuously differentiable on the convex set $D_0 \subset D$. If $F'(x)$ is a Ω-diagonally dominant matrix for each $x \in D_0$, then F is a weakly

Ω-diagonally dominant function on D_0.

Proof. Let $x \in D_0$ be given and set $A = F'(x)$, $\Lambda_x = \{(i,j): i \neq j,\ a_{ij} \neq 0\}$, $\Omega_x = (N, \Lambda_x)$, and $J_x = \{i: |a_{ii}| > \sum_{j \neq i} |a_{ij}|\}$. Then for each $i \notin J_x$ there is a path in Λ_x from i to some $j = j(i) \in J_x$. We now show that F is diagonally dominant with respect to the family of networks $\{\Omega_x: x \in D_0\}$. Assume thus, that $f_k(y) = f_k(x)$ for some $y \neq x$ in D_0 and $k \in N$, and that $|y_k - x_k| = \|y-x\|_\infty$. Since the diagonal entries of $F'(x)$ do not vanish for any $x \in D_0$ by Theorem 2.12 c), the proof of Theorem 2.6 shows that if $\psi(t) = f_k(x+t(y-x))$ for $t \in [0,1]$, then $\psi'(t) = 0$ for each $t \in [0,1]$. In particular,

$$\psi'(0) = \sum_{j=1}^{n} \partial_j f_k(x)\,(y_j - x_j) = 0$$

and by Lemma 2.8 it follows that $|y_j - x_j| = \|y-x\|_\infty$ whenever $(k,j) \in \Lambda_x$. To conclude the proof note that if $k \in J_x$, $\psi'(0) = 0$ implies that $|y_k - x_k| < \|y-x\|_\infty$. Hence, if $f_k(y) = f_k(x)$ for some $y \neq x$ in D_0 and $k \in J_x$, then we must have $|y_k - x_k| < \|y-x\|_\infty$.

Under the assumptions of Theorem 2.14 it does not follow that F is Ω-diagonally dominant.

Example 2.15 Let $g: R^1 \to R^1$ be any continuously differentiable function such that $g'(t) = 0$, $t \leq -1$, $g'(t) = 1$, $t \geq 1$, and $0 < g'(t) < 1$, $|t| < 1$, and define $F: R^2 \to R^2$ by

$$F(x_1, x_2) = \begin{pmatrix} x_1 - g(x_2) \\ x_2 + g(x_2) - x_1 \end{pmatrix}$$

Then $F'(x_1,x_2)$ is Ω-diagonally dominant for all (x_1,x_2) but F is not a Ω-diagonally dominant function since none of its component functions is strictly diagonally dominant with respect to the appropriate variable.

3. Basic Properties of Weakly Ω-diagonally Dominant Functions. In this section we will generalize some of the facts known about strictly and irreducibly diagonally dominant matrices (see, for example, Varga [19]) to functions that satisfy Definitions 2.3, 2.9 and 2.10. For this purpose, we will need the notion of a subfunction.

Definition 3.1 Consider $F:D \subset R^n \to R^m$, $n \geq m$, and let $L = \{i_1,\ldots,i_k\}$ be a non-empty subset of N. For fixed x in D, define

$$D_G = \{(t_{i_1},\ldots,t_{i_n}): \sum_{j=1}^{k} t_{i_j} e^{i_j} + \sum_{j \notin L} x_j e^j \text{ belongs to } D\}.$$

Then $G:D_G \subset R^k \to R^k$ is a subfunction of F at x belonging to L if

$$g_\ell(t_{i_1},\ldots,t_{i_k}) = f_{i_\ell}\left(\sum_{j=1}^{k} t_{i_j} e^{i_j} + \sum_{j \notin L} x_j e^j\right), \quad \ell=1,\ldots,k.$$

If $F:R^n \to R^n$ is linear, then a subfunction of F with $x = 0$ is a principal submatrix. If F is nonlinear this concept of subfunction has been used implicitly by many authors, but Rheinboldt [16] seems to be the first one to make explicit use of this definition. We also remark that the subfunction G depends on a specific value of x in D, but since it will always be clear which x is being used, this x has not been made an explicit part of the notation.

Theorem 3.2 Let $F:D \subset R^n \to R^n$ be (weakly) Ω-diagonally dominant on D. Then each subfunction of F is also (weakly) Ω-diagonally dominant.

Proof: We only carry out the proof for the weakly Ω-diagonally dominant case; the proof is analogous in the other case.

Let $L \subset N$ be a non-empty proper subset, and for ease of notation assume that $L = \{1,\ldots,k\}, 1 \le k < n$. Let $G: D_G \subset R^k \to R^k$ be the subfunction of F belonging to L with components

$$g_i(x_1,\ldots,x_k) = f_i(x_1,\ldots,x_k,z_{k+1},\ldots,z_n), i \in L.$$

In order to show that G is weakly Ω-diagonally dominant, we must exhibit for each $x = (x_1,\ldots,x_k)^T \in D_G$ a network Ω_x and a non-empty J_x of L which satisfy Definition 2.10. Set $\hat{x} = (x_1,\ldots,x_k, z_{k+1},\ldots,z_n)^T$ and define $J_x = \{i \in L: (i,j) \in \Lambda_{\hat{x}}$ for some $j \notin L\} \cup (L \cap J_{\hat{x}})$, and set $\Omega_x = (L, \Lambda_x)$ where $\Lambda_x = \Lambda_{\hat{x}} \cap (L \times L)$. We show first that J_x is not empty. For this assume that $L \cap J_{\hat{x}}$ is empty; otherwise there is nothing to prove. Then there exists an $i_0 \in L$ such that $i_0 \notin J_{\hat{x}}$ and a path in $\Lambda_{\hat{x}}$

$$(3.1) \qquad (i_0,i_1),(i_1,i_2),\ldots,(i_{r-1},i_r)$$

connecting i_0 to some $i_r \in J_{\hat{x}}$. If p is the first integer such that $i_p \notin L$, then $1 \le p \le r$, since $i_0 \in L$ and $i_r \notin L$. Therefore, J_x is not empty, since necessarily $i_{p-1} \in J_x$. Assume now that $i_0 \in L$ is any index for which $i_0 \notin J_x$; then $i_0 \notin J_{\hat{x}}$ and thus there is a path (3.1) in $\Lambda_{\hat{x}}$ connecting i_0 to some $i_r \in J_{\hat{x}}$. If path (3.1) is also in Λ_x, then $i_r \in L$ and consequently, $i_r \in J_x$. Hence, (3.1) is a path in Λ_x connecting i_0 to $i_r \in J_x$. Otherwise, $1 \le p \le r$ where p is defined as above, and, since $i_0 \notin J_x$, we have $p \ne 1$. Hence, $(i_0,i_1),\ldots,(i_{p-2},i_{p-1})$ is a path in Λ_x connecting i_0 to $i_{p-1} \in J_x$. Clearly, G is diagonally dominant with respect to $\{\Omega_x: x \in D\}$, and hence, we only need to show that for $i \in J_x$, $g_i(y) = g_i(x)$ for some $y \ne x$ in D_G implies that $|y_i - x_i| < \|y-x\|_\infty$. If $i \in J_{\hat{x}}$ this is clear; hence assume that

For the remainder of this section we will assume that F is continuous. In this connection recall that if $f: J \subset R^1 \to R^1$ is continuous and injective on the interval J, then f is either <u>strictly isotone</u> ($s > t$ implies $f(s) > f(t)$) or <u>strictly antitone</u> ($s > t$ implies $f(s) < f(t)$) on J.

