STATISTICAL ANALYSIS OF MIXED POPULATION FLOOD SERIES

A Thesis

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Master of Science

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ABSTRACT

This thesis considers several models for representing the distribution of the annual maximum flood at a site when the annual maximum flood series corresponds to the maximum of two distinct annual flood series. A joint model, that incorporates correlation between the two series, is theoretically correct. However, in most cases, a simpler mixture model essentially does as well. The simple mixture model, which assumes independence between the two distinct series, is a special case for the joint model; it provided a good description of the distribution of the annual maxima above some critical probability, when in our case the risk due to rainfall floods dominated. That critical probability is determined by the distribution of each population, and also the correlation between them.

A method recommended by William Kirby uses only the events recorded in the annual maximum series to develop conditional flood risk models for both processes; the resulting flood risk model for the annual maximum series is a weighted function of the two. When compared to use of a single 3-parameter lognormal model, the Kirby Method usually provide more precise estimates of flood quantiles.

Overall, for many cases and for extreme quantiles, both the just-rainfall model that models only rainfall, and the mixture model that assumes rainfall and snowmelt maxima are independent, provide flood quantile estimates that are as accurate as use of the full Joint model.

BIOGRAPHICAL SKETCH

Jiajia Lu was born and raised in Dongying, Shandong, China. In 2006, she started her undergraduate studies in the Department of Environmental Science at Jilin University, where she gained a strong background in chemistry, physics, biology, mathematics, and computer science. Soon after graduating, she began studies at Cornell University in the area of Environmental and Water Resources Systems Engineering (EWRS). In August 2012, she defended her MS thesis.

DEDICATION

The document is dedicated to all Cornell graduate students.

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CHAPTER 1

1 INTRODUCTION

A common problem in hydrology is that annual maximum series are composed of events that may arise from distinctly different processes, such as rainfall and snowmelt (Stedinger et al., 1993). Cudworth (1989) indicates that a mixture could also be the result of distinctly different hydrological factors, such as infiltration, cover, channel roughness, and antecedent conditions, with differences possibilities related to time of year. In areas where high flows are generated by more than one distinct hydrologic process, peak discharge data can be considered to be drawn from populations with different statistical characteristics. Elliot et al. (1982) provide data and illustrate the identification of snowmelt and rainfall peak floods in the Rocky Mountains of the Western United States. Waylen and Woo (1982) provide an example of a flood record that is composed of distinct rainfall and snowmelt events in British Columbia, Canada. Watt et al. (1989) discusses different types of floods and the separation of floods in British Columbia according to the generating mechanism: spring snowmelt, autumn/winter rain and rain-on-snow, summer rain in smaller basins and particularly unusual flood caused by glacier melt, ice-jam break-up, and dam breaks. U.S. Army Corps of Engineers (1958) describe separating the data into distinct and independent populations using different criteria. IACWD (1982) describes flood frequency procedures adopted by federal agencies in the United States. They note that when annual maximum series are the result of distinct series, separate frequency curves for each series can be combined. The three examples given are rain and snowmelt, tropical storms and general cyclonic storms, and along the Gulf Coast hurricane and non-hurricane storms.

Frequency analysis for such situations can be conducted using several methods. Watt et al. (1989) discussed 3 methods for estimating flood flows, water levels, or frequencies for design purpose when flows and water levels change relatively slowly; Monte Carlo simulation and an engineering judgment approach are recommended. Jarrett and Costa (1982) investigate the flood hydrology of foothill streams in Colorado using a multidisciplinary study, where the peak flows were classified into to populations according to metrological causes and then combined assuming populations are independent. ASCE (1996, p. 490-91) also suggests that when the mixed population is the combination of two distinct and independent processes, the combined flood risk would be the summation of the risks of each population minus their product; that is equivalent to estimating their non-exceedance probability by the product of the non-exceedance probability for each series. Cudworth (1989, pp. 2057; 219-20) provide a general discussion of mixed populations in the Western United States and suggest developing a combined frequency relationship as was suggested by ASCE (1996)

This thesis focuses on several methods for performing flood frequency analysis with such records: the Mixture Method, Joint Distribution Method, Kirby Method, One Single Distribution Method, and Just-Rainfall Method for extreme events. The Mixture Method is an easy way to do flood frequency analysis when we assume that the values drawn from each population are statistically independent, which may not be true (Cudworth, 1989; Stedinger et al., 1993). One advantage of the Mixture Method is that with an N-year record, N observations are available to fit each distribution, thus there are 2N observations available in total. However, when the flood peaks in each year from each population are dependent, a fundamental assumption employed by the Mixture Method is incorrect. Chapters 2 and 4 explores the range of the cross-correlation between the two series for which the Mixture Method provides an accurate representation of the joint distribution.

The Joint Distribution Method provides a correct and general framework for describing a mixture of two distributions that allows the values in each population to be cross-correlated. When the values drawn from each population are selected independently, the calculation process is the same with the Mixture Method. Thus, the Mixture Method is a special case for the Joint Distribution Method. The calculation process for the Joint Distribution Method is more complicated than that for the mixture method because it requires a representation of the joint distribution of the two flood series.

The Kirby Method is a third approach to derive a flood frequency distribution describing a mixture of two different and potentially dependent processes. It was developed by William Kirby, and was described by Charles Parrot and Jery Stedinger in their correspondence (Parrot, personal communication, 2011) It does not generate the joint distribution of the two series. Rather it employs the conditional distributions of rainfall and snowmelt floods given that they are also annual maxima. Because the Kirby Method uses conditional probabilities, we can use the Kirby Method even when the values drawn from different populations are dependent. The Kirby Method divides the annual maximum flood series into subsets corresponding to the annual maxima that came from each population. Thus for an N-year record, the number of observations available in total is just N instead of 2N. Moreover, as will be shown, conditional distributions of the rainfall or snowmelt floods that are also the annual maxima can be significantly different from the complete data sets; this can make it difficult to specify the appropriate conditional distributions to use to describe the rainfall or snowmelt floods that are also the annual maxima even that we know the distributions of the complete data sets.

We are also interested in describing the annual maximum series with one simple single distribution. Although the mixed population is composed of events that arise from distinctly different processes, our study shows that one simple single distribution (3-parameter lognormal distribution in our case) works when describing the mixed population.

Moreover, we also explore the performance of just modeling the rainfall data as approximation of annual maximum distribution. Because the really large floods are almost always rainfall events, the Just-Rainfall Method can be reasonable.

This thesis is organized as follows: Chapter 2 describes the frequency procedures of each method for mixed populations. Chapter 3 explores the range within which the Mixture Method or the Just-Rainfall Method works. Chapter 4 provides a more detailed analysis of these methods using Monte Carlo simulation. Chapter 5 provides conclusions.

CHAPTER 2

2 Different Procedures for Mixed Populations

2.1 Mixture Method

This section introduces the theory and computation process of the Mixture Method and provides an example.

2.1.1 Theory

Stedinger et al. (1993) indicate that for an N-year record, the annual maximum flood for the t^{th} year Q_t can be viewed as the maximum of the maximum rainfall event for the t^{th} year R_t and the maximum snowmelt event for the t^{th} year S_t:

$$Q_t = \max\{R_t, S_t\} \quad \text{For t=1, 2… N} \tag{1}$$

Let the cumulative distribution functions (CDFs) of the rainfall and snowmelt variables be denoted $F_R(R)$ and $F_S(S)$. Then if the magnitudes of the rainfall and snowmelt events R and S are statistically independent, meaning that knowing one has no effect on the probability distribution of the other, the CDF of the annual maxima Q

is

$$F_Q(q) = P\{Q < q\} = F_R(q)F_S(q) \tag{2}$$

Modeling the two component series separately is most attractive when the annual maximum series is composed of components with distinctly different distributions which are individually easy to model because classical two-parameter Gumbel or lognormal distributions describe them well, and such a simple model provides a poor description of the composite annual maximum series.

A concern is that this method as formulated assumes the rainfall and snowmelt events in each year are statistically independent. Experiments reported below explore the performance of this method when the assumption above is not valid. The examples indicate that when the correlation is weak between the values selected from each population, the approximation works well. Moreover, because the really large floods are dominated by the rainfall events, the Mixture Method would still be attractive as long as the snowmelt events don't significantly influence the certain range of the mixed population within which we are interested.

2.1.2 Computation Method

To use equation (2), one needs to fit the CDFs F_R and F_S given the rainfall and snowmelt samples by computing estimates of F_R and F_S using some reasonable distribution (here we use lognormal distribution) to obtain an estimator of distribution of the annual maximum. Although not requited, for an N-year record, we generally assume both the rainfall and snowmelt series have the same sample size of N.

Consider how to fit $\mathrm{F_R}$ and $\mathrm{F_S}$ with lognormal distribution. Lognormal distribution is an easy and convenient model which has been long and widely used in water resources. An early study of lognormal distribution can be dated back to Hazen (1914). And Chow (1954) reviews many of the applications of this distribution. Studies about fitting 2- and 3-parameter lognormal distributions can be found in Wilson and Worcester (1945), Cohen (1951), Aitchison and Brown(1957), Sangal and Biswas (1970), Burges et al. (1975), Giesbrechat and Kempthorne (1976), Charbeneau (1978), Stedinger (1980), etc.

In general, lognormal distribution can be described as follows:

Let μ_R and σ^2_R be the real-space moments of the rainfall events, we have:

$$F_R(r) = \Phi\left(\frac{\ln(r) - \mu_{RL}}{\sigma_{RL}}\right) \tag{3}$$

$$\sigma_{RL}^{\ 2} = \ln\left(1 + \frac{\sigma_R^{\ 2}}{\mu_R^{\ 2}}\right) \tag{4}$$

$$\mu_{RL} = \ln(\mu_R) - \frac{1}{2}\sigma_{RL}^{\ 2} \tag{5}$$

And F_S can be obtained in the same way. Then one can use the simple expression in (2) to obtain an estimator of distribution of the annual maximum, where

$$F_Q(q) = F_R(q)F_S(q) = \Phi\left(\frac{\ln(q) - \mu_{RL}}{\sigma_{RL}}\right)\Phi\left(\frac{\ln(q) - \mu_{SL}}{\sigma_{SL}}\right) \tag{6}$$

Plots of the CDFs F_R, F_S, and F_Q are provided in the following section.

2.1.3 Example

We use the Gumbel distribution parameters provided by Waylen and Woo (1982) to get our lognormal distribution parameters in order to simulate a real situation. Waylen and Woo (1982) model the snowmelt and rainfall floods in the Cascade Mountains separately using simple Gumbel distributions and use them to provide a good fit to the annual floods which are produced by mixed processes, where

$$F(x) = \exp\left\{-\exp\left[-\alpha(x - \beta)\right]\right\} \tag{7}$$

And their parameters are shown in Table 1.

Table 1. Gumbel Parameters given by Waylan and Woo (1982)

Gumbel	Rainfall	Snowmelt
α	0.0098	0.030
β	140.1	142.2

Use the parameters provided by Waylen and Woo (1982) , we have (Lowery and Nash, 1970):

$$\sigma = \frac{1.281}{\alpha} \tag{8}$$

$$\mu = \beta + 0.45\sigma \tag{9}$$

where μ and σ are the real-space moments.

Then use the estimation method provided by equation (4) and (5), we have the lognormal parameters (to four digits) reported in Table 2.

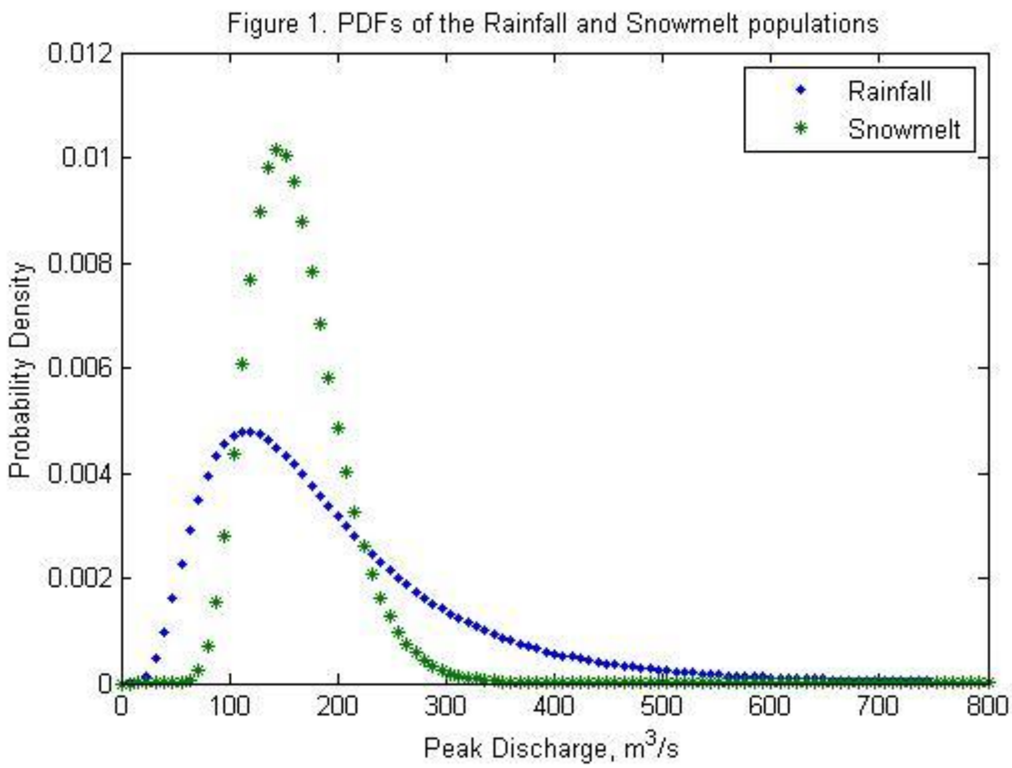
Table 2 Parameters of LN Distributions R and S

lognormal	Rainfall	Snowmelt
μ	198.9	161.4
σ	130.7	42.70
μ_L	5.113	5.050
σ_L	0.5991	0.2601

To better understand the characteristics of the rainfall and snowmelt probability distributions, Figure 1 provides plots of the PDFs of the populations, where

$$f_R(r) = \frac{1}{r\sigma_{RL}\sqrt{2\pi}} \exp\left[-\frac{\left(\ln(r)-\mu_{RL}\right)^2}{2\sigma_{RL}^{\ 2}}\right] \tag{10}$$

$$f_S(s) = \frac{1}{s\sigma_{SL}\sqrt{2\pi}} \exp\left[-\frac{\left(\ln(s)-\mu_{SL}\right)^2}{2\sigma_{SL}^{\ 2}}\right] \tag{11}$$



Apparently most snowmelt floods fall within [50m^3/s 350m^3/s]. Because the rainfall population has a much larger variance than the snowmelt population, one can see in Figure 1 that really large annual maximum floods will be rainfall events. On the other hand, because snowmelt floods will not be less than 50m^3/s, annual maximum floods will almost always exceed that value.

Given the lognormal parameters reported in Table 2, we provide plots of the CDFs F_R, F_S, and F_Q in Figure 2, with F_R, F_S obtained using (3) and F_Q obtained using (6).

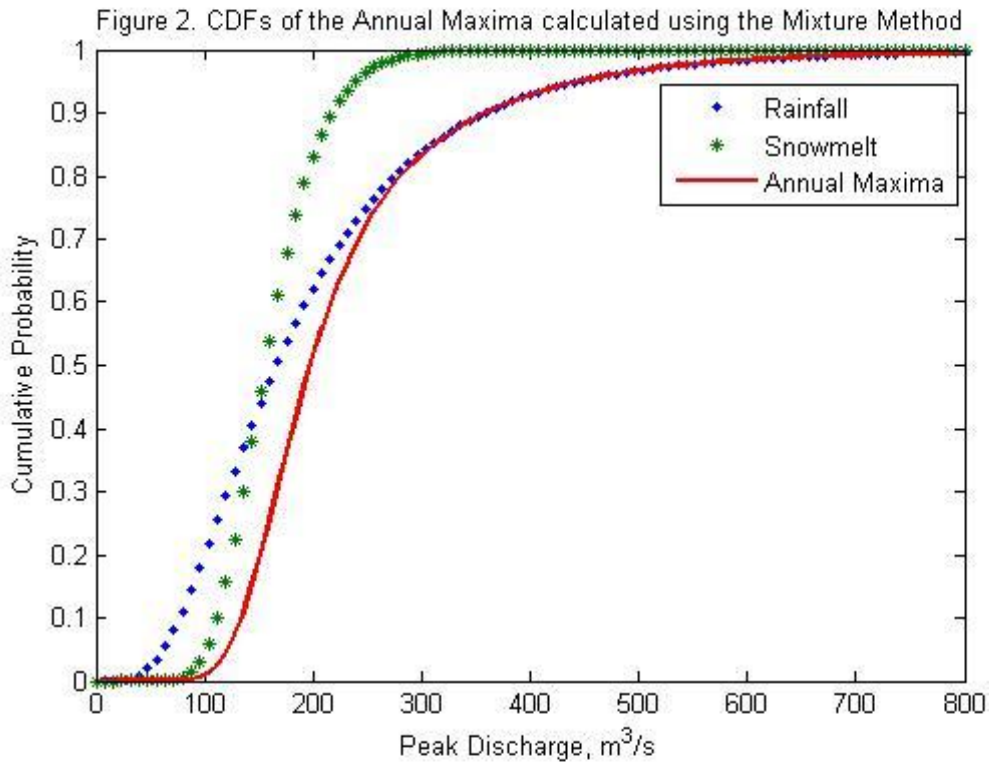


Figure 2 shows that the annual maximum curve is a little to the right of the snowmelt curve when q is small while the annual maximum curve is almost identical with the rainfall curve when q is large (above 300 m^3/s).

Because the cross-correlation between same-year rainfall and snowmelt floods is not considered in the Mixture method, the results obtained using Equation (6) are independent of any correlation between rainfall and snowmelt floods; the two are assumed to be independent. A concern is the accuracy of the Mixture Method when rainfall and snowmelt floods in the same year are correlated. The next section introduces that concern while Chapter 3 provides a detailed study of that issue.

2.2 Joint Distribution Method

This section introduces the theory and computation process of the Joint Distribution Method and provides an example.

2.2.1 Theory

A Joint Distribution Method can make use of a joint probability distribution for the two series. For the situation when the values drawn from each population are dependent, the probability that the annual maximum flood is less than q is:

$$F_Q(q) = P\{R < q, S < q\} = \int_{-\infty}^{q}\int_{-\infty}^{q} f_{R,S}(r,s)drds \tag{12}$$

However, it is usually difficult or even impossible to do the integration in (12) analytically. So we have to use numerical methods to calculate the CDF and the PDF of the annual maxima.

2.2.2 Computation Method

Based on the conditional density function for R given S, we have:

$$F_Q(q) = P\{R < q, S < q\} = \int_{-\infty}^{q}\int_{-\infty}^{q} f_{R|S}(r\,|\,s) f_S(s)drds \tag{13}$$

For a joint normal distribution, the conditional distribution of ln(R) given S=s is:

$$\{\ln(R)\,|\,S = s\} \sim N\left[\mu_{RL} + \frac{\sigma_{RL}}{\sigma_{SL}}\rho(\ln(s) - \mu_{SL}), (1-\rho^2)\sigma_{RL}{}^2\right](s > 0) \tag{14}$$

where ρ is the correlation between the log-space rainfall and log-space snowmelt floods (not of the correlation between the real-space rainfall and snowmelt data). The correlation between the real-space rainfall and snowmelt floods is (Stedinger, 1981)

$$Corr(R,S)=\frac{\exp(\rho\sigma_{RL}\sigma_{SL})-1}{\sqrt{\left[\exp(\sigma_{RL}{}^{2})-1\right]\left[\exp(\sigma_{SL}{}^{2})-1\right]}} \quad (15)$$

thus

$$F_{R|S}(r\mid s)=\Phi\left\{\frac{\ln(r)-\left[\mu_{RL}+\frac{\sigma_{RL}}{\sigma_{SL}}\rho(\ln(s)-\mu_{SL})\right]}{\sqrt{(1-\rho^{2})\sigma_{RL}{}^{2}}}\right\}(r>0,s>0) \quad (16)$$

For the rainfall-snowmelt problem, when R and S have a bivariate lognormal distribution, we can transform the real-space data sets into their log-space values. However, the integration for the Joint Normal Distributions is still impossible analytically even though integration of both the transformed rainfall and snowmelt distributions is possible analytically. So a numerical method, Simpson's rule, is applied.

To use Simpson's rule, we need to break up the interval of the snowmelt floods [0, s] into 2m (m is a integer) subintervals, where

$$\Delta s=\frac{s}{2m} \quad (17)$$

$$s_j = j\Delta s \text{ j=0, 1 ... 2m} \quad (18)$$

$$\begin{aligned}F_Q(q) &= \int_{-\infty}^{q}\int_{-\infty}^{q} f_{R|S}(r\mid s)f_S(s)drds=\int_{-\infty}^{q}F_{R|S}(r\mid s)f_S(s)ds\\ &=\frac{\Delta s}{3}\Big\{F_{R|S}\left(q\mid s_0\right)f_S\left(s_0\right)-F_{R|S}\left(q\mid s_{2m}\right)f_S\left(s_{2m}\right)\\ &+\sum_{j=1}^{m}\left[4F_{R|S}\left(q\mid s_{2j-1}\right)f_S\left(s_{2j-1}\right)+2F_{R|S}\left(q\mid s_{2j}\right)f_S\left(s_{2j}\right)\right]\Big\}\end{aligned} \quad (19)$$

We can also solve equation (12) by discretizing R instead of S. The reason why we discretize S instead of R is that R is the more critical distribution for large-flood risk, so handling that dimension analytically can yield a more accurate result for less effort. If S-distribution had no impact on the result, then our numerical computation would be exact because it handles R analytically. However, as in our case, S-distribution does have an impact on the result, especially when the annual maximum is small, so we may get a better result if we use a coarser grid for S when dealing with the situation where the annual maximum is small.

Using the conditional distribution of R given S as given in equation (16) and the approximation method provided by equations (19), we can calculate the CDFs of the annual maxima use the Joint distribution.

2.2.3 Example

A Joint Bivariate Lognormal Distribution is used to construct a joint distribution model that allows computation of the distribution of the annual maximum as a function of the distributions and the cross-correlation between the two series. When the cross-correlation between the two series is zero, the joint distribution model would yield the same answer as the mixture model: the mixture model is actually a special case for the joint distribution model.