<u>Theorem 3.4</u> Let $K \subset R^n$ be a convex set and $J \subset R^1$ an aribitrary interval, and assume that $H: K \times J \to R^1$ satisfies the following conditions:

a) For each $t \in J$, $H(\cdot,t)$ is continuous on K, and

b) For each $z \in K$, $H(z,\cdot)$ is continuous and injective on J.

Then either $H(z,\cdot)$ is strictly isotone on J for all z in K, or $H(z,\cdot)$ is strictly antitone on J for all z in K.

<u>Proof.</u> Since $H(z,\cdot)$ is continuous and injective on an interval J for any fixed $z \in K$, it is either strictly isotone or strictly antitone on J. Assume $H(z,\cdot)$ is strictly isotone for some $z \in K$, and define

$$K_0 = \{z \in K: H(z,\cdot) \text{ is strictly isotone on } J\}.$$

Then K_0 is a nonempty subset of K and since $z \in K_0$ if and only if for some t, s in J, $t > s$ and $H(z,t) > H(z,s)$, the continuity of a) implies that K_0 is open relative to K. Similarly, if $z \in K$ is a limit point of K_0, then for any $t > s$ in J, $H(z,t) \geq H(z,s)$ and by the injectivity of b), $H(z,t) > H(z,s)$. Hence, K_0 is also closed relative to K, and since K is connected, $K_0 = K$ as desired.

To state the consequences of the previous result, we will need the following definition.

<u>Definition 3.5</u> Let $F:D \subset R^n \to R^n$ be given. The kth diagonal subfunction of F at x is the subfunction of F at x belonging to $L = \{k\}$.

We can now prove the main result of this section which, for example, will help us analyze the Jacobi and Gauss-Seidel iterates.

<u>Corollary 3.6</u> Let $F:Q \subset R^n \to R^n$ be continuous and weakly Ω-diagonally dominant on the rectangle Q. Then for any $k \in N$, the kth diagonal subfunction ψ_k of F at x is either strictly isotone for all x in Q, or strictly antitone for all x in Q. Moreover, if for some $y \neq x$ in Q, $|y_k - x_k| = \|y-x\|_\infty$, then $(x_k-y_k)[f_k(x) - f_k(y)] \geq 0$ if ψ_k is isotone, and $(x_k-y_k)[f_k(x)-f_k(y)] \leq 0$ if ψ_k is antitone.

<u>Proof</u>. Since Q is a rectangle, $Q = \prod_{i=1}^{n} I_i$ where I_i is an interval. Now let $x \in Q$ and $k \in N$ be given and define

$$K = \prod_{\substack{i=1\\ i\neq k}}^{n} I_i, \quad J = I_k, \text{ and } H: K \times J \to R^1 \text{ by}$$

$$H(x,t) = f_k(x_1,\ldots,x_{k-1}, t, x_{k+1},\ldots,x_n).$$

Since the hypotheses of Theorem 3.4 are satisfied, we have proved the first part of the theorem. For the second part, assume $|y_k-x_k| = \|y-x\|_\infty$ for some $y \neq x$ in Q. Without loss of generality, we take $y_k-x_k > 0$ and then define

$$K = \{z \in Q: z \neq x,\ z_k-x_k = \|z-x\|_\infty,$$
$$|z_j-x_j| < \|z-x\|_\infty \text{ whenever } (k,j)\in\Lambda_z\},$$

$J = [0,1]$, and $H: K \times J \to R^1$ by

$$H(z,t) = f_k(x+t(z-x)).$$

It is now easy to verify that K is convex and the the hypotheses of Theorem 3.4 are satisfied for this H. Thus, if ψ is strictly isotone, $H(x+\tilde{c}e^k,\cdot)$ is strictly isotone for

$\epsilon = y_k - x_k$ and hence, $H(z,\cdot)$ is strictly isotone for every $z \in K$. If $f_k(y) < f_k(x)$, then there is a $\hat{y} \in K$ such that $f_k(\hat{y}) < f_k(x)$; but then

$$H(\hat{y},1) = f_k(\hat{y}) < f_k(x) = H(\hat{y},0)$$

which contradicts the fact that $H(\hat{y},\cdot)$ is strictly isotone.

An immediate consequence of the previous result is a partial converse of Theorems 2.16 and 2.14.

Corollary 3.7 If $F:D \subset R^n \to R^n$ is continuous, G-differentiable, and weakly Ω-diagonally dominant on the open, convex set $D_0 \subset D$, then $F'(x)$ is a diagonally dominant matrix whose diagonal entries do not change sign on D_0.

Proof. Let $x \in D_0$ be given. Since F is continuous on D_0, we can apply Corollary 3.6 to any rectangle $Q \subset D_0$ containing x. Hence, for any given $k \in N$, the kth diagonal subfunction ψ_k of F at x is either strictly isotone or strictly antitone, and--to be definite--assume that ψ_k is strictly isotone. Now let u be the vector with the components $u_k = 1$ and $u_j = -\operatorname{sgn} \partial_j f_k(x)$ for $j \neq k$, and let $\delta > 0$ be such that $x + tu \in [0,\delta)$. Then, for $t \in (0,\delta)$, $t = \|x+tu-x\|_\infty$, and therefore

$$t[f_k(x+tu)-f_k(x)] \geq 0.$$

Dividing by $t^2 > 0$ and passing to the limit as $t \to 0^+$, we obtain,

$$(3.2) \qquad f_k'(x)u = \sum_{j=1}^{n} \partial_j f_k(x) u_j \geq 0.$$

But $\partial_k f_k(x) \geq 0$ since ψ_k is strictly isotone, and hence, (3.2) is equivalent to

$$|\partial_k f_k(x)| \geq \sum_{j \neq k} |\partial_j f_k(x)|.$$

To conclude the proof, note that the diagonal entries of $F'(x)$ do not change sign in any rectangle $Q \subset D_0$ which contains x; by applying this argument to a sequence of overlapping rectangles lying along the line segment $[x,y]$, we obtain the desired result.