Given the lognormal parameters reported in Table 2, we provide plots of the CDFs F_R, F_S, and F_Q in Figure 3a and 3b as the correlation ρ between the log-space rainfall and snowmelt floods changes, with F_Q obtained using (19). Note that Figure 3b is just an expanded scale of Figure 3a:

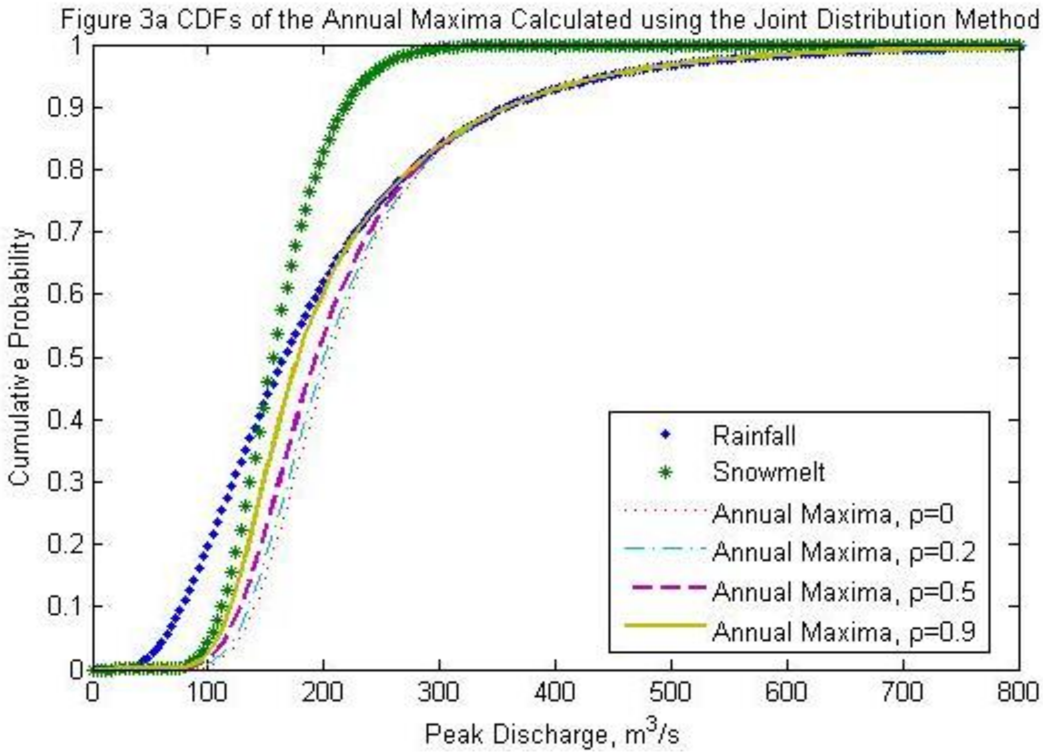


Figure 3a CDFs of the Annual Maxima Calculated using the Joint Distribution Method
Cumulative Probability
Peak Discharge, m3/s
Rainfall
Snowmelt
Annual Maxima, ρ=0
Annual Maxima, ρ=0.2
Annual Maxima, ρ=0.5
Annual Maxima, ρ=0.9

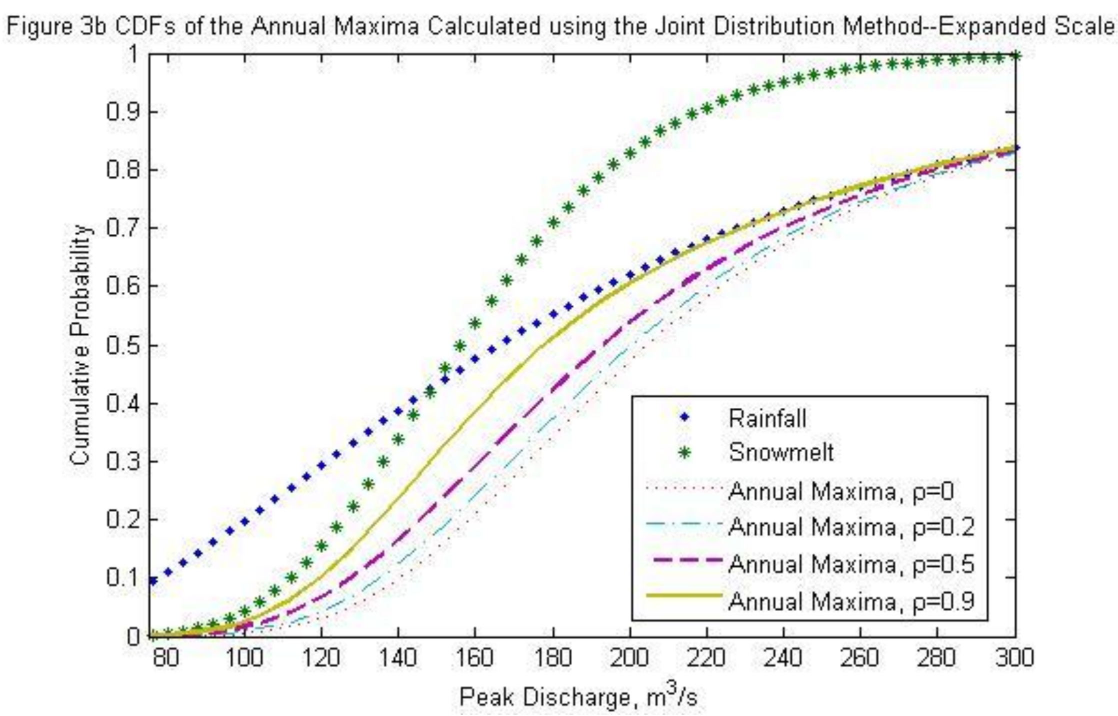


Figure 3b CDFs of the Annual Maxima Calculated using the Joint Distribution Method--Expanded Scale
Cumulative Probability
Peak Discharge, m3/s
Rainfall
Snowmelt
Annual Maxima, ρ=0
Annual Maxima, ρ=0.2
Annual Maxima, ρ=0.5
Annual Maxima, ρ=0.9

Figure 3 displays the annual maximum curves. Those curves follow the snowmelt curve when q is small, and are almost identical with the rainfall curve when q is larger than $300m^3/s$. When $\rho=0$, the annual maximum curve is identical with the curve calculated using the Mixture Method. For $\rho\neq0$, the Mixture Method is no longer correct, and there is a noticeable difference for ρ as small as 0.2. For a large correlation, the annual maximum distribution almost equals the snowmelt distribution for small q. When $\rho=1$, the annual maximum curve is identical with the snowmelt curve when q is small and is identical with the rainfall curve when q is large, and the change point is at the intersection of the rainfall and snowmelt cumulative distribution curves.

Chapter 3 explores the range within which the Mixture Method provides a good approximation of the distribution of the annual maximum.

2.3 Kirby Method

This section introduces for the Kirby Method and develops the theory, provides parameter estimators, and illustrates the character of the conditional distributions. Examples are provided. In particular section 2.3.3 shows how the conditional distributions for rainfall and snowmelt maxima that are also annual maxima differ from the distribution for rainfall and snowmelt maxima.

2.3.1 Theory

When we only have the annual maxima series, and not the individual rainfall and snowmelt series, a method developed by W. Kirby can be adopted. The Kirby model fits a conditional probability distribution to snowmelt maxima and rainfall maxima that are also the annual maxima for their year. Thus it obtains the distribution of the overall maxima by weighting the two CDFs by the probably of each. It does not assume that the two separate series are independent: it only needs to estimate the probability that an annual maximum flood is a snowmelt or rainfall event.

Charles Parrot and Jery Stedinger described the procedure in their correspondence(Parrot, personal communication, 03/07/2011). Useful definitions include:

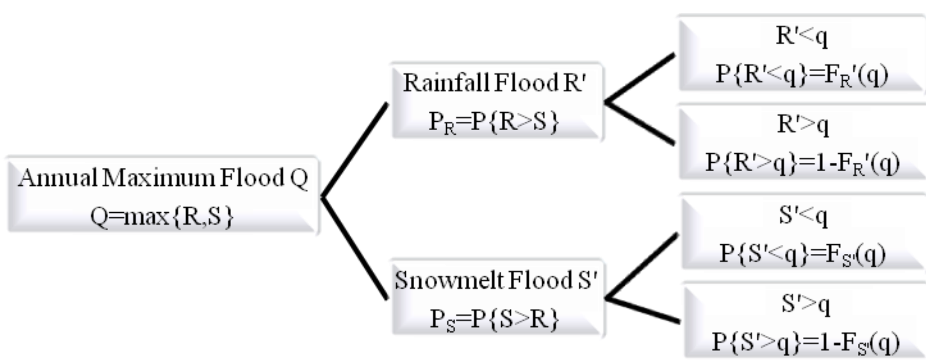
R' is a rainfall maxima that is also an annual maxima

P_R = probability that a rainfall maxima is also an annual maximum = P{R>S}

S' is a snowmelt maxima that is also an annual maxima

P_S = probability that a snowmelt maxima is also an annual maximum =P{S>R}

For the Kirby Method, the key conceptual relationship, which allows computation of non-exceedance and exceedance probabilities for any threshold q, is described by the following event tree:



Using the Total Probability Theorem with the partitions R>S and R< S, yields

$$F_Q(q) = P\{Q < q\} = P\{R' < q\}P_R + P\{S' < q\}P_S = F_{R'}(q)P_R + F_{S'}(q)P_S \quad (20)$$

One can derive P_R, P_S and the conditional distributions $F_{R'}(q)$ and $F_{S'}(q)$ from the joint distribution of R and S in section 2.2. In particular, using $f_{RS}(r,s) = f_{R|S}(r|s)f_S(s)$, one can organize the computations as follows

$$\begin{aligned} P_R &= P\{R > S\} = \int_{-\infty}^{\infty}\int_{s}^{\infty} f_{RS}(r,s)drds = \int_{-\infty}^{\infty}\int_{s}^{\infty} f_{R|S}(r \mid s) f_S(s)drds \\ &= \int_{-\infty}^{\infty}[1 - F_{R|S}(s \mid s)] f_S(s)ds \end{aligned} \tag{21}$$

where integration first over r yields the conditional CDF $F_{R|S}(r,s)$. Because $P_R + P_S = 1$, P_S is easily obtained as:

$$P_S = 1 - P_R \tag{22}$$

To obtain $F_Q(q)$ using equation (20), we need $F_{R'}(q)$ and $F_{S'}(q)$. $F_{R'}(q)$ can be computed using

$$\begin{aligned} F_{R'}(q) &= P\{R' < q\} = P\{R < q \text{ and } R > S\} / P\{R > S\} \\ &= \frac{1}{P_R}\int_{-\infty}^{q}\int_{s}^{q} f_{R|S}(r \mid s) f(s)drds = \frac{1}{P_R}\int_{-\infty}^{q}\left[F_{R|S}(q \mid s) - F_{R|S}(s \mid s)\right] f(s)ds \end{aligned} \tag{23}$$

The PDF for R' is obtained by differentiating equation (23) yielding

$$f_{R'}(q) = \frac{1}{P_R}\int_{-\infty}^{q} f_{R|S}(r \mid s) f(s)ds \tag{24}$$

The CDF and PDF for S' are obtained by similar computations. Thus

$$\begin{aligned} F_{S'}(q) &= P\{S' < q\} = P\{S < q \text{ and } S > R\} / P\{S > R\} \\ &= \frac{1}{P_S}\int_{-\infty}^{q}\int_{r}^{q} f_{S|R}(s \mid r) f(r)dsdr = \frac{1}{P_S}\int_{-\infty}^{q}\left[F_{S|R}(q \mid r) - F_{S|R}(r \mid r)\right] f(r)dr \end{aligned} \tag{25}$$

$$f_{S'}(q) = \frac{1}{P_S}\int_{-\infty}^{q} f_{S|R}(s \mid r) f(r)dr \tag{26}$$

Equations (20)-(26) show that the log-space moments (μ_{RL}, σ_{RL}, μ_{SL}, σ_{SL}) of the two populations and the correlation ρ between the log-space variates influence the distributions of the Rainfall and Snowmelt floods that are also the annual maxima, denoted R' and S', the next two sections explore those relationships .

2.3.2 Estimation of parameters

Given an annual maxima flood series $\{Q_t\}$, one can estimate P_R, P_S, $F_{R'}(q)$, and $F_{S'}(q)$ from the sample observations, rather than using the theoretical relationships above.

Let n_r and n_s be the number of rainfall and snowmelt floods that are also the annual maxima in $\{Q_t\}$ ($n = n_r + n_s$), one can estimate $\hat{P}_R$, $\hat{P}_S$ as:

$$\hat{P}_R = \frac{n_r}{n} \tag{27}$$

$$\hat{P}_S = \frac{n_s}{n} = 1 - \hat{P}_R \tag{28}$$

One can get the rainfall and snowmelt floods that are also the annual maxima, $\{R_r'\}$ and $\{S_s'\}$, by extracting the rainfall and snowmelt floods that are also the annual maxima from $\{Q_t\}$. Then by fitting $\{R_r'\}$ and $\{S_s'\}$ using a reasonable distribution, one obtains an approximations for $F_{R'}(q)$ and $F_{S'}(q)$.

The distribution of $\{R_r'\}$ and $\{S_s'\}$ can be appreciably different from the distribution of the individual annual maximum series $\{R_t\}$ and $\{S_t\}$. Thus one may have difficulty specifying the appropriate distribution for R' and S', even when they know the distribution of the complete data. Examples in the next section illustrate this concern.

2.3.3 Illustration of Kirby Conditional distributions

To illustrate the concerns voiced above, we compare the distributions of the complete rainfall and snowmelt series R and S, with the distributions of the rainfall and snowmelt floods R' and S' that are also annual maxima. We also compute analytically the means and variances of the rainfall and snowmelt floods R' and S' that are also the annual maxima. A comparison of 2-parameter lognormal distributions that have those log-space moments with the actual distribution of R' and S', demonstrates that rainfall floods that are also the annual maxima do not have a 2-parameter lognormal distribution, even though R and S did. The distribution of snowmelt floods that are also annual maxima is very close to a 2-parameter distribution in our examples. We also illustrate the precision of approximations of $F_{R'}$ and $F_{S'}$ provided by a 3-parameter lognormal distribution.

First, consider the comparisons of the complete rainfall and snowmelt maximum flood distributions for R and S, versus the rainfall and snowmelt flood distributions for R' and S' that are also the annual maxima. Equations (10)-(11) in section 2.1.3 provide the PDFs of the complete rainfall and snowmelt floods R and S. Equations (20)-(26) above provide the distributions of the rainfall and snowmelt floods R' and S' that are also the annual maxima. Analytical integration in (20)-(26) is in general impossible. However, we can integrate numerical using Simpson's rule by discretizing R and S into $\{r_i\}$ and $\{s_i\}$ with the length of each interval having values Δr and Δs. In our example, we use Δr=Δs=q/2m, thus we have:

$$\begin{aligned} P_R &= \int_{-\infty}^{\infty} [1 - F_{R|S}(s\,|\,s)] f_S(s) ds \\ &= \frac{\Delta s}{3}\left\{[1 - F_{R|S}(s_0\,|\,s_0)] f_S(s_0) - [1 - F_{R|S}(s_{2m}\,|\,s_{2m})] f_S(s_{2m})\right\} \\ &+ \frac{\Delta s}{3}\sum_{j=1}^{m}\left\{4[1 - F_{R|S}(s_{2j-1}\,|\,s_{2j-1})] f_S(s_{2j-1}) + 2[1 - F_{R|S}(s_{2j}\,|\,s_{2j})] f_S(s_{2j})\right\} \end{aligned} \tag{29}$$

In practice we cannot have an infinite number of points, but a large upper bound can be selected so that, $[1-F_{R|S}(s_{2j}|s_{2j})]f_S(s_{2j})\approx 0$. For our examples, we use q=1200m^3/s as an upper bound.

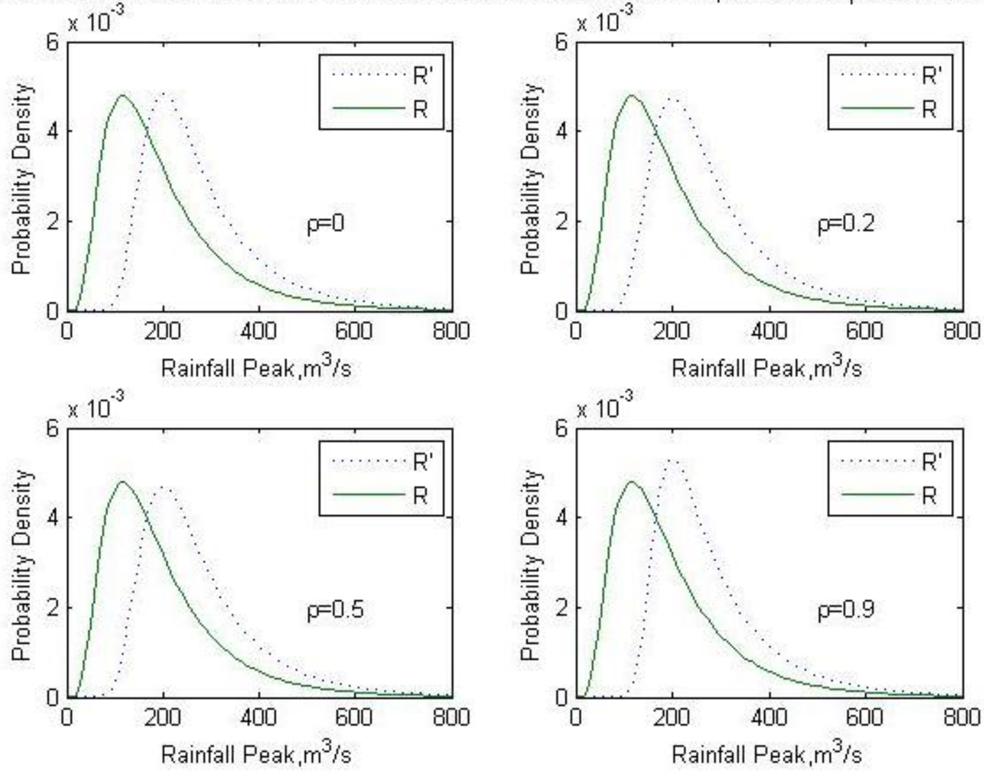
The CDFs and PDFs of the rainfall floods that are also the annual maxima can be obtained as follows:

$$\begin{aligned}
&F_{R'}(q) = P\{R' < q\} = \frac{1}{P_R}\int_{-\infty}^{q}\left[F_{R|S}(q\,|\,s) - F_{R|S}(s\,|\,s)\right]f_S(s)ds \\
&= \frac{\Delta s}{3P_R}\left\{\left[F_{R|S}(q\,|\,s_0) - F_{R|S}(s_0\,|\,s_0)\right]f_S(s_0) - \left[F_{R|S}(q\,|\,s_{2m}) - F_{R|S}(s_{2m}\,|\,s_{2m})\right]f_S(s_{2m})\right\} \\
&= \frac{\Delta s}{3P_R}\sum_{i=1}^{m}\left\{4\left[F_{R|S}(q\,|\,s_{2i-1}) - F_{R|S}(s_{2i-1}\,|\,s_{2i-1})\right]f_S(s_{2i-1}) + 2\left[F_{R|S}(q\,|\,s_{2i}) - F_{R|S}(s_{2i}\,|\,s_{2i})\right]f_S(s_{2i})\right\}
\end{aligned} \tag{30}$$

$$\begin{aligned}
&f_{R'}(r) = \frac{1}{P_R}\int_{-\infty}^{r} f_{R|S}(r\,|\,s)\,f(s)ds \\
&= \frac{\Delta s}{3P_R}\Big\{ f_{R|S}\left(r\,|\,s_0\right) f_S\left(s_0\right) - f_{R|S}\left(r\,|\,s_{2m}\right) f_S\left(s_{2m}\right) \\
&+\sum_{j=1}^{m}\left[4 f_{R|S}\left(r\,|\,s_{2j-1}\right) f_S\left(s_{2j-1}\right) + 2F_{R|S}\left(r\,|\,s_{2j}\right) f_S\left(s_{2j}\right)\right]\Big\}
\end{aligned} \tag{31}$$

Figure 4 provides comparisons of the PDFs of the complete rainfall and snowmelt series and the PDFs of the floods that are also the annual maxima, as the correlation ρ between the two annual maxima series changes. The PDFs of the complete rainfall and snowmelt series are obtained using equation (10)-(11). The PDFs of the rainfall and snowmelt floods that are also the annual maxima are obtained using equation (31).

Figure 4a. PDFs of the Rainfall floods that are also the Annual Maxima R',and the complete Rainfall data R



In Figure 4a rainfall floods that are also the annual maximum R’ have a larger mean and a smaller variance than the complete rainfall data; moreover the lower tail of the distribution is very different from the complete data set R-distribution, and essentially has a lower bound provided by the snowmelt floods in each year. As the correlation ρ between the log-space Rainfall and Snowmelt floods increases from 0.5 to 0.9, the mean μ_R of R’, increases, and the PDF of the rainfall floods R’ that are also the annual maximum becomes peakier. With a large ρ, the larger rainfall events are with high probability the annual maximum, and the smaller rainfall events, are less than the snowmelt flow in that year; this moves the R’ distribution to the right of the R distribution. There is a similar shift as ρ increases from 0 to 0.5, but the change is relatively modest.

Figure 4b. PDFs of the Snowmelt floods that are also the Annual Maxima S',and the complete Snowmelt data S

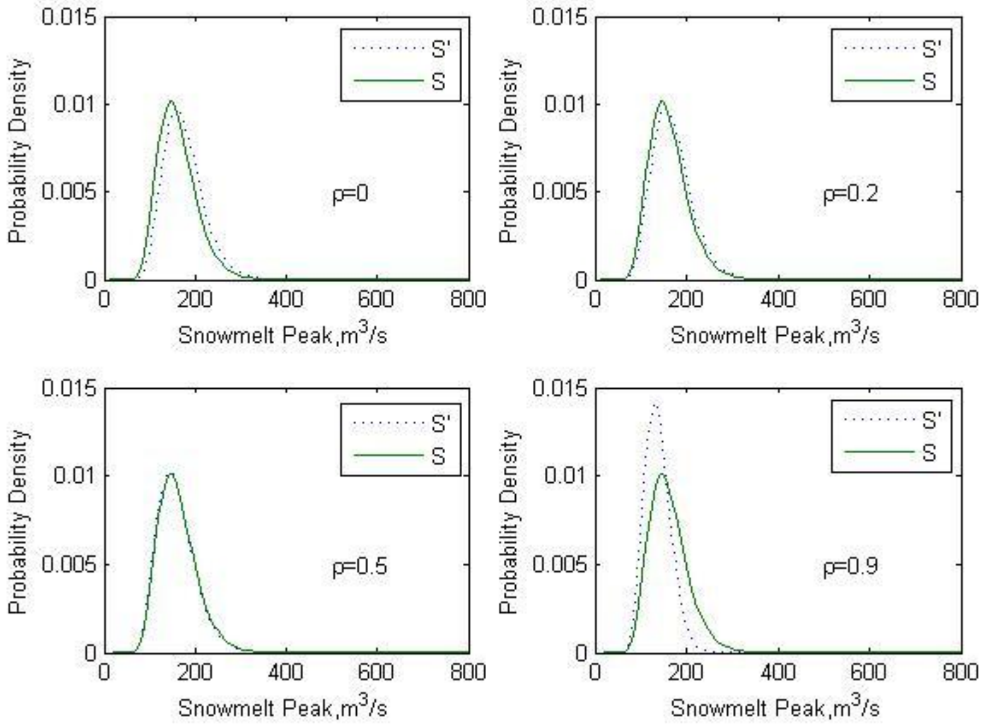


Figure 4b shows that the snowmelt floods S' that are also the annual maximum have nearly the same mean and similar variance as the complete snowmelt data when ρ is not large. As the correlation ρ between the log-space Rainfall and Snowmelt floods becomes larger (ρ = 0.9 in the example), the PDF of the snowmelt floods S' that are also the annual maximum becomes peakier, with a smaller variance and smaller the mean. In this high correlation case, the larger snowmelt events generally occur in years with large rainfall events, which are larger and thus are the annual maximum; however, the smaller snowmelt events are generally with very small rainfall events, so that the snowmelt value is the annual maximum. Thus the distribution of S' is to the left of the S distribution. There is a similar shift as ρ increases from 0 to 0.5, but the change is relatively modest.

Because we are interested the moments of both populations, and the relationship between them (measured by the log-space correlation), we computed the ratio of the means, standard deviations, and skewnesses of the floods that are also the annual maxima R' and S' and those of the complete rainfall and snowmelt series R and S ($\mu_{R'}/\mu_R$, $\sigma_{R'}/\sigma_R$, $\gamma_{R'}/\gamma_R$, μ_S'/μ_S, $\sigma_{S'}/\sigma_S$, $\gamma_{S'}/\gamma_S$) as the correlation ρ.

To calculate the moments of the rainfall and snowmelt floods that are also the annual maxima by Monte Carlo simulation, we can generate samples of the rainfall and snowmelt series, $\{R_t\}$ and $\{S_t\}$. We know that the complete snowmelt floods can be described as

$$\ln(s) \sim N\left[\mu_{SL}, \sigma_{SL}{}^2\right] \tag{32}$$

And given the snowmelt floods, the rainfall floods can be described as:

$$\left\{\ln(R) \mid S = s\right\} \sim N\left[\mu_{RL} + \frac{\sigma_{RL}}{\sigma_{SL}}\rho(\ln(s) - \mu_{SL}), (1-\rho^2)\sigma_{RL}{}^2\right] \tag{33}$$

where ρ is the correlation between the log-space rainfall and snowmelt populations instead of the correlation between the real-space rainfall and snowmelt data.

Then we determine the annual maxima, $\{Q_t\}$, and the rainfall and snowmelt maxima that are also the annual maxima, $\{R_r'\}$ and $\{S_s'\}$, for their year. The log-space moments of the rainfall floods that are also the annual maxima can be estimated as (we use log-space moments because our research shows that for lognormal distribution, use the log-space moments is more accurate and stable than the real-space moments):

$$\hat{\mu}_{R'L} = \frac{1}{n_r}\sum_{i=1}^{n_r} \ln(r_i') \tag{34}$$

$$\hat{\sigma}^2_{R'L} = \frac{1}{n_r - 1}\sum_{i=1}^{n_r}\left[\ln(r_i') - \hat{\mu}_{R'L}\right]^2 \tag{35}$$

And $\hat{\mu}_{R'}$ and $\hat{\sigma}^2_{R'}$ can be obtained by solving equation (4)-(5), then $\hat{\gamma}_{R'} = \frac{1}{n_r - 1}\sum_{i=1}^{n_r}\frac{\left[r_i' - \hat{\mu}_{R'}\right]^3}{\hat{\sigma}^3_{R'}}$

We can also get those moments theoretically:

$$\mu_{R'} = \int_{-\infty}^{\infty} r f_{R'}(r)dr \tag{36}$$

$$\sigma^2_{R'} = \int (r - \mu_{R'})^2 f_{R'}(r)dr \tag{37}$$

And $\gamma_{R'} = \frac{\int (r - \mu_{R'})^3 f_{R'}(r)dr}{\sigma^3_{R'}}$. Use equation (4)-(5), we can obtain the log-space moments of R’ theoretically. Using equation (10)-(11), we can find the 2-parameter lognormal distributions that has those log-space moments.

The moments of the logarithm of the snowmelt floods that are also the annual maxima can be obtained in the same way.

The analysis is repeated for different values of the real-space and log-space moments of the complete snowmelt population resulting in a different relationship between snowmelt and rainfall in their competition to be the annual maximum (in our case, we get the moments theoretically). We kept the real-space moments of the complete rainfall population and the coefficient of variation of the complete snowmelt population CV=σ_S/μ_S constant, and varied the median ratio $M_{R/S}$=Med[R]/Med[S]=1.5, 1.07, 0.75 to get different moments of the snowmelt population, where Med[R]=exp(μ_{RL}) and Med[S]=exp(μ_{SL}). (See values listed Table 2, $M_{R/S}$=1.07.) This changes the probability P_R that the annual maximum is a rainfall event. In particular, $M_{R/S}$=1.5, 1.07, 0.75 correspond to $P_R \approx$ 0.8, 0.5, 0.3 with the exact value dependent upon ρ, given the adopted coefficients for R and S. See results in Figure 5 which displays $\mu_{R'}/\mu_R$, $\sigma_{R'}/\sigma_R$, and $\gamma_{R'}/\gamma_R$, which is the ratio of the R' moment to the R moment.

Figure 5a. $\mu_{R'}/\mu_R$ as ρ and the complete snowmelt population moments change

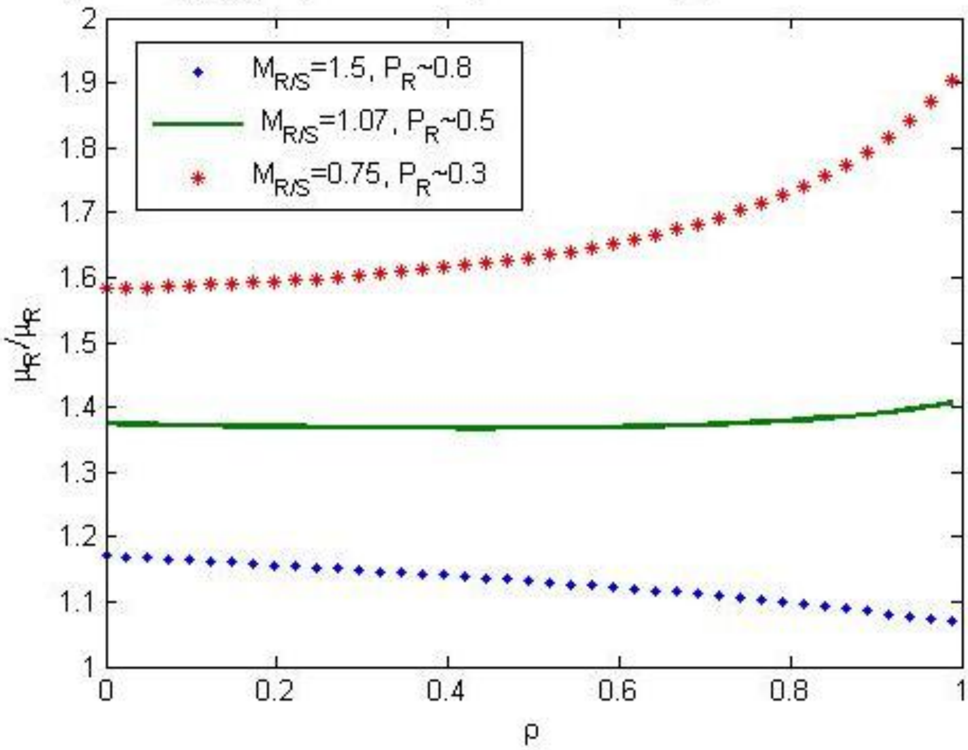


Figure 5b. $\sigma_{R'}/\sigma_R$ as ρ and the complete snowmelt population moments change

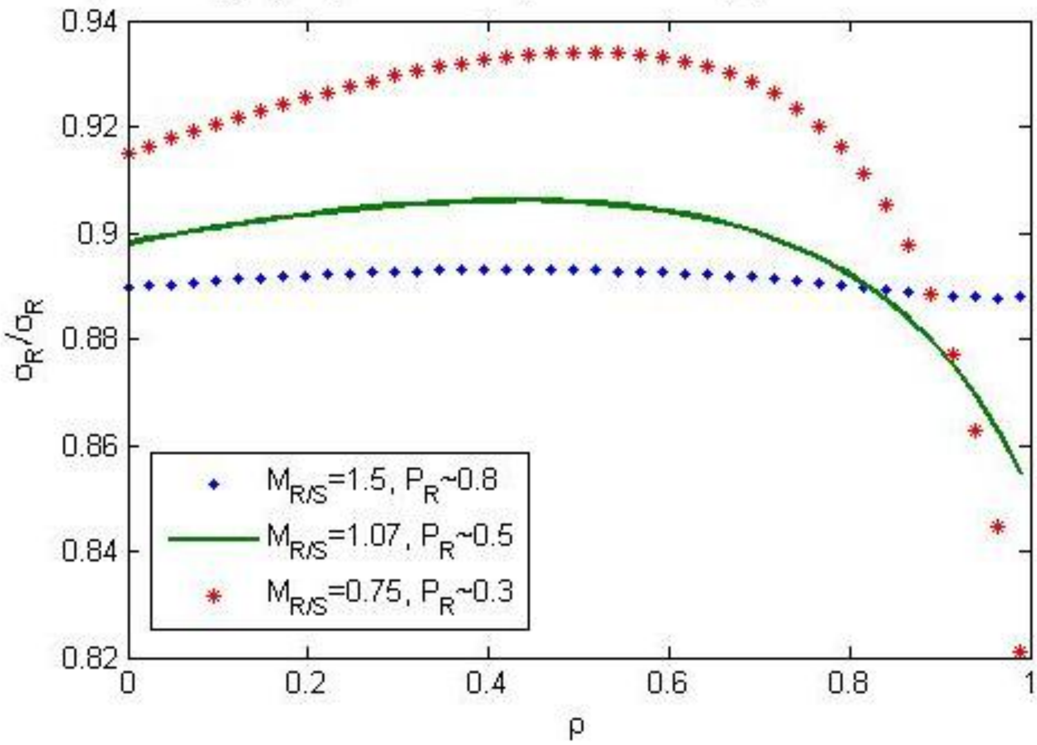
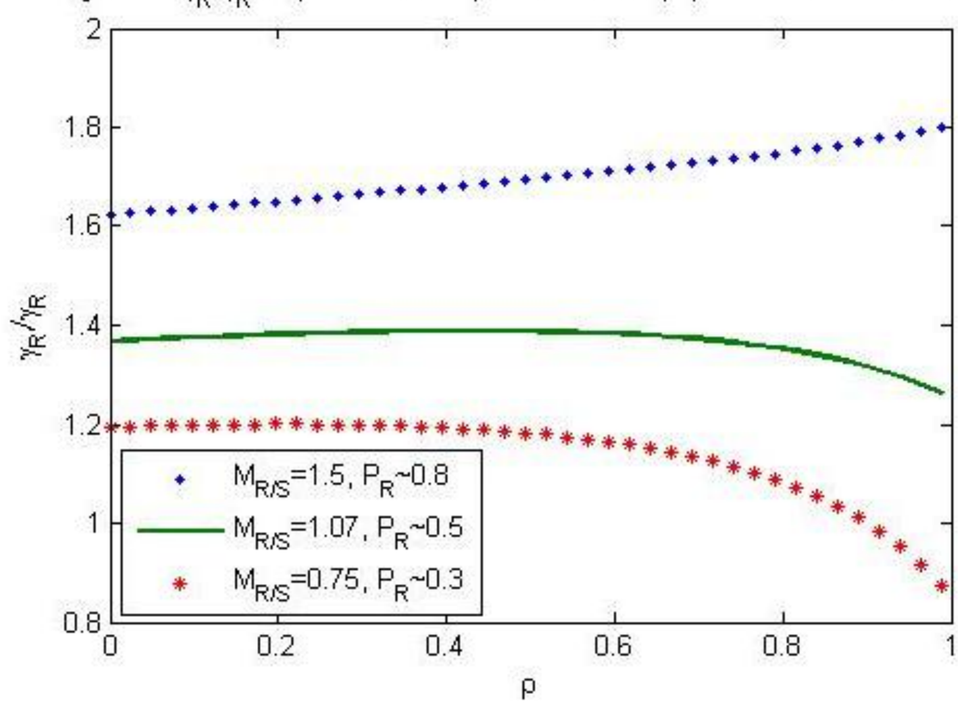


Figure 5c. $\gamma_{R'}/\gamma_R$ as ρ and the complete snowmelt population moments change



In figure 5, because the snowmelt floods dominate the lower tail of the mixed population, R' lacks many of the smaller values in the R distribution, so R' has a larger mean, with a smaller standard deviation, than the complete rainfall data. Figure 5 also shows that R' generally has a larger skewness as well. When $P_R>0.5$, the larger ρ, the smaller the $\mu_{R'}$; for $P_R\leq 0.5$, the larger ρ, the smaller $\mu_{R'}$. No matter how the median ratio changes, the largest $\sigma_{R'}$ occurs when ρ=0.5. No matter how the median ratio changes, the $\gamma_{R'}$ is relatively constant for ρ≤0.9.

Just as figure 5 showed how the moments of R' changed relative to those of R, figure 6 shows how the moments of S' change relative to S.

Figure 6a. $\mu_{S'}/\mu_S$ as ρ and the complete snowmelt population moments change

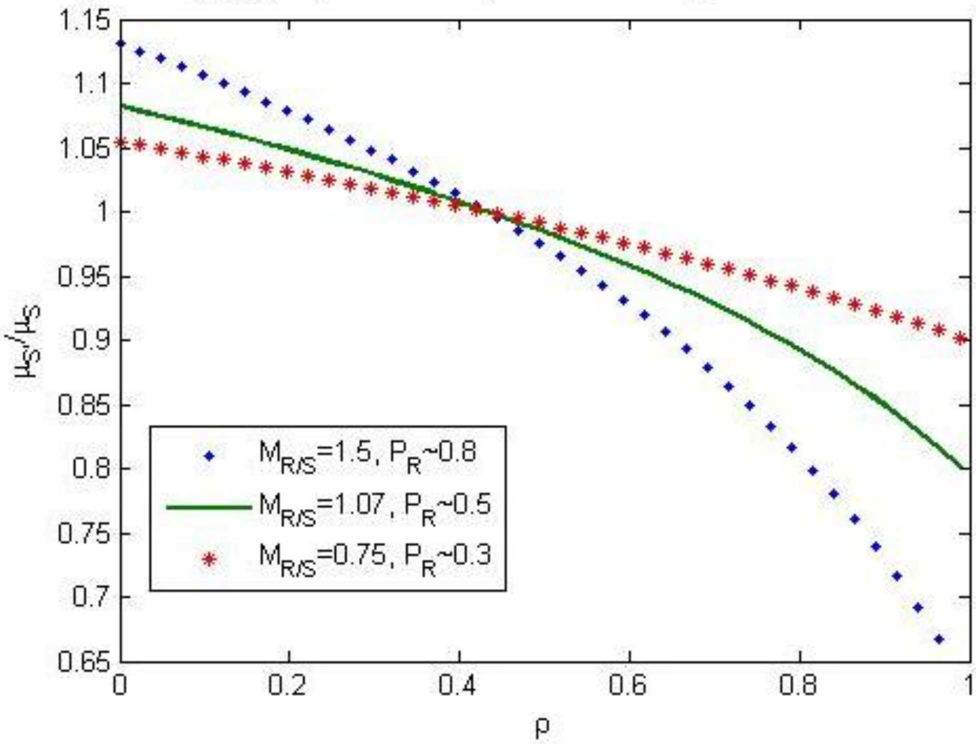


Figure 6b. $\sigma_{S'}/\sigma_S$ as ρ and the complete snowmelt population moments change

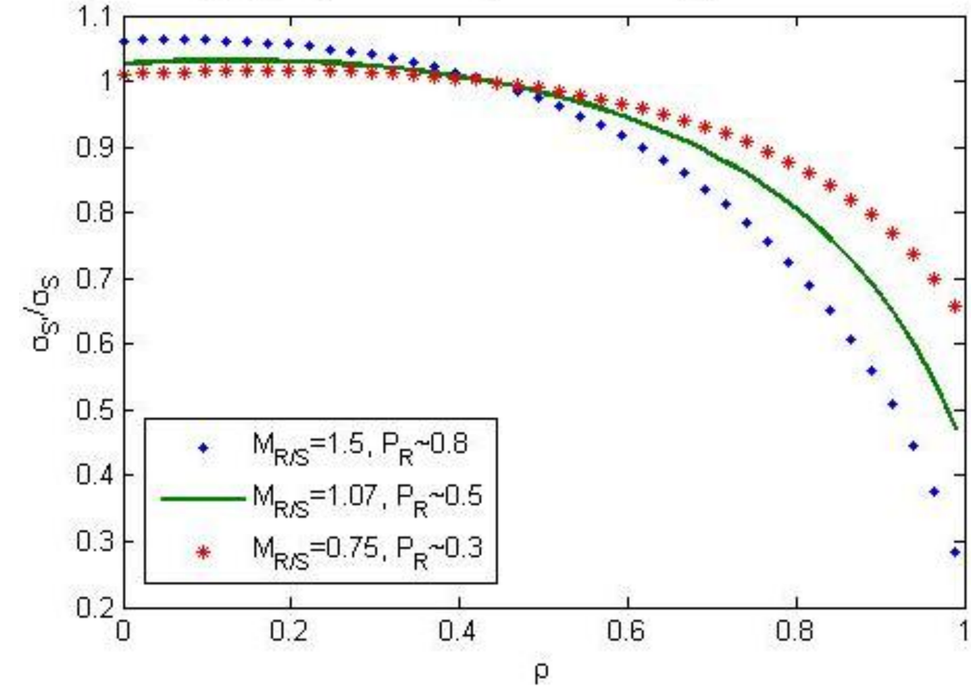


Figure 6c. $\gamma_{S'}/\gamma_S$ as ρ and the complete snowmelt population moments change

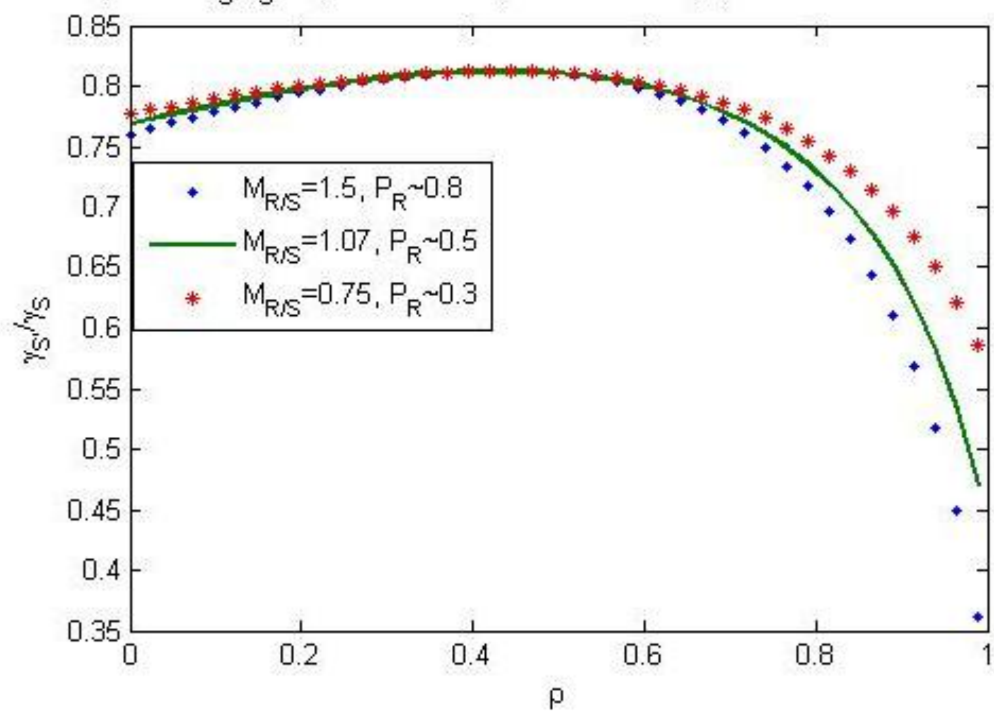


Figure 6 shows that the S' distribution doesn't always have a larger mean and standard deviation than S; the relationship depends on ρ and P_R. For the cases in the figures, the values of $\mu_{S'}$ and $\sigma_{S'}$ decrease with ρ . For these cases in the figures, $\gamma_{S'}$ is always smaller than γ_S. When ρ>0.7, $\gamma_{S'}/\gamma_S$ decreases rapidly as ρ increases.

Now consider if the rainfall and snowmelt floods that are also the annual maxima have 2-parameter lognormal distributions. The moments of the rainfall and snowmelt floods series that are also the annual maxima and of their logarithm can be obtained either numerically by integrating the appropriate analytical expressions as in figures 5 and 6, or by Monte Carlo simulation.

With the moments calculated above, Figure 7 provides the comparison of the lognormal distributions which have the log-space moments of the rainfall and snowmelt data that are also the annual maxima and the PDFs of the rainfall and snowmelt data that are also the annual maxima.

Figure 7a. PDFs of the Rainfall floods that are also the Annual Maxima R',
and the lognormal distributions that have their log-space moments

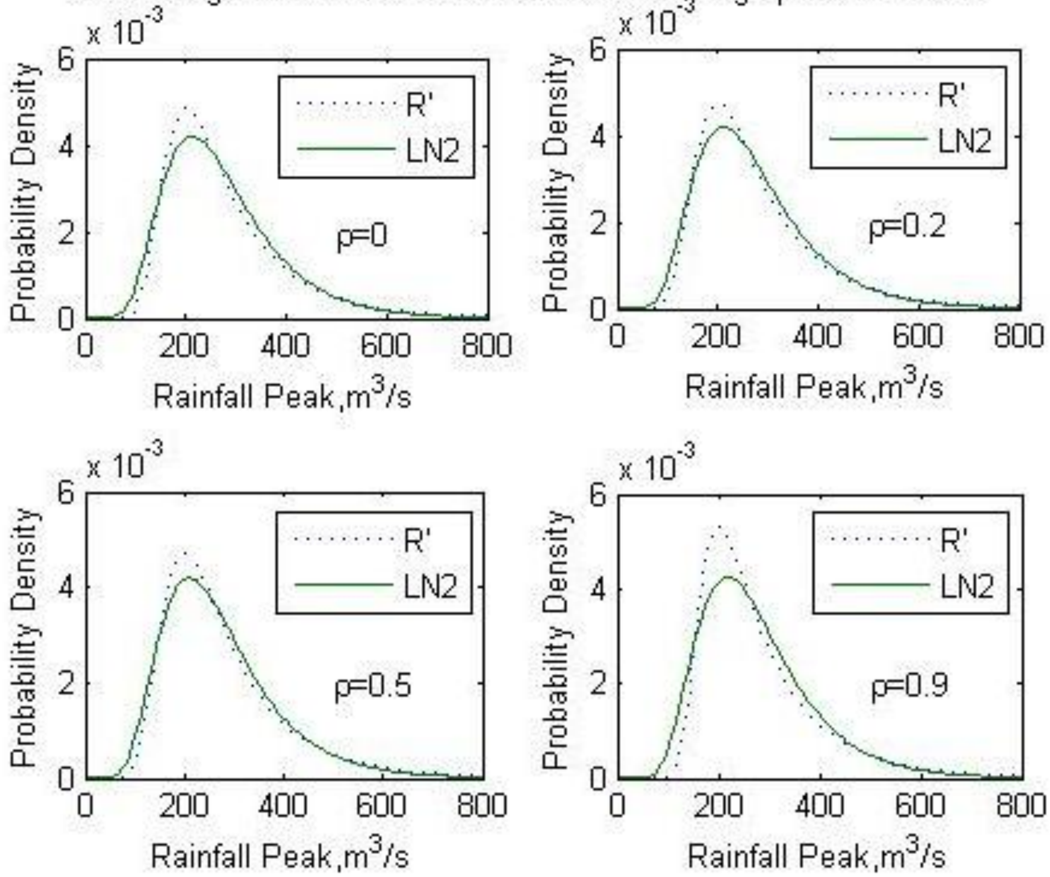


Figure 7b. PDFs of the Snowmelt floods that are also the Annual Maxima S', and the lognormal distributions that have their log-space moments

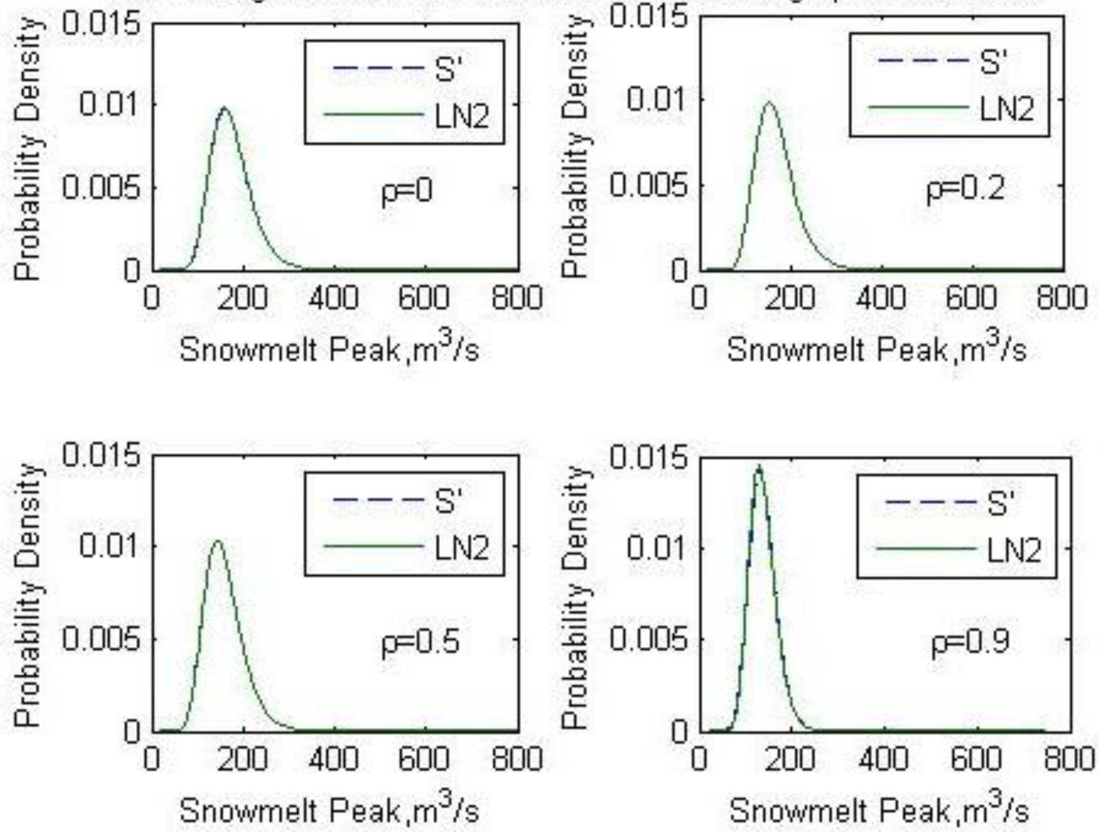
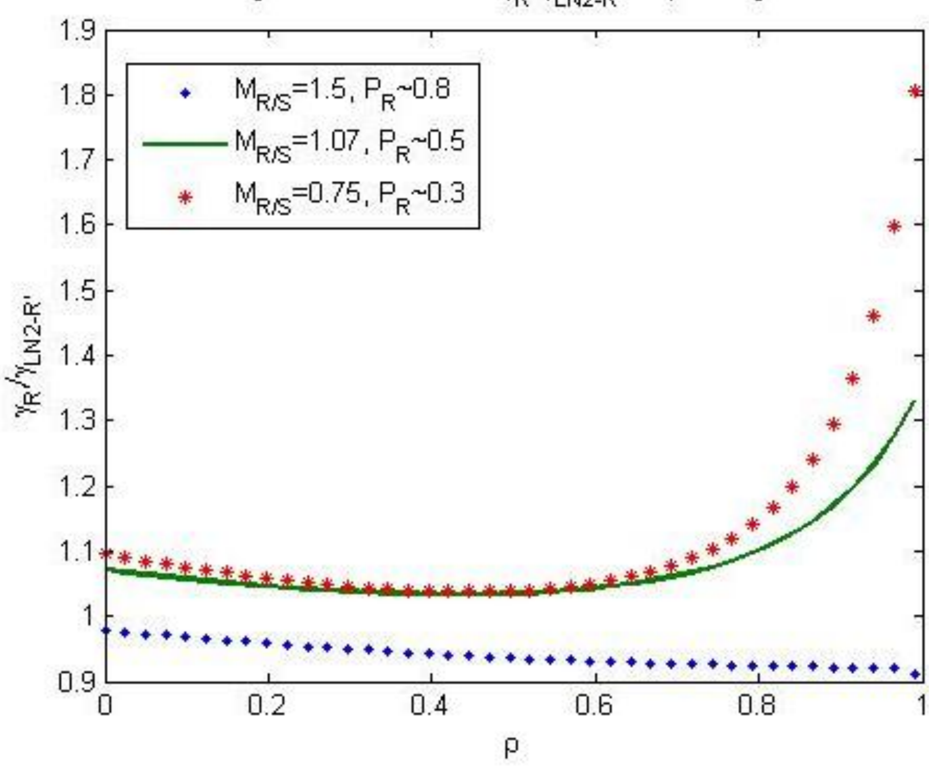


Figure 7 shows the differences between the distributions of the rainfall and snowmelt data that are also the annual maxima, and the lognormal distributions that have the same real and log-space moments: One can see that the distributions of the rainfall data that are also the annual maxima does not match the lognormal distributions with the same real and log-space moments. However, the distributions of the snowmelt data that are also the annual maxima closely match the lognormal distributions with the same moments. The reason the lognormal distributions can provide a good description of S' is that for $\rho \leq 0.5$, the distributions of S and S' are almost identical' for $\rho = 0.9$, the S' distribution is a little more symmetric but is still well described by a 2-parameter lognormal distribution. However for ρ near one, an unrealistic case for floods, the S' skew goes negative and a lognormal distribution would be an unsatisfactory model.