The last result can be improved if F is strictly diagonally dominant.

Corollary 3.8 Assume $f: D \subset R^n \to R^1$ is continuous on the open, convex set $D_0 \subset D$, and for some $k \in N$ let ψ be the subfunction of f at x belonging to $L = \{k\}$. Then f is strictly diagonally dominant on D_0 with respect to the kth variable if and only if the following two conditions hold:

a) ψ is either strictly isotone for all x in D_0, or strictly antitone for all x in D_0.

b) If $|y_k - x_k| = ||y-x||_\infty$ for $y \neq x$ in D_0, then
$(x_k - y_k)\ [f(x) - f(y)] > 0$ if ψ is isotone, and
$(x_k - y_k)\ [f(x) - f(y)] < 0$ if ψ is antitone.

Proof. If the two conditions hold, then f is clearly strictly diagonally dominant on D_0 with respect to the kth variable. For the converse, let $x \neq y$ in D_0 be given. Since D_0 is open and convex, there is an $\epsilon > 0$ such that $x + z(y-x) + te^k$ belongs to D_0 for each $z \in [0,1]$ and $t \in [-\epsilon, \epsilon]$. If we now define $H: [0,1] \times [-\epsilon,\epsilon] \to R^1$ by

$$H(z,t) = f(x + z(y-x) + t\,e^k)$$

then the hypotheses of Theorem 3.4 are satisfied and hence, $H(0,\cdot)$ and $H(1,\cdot)$ are both strictly isotone, or both strictly antitone. This proves part a); part b) follows by the same

4. <u>Relationship between Weakly Ω-diagonally Dominant Functions and other Classes of Functions</u>. In this section we will explore the connection between Definition 2.10 and the M- and P- functions introduced in [16] and [11] respectively. We will need the following standard terminology; see Collatz [5] and Rheinboldt [16].

<u>Definition 4.1</u>

a) The mapping $F:D \subset R^m \to R^n$ is isotone (antitone) on D if $x \leq y$, $x,y \in D$, implies that $Fx \leq Fy$ ($Fx \geq Fy$), and strictly isotone (strictly antitone) if, in addition, it follows from $x < y$, x, $y \in D$, that also $Fx < Fy$ ($Fx > Fy$).

b) The function $F:D \subset R^m \to R^n$ is inverse isotone on D if $Fx \leq Fy$, x, $y \in D$, implies that $x \leq y$.

c) A mapping $F:D \subset R^n \to R^n$ is off-diagonally antitone on D if for any $x \in D$ and any $i \neq j$, i, $j \in N$, the functions

$$\psi_{ij}:\{t \in R^1: x + t\, e^j \in D\} \to R^1,\ \psi_{ij}(t) = f_i(x + t\, e^j)$$

are antitone.

d) The mapping $F:D \subset R^n \to R^n$ is an M- function on D if F is off-diagonally antitone and inverse isotone on D.

As stated, c) is somewhat awkward to use, but note that if D is a rectangle, then F is off-diagonally antitone on D if and only if for every $x,y \in D$ with $x \leq y$, $x_k = y_k$ implies that $f_k(x) \geq f_k(y)$.

<u>Definition 4.2</u> The mapping $F:D \subset R^n \to R^n$ is a P-function if, for any x, $y \in D$, $x \neq y$, there is an index $k = k(x,y) \in N$ such that $(x_k - y_k)[f_k(x) - f_k(y)] > 0$.

Clearly, if F is either an M- or P- function, then ([11], [16]) F is strictly diagonally isotone, where $F:D \subset R^n \to R^n$ is <u>(strictly) diagonally isotone on D</u> if for each $x \in D$ and $k \in N$, the kth diagonal subfunction of F at x is (strictly) isotone. The precise relationship between M- and P- functions is given in the following result whose proof can be found in [11].

<u>Theorem 4.3</u> The mapping $F:Q \subset R^n \to R^n$ is an M-function on the rectangle Q if and only if F is an off-diagonally antitone P-function.

The relationship between weakly Ω-diagonally dominant functions and P-functions is contained in the next result.

<u>Theorem 4.4</u> Let $F:Q \subset R^n \to R^n$ be continuous and weakly Ω-diagonally dominant on the rectangle Q. Then F is a P-function if and only if F is diagonally isotone on Q.

<u>Proof.</u> If F is a P-function, then F is clearly diagonally isotone on Q. Conversely, if F is diagonally isotone on Q and $x \neq y$ are given, we must show that there is a $k \in N$ such that $(x_k-y_k)[f_k(x) - f_k(y)] > 0$. Let

$$L = \{i \in N: f_i(x) = f_i(y), |x_i-y_i| = ||x-y||_\infty\},$$

and note that if L is empty, then there is necessarily a $k \in N$ such that $|x_k-y_k| = ||x-y||_\infty$ and $f_k(x) \neq f_k(y)$. Corollary 3.6 then implies that $(x_k-y_k)[f_k(x) - f_k(y)] > 0$. Otherwise, L is a non-empty subset of N such that $L \cap J_x$ is empty. Since F is weakly Ω-diagonally dominant, Lemma 2.13 yields an $(i,j) \in \Lambda_x$ with $i \in L$ and $j \notin L$. It follows that $|x_j-y_j| = ||x-y||_\infty$, and since $j \notin L$, we have $f_j(x) \neq f_j(y)$. Corollary 3.6 now shows that

$$(x_j-y_j)\ [f_j(x) - f_j(y)] > 0.$$

We will now investigate the relationship between weakly Ω-diagonally dominant functions and M-functions. The functions to be considered will be assumed to be defined on a rectangle of the form

$$Q = \prod_{i=1}^{n} I_i, \tag{4.1}$$

where each I_i is an interval of the form $(\alpha_i, +\infty)$ or $[\alpha_i\ +\infty)$. In the first case, $\alpha_i = -\infty$ is permitted; otherwise, α_i is real. We set $R^1_+ = \{t \in R^1: t \geq 0\}$.

<u>Theorem 4.5</u> Let $F:Q \subset R^n$ be a continuous, off-diagonally antitone function on a rectangle Q of the form (4.1). The following three statements are then equivalent, and in each case F is an M-function.

a) F is a diagonally isotone, strictly diagonally dominant function on Q.

b) For each x in Q, $F(x + t\ e)$ is strictly isotone as a function of $t \in R^1_+$.

c) For any $x \neq y$ in Q, $(x_k-y_k)[f_k(x) - f_k(y)] > 0$ whenever $|x_k-y_k| = ||x-y||_\infty$.