There is another way to check if R' or S' has 2-parameter lognormal distribution. Consider the skew defined as $\gamma = \frac{E(x-\mu)^3}{\sigma^3}$. For 2-parameter lognormal distribution we have the relationship (Stedinger (1980): $\gamma_{LN2} = 3\ CV + CV^3$, with $CV = \sqrt{\exp(\sigma_L^2)-1}$.

Thus if the skew γ of R' and S' equal γ_{LN2}, a 2-parameter lognormal distribution is likely to be a good model. Figure 8 displays the skew ratios $\gamma_{R'}/\gamma_{LN2-R'}$, $\gamma_{S'}/\gamma_{LN2-S'}$, as the correlation ρ between log-space rainfall and snowmelt populations changes with $M_{R/S}$=1.5, 1.07, 0.75.

Figure 8a. Skew Ratio $\gamma_{R'}/\gamma_{LN2-R'}$ as ρ changes



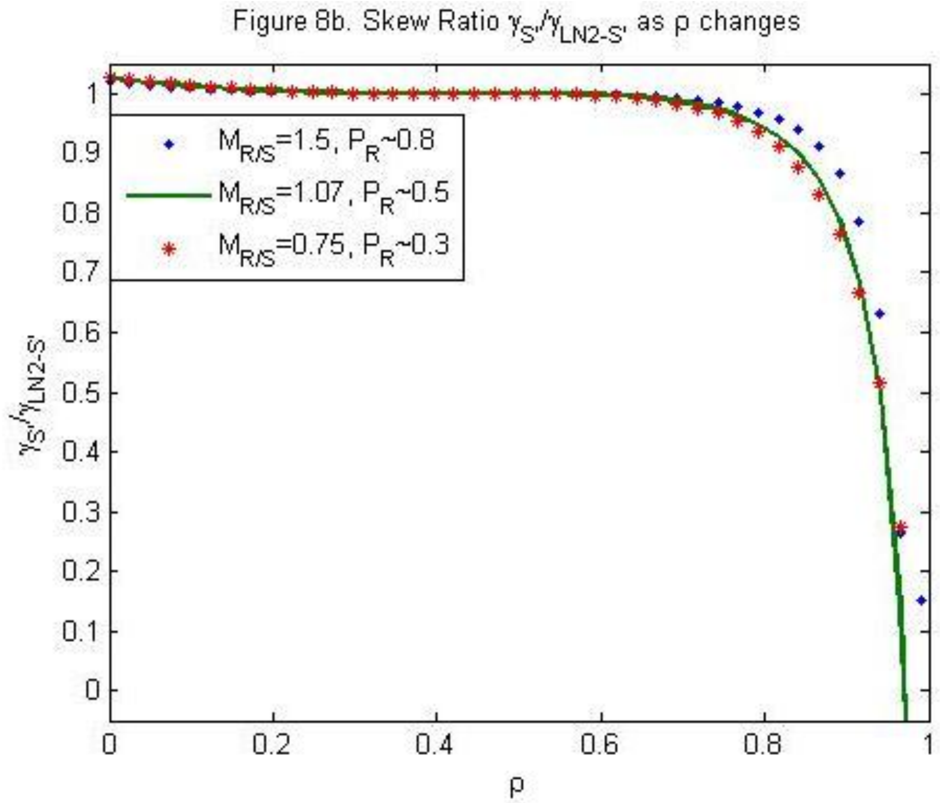


Figure 8a shows that $\gamma_{R'}/\gamma_{LN2-R'}<0.98$ when $M_{R/S}=1.5$, $P_R\approx0.8$ for all ρ, otherwise, $\gamma_{R'}/\gamma_{LN2-R'}>1.02$. Figure 8b shows that $0.98<\gamma_{R'}/\gamma_{LN2-R'}<1.02$ when $\rho<0.7$. In practice the correlations ρ between the log-space population is usually smaller than 0.7, thus we may use 2-parameter lognormal distribution to describe S' in most cases. When $\rho>0.7$, $\gamma_{R'}/\gamma_{LN2-R'}$ drops rapidly and eventually goes to negative. when $\rho=0.9$, $\gamma_{R'}/\gamma_{LN2-R'}=0.75$, yet Figure 7b shows 2-parameter lognormal distribution provides a good approximation to S', we assume that is because $\gamma_{S'}$ is small, and the difference between $\gamma_{S'}$ and $\gamma_{LN2\text{-}S'}$ is not as visible as $\gamma_{R'}$ and $\gamma_{LN2\text{-}R'}$; see Table 3.

Table 3 Comparisons of real Skews and CV^3+3CV when $M_{R/S}$ =1.07

Skewness	$\gamma_{R'}$	$\gamma_{LN2\text{-}R'}$	$\gamma_{S'}$	$\gamma_{LN2\text{-}S'}$
ρ=0	1.49	1.41	0.79	0.77
ρ=0.2	1.48	1.42	0.80	0.80
ρ=0.5	1.47	1.44	0.81	0.81
ρ=0.9	1.59	1.39	0.49	0.63

Because we are concern more about the upper tail of the mixed population, which has little to do with snowmelt population, we may still use a 2-parameter lognormal distribution to fit S' even when ρ is large. Another advantage of using a 2-parameter instead of 3-parameter lognormal distribution to fit S' is that we would have one parameter fewer to estimate when dealing with small samples. An example of choosing the proper number of parameters is provided in Lu and Stedinger (1992).

Because we are concern more about large floods, which most are rainfall events, a more accurate estimator of R' is attractive.

Consider the situation when fitting $\{R_r'\}$ and $\{S_s'\}$ using 3-parameter lognormal distribution. A 3-parameter lognormal distribution is a more general form of the lognormal distribution, which includes an additional shift parameter τ. Stedinger (1980) recommends a procedure that combines the moment method already studied for the 2-parameter distribution with a quantile estimator of the lower bound τ. The lower bound estimator is

$$\hat{\tau} = \begin{cases} \frac{\hat{X}_p \hat{X}_{1-p} - \hat{X}_{0.5}{}^2}{\hat{X}_p + \hat{X}_{1-p} - 2\hat{X}_{0.5}} & if \ \hat{X}_p + \hat{X}_{1-p} - 2\hat{X}_{0.5} > 0 \ and \ \hat{X}_p \hat{X}_{1-p} - \hat{X}_{0.5}{}^2 > 0 \\ 0 & otherwise \end{cases} \quad (38)$$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad (39)$$

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \hat{\mu}\right)^2 \quad (40)$$

$$\hat{\mu}_L = \ln\left[\frac{\hat{\mu} - \hat{\tau}}{1 + \hat{\sigma}^2 / \left(\hat{\mu} - \hat{\tau}\right)^2}\right] \quad (41)$$

$$\hat{\sigma}_L^2 = \ln\left[1 + \hat{\sigma}^2 / \left(\hat{\mu} - \hat{\tau}\right)^2\right] \quad (42)$$

Here $\hat{X}_p$ and $\hat{X}_{1-p}$ are the largest and smallest values of the observations, and $\hat{\mu}_L$ and $\hat{\sigma}_L^2$ are the mean and variance of the log-space data, $\hat{\mu}$ and $\hat{\sigma}^2$ are the mean and variance of the real-space data. When the sample size is larger than 100, we use p=0.05.

If R' can be described using 3-parameter lognormal distribution, we can also get the 3 parameters theoretically. In our case, we store the cumulative probability p_i with respect to each value $F_{R'}\left(r\right)$ obtained using (30), and we can find a value r_a such that

$$F_{R'}\left(r_a\right) < p < F_{R'}\left(r_{a+1}\right) \quad (43)$$

Use the secant method, we search for a root using

$$r_p = \left[p - F_{R'}\left(r_a\right)\right]\frac{r_{a+1} - r_a}{F_{R'}\left(r_{a+1}\right) - F_{R'}\left(r_a\right)} + r_a \quad (44)$$

We also know that for a 3-parameter lognormal distribution,

$$r_p = \tau + \exp(\mu_{RL'} + z_p \sigma_{Rl'}) \tag{45}$$

$$r_{0.5} = \tau + \exp(\mu_{RL'}) \tag{46}$$

Thus we can get all 3 parameters with r_p, $r_{0.5}$, r_{1-p}. And the parameters for snowmelt population can be obtained in the same way. In our case, R' can be well described by LN3, and the parameters are obtained using equations (43)-(46).

Figure 9 displays the comparisons of the 3-parameter lognormal distributions which have the moments rainfall floods R' that are also the annual maxima, and the distributions of the rainfall floods R' that are also the annual maxima as the correlation ρ between the log-space rainfall and snowmelt floods and the moments of the populations change, where the moments of the rainfall floods R' that are also the annual maxima are obtained use equation (36)-(37), (45)-(46), and the PDFs of the rainfall data R' that are also the annual maxima are obtained with equation (31).

Figure 9a PDFs of the Rainfall floods that are also the Annual Maxima R',
and the 3-parameter lognormal distributions fitted with their moments, $M_{R/S}$=1.50

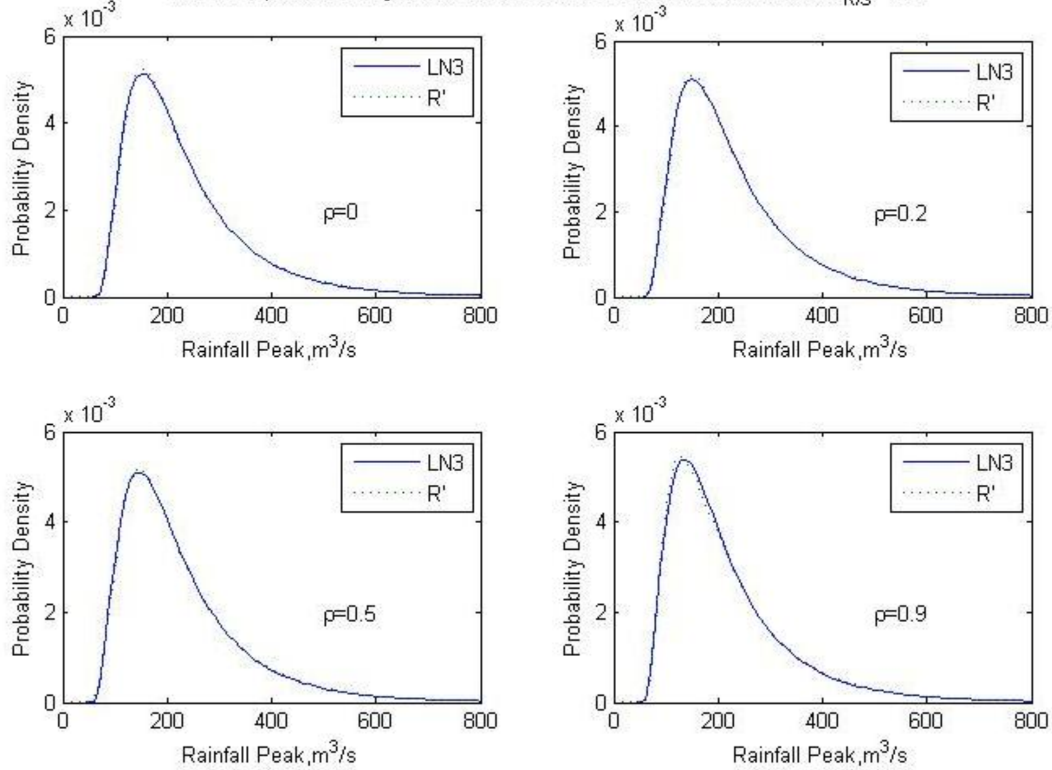
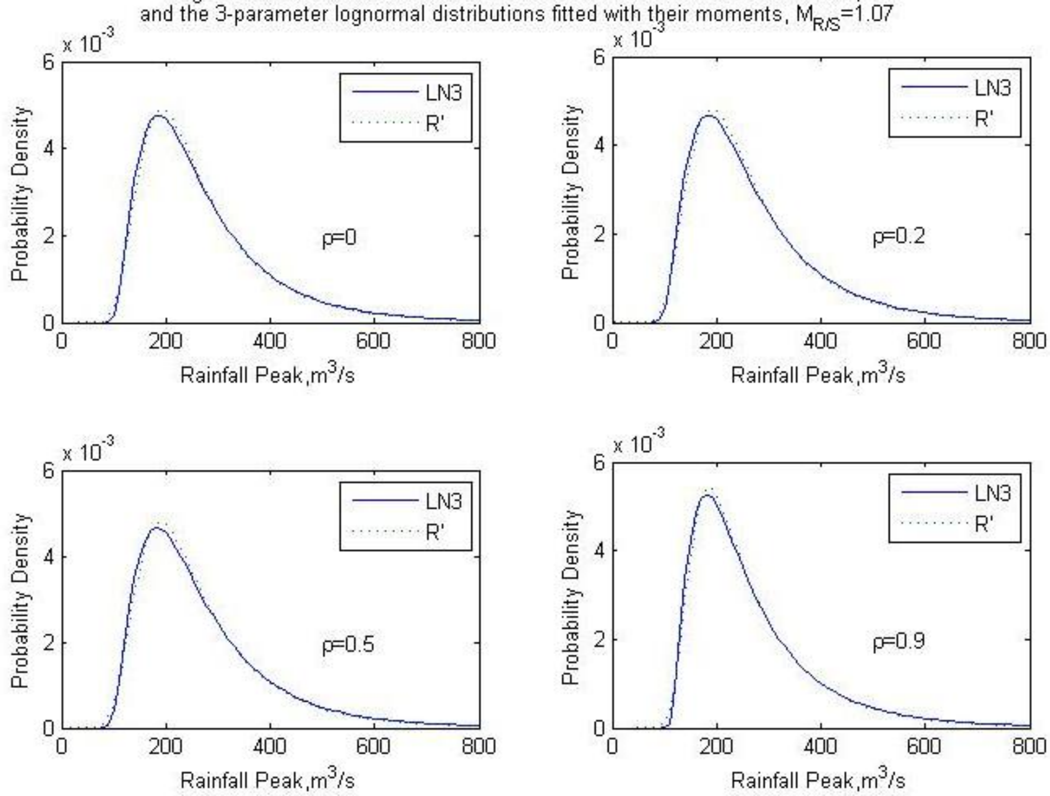


Figure 9b PDFs of the Rainfall floods that are also the Annual Maxima R',
and the 3-parameter lognormal distributions fitted with their moments, $M_{R/S}$=1.07



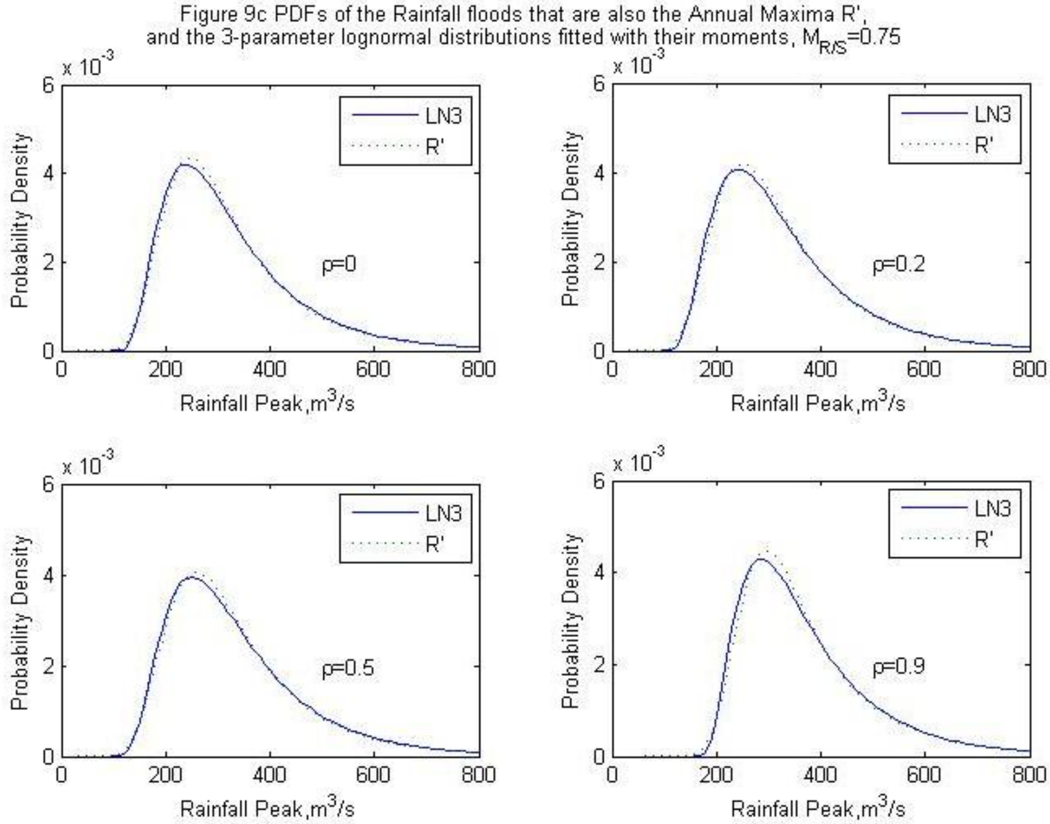
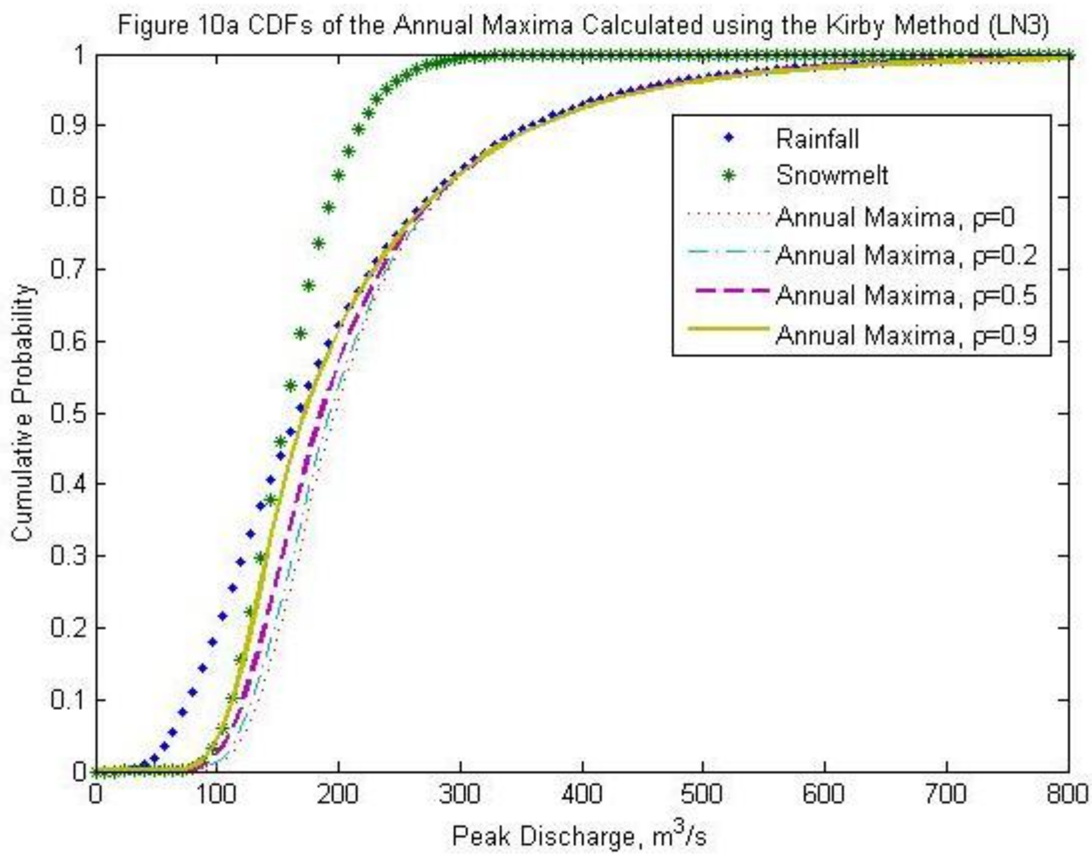


Figure 9 shows that 3-parameter lognormal distribution works much better than 2-parameter lognormal distribution when describing the rainfall data that are also the annual maxima. And as ρ and $M_{R/S}$ change, 3-parameter lognormal distribution can still well describe the rainfall data that are also the annual maxima.

In this analysis, a 3-parameter lognormal distribution is employed when describing the rainfall data that are also the annual maxima R', while 2-parameter lognormal distribution is employed when describing the snowmelt data that are also the annual maxima S'. The values of P_R, P_S, $F_{R'}$, and $F_{S'}$ were computed using equation (30). The results should be identical with the results calculated using the Joint Distribution Method. The two are compared in figure 10.

Figure 10a displays the CDFs F_Q as the correlation between the log-space rainfall and snowmelt floods ρ changes, where F_Q is obtained using (20), $F_{R'}$ is described by 3-parameter lognormal distribution, and $F_{S'}$ is described by a using 2-parameter lognormal distribution.



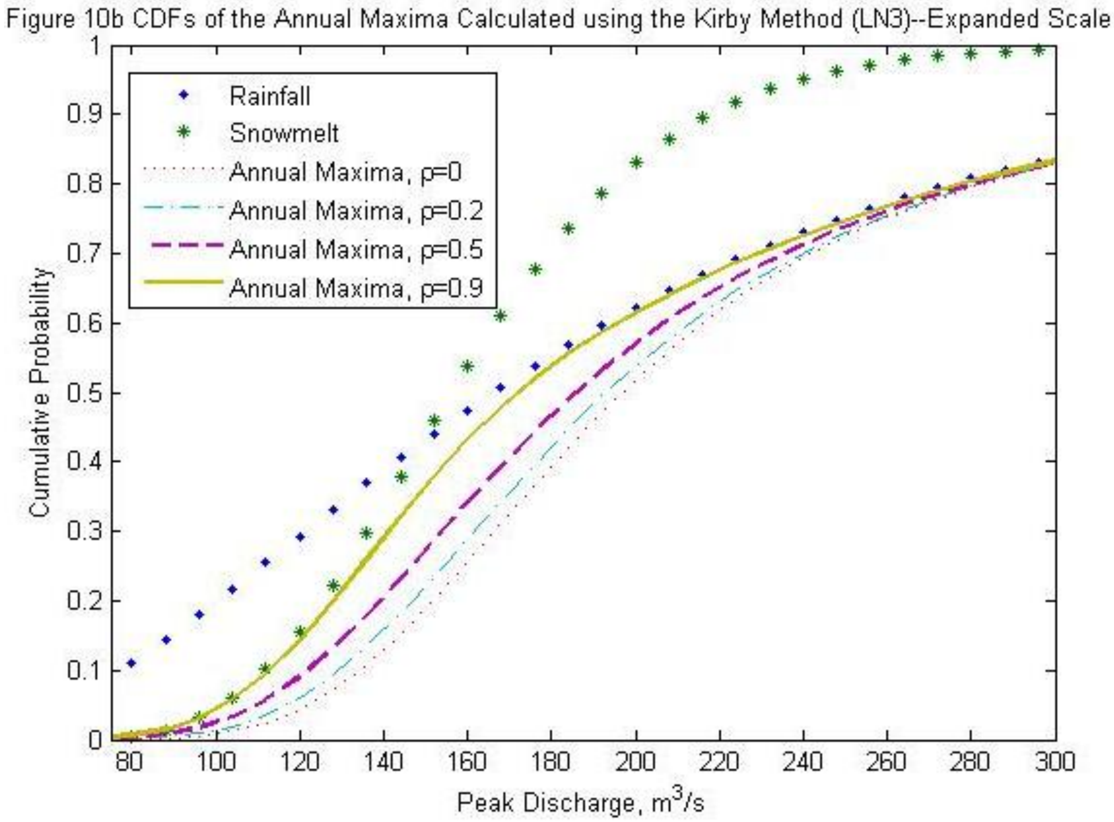


Figure 10c compares the results calculated using the Kirby Method (LN3), and the results calculated using the Joint Distribution Method.

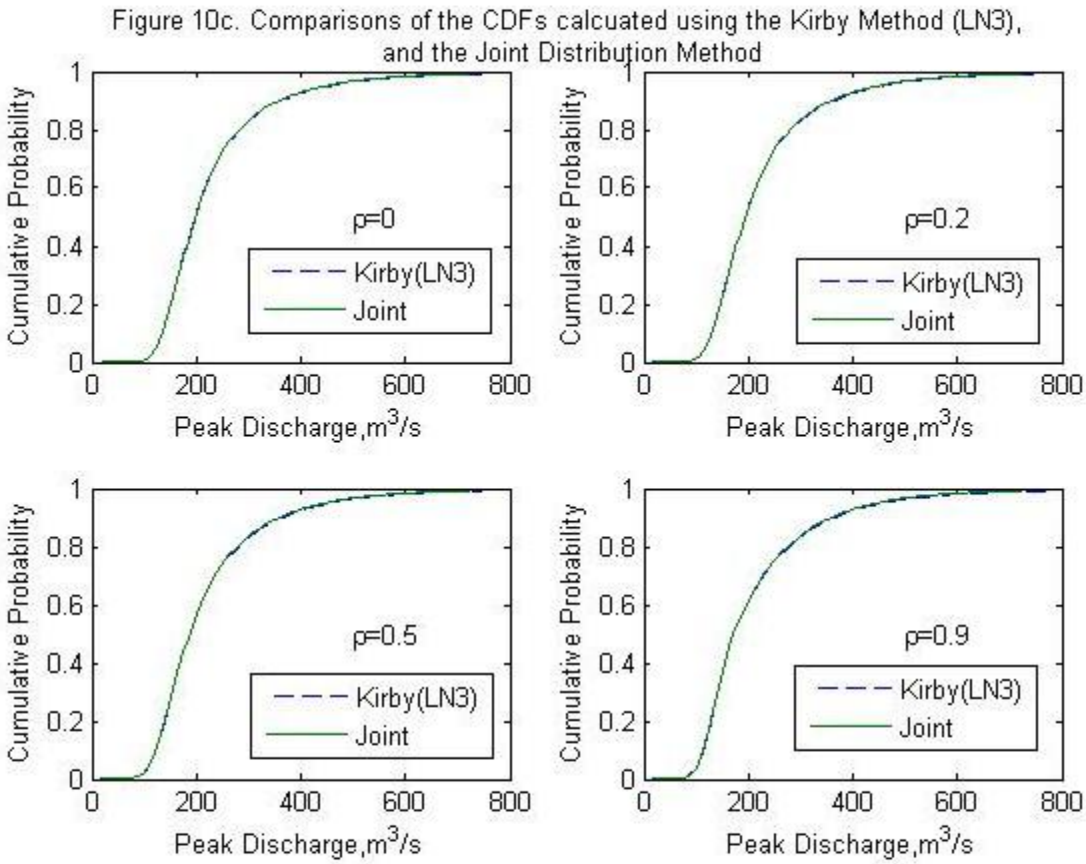


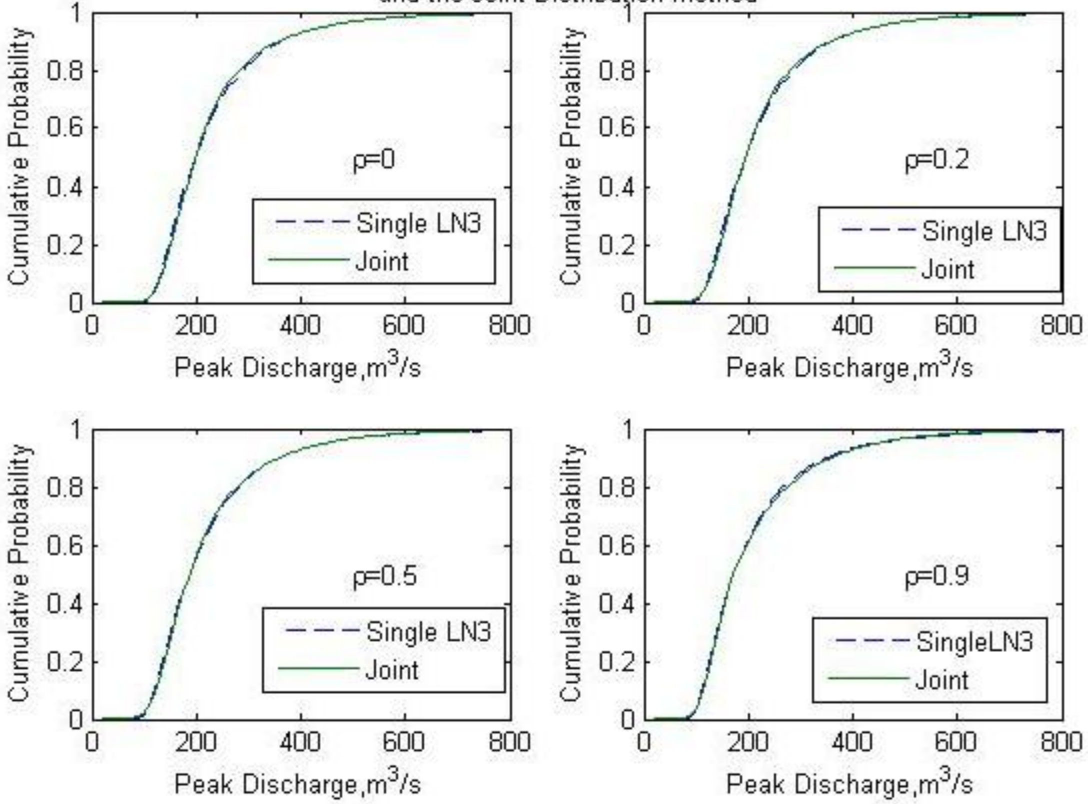
Figure 10c shows that use 3-parameter lognormal distribution to describe the rainfall floods that are also the annual maxima R' and use 2-parameter lognormal distribution to describe snowmelt data that are also the annual maxima S' provides a very accurate description of the annual maximum distribution obtained directly using the joint distribution method.

2.4 Fitting the Annual Maximum Floods using a single 3-parameter distribution

This section explores the opportunity of describing the annual maxima using a single distribution. Here a 3-parameter -parameter lognormal distribution is adopted. Section 2.3.3 provides equations for fitting a LN3 to flood series; see equations (38)-(42) . When the true quantiles are known, equation (43)-(46) are recommended.

Figure 11 compares of the CDFs for the annual maximum series calculated using a single 3-prarameter lognormal distribution using equation (43)-(46), and the CDFs calculated using the Joint Method.

Figure 11. Comparisons of the CDFs calcuated using one 3-paremeter lognormal distributions and the Joint Distribution Method



The CDFs calculated using a single 3-parameter lognormal distribution are very similar, but not quite identical, with the CDFs calculated using the Joint Distribution Method.

2.5 Use of a Just - Rainfall Distribution

When we are only concern with large floods, which are almost always rainfall events, modeling just the rainfall events as an approximation to the annual maximum distribution may be easy, simple, and accurate solution.

To apply the Just-Rainfall Method, we fit the rainfall maximum series with a 2-parameter lognormal distribution (in our case, the moments of the 2-patameter lognormal rainfall population are listed in Table 2.), and use the rainfall quantile to approximate the mixed population quantile.

Figure 3 shows the CDFs of the Rainfall, Snowmelt and mixed population as the correlation ρ between the log-space rainfall and snowmelt events changes. The figure shows that the Just-Rainfall Method does not represent well the lower quantiles; however, there is to be a critical probability, which here is denoted P_C, above which the just-rainfall model provides a very good approximation of flood risk. As shown below, that threshold varies with $M_{R/S}$, as one should expect. If the median ratio $M_{R/S}$ is much larger than 1, then the rainfall process should dominate, however, if $M_{R/S}$ is on the order of 1 or less, the largest annual maxima would generally be a mixture of rain and snowmelt events.

Figure 12 is a repeat of Figure 3, except that the $M_{R/S}$ ratio has 3 values, 0.75, 1.00, and 1.51. The moments of the rainfall and snowmelt populations are listed in Table 2; the CDFs of the annual maximum population are calculated using the correct Joint Distribution Method.

Figure 12a CDFs of the Rainfall, Snowmelt, and mixed population as ρ changes, $M_{R/S}$=1.5

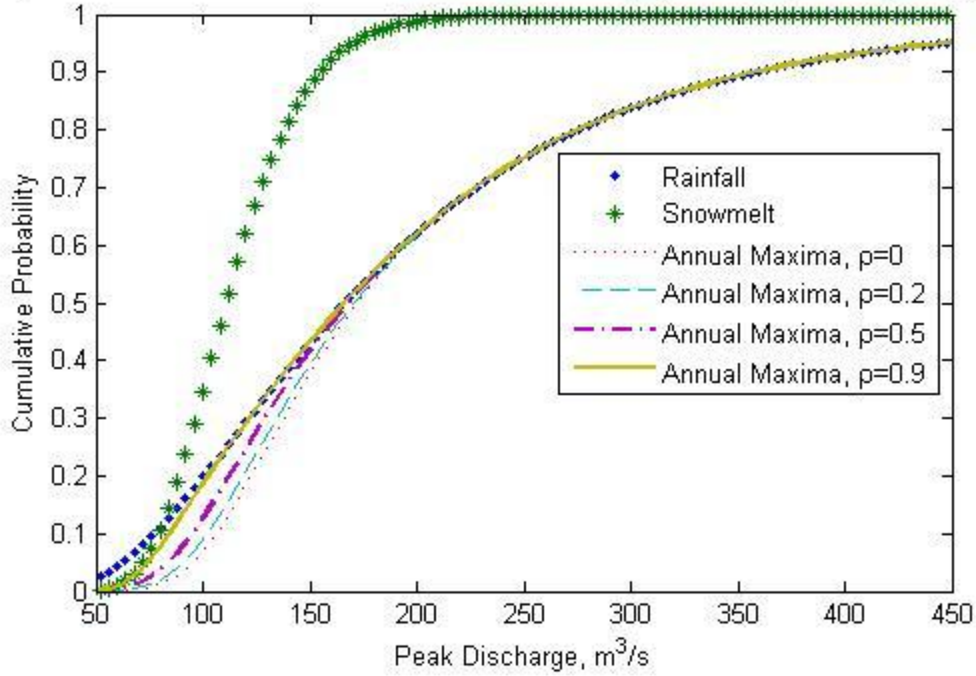
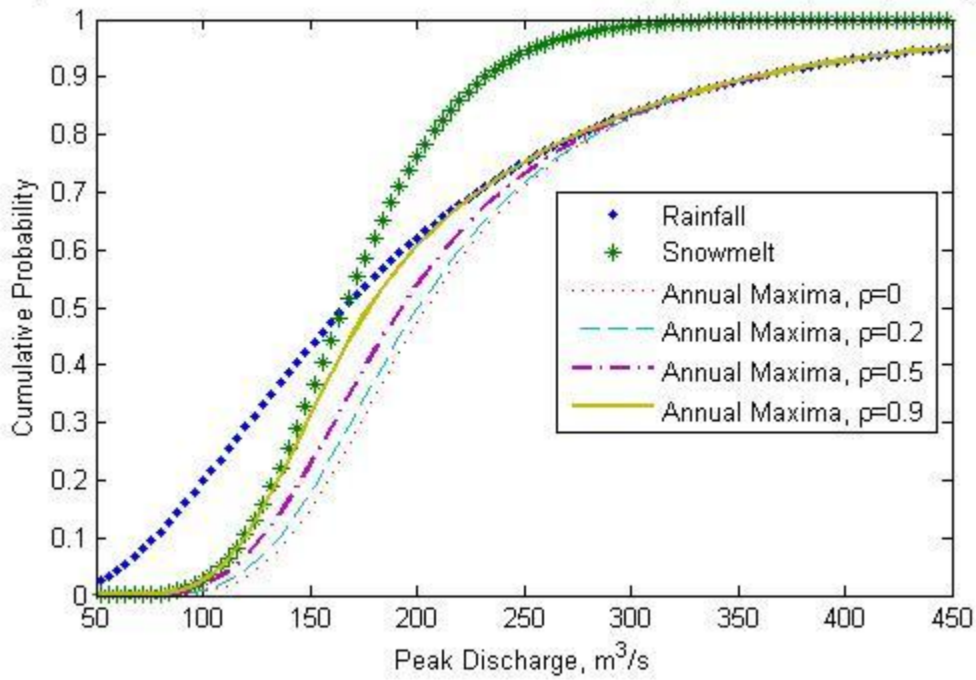
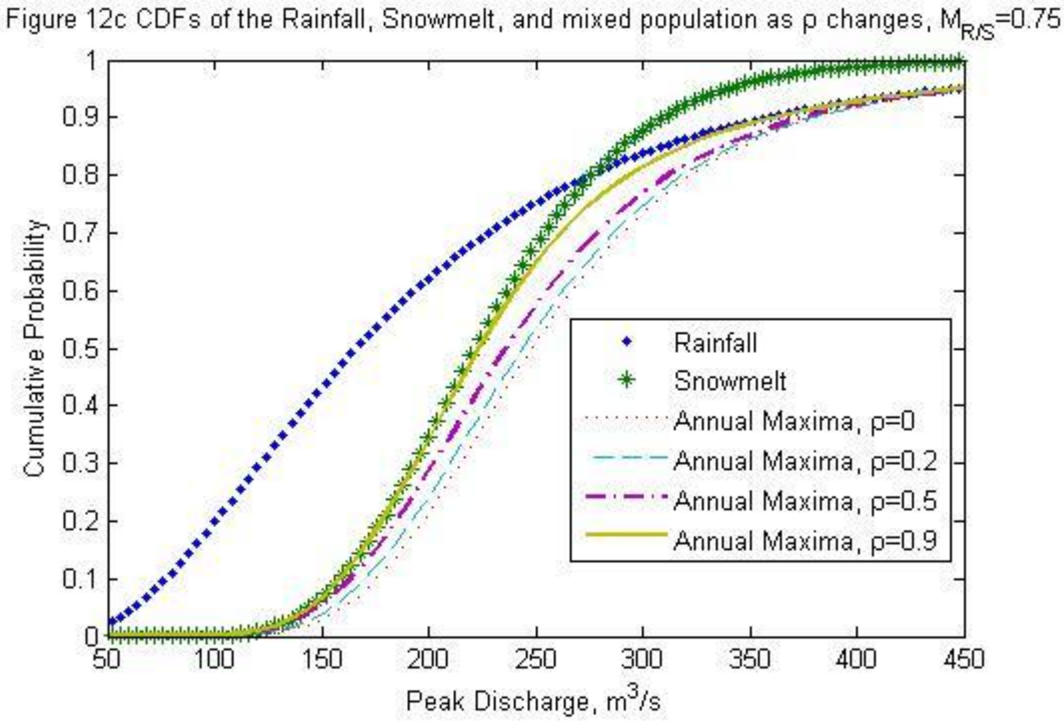


Figure 12b CDFs of the Rainfall, Snowmelt, and mixed population as ρ changes, $M_{R/S}$=1





Consider Figure 12a with $M_{R/S}$=1.5, so the by eye the critical probability appears to be P_C≈0.6 because the rainfall distribution dominates above 175 m^3/s. In Figure 12b, $M_{R/S}$=1, and the critical probability is about P_C≈0.85. Finally in figure 12c, $M_{R/S}$=0.75, so the critical probability is P_C≈0.95 because the snowmelt process dominates flood risk over a wide range of flows, but not the very largest extremes. Figure 12abc shows that $M_{R/S}$ is the dominate factor that influences P_C. The correlation ρ has a modest impact on the critical probability P_C above which the joint model CDF cannot be distinguished from the rainfall-on model. Chapter 3 explores the computation of P_C.

CHAPTER 3

3 Range of Applicability

The Mixture model gives the correct probabilities of the annual maximum flood when the rainfall and snowmelt series are independent, because that is what it assumes. When the two series are correlated, the mixture model may be inaccurate, when the correlation ρ between the log-space populations is large. However, as shown in Figure 12, there is a critical probability P_C, dependent upon the values of the parameters, above which the Mixture Model is still accurate regardless of the value of ρ because in this example, the rainfall risk dominates snowmelt floods above that frequency. In fact, the mixture model in these cases can be equivalent to a model that considers only the risk from rainfall floods, which in this thesis is called the "Just-Rainfall Model." This chapter uses the Mixture Model to develop a simple formula for P_c.

3.1 A Formula for P_C using the mixture model

Consider the PDF for floods that results from use of the mixture model:

$$\frac{\partial F_Q(q)}{\partial q} = \frac{\partial F_R(q) F_S(q)}{\partial q} = F_S(q)\frac{\partial F_R(q)}{\partial q} + F_R(q)\frac{\partial F_S(q)}{\partial q} = F_S(q) f_R(q) + F_R(q) f_S(q) \quad (47)$$

where $F_R(q)$ and $F_S(q)$ are the CDFs of the rainfall and snowmelt populations, which are defined in equation (3); $f_R(q)$ and $f_S(q)$ are the PDFs of the rainfall and snowmelt floods, which are defined in equation (10) and (11). Equation (47) indicates that the upper tail of the mixed population, describing the risk of flooding at level q, is the weighted average of the rainfall risk $f_R(q)$ and the snowmelt risk $fs(q)$, where the CDFs are the weights. Thus the critical probability P_C describes the point beyond which $F_S(q) f_R(q) \gg F_R(q) f_S(q)$. If beyond

$q,\ F_S(q) f_R(q) \gg F_R(q) f_S(q)$, then the rainfall risk is much greater than the snowmelt risk at and beyond q, . Using a 2-parameter LN model for rainfall,

$$p = F_Q(q_p) = F_R(q_p) F_S(q_p) = \Phi\left[\frac{\ln(q_p) - \mu_{RL}}{\sigma_{RL}}\right] F_S(q_p) \tag{48}$$

Thus

$$\Phi\left[\frac{\ln(q_p) - \mu_{RL}}{\sigma_{RL}}\right] = \frac{F_Q(q_p)}{F_S(q_p)} = \frac{p}{F_S(q_p)} \tag{49}$$

$$\frac{\ln(q_p) - \mu_{RL}}{\sigma_{RL}} = \Phi^{-1}\left[\frac{p}{F_S(q_p)}\right] \tag{50}$$

$$\ln(q_p) = \mu_{RL} + \Phi^{-1}\left[\frac{p}{F_S(q_p)}\right]\sigma_{RL} \approx \mu_{RL} + \Phi^{-1}(p)\sigma_{RL} \tag{51}$$

Equation (51) assumes that when rainfall is the dominant risk for flows great then q, which means we can assume $F_S(q) \approx 1$. This is generally the case when rainfall is the dominant risk of flooding, and the rainfall distribution has a thicker more extreme upper tail than the snowmelt distribution. In these case, the Just-Rainfall model will also be attractive because the mixed population is dominated by rainfall events in the upper tail, which is when $F_S(q_p)$ is almost 1. Use of a Just-Rainfall Method is illustrated by Lamontagne et al. (2012).

For our case, we define the critical non-exceedance probability $P_C = F_R(q) F_S(q)$ to be the point where $F_S(q) f_R(q) = 10 F_R(q) f_S(q)$. Here a factor of 10 seems sufficient in a practical definition of dominance. The next section explores the behavior of PC for different sets of model parameters.

3.2 Value of P_C with different parameters

To explore the behavior of P_C, this section considers a range of cases that have rainfall and snowmelt distributions with different parameters. To define different cases, we fix the mean and variance of the rainfall population R, and the coefficient of variation of the snowmelt population $CV=\sigma_S/\mu_S$. The median ratio $M_{R/S}$ was assigned values between 0 to 4 to generate different values of the mean and variance of the snowmelt population. For a LN2 distribution, $Med[R]=exp(\mu_{RL})$, $Med[S]=exp(\mu_{SL})$. In Table 2, and $M_{R/S}=1.07$. Changing $M_{R/S}$ changes the probability P_R that the annual maximum is a rainfall event.

Figure 13 displays the relationship between P_R and the median ratio $M_{R/S}$ and correlation ρ between the rainfall and snowmelt maximum series. P_R is least sensitive to $M_{R/S}$ when $\rho = 0$, and most sensitive with $\rho = 0.9$. For a high correlation, P_R is simply the point where the two CDFs cross, whereas for low cross-correlation, P_R depends upon the point where the two CDFs cross and the variance of each series describing the probability that one by chance happens to be larger than the other.