<u>Proof</u>. If a) holds, then F is an M-function by Theorems 4.3 and 4.4. Moreover, for any k, $f_k(x + t\ e)$ is injective as a function of t and, by Corollary 3.6, $(s - t)\ [f_k(x + s\ e) - f_k(x + t\ e)] \geq 0$ for any $s \neq t$ in R^1_+. Hence, b) holds, and we only need to show now that b) implies c), since c) trivially yields a).

If F satisfies b), and for some $x \neq y$ in Q we have $|x_k-y_k| = ||x-y||_\infty$, then $x_k \neq y_k$. If $y_k > x_k$, then b) together with the off-diagonal antitonicity of F implies that $f_k(x) < f_k(x+(y_k-x_k)e) \leq f_k(y)$. Similarly, if $x_k > y_k$ we obtain that $f_k(y) < f_k(y+(x_k-y_k)e) \leq f_k(x)$. In either case, $(x_k-y_k)[f_k(x) - f_k(y)] > 0$ and c) is satisfied.

Note that if b) holds, F is an M-function [16] even if F is not continuous; in fact, a weakening of the hypothesis allows us to obtain a sufficient condition for F to be a weakly Ω-diagonally dominant M-function.

<u>Theorem 4.6</u> Let $F:Q \subset R^n \to R^n$ be an off-diagonally antitone function on a rectangle Q of the form (4.1), and suppose that for any $x \in Q$, $F(x+te)$ is an isotone function for t on R^1_+. Assume further that for each $x \in Q$,

$J_x = \{j \in N$: There is a relative neighborhood N_x of x such that $f_j(z+te) > f_j(z)$ for all $t>0$ and $z \in N_x\}$

is not empty, and define $\Omega_x = (N, \Lambda_x)$ by

$\Lambda_x = \{(i,j)$: There is a relative neighborhood N_x of x such that $f_i(z+te^j) < f_i(z)$ for all $t>0$ and $z \in N_x\}$.

If for any $i \notin J_x$ there is a path in Λ_x from i to some $j \in J_x$, then F is a weakly Ω-diagonally dominant M-function on Q.

<u>Proof</u>. We prove first that F is diagonally dominant on Q with respect to the family of networks $\{\Omega_x : x \in Q\}$. Let $x \neq y$ in Q be given, and suppose $f_k(x) = f_k(y)$. For $|x_k-y_k| < ||x-y||_\infty$ there is nothing to prove; thus, assume that $|x_k-y_k| = ||x-y||_\infty$. Assume first that $x_k-y_k = ||x-y||_\infty$. If $(k,j) \in \Lambda_x$ but $|x_k-y_k| > |x_j-y_j|$, then $x_k-y_k > x_j-y_j$ and $x_k-y_k \geq x_i-y_i$ for $i \neq j$. Since $(k,j) \in \Lambda_x$,

$$f_k(y) \leq f_k(y+ (x_k-y_k)e) < f_k(x),$$

which is a contradiction. If however, $y_k-x_k = ||y-x||_\infty$, then for $0<t<1$ small enough,

$$f_k(x) \leq f_k(x+t(y_k-x_k)e) < f_k(x+t(y-x)) \leq f_k(y).$$

To show that F is weakly Ω-diagonally dominant on Q, we only need to prove that if $k \in J_x$ and $f_k(y) = f_k(x)$ for some $y \neq x$ in Q, then $|y_k-x_k| < ||y-x||_\infty$. If $|y_k-x_k| = ||y-x||_\infty$ and $y_k>x_k$, then

$$f_k(x) < f_k(x + (y_k-x_k)e) \leq f_k(y)$$

which is again a contradiction, while if $x_k > y_k$ then for $0<t<1$ sufficiently close to 1,

$$f_k(y) \le f_k(y + t(x_k-y_k)e) \le f_k(y + t(x-y)) < f_k(x).$$

To conclude the proof, we must show that F is an M-function. For this note that the above argument yields that if $|y_k-x_k| = ||y-x||_\infty > 0$ and $f_k(y) \ne f_k(x)$ then

$$(x_k-y_k)\,[f_k(x) - f_k(y)] > 0.$$

We can now proceed as in the proof of Theorem 4.4 to conclude that F is a P-function and hence, by Theorem 4.3, an M-function.

Note that if F is G-differentiable on Q, then the above result provides a different proof of Theorem 2.14 for convex sets of the form (4.1). On the other hand, if $Q = R^n$ and the networks Ω_x and sets J_x are independent of x so that F is Ω-diagonally dominant, then this theorem shows that certain mappings considered by Duffin [7] and Rheinboldt [16] are Ω-diagonally dominant functions.

To conclude this section we present two results which show that Ω-diagonally dominant matrices form an important subclass of the diagonally dominant matrices.

<u>Theorem 4.7</u> Let $A \in L(R^n)$, and assume that $a_{ij} \le 0$, $i \ne j$, and $\sum_{j=1}^{n} a_{ij} \ge 0$ for each $i \in N$. Then A is an M-matrix if and only if A is Ω-diagonally dominant.

<u>Proof</u>. If A is Ω-diagonally dominant then A is an M-matrix by Theorems 4.3 and 4.4. For the converse, note that since A is nonsingular the set

$$J = \{i: \sum_{j=1}^{n} a_{ij} > 0\}$$

cannot be empty. To finish the proof, assume that there is a set L such that $L \cap J$ is empty and if $a_{ij} \neq 0$ with $i \in L$, then $j \in L$. For each $i \in L$ we have

$$\sum_{j=1}^{n} a_{ij} = 0 = \sum_{j \in L} a_{ij}$$

while if $i \notin L$ then $\sum_{j \in L} a_{ij} \leq 0$. Thus, if $v = \sum_{j \in L} e^j$, $A v \leq 0$ which implies that $v \leq 0$ and gives us the desired contradiction.

The sufficiency of the above condition was proven by Bramble and Hubbard [3], while the necessity is essentially due to Schäfke [17]. In fact, Schäfke considered six equivalent formulations of Walter's [20] condition Z_2; the next result states part of his results in our terminology with the spectral radius of a matrix C being denoted by $\rho(C)$.

<u>Corollary 4.8</u> Assume $B \in L(R^n)$ satisfies $||B||_\infty \leq 1$. Then $\rho(|B|) < 1$ if and only if $A = I - |B|$ is Ω-diagonally dominant.