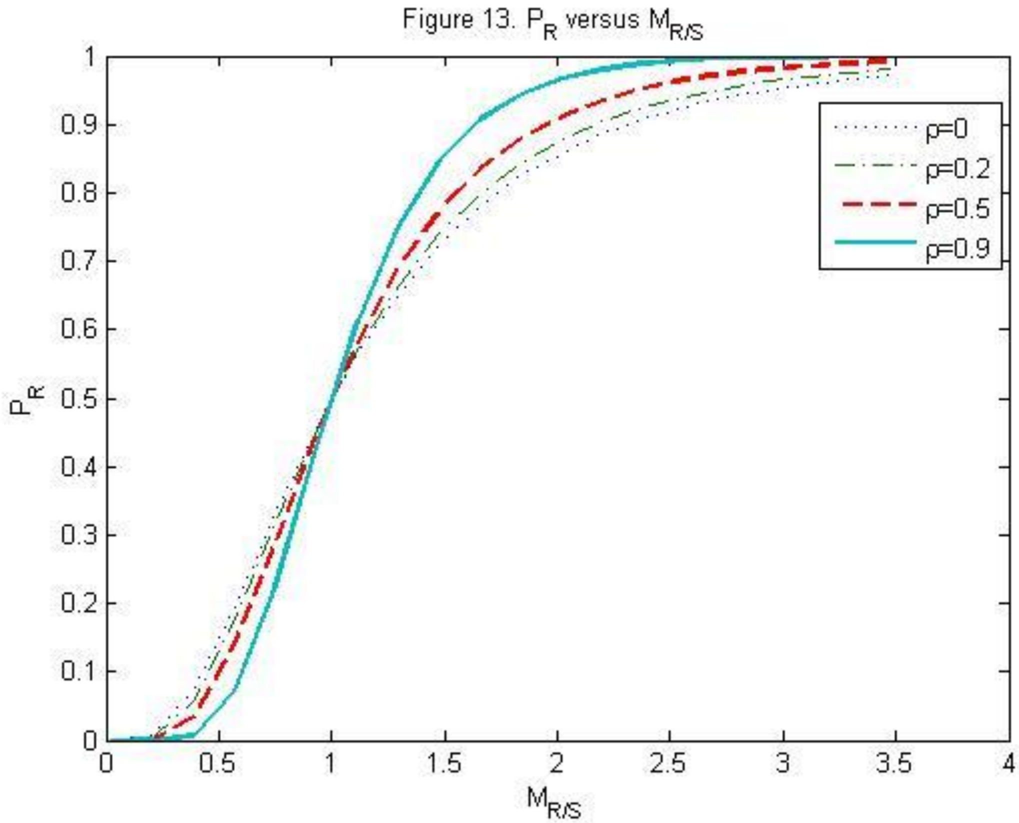
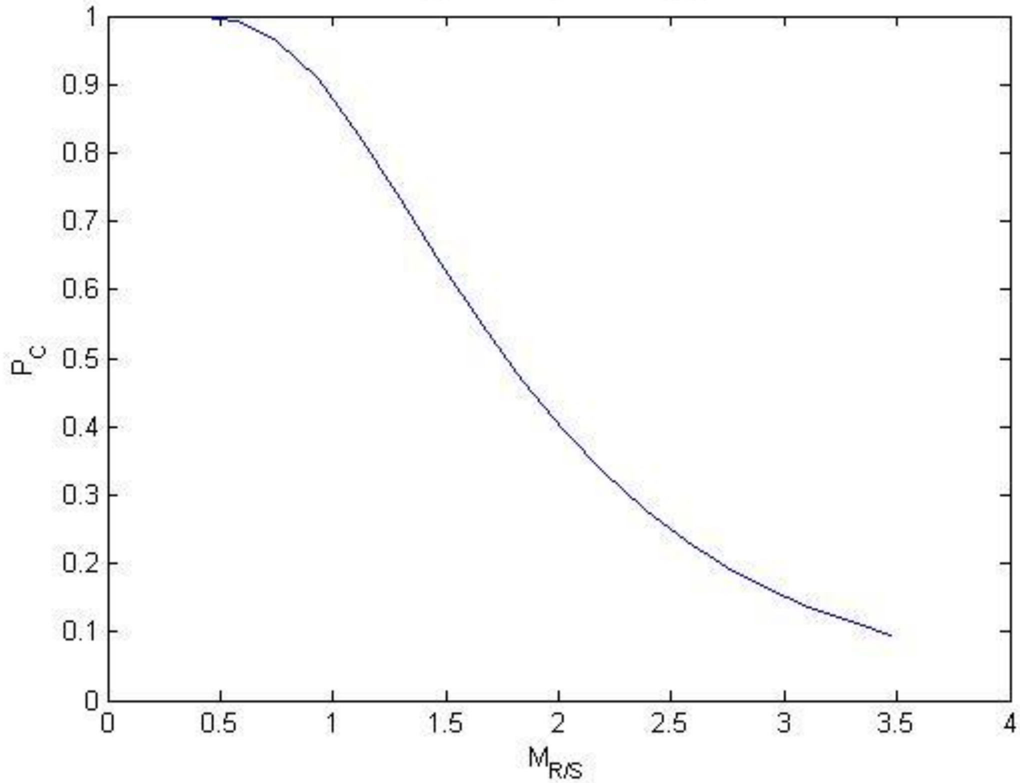


Figure 13 shows that both ρ and $M_{R/S}$ influence P_R. In our lognormal case, when $M_{R/S}$=1, $\mu_{RL}=\mu_{SL}$ and $P_R=P_S=0.5$ for all ρ; this can be understood by thinking about the values in log space where the difference log(R) – log(S) would have a normal distribution with zero mean. Thus the probability that R > S will be 50% for any cross-correlation and different values of the variance of each series.

We identified the points P_C where $F_S(q)\, f_R(q) = 10 F_R(q)\, f_S(q)$ for different cases numerically using the secant method (see in Section 2.3.3). Figure 14 displays P_C as a function of $M_{R/S}$. The correlation ρ between the log-space rainfall and snowmelt populations is not considered in the definition of P_C. It should not be necessary to include ρ, because beyond a cumulative probability corresponding to P_C, the rainfall series should dominate the flood risk, regardless of the value of ρ.

Figure 14. P_C versus $M_{R/S}$



Values of P_C describe the CDF value below which the Mixture Method or Just-Rainfall Model may provide invalid results. As shown in Figure 12abc, if P_C were defined using the Joint Model, the largest values of P_C for $\rho \geq 0$, would be obtained with $\rho = 0$. Thus use of the mixture model (Joint model with $\rho = 0$) is conservative in that it defines a smaller interval (P_C to 1) within which the mixture or just-rainfall model is expected to be accurate.

Although the Joint Distribution Method is theoretically correct, when the parameters must be estimated from small samples, the Mixture Method can give more accurate results than the Joint Distribution Method because the Joint Distribution Method requires estimation of an additional parameter: the cross-correlation between the two series. Cross correlation estimators are relatively inaccurate (Stedinger, 1981). And when the mixed population is completely dominated by the rainfall floods, the Just-Rainfall Model should provide the same or even better results than the Mixture Method. These issues are explored in a Monte Carlo study in Chapter 4.

CHAPTER 4

4 Monte Carlo Study

This Chapter provides an evaluation of the performances of different flood frequency estimators in small samples with a mixed population. The performance of each method is considered as a function of the correlation ρ between the maximum log-space rainfall and snowmelt events, and the distributions of the two series.

Stedinger (1980) suggests that performance criteria should reflect the impact that misspecification of those parameters might have on the planning process, its recommendations, and the social benefits achieved. Following Griffis and Stedinger (2007) who discuss different performance criteria, the Mean Squared Error $MSE[ln(\hat{q}_p)]$, for p=0.90, 0.99, is the primary statistic employed in this study, though other probabilities were examined. The MSE was computed using the formula

$$MSE[\ln(q_p)] = \frac{1}{N_{\text{Re}}}\sum_{i=1}^{N_{\text{Re}}}\left[\ln(\hat{q}_p) - \ln(q_p)\right]^2 \tag{52}$$

For a lognormal distribution , $MSE[ln(q_p)]$ is independent of μ_L (corresponding to a scaling of both flood series), and $RMSE[ln(q_p)]/\sigma_L$ would be independent of σ_L, where σ_L, is a shape parameter of the real-space lognormal distribution.

4.1 Experiment

A Monte Carlo study determined the $MSE[ln(q_p)]$ (p=0.90, 0.99) of each methods when the flood distributions have different moments ($M_{R/S}$ =1.5, 1.0, 0.75), and different ρ=0, 0.2, 0.5, 0.9; considering several sample sizes N=25, 50, 100. We also report in an appendix

MSE[ln(q_p)] (p=0.63, 0.88, 0.96, which are the corresponding critical probability when $M_{R/S}$ =1.5, 1.0, 0.75, and P_R≈0.8, 0.5, 0.3). The reason we choose those values is that by calculating *MSE[ln(q_{pc})]* when P_R≈0.8, 0.5, 0.3, we can check whether our estimator for P_C is valid.

Table 4a provides the real and log-space moments for the theoretical rainfall and snow melt distributions consider here as a function of the median ratio of the two populations distributions $M_{R/S}$ (to 4 digits).

Table 4a. Real and Log-space moments as a function of $M_{R/S}$

$M_{R/S}$	1.5		1.007		0.75	
Moments	Rainfall	Snowmelt	Rainfall	Snowmelt	Rainfall	Snowmelt
μ	198.9	114.6	198.9	172.0	198.9	229.3
σ	130.7	30.33	130.7	45.49	130.7	60.65
μ_L	5.113	4.708	5.113	5.113	5.113	5.401
σ_L	0.5991	0.2601	0.5991	0.2601	0.5991	0.2601
P_C	0.63		0.88		0.96	

Table 4b provides the calculated probability the annual maximum is a rainfall event P_R as a function of $M_{R/S}$ ($M_{R/S}$ =1.5, 1.00, 0.75, P_C=0.63, 0.88, 0.96) and ρ (ρ=0, 0.2, 0.5, 0.9).

Table 4b. P_R with respect to $M_{R/S}$ and ρ

Med[R]/Med[S]	1.5	1.00	0.75
ρ	P_R		
0	0.73	0.50	0.33
0.2	0.75	0.50	0.32
0.5	0.78	0.50	0.29
0.9	0.86	0.50	0.23

The Monte Carlo analysis considered the estimation of the model parameters with small samples. When fitting the mixture model employing two-parameter lognormal distributions for R and S, we need to estimate 4 parameters μ_{RL}, σ_{RL}, μ_{SL}, and σ_{SL}. Maximum likelihood estimators were employed corresponding to the log-space sample mean and variance. When fitting the Joint Distribution Method, we also need to estimate the cross-correlation of the logarithms ρ, so there are 5 parameters in total. Equation (53)-(54) was used to get ρ.

When fitting the Kirby Method, we fit R' with 3-parameters lognormal distribution using the quantile lower bound estimator and the real-space sample mean and variance as suggested in Stedinger (1980). See equations (34)-(35) in Chapter 2. The distribution of S' was again described with a 2-parmameter lognormal distribution using maximum likelihood estimators., P_R was estimated using the observed frequency, as in equation (27). So there are 6 parameters in total. When fitting a Single LN3 Method, we just need to fit the annual maxima with 3-pamrameter lognormal distribution, so there are 3 parameters. Again the quantile lower bound estimator was used with the real-space sample mean and variance. When fitting the Just-Rainfall Method, we need to fit the complete rainfall series with a 2-parameter lognormal distribution, so there are 2 parameters, fit using the maximum likelihood estimators.

To estimate the cross-correlation ρ between the log-space rainfall and log-space snowmelt data, where the rainfall or snowmelt series are each independent random variables, the maximum likelihood estimator in Stedinger (1980) was employed, which equals

$$\hat{\rho} = \frac{\hat{\sigma}_{RS}^2}{\hat{\sigma}_{RL}\hat{\sigma}_{SL}} \tag{53}$$

$$\hat{\sigma}_{RS} = \frac{1}{n-1}\sum_{i=1}^{n}\left[\ln(r_i) - \hat{\mu}_{RL}\right]\left[\ln(s_i) - \hat{\mu}_{SL}\right] \tag{54}$$

Equation (38)-(42) in Chapter 2 discussed how to fit 3-parameter lognormal distribution; we used the quantile lower bound estimator with the sample mean and variance of the real data. This method is applied when fitting the rainfall events that are also the annual maxima R', and the annual maximum floods. Note that when the sample size of {R'} or {S'} is smaller than 10, we substitute a 3-parameter lognormal distribution to fit the annual maxima instead of fitting them separately. This avoided the unstable results that could result from trying to describe the R' and S' distributions with too few observations.

4.2 Results

$MSE[ln(q_p)]$ (p=0.90, 0.99, and P_C) will be used to compare different methods when the real flood distributions have different rainfall maximum probabilities P_R ($P_R \approx 0.3, 0.5, 0.8$) and correlations ρ (ρ=0, 0.2, 0.5, 0.9) with different sample size N (N=25, 50, 100). The true q_p (p=0.90, 0.99, P_C) was computed using the Joint Distribution Method with the real parameters;

q_p values are reported in Table 5 (to 4 digits). One can observe that across the ρ values considered, the quantile vary very little, if at all; in particularly, for $p > P_C$, ρ should have no visible effect on q_p. That is the case for the examples in Table 5. The results of the Monte Carlo analysis are listed in Tables 6-11 and Figures 15-20.

Table 5 q_p as a function of $M_{R/S}$ and ρ

$M_{R/S}$=1.5, Pc=0.63					
ρ	$q_{0.63}$(m³/s)	$q_{0.88}$(m³/s)	$q_{0.96}$(m³/s)	$q_{0.90}$(m³/s)	$q_{0.99}$(m³/s)
0	204.7	336.1	474.5	358.3	669.9
0.2	204.1	336.1	474.5	358.3	669.9
0.5	203.2	336.1	474.4	358.3	669.9
0.9	202.8	336.1	474.5	358.3	669.9
$M_{R/S}$ =1.00, Pc=0.88					
ρ	$q_{0.63}$(m³/s)	$q_{0.88}$(m³/s)	$q_{0.96}$(m³/s)	$q_{0.90}$(m³/s)	$q_{0.99}$(m³/s)
0	229.7	338.9	474.6	359.9	669.9
0.2	226.5	338.4	474.6	359.6	669.9
0.5	220.1	337.3	474.6	358.9	669.9
0.9	206.5	336.1	474.5	358.3	669.9
$M_{R/S}$ =0.75, Pc=0.96					
ρ	$q_{0.63}$(m³/s)	$q_{0.88}$(m³/s)	$q_{0.96}$(m³/s)	$q_{0.90}$(m³/s)	$q_{0.99}$(m³/s)
0	273.5	364.7	479.4	381.2	670.1
0.2	269.6	362.6	479.0	379.5	670.1
0.5	262.0	357.2	477.5	374.7	670.0
0.9	246.4	341.3	474.6	361.0	669.9

Table 6 and Figure 15 report the value of $MSE[ln(q_{0.99})]$ of each of five method: mixture using two LN2 distributions, the joint distribution using two LN2 distributions and their cross-correlation, the Kirby method using an LN3 distribution to describe rainfall, a single LN3 distribution for the annual maximums, and a Just-rainfall LN2 distribution. Because we use a 3-parameter lognormal distribution to fit the annual maxima instead of fitting R' and S' separately when the sample size of {R'} or {S'} is smaller than 10, we don't report a *MSE[ln(* $q_{0.99}$ *)]* of the Kirby method when N=25 in Figure 15a.

Table 6 MSE[ln($q_{0.99}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0547	0.0544		0.0742	0.0547
0.2	0.0542	0.0557		0.0708	0.0542
0.5	0.0554	0.0532		0.0699	0.0554
0.9	0.0553	0.0539		0.0627	0.0553
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0541	0.0543		0.0869	0.0549
0.2	0.0524	0.0531		0.0823	0.0531
0.5	0.0540	0.0556		0.0821	0.0544
0.9	0.0535	0.0539		0.0785	0.0535
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0485	0.0476		0.0760	0.0548
0.2	0.0497	0.0475		0.0747	0.0554
0.5	0.0481	0.0505		0.0750	0.0519
0.9	0.0543	0.0537		0.0923	0.0553
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0269	0.0275	0.0435	0.0425	0.0269
0.2	0.0264	0.0266	0.0408	0.0397	0.0264

0.5	0.0270	0.0268	0.0410	0.0394	0.0270
0.9	0.0273	0.0276	0.0374	0.0362	0.0273
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0267	0.0267	0.0466	0.0504	0.0268
0.2	0.0268	0.0262	0.0465	0.0497	0.0268
0.5	0.0261	0.0268	0.0464	0.0483	0.0262
0.9	0.0274	0.0265	0.0500	0.0456	0.0274
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0247	0.0249	0.0431	0.0462	0.0261
0.2	0.0253	0.0259	0.0446	0.0474	0.0268
0.5	0.0260	0.0257	0.0452	0.0477	0.0270
0.9	0.0269	0.0265	0.0562	0.0552	0.0271
n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0131	0.0133	0.0241	0.0227	0.0131
0.2	0.0130	0.0132	0.0237	0.0223	0.0130
0.5	0.0136	0.0129	0.0240	0.0223	0.0136
0.9	0.0130	0.0134	0.0229	0.0205	0.0130
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0134	0.0133	0.0270	0.0293	0.0134
0.2	0.0133	0.0133	0.0267	0.0294	0.0133
0.5	0.0133	0.0139	0.0272	0.0282	0.0133
0.9	0.0134	0.0134	0.0278	0.0252	0.0134
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0132	0.0128	0.0268	0.0301	0.0137
0.2	0.0128	0.0128	0.0261	0.0297	0.0132
0.5	0.0129	0.0133	0.0266	0.0297	0.0131
0.9	0.0136	0.0136	0.0303	0.0344	0.0137

*Table 6 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

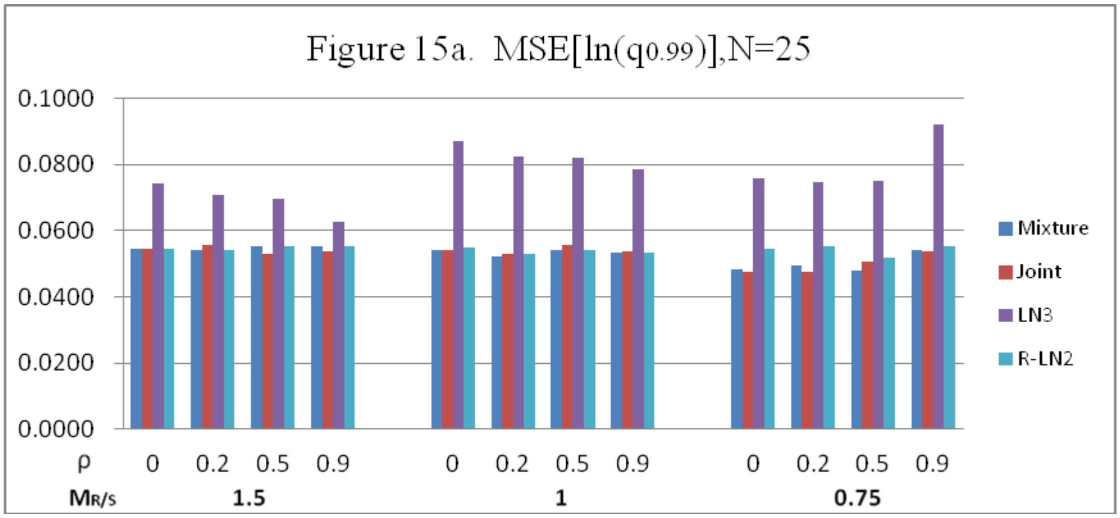


Figure 15a. MSE[ln(q0.99)],N=25
0.1000
0.0800
0.0600
0.0400
0.0200
0.0000
ρ
0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
LN3
R-LN2

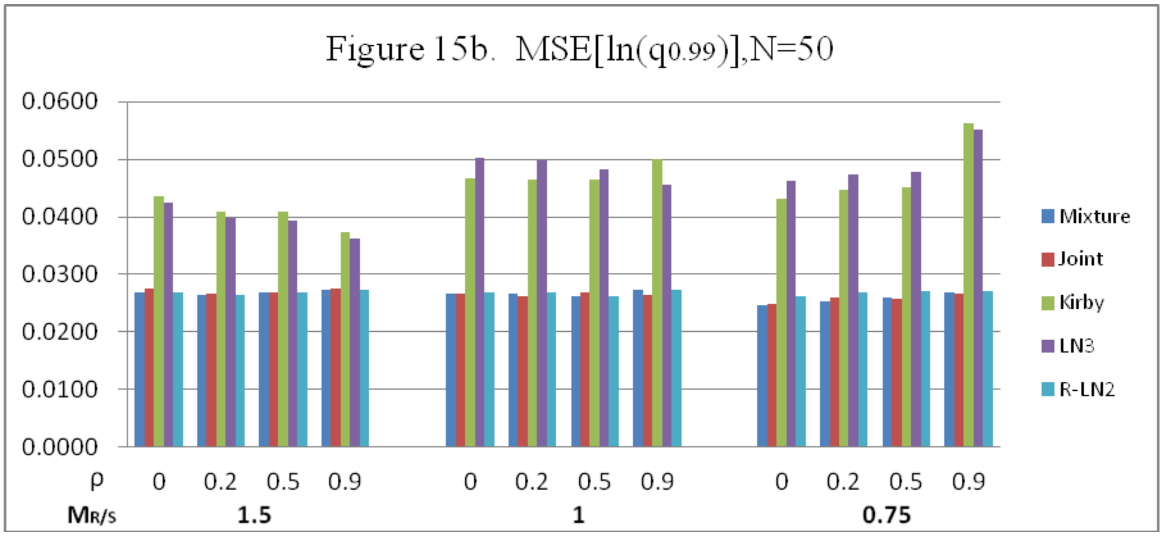


Figure 15b. MSE[ln(q0.99)],N=50
0.0600
0.0500
0.0400
0.0300
0.0200
0.0100
0.0000
ρ
0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
Kirby
LN3
R-LN2

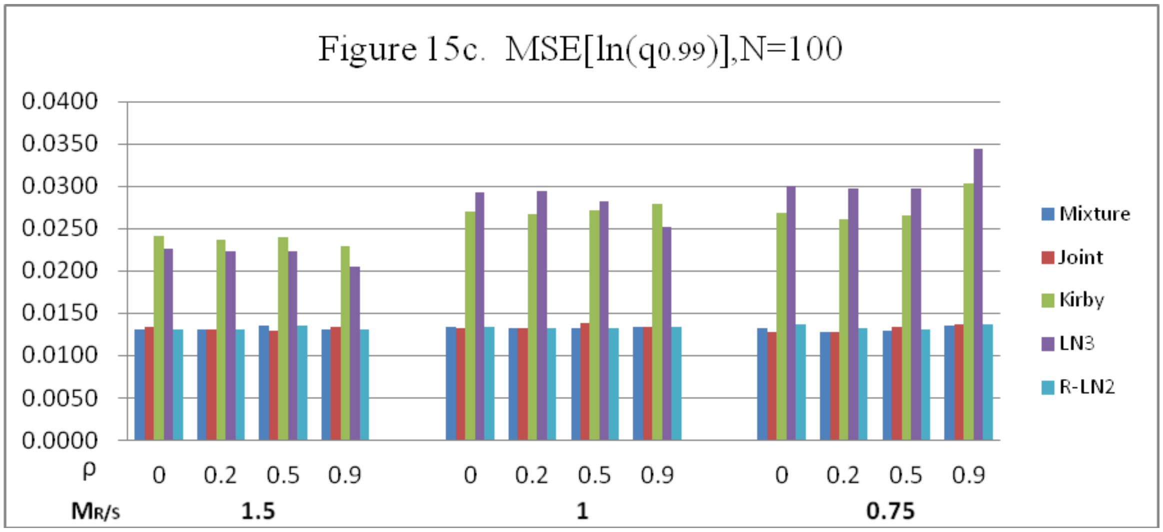


Figure 15c. MSE[ln(q0.99)],N=100
0.0400
0.0350
0.0300
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
ρ
0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
Kirby
LN3
R-LN2

Figure 15 shows that for p = 99%, the performance of the Mixture Method, Joint Distribution Method, and Just-Rainfall Method are almost identical, and they produce the most accurate estimators. When estimating really large quantiles, the simple Mixture Method or Just-Rainfall Method would be good choice because snowmelt distribution does not affect the quantile estimator.

We don't report $MSE[ln(q_{0.99})]$ of the Kirby method when N=25 in Figure 15a. However, in Figure 15b and 15c the LN3 and Kirby Methods do not do as well as the Mixture, Joint, or Just-Rainfall Method. When $M_{R/S}$ = 1.5, the single LN3 Method does a little better than the Kirby method; as shown in figure 12a, this is a situation where in the rainfall distribution dominates flood risk over most of the range of likely values, and in particular the large floods. However, for $M_{R/S}$ = 1 and 0.75, the Kirby method is a little better. When $M_{R/S}$ = 0.75, the annual maximum distribution shown in Figure 12c is mostly snowmelt events, with just the upper tail defined by the rainfall flood risk. As a result, the annual maximum series has a very large positive skew in real and in log space, making quantile estimates for the upper tail highly unreliable. A more physical representation of that argument is this. $M_{R/S}$= 0.75 illustrates the kind of situation wherein use of separate models of the two populations is highly advantageous because most of the annual maximum flood events are snowmelt floods, whereas the largest floods are rainfall events. Thus a model that separates the two phenomena, such as the Kirby method, can better resolve the rainfall flood risk and thus provide a more accurate estimate of upper quantiles, even though it uses more parameters with the same number of data points employed with the single LN3 distribution. The differences are modest, but important to understand. OF course, if one has the whole maximum series for the rainfall, the Just-Rainfall estimator does better because it has more information on the rainfall distribution.