<u>Proof</u>. It is well-known that (Varga [19]) $A^{-1} \geq 0$ if and only if $\rho(|B|) < 1$ so that the result follows immediately from Theorem 4.7.

5. <u>Convergence Theorem</u>. We now want to show how the nonlinear generalizations of diagonal dominance allow us to extend the following classical result: If $A \in L(R^n)$ is a strictly or irreducibly diagonally dominant matrix, then for every b in R^n, $Ax = b$ has a unique solution x^*, and for any x^0 in R^n, the Jacobi and Gauss-Seidel sequences converge to x^*.

The following convergence proofs are somewhat long, but the ideas behind them are rather simple. Specifically, we will define an iteration function H for the Jacobi and Gauss-Seidel sequences, which will allow us to represent these implicit iterative methods as explicit iterative schemes $x^{k+1} = Hx^k$, $k = 0,1,\ldots$. The iteration function H will then be shown to satisfy the hypotheses of the next result.

<u>Lemma 5.1</u> Let $H: D_0 \subset R^n \to R^n$ be a mapping of the closed set D_0 into itself, and suppose that H has a fixed point x^* in D_0. If for some $m \geq 1$.

(5.1) $||H^m x - x^*|| < ||x - x^*||$ $x \in D_0$, $x \neq x^*$,

with H^m continuous on D_0, then x^* is the only fixed point of H in D_0, and for any $x^0 \in D_0$, the sequence $x^{k+1} = Hx^k$ converges to x^*.

<u>Proof</u>. Assume for the moment that $m = 1$. Then (5.1) implies that x^* is the only fixed point of H in D_0, and that $\epsilon_k = ||x^k - x^*||$ is a decreasing sequence of nonnegative numbers and hence convergent. Thus $\{x^k\}$ is bounded, and if $\{x^{k_i}\}$ is any convergent subsequence such that $\lim_{i\to+\infty} x^{k_i} = y^* \neq x^*$, then

$$\lim_{i\to+\infty} \epsilon_{k_i} = ||Hy^* - x^*|| < ||y^* - x^*|| = \lim_{i\to+\infty} \epsilon_{k_i},$$

which contradicts the fact that $\{\varepsilon_i\}$ is convergent. Therefore, $\lim_{i\to+\infty} x^{k_i} = x^*$, and consequently, $\lim_{k\to+\infty} x^k = x^*$. If $m > 1$, then (5.1) implies that H^m and H have the same number of fixed points. Moreover, the previous argument applied to H^m implies that $y^{k+1} = H^m y^k$ converges to x^* for any $y^0 \in D_0$. Setting y^0 successively equal to $x^0,\ldots,H^{m-1}x^0$, we obtain the desired result.

If $m = 1$, then the previous theorem is a special case of a result of Diaz and Metcalf [6]. On the other hand, the next result was proved by Browder and Petryshyn [4] in the context of uniformly convex Banach spaces.

<u>Lemma 5.2</u> Let $H: D_0 \subset R^n \to R^n$ be a mapping of the closed, convex set D_0 into itself, and suppose that

$$\|Hx-Hy\| \leq \|x-y\| \qquad x, y \in D_0.$$

Then H has a fixed point in D_0 if and only if for some $x^0 \in D_0$ the sequence $x^{k+1} = Hx^k$ is bounded.

The proof of this result will be omitted; see, for example, Ortega and Rheinboldt [12] for a proof in our setting.

We now show that under suitable hypotheses, the Jacobi and Gauss-Seidel sequences (1.1) - (1.4) are well-defined and are given by an iteration function which satisfies (5.1).

<u>Theorem 5.3</u> Let $F:Q \subset R^n \to R^n$ be weakly Ω-diagonally dominant on the rectangle Q, and suppose that for each x in Q and $i \in N$, the one-dimensional equation

(5.2) $f_i(x_1,\ldots,x_{i-1}, t, x_{i+1},\ldots,x_n) = 0$

has a (necessarily unique) solution t_i^* with

$$(x_1,\ldots, x_{i-1}, t_i^*, x_{i+1},\ldots,x_n)^T \text{ in } Q.$$

Then the Jacobi and Gauss-Seidel sequences (1.1) - (1.4) with $\omega \in (0,1]$, $k = 0,1,\ldots$ are well-defined for any $x^0 \in Q$ and for either method there is an iteration function $H: Q \subset R^n \to R^n$ such that

a) The method is equivalent with $x^{k+1} = H x^k$, $k = 0, 1, \ldots,$

b) $H(Q) \subset Q$

c) $||Hx-Hy||_\infty \le ||x-y||_\infty$ for every x,y in Q.

Moreover, if $Fx = 0$ has a (necessarily unique) solution x^* in Q, then

d) $|| H^{n-\ell+1}x - x^* ||_\infty < ||x-x^*||_\infty$ for every $x \neq x^*$ in Q where ℓ denotes the number of elements in J_{x^*}.

Proof. We will first present the proof for the Gauss-Seidel method.

Let $x \in Q$ be given, and define the iteration function $H:Q \subset R^n \to R^n$ of the Gauss-Seidel method as follows: By assumption there is t_1^* -- which by Theorem 3.3 is unique -- such that

$$f_1(t_1^*, x_2,\ldots,x_n) = 0$$

and $(t_1^*,x_2,\ldots,x_n)^T \in Q$. Set $h_1(x) = (1-\omega)x_1 + \omega t_1^*$, and note that since Q is convex, $(h_1(x),x_2,\ldots,x_n)^T \in Q$. Assume that $h_j(x)$ for $j = 1,\ldots,i-1$ have been defined such that $(h_1(x),\ldots,h_{i-1}(x), x_i,\ldots,x_n)^T \in Q$. Once again, there is a unique t_i^* such that

$$f_i(h_1(x),\ldots,h_{i-1}(x), t_i^*,x_{i+1},\ldots,x_n) = 0,$$

and we set $h_i(x) = (1-\omega)x_i + \omega t_i^*$. In this way, we have defined $H:Q \subset R^n \to R^n$ such that $H(Q) \subset Q$, and that for any

$x^0 \in Q$, the Gauss-Seidel method is well-defined and equivalent with $x^{k+1} = Hx^k$.

To show that H satisfies c) and d), we first prove that for every $x \neq y$ in Q and $i \in N$, either

(5.3) $|h_i(x)-h_i(y)| < ||x-y||_\infty$,

or

$|h_i(x)-h_i(y)| = ||x-y||_\infty$, and,

(5.4) $|h_i(x)-h_i(y)| = |h_j(x)-h_j(y)|$ if $(i,j) \in \Lambda_x, i > j$,

$|h_i(x)-h_i(y)| = |x_j-y_j|$ if $(i,j) \in \Lambda_x, i < j$.