For N=50 and 100, when $M_{R/S}$=1.5, the Kirby Method works worse than a Single LN3 Method; when $M_{R/S}$=1, the Kirby Method works a little better than a Single LN3 Method except for ρ=0.9; when $M_{R/S}$=0.75, the Kirby Method generally works better than a Single LN3 Method. Although the Kirby Method has 6 parameters to fit while a Single LN3 Method has only 3, the Kirby Method may work better because the conditional distribution for R' has a smaller skewness than the distribution of the annual maxima, which has a large positive skew. Thus to determine which of the two is more accurate requires knowing the parameters of the problem.

As one would expect, all of the methods work better as the sample size N increases.

Table 7 reports, and figure 16 displays the $MSE[ln(\hat{q}_{0.90})]$ of the five estimators for different $M_{R/S}$, ρ and sample size N. Again we don't report $MSE[ln(\hat{q}_{0.9})]$ of the Kirby method when N=25.

Table 7. MSE[ln($\hat{q}_{0.90}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0265	0.0263		0.0288	0.0267
0.2	0.0262	0.0268		0.0280	0.0264
0.5	0.0266	0.0258		0.0289	0.0267

0.9	0.0267	0.0261		0.0289	0.0267
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0221	0.0220		0.0214	0.0273
0.2	0.0214	0.0225		0.0208	0.0258
0.5	0.0228	0.0244		0.0227	0.0261
0.9	0.0247	0.0262		0.0276	0.0263
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0129	0.0128		0.0157	0.0313
0.2	0.0139	0.0128		0.0159	0.0314
0.5	0.0149	0.0147		0.0158	0.0280
0.9	0.0213	0.0198		0.0188	0.0268
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0130	0.0134	0.0142	0.0148	0.0131
0.2	0.0127	0.0130	0.0136	0.0143	0.0128
0.5	0.0133	0.0130	0.0143	0.0151	0.0133
0.9	0.0133	0.0135	0.0147	0.0150	0.0134
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0113	0.0112	0.0124	0.0114	0.0131
0.2	0.0117	0.0113	0.0127	0.0117	0.0133
0.5	0.0115	0.0124	0.0130	0.0120	0.0128
0.9	0.0126	0.0129	0.0144	0.0148	0.0133
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0061	0.0062	0.0073	0.0093	0.0167
0.2	0.0066	0.0067	0.0079	0.0096	0.0168
0.5	0.0076	0.0075	0.0086	0.0097	0.0156
0.9	0.0123	0.0102	0.0116	0.0112	0.0136
n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				

0	0.0064	0.0065	0.0071	0.0077	0.0064
0.2	0.0064	0.0064	0.0070	0.0076	0.0064
0.5	0.0067	0.0063	0.0073	0.0080	0.0067
0.9	0.0064	0.0066	0.0075	0.0078	0.0064
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0056	0.0057	0.0063	0.0061	0.0065
0.2	0.0057	0.0059	0.0064	0.0061	0.0066
0.5	0.0059	0.0065	0.0068	0.0063	0.0065
0.9	0.0063	0.0066	0.0071	0.0075	0.0066
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0031	0.0030	0.0037	0.0063	0.0106
0.2	0.0032	0.0032	0.0040	0.0064	0.0100
0.5	0.0040	0.0037	0.0045	0.0064	0.0086
0.9	0.0075	0.0053	0.0066	0.0070	0.0068

*Table 7 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

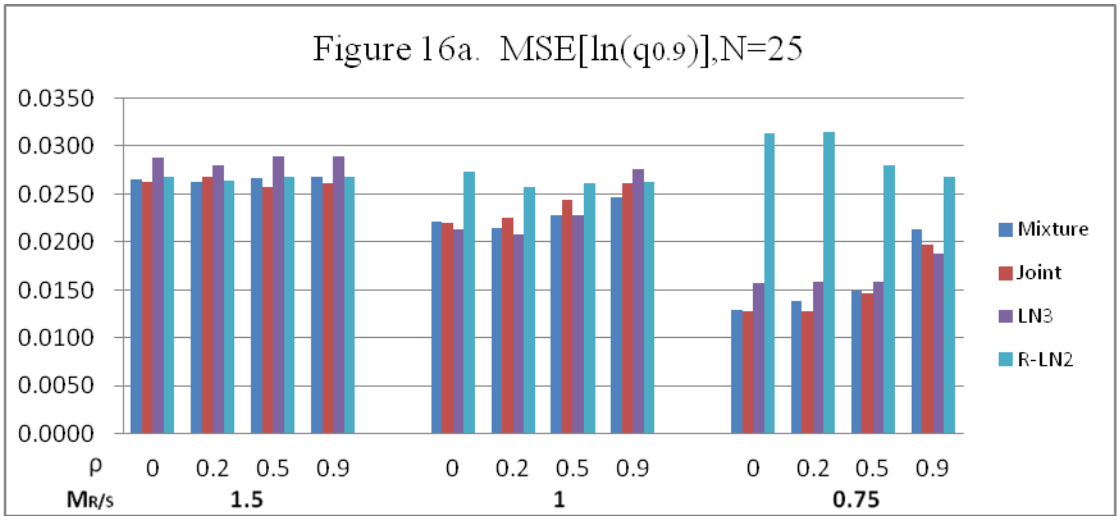


Figure 16a. MSE[ln(q0.9)],N=25
0.0350
0.0300
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
ρ 0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
LN3
R-LN2

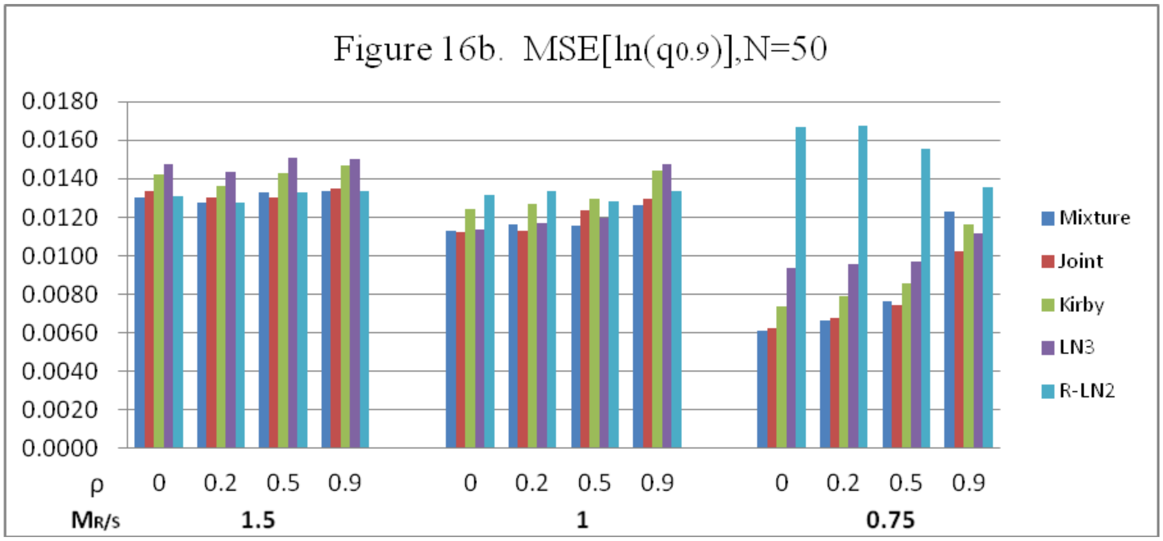


Figure 16b. MSE[ln(q0.9)],N=50
0.0180
0.0160
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
ρ 0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
Kirby
LN3
R-LN2

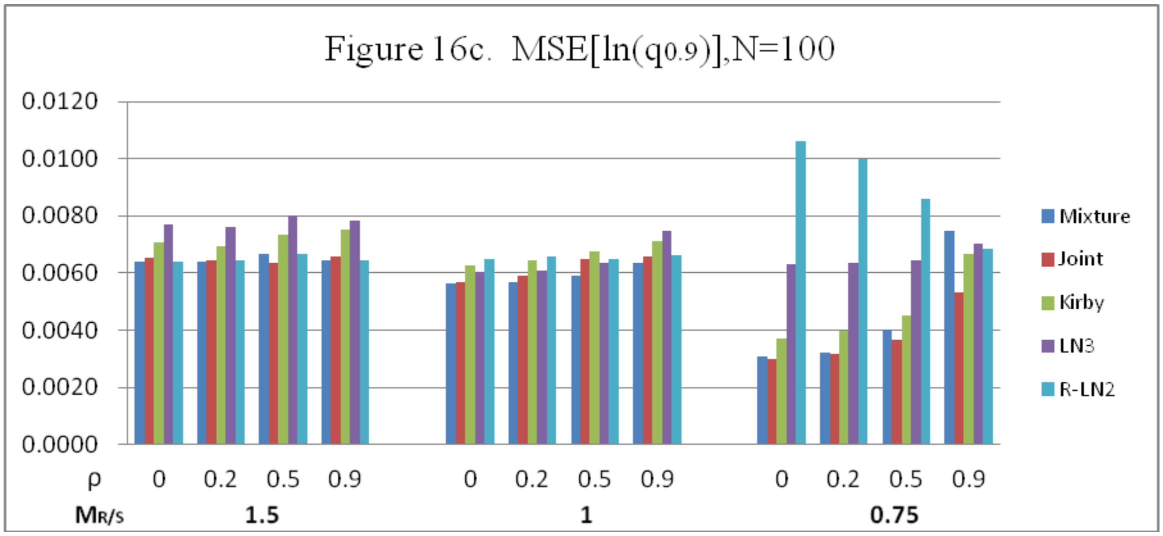


Figure 16c. MSE[ln(q0.9)],N=100
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
ρ 0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
Kirby
LN3
R-LN2

Figure 16 shows that when $M_{R/S}$=1.5, P_R≈0.8, P_C=0.63<0.9, the influence of snowmelt floods on $q_{0.9}$ would be invisible, thus the Mixture Method, Joint Distribution Method, and Just-Rainfall Method work almost identical with each other, and they all work well. For N=50 and 100, the Kirby Method works better than a Single LN3 Method, but not as well as the other 3 methods.

When $M_{R/S}$=1, P_R=0.5, P_C=0.88≈0.9, the influence of snowmelt floods on $q_{0.9}$ becomes visible. *MSE[ln($q_{0.9}$)]* for the Mixture Method and the Joint Distribution Method increases with ρ.. Except for ρ = 0.9, the Just-Rainfall Method doesn't work as well as the Mixture and Joint Distribution Method, particularly for small N. For N=50 and 100, a Single LN3 Method works a little better than the Kirby Method except for ρ=0.9, but not as well as the Mixture Method or the Joint Distribution Method.

When $M_{R/S}$=0.75, P_R≈0.3, P_C=0.96>0.9, the influence of snowmelt floods on $q_{0.9}$ is important. Surprisingly the Mixture method is a little better than the Joint Distribution Method except for ρ =0.9. The Just-Rainfall Method does poorly in this case. The Single LN3 Method does poorly, as expected. The Kirby Method does surprisingly well, and not much worse than the Joint method for ρ ≤ 0.5, even though it uses less data.

To better understand the results, we also provide the Bias and Variance of each method when estimating $q_{0.99}$ and $q_{0.9}$, where

$$Bias[\ln(q_p)] = \frac{1}{N_{\mathrm{Re}}}\sum_{i=1}^{N}\left[\ln(q_p) - \ln(q_p)\right] \quad (55)$$

$$MSE = Variance + Bias^2 \quad (56)$$

Table 8 and Figure 17 report *Bias[ln($q_{0.99}$)]* of each method.

Table 8 Bias[ln($q_{0.99}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0160	-0.0144		-0.0744	-0.0160
0.2	-0.0112	-0.0138		-0.0590	-0.0113
0.5	-0.0174	-0.0166		-0.0567	-0.0174
0.9	-0.0182	-0.0163		-0.0381	-0.0182
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	-0.0198	-0.0138		-0.1175	-0.0207
0.2	-0.0156	-0.0115		-0.1114	-0.0163
0.5	-0.0129	-0.0127		-0.1019	-0.0132
0.9	-0.0183	-0.0186		-0.0902	-0.0183
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	-0.0035	-0.0077		-0.1152	-0.0128
0.2	-0.0049	-0.0106		-0.1104	-0.0139
0.5	-0.0068	-0.0077		-0.1149	-0.0128
0.9	-0.0106	-0.0170		-0.1326	-0.0122
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0056	-0.0068	-0.0409	-0.0391	-0.0056
0.2	-0.0088	-0.0073	-0.0420	-0.0392	-0.0088
0.5	-0.0056	-0.0062	-0.0350	-0.0296	-0.0056
0.9	-0.0099	-0.0088	-0.0239	-0.0192	-0.0099
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	-0.0037	-0.0096	-0.0533	-0.0805	-0.0038
0.2	-0.0031	-0.0071	-0.0469	-0.0721	-0.0032
0.5	-0.0073	-0.0068	-0.0548	-0.0733	-0.0073
0.9	-0.0118	-0.0062	-0.0727	-0.0623	-0.0118

	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	-0.0007	-0.0010	-0.0485	-0.0901	-0.0039
0.2	-0.0022	-0.0048	-0.0468	-0.0891	-0.0054
0.5	-0.0082	-0.0037	-0.0526	-0.0933	-0.0106
0.9	-0.0047	-0.0091	-0.0872	-0.1027	-0.0054

n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0067	-0.0029	-0.0288	-0.0250	-0.0067
0.2	-0.0043	-0.0032	-0.0277	-0.0222	-0.0043
0.5	-0.0041	-0.0056	-0.0258	-0.0166	-0.0041
0.9	-0.0048	-0.0028	-0.0178	-0.0044	-0.0048
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	-0.0042	-0.0027	-0.0370	-0.0642	-0.0043
0.2	-0.0074	-0.0029	-0.0423	-0.0684	-0.0074
0.5	-0.0034	-0.0001	-0.0370	-0.0554	-0.0034
0.9	-0.0029	-0.0052	-0.0367	-0.0324	-0.0029
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	-0.0022	-0.0012	-0.0351	-0.0789	-0.0036
0.2	-0.0031	-0.0028	-0.0307	-0.0764	-0.0044
0.5	-0.0020	-0.0053	-0.0281	-0.0745	-0.0029
0.9	-0.0043	-0.0029	-0.0468	-0.0867	-0.0047

*Table 8 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

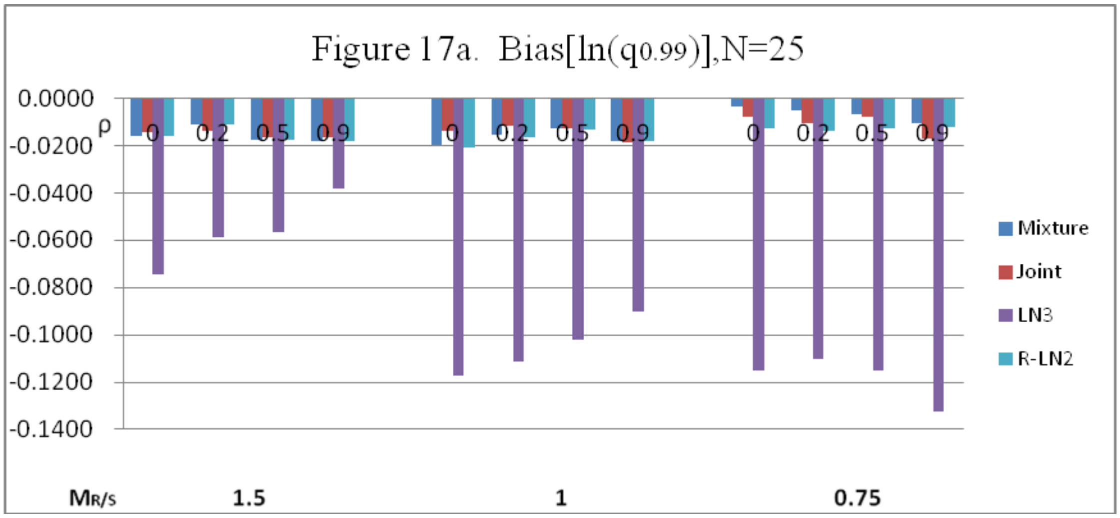


Figure 17a. Bias[ln(q0.99)],N=25
0.0000
-0.0200
-0.0400
-0.0600
-0.0800
-0.1000
-0.1200
-0.1400
ρ
0
0.2
0.5
0.9
Mixture
Joint
LN3
R-LN2
MR/S
1.5
1
0.75

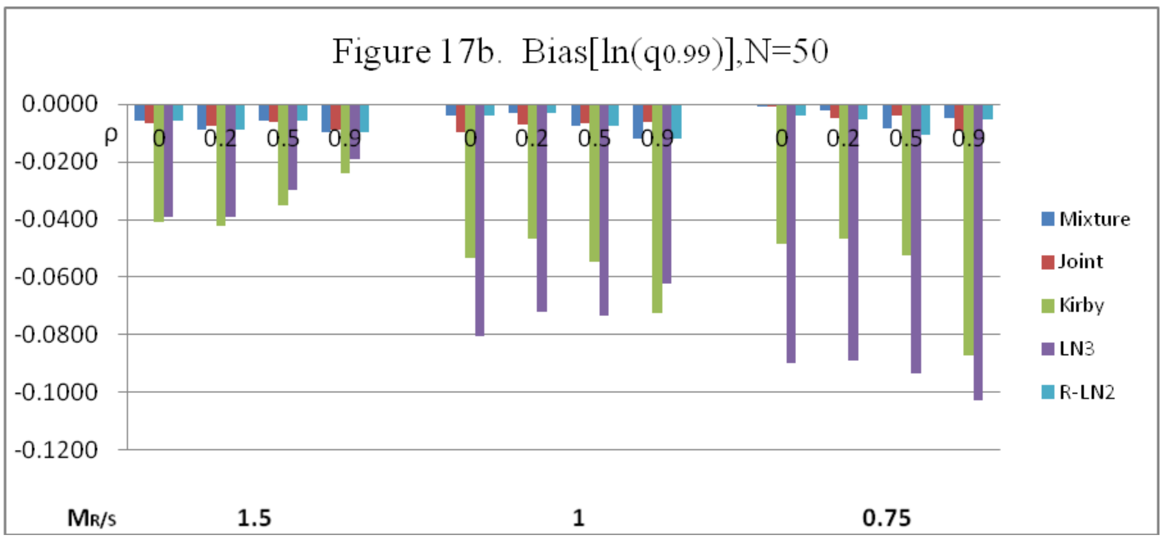


Figure 17b. Bias[ln(q0.99)],N=50
0.0000
-0.0200
-0.0400
-0.0600
-0.0800
-0.1000
-0.1200
ρ
0
0.2
0.5
0.9
Mixture
Joint
Kirby
LN3
R-LN2
MR/S
1.5
1
0.75

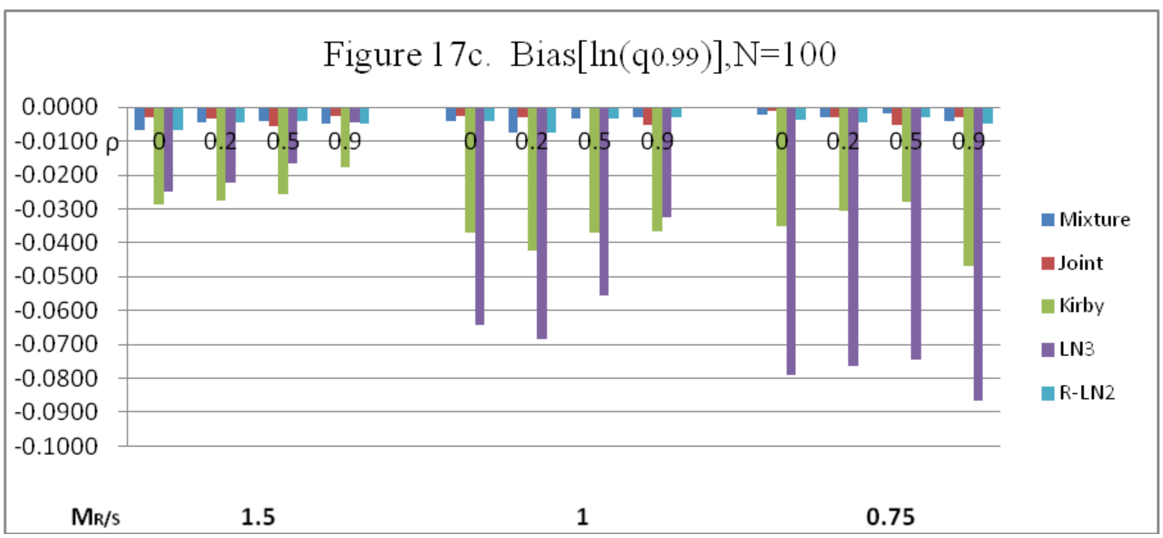


Figure 17c. Bias[ln(q0.99)],N=100
0.0000
-0.0100
-0.0200
-0.0300
-0.0400
-0.0500
-0.0600
-0.0700
-0.0800
-0.0900
-0.1000
ρ
0
0.2
0.5
0.9
Mixture
Joint
Kirby
LN3
R-LN2
MR/S
1.5
1
0.75

Figure 17 show that the Mixture, Joint, and Just-Rainfall Method yield much smaller Bias than the Kirby Method and a Single LN3 Method. Perhaps this should have been expected because fitting 3-parameter LN distribution to a highly skewed distribution is most likely to results in quantile estimators in at least modest biases. Mixture, Joint and Just-rainfall fit 2-parameter lognormal distributions which yield quantile estimators with much less if any appreciable bias (in log-space).

For N=50 and 100, when $M_{R/S}$=1.5, the Kirby Method yields larger Bias than a Single LN3 Method, it also has larger MSE than a Single LN3 Method; when $M_{R/S}$=1, the Kirby Method yields smaller Bias than a Single LN3 Method except for ρ=0.9, it also has smaller MSE than a Single LN3 Method except for ρ=0.9; when $M_{R/S}$=0.75, the Kirby Method yields smaller Bias than a Single LN3 Method, it also has smaller MSE than a Single LN3 Method most of the time.

Table 9 and Figure 18 report *Var[ln(*$q_{0.99}$*)]* of each method.