The proof is by induction. Set $\hat{x} = (t_1*(x), x_2, \ldots, x_n)^T$ where $f_1(\hat{x}) = 0$, and similarly for $\hat{y}$. Since

(5.5) $|h_1(x)-h_1(y)| \leq (1-\omega)\ |x_1-y_1| + \omega|\hat{x}_1-\hat{y}_1|$,

$\hat{x}_1 = \hat{y}_1$ implies that $|h_1(x)-h_1(y)| < ||x-y||_\infty$ and (5.3) applies. If $\hat{x}_1 \neq \hat{y}_1$, then $\hat{x} \neq \hat{y}$, and, since $f_1(\hat{x}) = f_1(\hat{y})$ and F is weakly Ω-diagonally dominant, either $|\hat{x}_1-\hat{y}_1| < ||\hat{x}-\hat{y}||_\infty \leq ||x-y||_\infty$ and (5.5) implies that (5.3) occurs; or $|\hat{x}_1-\hat{y}_1| = ||\hat{x}-\hat{y}||_\infty = |x_j-y_j|$ for $j > 1$, and the second part of (5.4) holds. Since $||\hat{x}-\hat{y}||_\infty = |x_j-y_j| \leq ||x-y||_\infty$, (5.5) yields that $|h_1(x)-h_1(y)| \leq ||x-y||_\infty$ as desired.

Assume now that (5.3) and (5.4) hold for $i = 1,\ldots,k-1$, and set $\hat{x} = (h_1(x),\ldots,h_{k-1}(x), t_k^* \quad, x_{k+1},\ldots,x_n)^T$ where $f_k(\hat{x}) = 0$, and similarly for $\hat{y}$. If $\hat{x}_k = \hat{y}_k$, the result follows from

(5.6) $|h_k(x)-h_k(y)| \leq (1-\omega)|x_k-y_k| + \omega|\hat{x}_k-\hat{y}_k|$,

and if $\hat{x}_k \neq \hat{y}_k$, then $\hat{x} \neq \hat{y}$. Since $f_k(\hat{x}) = f_k(\hat{y})$ and F is weakly Ω-diagonally dominant, either $|\hat{x}_k-\hat{y}_k| < ||\hat{x}-\hat{y}||_\infty \leq ||x-y||_\infty$

and (5.6) implies that (5.3) holds, or $|\hat{x}_k-\hat{y}_k| = ||\hat{x}-\hat{y}||_\infty = |\hat{x}_j-\hat{y}_j|$ for any $(k,j) \in \Lambda_x$. If $k < j$ the third part of (5.4) takes place, while if $k > j$, the second part holds. In either case, $|\hat{x}_k-\hat{y}_k| \leq ||x-y||_\infty$, and (5.6) yields $|h_k(x)-h_k(y)| \leq ||x-y||_\infty$.

Note that (5.3) and (5.4) together imply that $||Hx-Hy||_\infty \leq ||x-y||_\infty$, and thus, we only need to verify that d) holds. For the proof we will use the notation $h_i^k(x) = h_i(H^{k-1}(x))$ where $k \geq 1$ and $i \in N$, and equations (5.3) and (5.4) with $x = x^*$; it will also be important to note that (5.3) applies whenever $i \in J_{x^*}$.

Assume for the moment that $\ell=1$, and let $x \neq x^*$ in Q be given. If $H^{n-1}x = x^*$, then $||H^n x-x^*||_\infty < ||x-x^*||_\infty$; otherwise $H^{n-1}x \neq x^*$ and we proceed to prove that $|h_i^n(x)-x_i^*| < ||x-x^*||_\infty$ for each $i \in N$. If $|h_i^n(x)-x_i^*| < ||H^{n-1}x-x^*||_\infty$ there is nothing to prove; otherwise $i \notin J_{x^*}$ and there is a path

(5.7) $(i,i_1),(i_1,i_2), \ldots,(i_{r-1},i_r)$

in Λ_{x^*}, with $i_r = j \in J_{x^*}$ and $r \leq n-1$. Hence, by (5.4) with $y = H^{n-1}x$,

$$|h_i^n(x) - x_i^*| = |h_{i_1}^n(x)-x_i^*| \quad \text{if } i > i_1,$$

or

$$|h_i^n(x) - x_i^*| = |h_{i_1}^{n-1}(x)-x_i^*| \quad \text{if } i < i_1.$$

Repeat this procedure until $|h_i^n(x)-x_i^*| < ||x-x^*||_\infty$, or $|h_i^n(x)-x_i^*| = |h_{i_r}^k(x) - x_i^*|$ for some k with

$1 \leq n-r \leq k \leq n$. Since $i_r = j \in J_x^*$,

$$|h_j^k(x) - x_j^*| < ||H^{k-1}x - x^*||_\infty \leq ||x-x^*||_\infty .$$

Hence, $|h_i^n(x)-x_i^*| < ||x-x^*||_\infty$ for each $i \in N$, and thus, $||H^n x-x^*||_\infty < ||x-x^*||_\infty$.

If $1 < \ell \leq n$, and $H^{n-1}x = x^*$, there is nothing to prove; otherwise, $H^{n-1}x \neq x^*$, and the proof proceeds as before by noting that each $i \notin J_x^*$ can be joined to an $i_r \in J_x^*$ by a path (5.7) with $r \leq n-\ell$.

The proof for the Jacobi method is very similar, but now full use is made of the assumption that Q is a rectangle. The distinction occurs in the definition of the iteration function H for the Jacobi method. The first component function of H is defined by $h_1(x) = (1-\omega)x_1 + \omega t_1^*$ where

$$f_1(t_1^*, x_2, \ldots, x_n) = 0$$

and $(t_1^*, x_2, \ldots, x_n)^T \in Q$. Assume that $h_j(x)$ for $j = 1, \ldots, i-1$ have been defined such that $(x_1, \ldots, x_{j-1}, h_j(x), x_{j+1}, \ldots, x_n)^T \in Q$ for $j = 1, \ldots, i-1$. If we set $h_i(x) = (1-\omega)x_i + \omega t_i^*$ where

$$f_i(x_1, \ldots, x_{i-1}, t_i^*, x_{i+1}, \ldots, x_n) = 0$$

and $(x_1, \ldots, x_{i-1}, h_i(x), x_{i+1}, \ldots, x_n)^T \in Q$, then $(x_1, \ldots, x_{i-1}, h_i(x), x_{i+1}, \ldots, x_n)^T \in Q$ since Q is convex, and, since Q is a rectangle, $(h_1(x), \ldots, h_i(x), x_{i+1}, \ldots, x_n)^T \in Q$. In this way the iteration function for the Jacobi method is defined, and it satisfies b). The rest of the proof proceeds along steps similar to those for the Gauss-Seidel sequence. This completes the proof.

If $F:R^n \to R^n$ is linear and irreducibly diagonally dominant, Mařík and Pták [10] have proved a result related to d) of Theorem 5.3 in the sense that their theorem can be used to prove that for the iteration matrix H of the Jacobi method, $||H^{n-\ell+1}||_\infty < 1$. A consequence of the next example is that $n - \ell$ is the largest number k such that $||H^k||_\infty = 1$.

Example 5.4 Consider the matrix of order $n > 1$:

$$A = \begin{bmatrix} 1 & -1 & & & & & & \\ 0 & 1 & -1 & & & & \bigcirc & \\ & & \ddots & \ddots & & & & \\ & & & 1 & -1 & & & \\ & \bigcirc & & & 1 & -\alpha & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & -\alpha \\ -\alpha & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}, \quad |\alpha| < 1,$$

where there are ℓ alphas. Then J has ℓ elements and if H denotes the iteration matrix of the Jacobi method then, by direct computation, $||H^k||_\infty = 1$ for $0 \le k \le n - \ell$, but $||H^{n-\ell+1}||_\infty = |\alpha| < 1$. Note that if $\alpha = 0$ A is Ω-diagonally dominant but not irreducibly diagonally dominant.

We can now prove our first convergence result.

<u>Corollary 5.5</u> Let $F: Q \subset R^n \to R^n$ satisfy the hypotheses of Theorem 5.3 on the closed rectangle Q. Then $Fx = 0$ has a (necessarily unique) solution x^* in Q if and only if for some $x^0 \in Q$ the Jacobi or Gauss-Seidel iterates (1.1) - (1.4) with $\omega \in (0,1]$ are bounded. In particular, this occurs if Q is bounded; in any case, the iterates will converge to x^* for any $x^0 \in Q$.

<u>Proof</u>. We only carry out the proof for the Jacobi method; the proof for the Gauss-Seidel method is similar.

By Theorem 5.3, the Jacobi method has a well-defined iteration function $H: Q \subset R^n$ which satisfies the hypotheses of Lemma 5.2 on the closed set $D_0 = Q$. The first part of the theorem now follows from the fact that

$$f_i(x_1, \ldots, x_{i-1}, (1 - \frac{1}{\omega})x_i + \frac{1}{\omega} h_i(x), x_{i+1}, \ldots, x_n) = 0$$

for each $i \in N$, while the convergence of the Jacobi iterates to x^* is a consequence of Lemma 5.1 and Theorem 5.3.

An important case of Corollary 5.5 occurs when Q is unbounded; in this case the next example shows that $Fx=0$ does not necessarily have a solution.

<u>Example 5.6</u> Define $F: R^2 \to R^2$ by

$$F(x_1, x_2) = \begin{bmatrix} x_1 - x_2 + g(x_1) \\ \\ x_2 - x_1 + g(x_2) \end{bmatrix},$$

where $g(t) = \arctan t - \pi/2$. By Theorem 4.5, F is a strictly diagonally dominant M-function. However, $Fx = 0$ does not have a solution, for otherwise, there would be an $x = (x_1,x_2)^T$ such that $F(t,t) \leq 0 = F(x_1,x_2)$ and, since F is inverse isotone, $t \leq x_1$, $t \leq x_2$ for every $t \in R^1$. This is clearly impossible.

Note that in this example the one-dimensional equations (5.2) are not only solvable, but the first and second diagonal subfunctions are surjective for each $x \in R^n$. On the other hand, the next result shows, in particular, that if $F:R^n \to R^n$ is continuous and weakly Ω-diagonally dominant on all of R^n, and $Fx=0$ has a solution, then the one-dimensional equations (5.2) are solvable.

<u>Theorem 5.7</u> Let $F:D \subset R^n \to R^n$ be continuous on the set D, and assume that $Fx = 0$ has a solution x^* in D, and that for some $r \geq 0$, $Q = \{x \in R^n : ||x-x^*||_\infty \leq r\} \subset D$. If F is weakly Ω-diagonally dominant on Q, then x^* is unique in Q, and for any x^0 in Q the Jacobi and Gauss-Seidel sequences (1.1) - (1.4) with $\omega \in (0,1]$, are well-defined and converge to x^*.

<u>Proof</u>. The result will follow from Corollary 5.5 if we prove that for each x in Q and $i \in N$, the equation (5.2) has a unique solution t_i^* with $(x_1,\ldots,x_{i-1},t_i^*,x_{i+1},\ldots,x_n)^T \in Q$. To show this, let $x \in Q$ and $i \in N$ be given, and define $\psi:R^1 \to R^1$ by

$$\psi(t) = f_i(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n).$$

Clearly, ψ is defined for $|t-x_i^*| \leq r$, and by Corollary 3.6, ψ is either strictly isotone or strictly antitone. In either case, $\psi(t) = 0$ has at most one solution. We now proceed to show that such a solution exists if ψ is strictly isotone; for the strictly antitone case the proof is analogous. Since $x \in Q$, $\|x-x^*\|_\infty \equiv \rho \leq r$, and if $\rho = 0$ there is nothing to prove; hence assume that $\rho > 0$ and let $t_i^+ = x_i^* + \rho$. If $v = (x_1,\ldots,x_{i-1},t_i^+,x_{i+1},\ldots,x_n)^T$, then $|t_i^+-x_i^*| = \|v-x^*\|_\infty$, and by Corollary 3.6,

$$(t_i^+-x_i^*)[f_i(v)-f_i(x^*)] = \rho\psi(t_i^+) \geq 0,$$

or $\psi(t_i^+) \geq 0$. Similarly, if $t_i^- = x_i^* - \rho$, then $\psi(t_i^-) \leq 0$. The continuity of ψ yields a $t_i^* \in [t_i^-,t_i^+]$ with $\psi(t_i^*) = 0$, and since $|t_i^*-x_i^*| \leq r$, it follows that $(x_1,\ldots,x_{i-1},t_i^*,x_{i+1},\ldots,x_n)^T \in Q$. This completes the proof.

If $F:R^n \to R^n$ is linear and Ω-diagonally dominant on R^n then F is necessarily surjective; and in the particular case where $\omega = 1$, the previous theorem is due to Walter [20]. If F is not defined on all of R^n, then, in general, it is very difficult to find a set Q which satisfies the hypotheses of the last two results; however, if F is off-diagonally antitone and for some u,v, $Fu \leq 0 \leq Fv$, then Q can be taken to be the set $\langle u,v\rangle = \{z \in R^n : u \leq z \leq v\}$.

<u>Theorem 5.8</u> Let $F:D \subset R^n \to R^n$ be continuous, off-diagonally antitone, and weakly Ω-diagonally dominant on the set D. If there are u,v in D such that $\langle u,v\rangle \subset D$ with $Fu \leq 0 \leq Fv$ and $u \leq v$, then $Fx = 0$ has a solution x^* in $\langle u,v\rangle$ which is unique in D, and for any x^0 in $\langle u,v\rangle$, the Jacobi and Gauss-Seidel sequences (1.1) - (1.4) with $\omega \in (0,1]$ are well-defined and converge to x^*.

<u>Proof</u>. By Corollary 5.5 we only need to verify that for each $x \in \langle u,v\rangle$, $\psi(t) = f_i(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n) = 0$ has a unique solution $t_i^* \in [u_i,v_i]$. Note that ψ is defined on $[u_i,v_i]$. Moreover, since F is off-diagonally antitone,

$$f_i(v_1,\ldots,v_{i-1},t,v_{i+1},\ldots,v_n) \le \psi(t) \le$$

$$f_i(u_1,\ldots,u_{i-1},t,u_{i+1},\ldots,u_n),$$

and hence, $\psi(u_i) \le 0 \le \psi(v_i)$. The continuity of ψ on $[u_i,v_i]$ then implies that there is a $t_i^* \in [u_i,v_i]$ with $\psi(t_i^*) = 0$.

It is of interest to note that the conclusions of Theorem 5.8 hold if F is a continuous M-function on D. In this form the theorem is implicit in the work of Rheinboldt [16]. Note, however, that under the hypotheses of Theorem 5.8, F is not necessarily an M-function.

To conclude, we present an application of our convergence results to finding nonnegative solutions of two-point boundary value problems.

Consider

(5.8) $u''(t) = g(t,u(t),u'(t))$ for $a < t \le b$; $u(a) = \alpha$, $u(b) =$

where g is continuously differentiable on

(5.9) $S = \{(t,u,u') \in R^3 : a \le t \le b,\ 0 \le u < +\infty,\ -\infty < u' < +\infty\}$,

and

(5.10) $g_u(t,u,u') \ge 0$, $|g_{u'}(t,u,u')| \le M < +\infty$,

for all $(t,u,u') \in S$.

Then, as shown, for instance, by Bailey, Shampine, and Waltman [1968] it is known that (5.8) - (5.10) has a unique,

twice-continuously differentiable, non-negative solution provided that $\alpha,\beta \geq 0$, and $g(t,0,0) \leq 0$ for $t \in [a,b]$.
To obtain a numerical solution of this problem, we introduce the partition $t_j = a + jh$, $j = 0,\ldots,n+1$, $h = \frac{b-a}{n+1}$, of $[a,b]$, and use the standard finite difference approximation of (5.8)

$$(5.11) \qquad Ax + \phi(x) - c = 0$$

where

$$(5.12) \quad A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & 0 \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & 0 & & -1 & 2 \end{bmatrix}, \quad c = (\alpha,0,\ldots,0,\beta)^T$$

and $\phi: R^n \to R^n$ is defined by

$$(5.13) \quad \phi_i(x) = h^2 g(t_i, x_i, \frac{x_{i+1}-x_{i-1}}{2h}) \quad i = 1,\ldots,n.$$

We want to find a solution of (5.11) in $R^n_+ = \{x \in R^n : x \geq 0\}$.

Theorem 5.9 Consider the mapping $Fx = Ax + \phi(x) - c$ defined by (5.11) - (5.13) where g is continuously differentiable on the set S of (5.9) and satisfies (5.10). If $\alpha,\beta \geq 0$, $g(t,0,0) \leq 0$ for $t \in [a,b]$, and $h = \frac{b-a}{n} \in (0,\frac{2}{M})$, then the equation (5.11) has a unique nonnegative solution $x^* \in R^n_+$, and for any $x^0 \in R^n_+$, the Jacobi and Gauss-Seidel iterates (1.1) - (1.4) with $Fx = Ax + \phi(x) - c$ and $\omega \in (0,1]$ are well-defined and converge to x^*. Moreover, if $g(t,0,0) \equiv 0$

in $[a,b]$, $x^* \in \langle 0, s_0 e\rangle$ where s_0 satisfies

$$s_0 + h^2 g(t_1, s_0, \frac{s_0-\alpha}{2h}) \geq \alpha, \quad s_0 + h^2 g(t_n, s_0, \frac{\beta-s_0}{2h}) \geq \beta.$$

In particular, s_0 can be chosen to be $\max(\alpha,\beta)$.

Proof. Since $h < 2/M$ and (5.10) holds, $F'(x)$ is a Ω-diagonally dominant matrix with nonpositive off-diagonal entries for each $x \in R^n_+$. It follows from Theorem 2.14 that F is weakly Ω-diagonally dominant on R^n_+. On the other hand, Theorem 4.7 applied to A yields that A is an M-matrix and thus, that $A^{-1} \geq 0$. It follows that if $u = A^{-1}e$, then $F(su) \geq 0$ for $s \geq 0$ large enough, and since, in addition, F is off-diagonally antitone and $F(0) = \phi(0) - c \leq 0$, the first part of the heorem follows directly from Theorem 5.8. If now $g(t,0,0) \equiv 0$, $F(se) \geq 0$ for $s \geq s_0$ and the result again follows from Theorem 5.8.

Although the previous result used Theorem 2.14 to assert that F is weakly Ω-diagonally dominant, we could have also used Theorem 4.6 to prove a somewhat stronger result where instead of (5.10) we would make the corresponding assumptions about the difference quotients.

The use of the approximation (5.11) is of course standard, but it is usually assumed that (5.10) or the corresponding difference quotients hold for all $(t, u,u') \in [a,b] \times R^1 \times R^1$. Under these assumptions, many authors have treated (5.8); in particular, Rheinboldt [16] showed that the corresponding F was a surjective M- functions. Similarly, discrete analogues of mildly nonlinear elliptic boundary value problems of the form $\Delta u = g(t,u)$ with $g(t,0) \equiv 0$ and $g(t,u) \geq 0$ for $u \geq 0$, have been considered, for example, by Greenspan and Parter [9], and these authors obtain an existence result similar to ours. However, they do not treat either the (nonlinear) Jacobi or Gauss-Seidel method.

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