Table 9 Var[ln($q_{0.99}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0545	0.0542		0.0687	0.0545
0.2	0.0541	0.0555		0.0674	0.0541
0.5	0.0551	0.0529		0.0666	0.0551
0.9	0.0550	0.0536		0.0612	0.0550
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0537	0.0541		0.0731	0.0545
0.2	0.0522	0.0530		0.0699	0.0528

0.5	0.0539	0.0554		0.0717	0.0542
0.9	0.0532	0.0535		0.0704	0.0532
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0485	0.0476		0.0627	0.0546
0.2	0.0497	0.0474		0.0625	0.0552
0.5	0.0480	0.0505		0.0618	0.0517
0.9	0.0542	0.0534		0.0747	0.0552
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0269	0.0274	0.0418	0.0410	0.0269
0.2	0.0263	0.0265	0.0391	0.0382	0.0263
0.5	0.0269	0.0267	0.0398	0.0385	0.0269
0.9	0.0272	0.0275	0.0368	0.0358	0.0272
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0267	0.0266	0.0438	0.0439	0.0268
0.2	0.0267	0.0262	0.0443	0.0445	0.0268
0.5	0.0261	0.0267	0.0434	0.0429	0.0261
0.9	0.0272	0.0265	0.0447	0.0417	0.0272
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0247	0.0249	0.0408	0.0381	0.0261
0.2	0.0253	0.0259	0.0424	0.0395	0.0268
0.5	0.0259	0.0257	0.0424	0.0390	0.0269
0.9	0.0269	0.0265	0.0486	0.0447	0.0271
n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0131	0.0133	0.0233	0.0221	0.0131
0.2	0.0130	0.0131	0.0229	0.0218	0.0130
0.5	0.0135	0.0129	0.0234	0.0220	0.0135
0.9	0.0130	0.0134	0.0225	0.0205	0.0130

	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0133	0.0133	0.0256	0.0252	0.0134
0.2	0.0132	0.0133	0.0250	0.0247	0.0132
0.5	0.0133	0.0139	0.0258	0.0252	0.0133
0.9	0.0134	0.0134	0.0265	0.0241	0.0134
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0132	0.0128	0.0256	0.0239	0.0137
0.2	0.0128	0.0128	0.0251	0.0239	0.0132
0.5	0.0129	0.0133	0.0258	0.0241	0.0131
0.9	0.0136	0.0136	0.0281	0.0269	0.0137

*Table 9 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

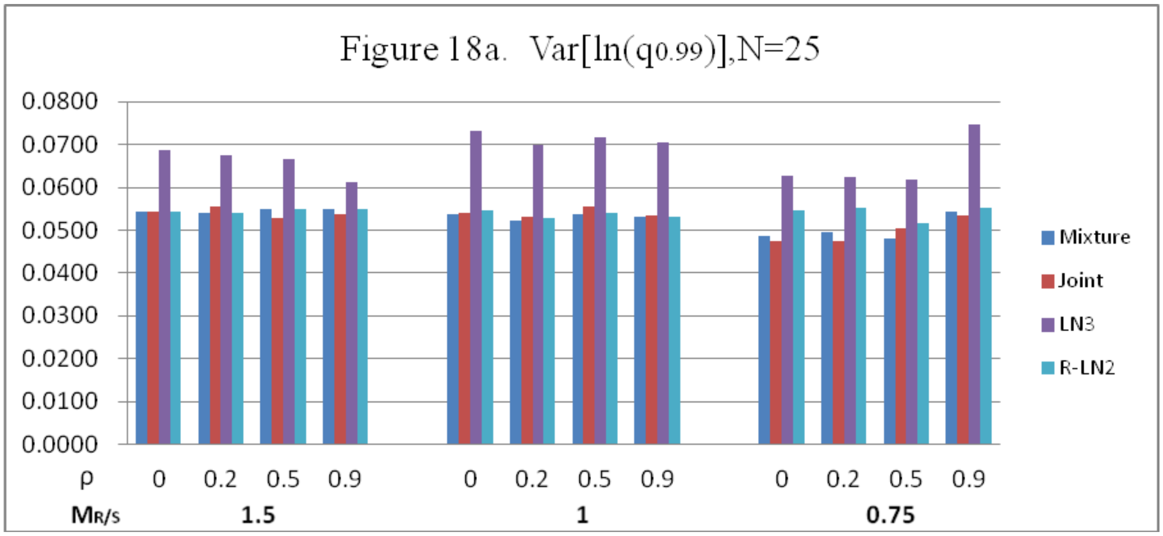


Figure 18a. Var[ln(q0.99)],N=25
0.0800
0.0700
0.0600
0.0500
0.0400
0.0300
0.0200
0.0100
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
LN3
R-LN2

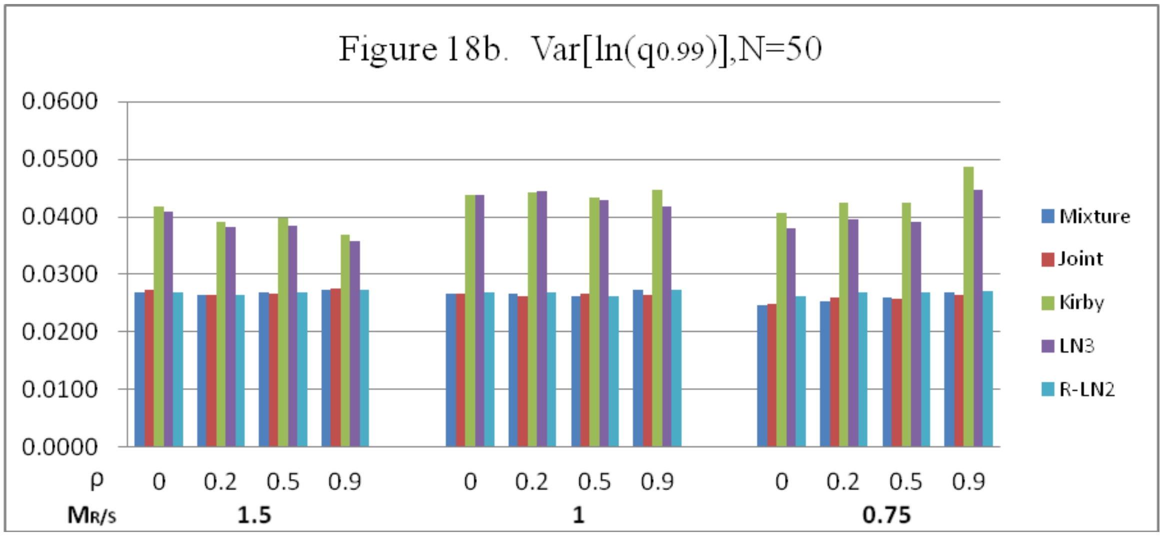


Figure 18b. Var[ln(q0.99)],N=50
0.0600
0.0500
0.0400
0.0300
0.0200
0.0100
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3
R-LN2

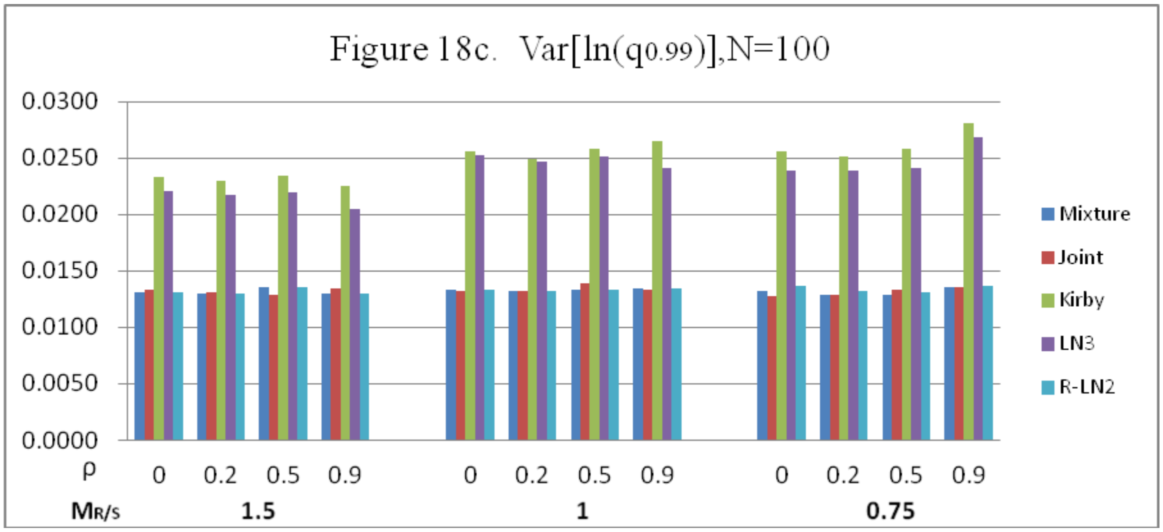


Figure 18c. Var[ln(q0.99)],N=100
0.0300
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3
R-LN2

Table 8-9 and Figure 17-18 show that the Mixture, Joint, and Just-Rainfall Method yield much smaller bias and variances than the Kirby and a Single LN3 Method. The differences between the *MSE[ln($q_{0.99}$)]* of the Kirby Method and a Single LN3 Method are mostly caused by different Bias.

Table 10 and Figure 19 show *Bias[ln(*$q_{0.90}$*)]* of each method.

Table 10 Bias[ln($q_{0.90}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0098	-0.0067		-0.0610	-0.0102
0.2	-0.0044	-0.0076		-0.0531	-0.0048
0.5	-0.0094	-0.0086		-0.0551	-0.0096
0.9	-0.0099	-0.0087		-0.0491	-0.0100
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	-0.0009	0.0015		-0.0239	-0.0166
0.2	-0.0001	0.0017		-0.0263	-0.0137
0.5	0.0026	-0.0006		-0.0336	-0.0090
0.9	-0.0030	-0.0106		-0.0638	-0.0104
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0117	0.0099		0.0141	-0.0691
0.2	0.0158	0.0067		0.0134	-0.0650
0.5	0.0245	0.0081		0.0099	-0.0509
0.9	0.0545	0.0018		0.0045	-0.0143
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0028	-0.0034	-0.0238	-0.0404	-0.0029
0.2	-0.0053	-0.0037	-0.0256	-0.0417	-0.0054
0.5	-0.0035	-0.0035	-0.0263	-0.0390	-0.0035

0.9	-0.0048	-0.0046	-0.0345	-0.0372	-0.0048
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0027	-0.0004	-0.0077	-0.0025	-0.0068
0.2	0.0043	-0.0001	-0.0060	-0.0042	-0.0046
0.5	0.0022	-0.0011	-0.0102	-0.0165	-0.0058
0.9	-0.0021	-0.0024	-0.0246	-0.0472	-0.0082
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0079	0.0069	0.0029	0.0294	-0.0628
0.2	0.0105	0.0042	0.0023	0.0266	-0.0602
0.5	0.0188	0.0043	0.0025	0.0229	-0.0506
0.9	0.0553	0.0018	0.0190	0.0221	-0.0099

n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0046	-0.0014	-0.0161	-0.0327	-0.0046
0.2	-0.0025	-0.0015	-0.0138	-0.0304	-0.0026
0.5	-0.0026	-0.0033	-0.0152	-0.0305	-0.0026
0.9	-0.0032	-0.0011	-0.0255	-0.0306	-0.0032
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0002	0.0009	-0.0014	0.0080	-0.0070
0.2	-0.0012	0.0005	-0.0036	0.0022	-0.0085
0.5	0.0028	0.0014	-0.0006	-0.0040	-0.0035
0.9	0.0040	-0.0037	-0.0078	-0.0280	-0.0012
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0036	0.0039	0.0026	0.0367	-0.0634
0.2	0.0070	0.0026	0.0041	0.0361	-0.0600
0.5	0.0193	0.0007	0.0033	0.0342	-0.0462
0.9	0.0538	0.0014	0.0122	0.0329	-0.0102

*Table 10 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

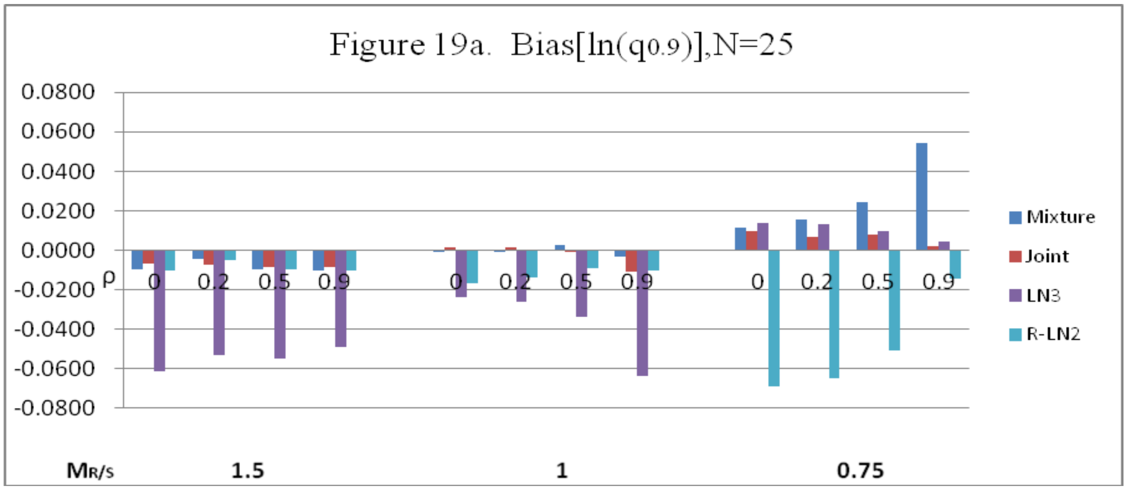


Figure 19a. Bias[ln(q0.9)],N=25
0.0800
0.0600
0.0400
0.0200
0.0000
-0.0200
-0.0400
-0.0600
-0.0800
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
LN3
R-LN2

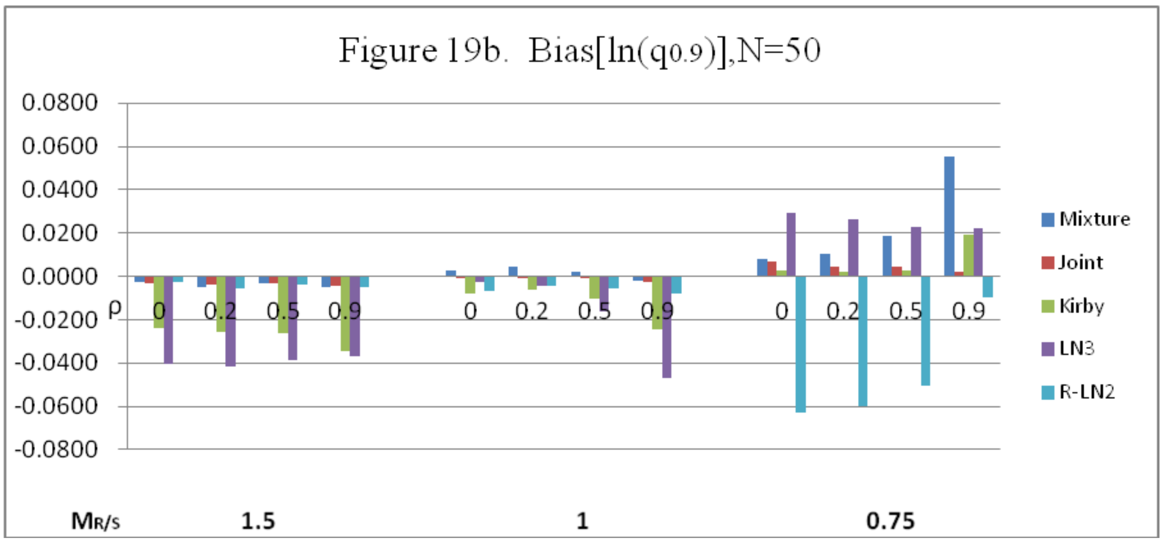


Figure 19b. Bias[ln(q0.9)],N=50
0.0800
0.0600
0.0400
0.0200
0.0000
-0.0200
-0.0400
-0.0600
-0.0800
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3
R-LN2

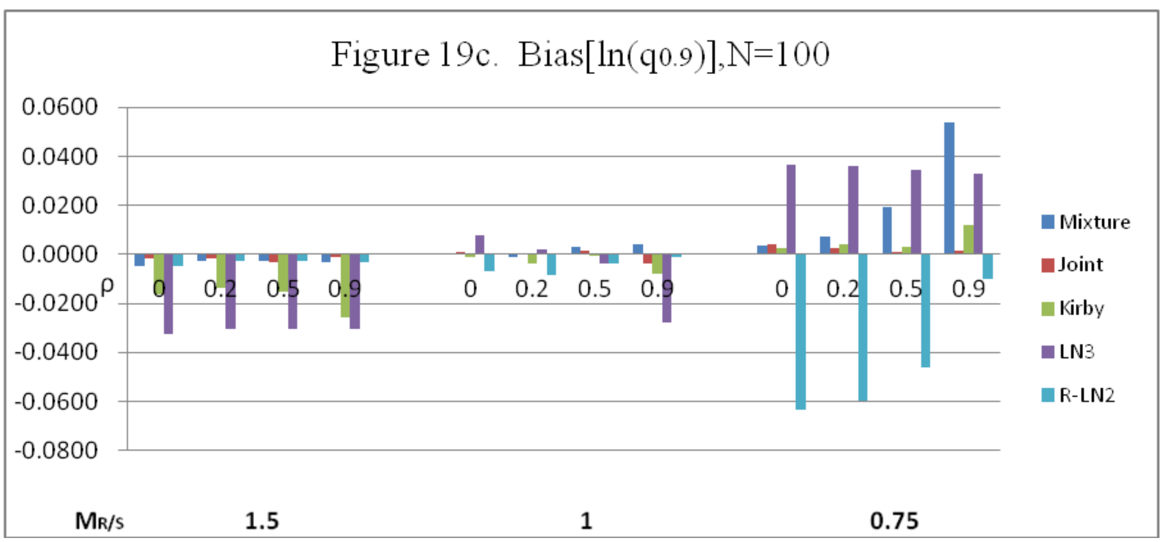


Figure 19c. Bias[ln(q0.9)],N=100
0.0600
0.0400
0.0200
0.0000
-0.0200
-0.0400
-0.0600
-0.0800
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3
R-LN2

As was expected, Figure 19 shows that when $M_{R/S}$=1.5, P_R≈0.8, P_C=0.63<0.9, the Mixture Method, Joint Distribution Method, and Just-Rainfall Method yield much smaller |Bias| than the Kirby Method and a Single LN3 Method. For N=50 and 100, the Kirby Method yields smaller Bias than a Single LN3 Method.

When $M_{R/S}$=1, P_R=0.5, P_C=0.88≈0.9, for N=50, $|Bias[ln(q_{0.9})]|$ of a Single LN3 Method is a little smaller than that of the Kirby Method when ρ<0.5. But $|Bias[ln(q_{0.9})]|$ of a Single LN3 Method is a little larger than that of the Kirby Method for N=100 for all ρ.

When $M_{R/S}$=0.75, P_R≈0.3, P_C=0.96>0.9, there is a clearly positive relationship between $Bias[ln(q_{0.9})]$ of the Mixture Method and ρ. the Joint Distribution Method yields the smallest Bias of all the methods. The Just-Rainfall Method can have large biases because it neglects the snowmelt process. For N=50 and 100, the Kirby Method yields smaller Bias than a Single LN3 Method.

Table 11 and Figure 20 shows $Var[ln(q_{0.9})]$ of each method.

Table 11 Var[ln($q_{0.9}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0264	0.0262		0.0251	0.0266
0.2	0.0262	0.0268		0.0251	0.0263
0.5	0.0266	0.0257		0.0259	0.0266
0.9	0.0266	0.0260		0.0265	0.0266
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0221	0.0220		0.0208	0.0271
0.2	0.0214	0.0225		0.0201	0.0256

0.5	0.0228	0.0244		0.0216	0.0260
0.9	0.0247	0.0261		0.0236	0.0262
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0128	0.0127		0.0155	0.0265
0.2	0.0136	0.0128		0.0157	0.0272
0.5	0.0143	0.0146		0.0157	0.0254
0.9	0.0183	0.0197		0.0188	0.0266
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0130	0.0133	0.0136	0.0131	0.0131
0.2	0.0127	0.0130	0.0129	0.0126	0.0127
0.5	0.0133	0.0130	0.0136	0.0136	0.0133
0.9	0.0133	0.0135	0.0135	0.0136	0.0133
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0113	0.0112	0.0124	0.0114	0.0131
0.2	0.0116	0.0113	0.0126	0.0117	0.0133
0.5	0.0115	0.0124	0.0129	0.0117	0.0128
0.9	0.0126	0.0129	0.0138	0.0125	0.0133
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0060	0.0062	0.0073	0.0085	0.0127
0.2	0.0065	0.0067	0.0079	0.0089	0.0131
0.5	0.0073	0.0074	0.0086	0.0092	0.0130
0.9	0.0092	0.0102	0.0113	0.0107	0.0135
n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0064	0.0065	0.0068	0.0067	0.0064
0.2	0.0064	0.0064	0.0068	0.0067	0.0064
0.5	0.0067	0.0063	0.0071	0.0071	0.0067
0.9	0.0064	0.0066	0.0069	0.0069	0.0064

	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0056	0.0057	0.0063	0.0060	0.0064
0.2	0.0057	0.0059	0.0064	0.0061	0.0065
0.5	0.0059	0.0065	0.0068	0.0063	0.0065
0.9	0.0063	0.0066	0.0071	0.0067	0.0066
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0031	0.0030	0.0037	0.0050	0.0066
0.2	0.0032	0.0032	0.0040	0.0051	0.0064
0.5	0.0036	0.0037	0.0045	0.0052	0.0065
0.9	0.0046	0.0053	0.0065	0.0059	0.0067

*Table 10 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

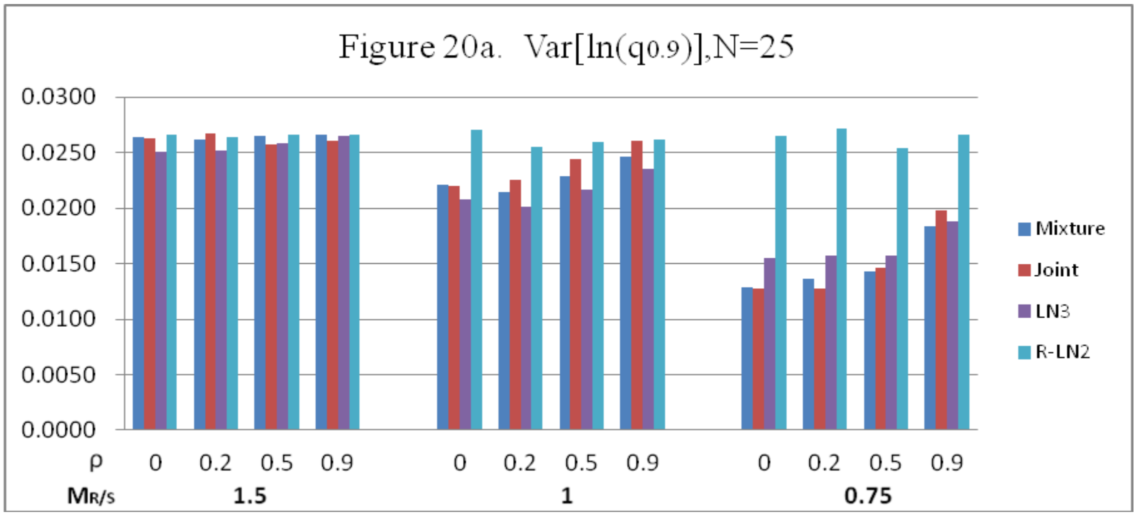


Figure 20a. Var[ln(q0.9)],N=25
0.0300
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
ρ 0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
LN3
R-LN2

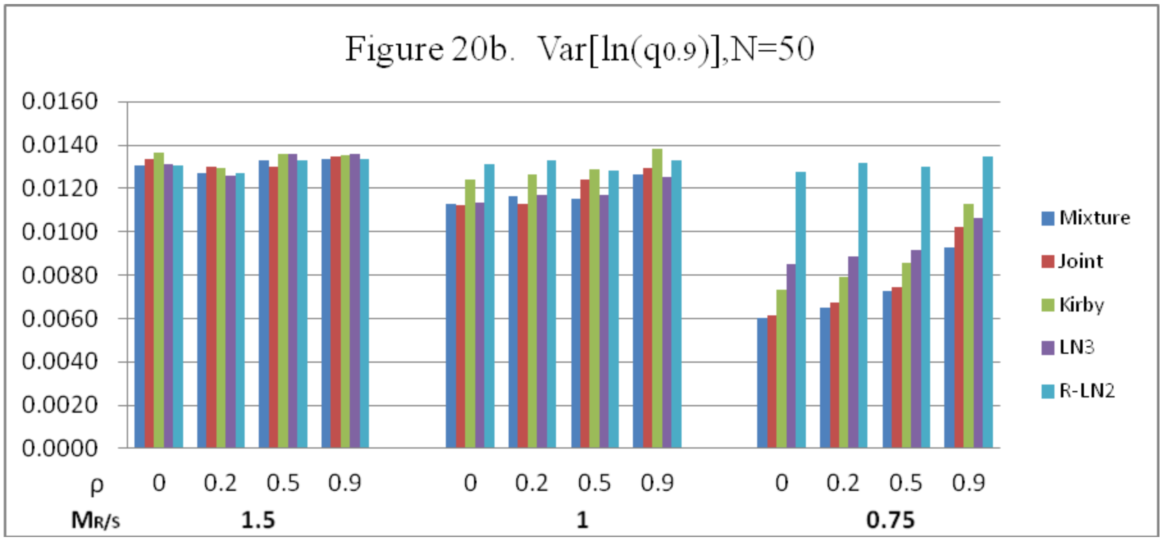


Figure 20b. Var[ln(q0.9)],N=50
0.0160
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
ρ 0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
Kirby
LN3
R-LN2

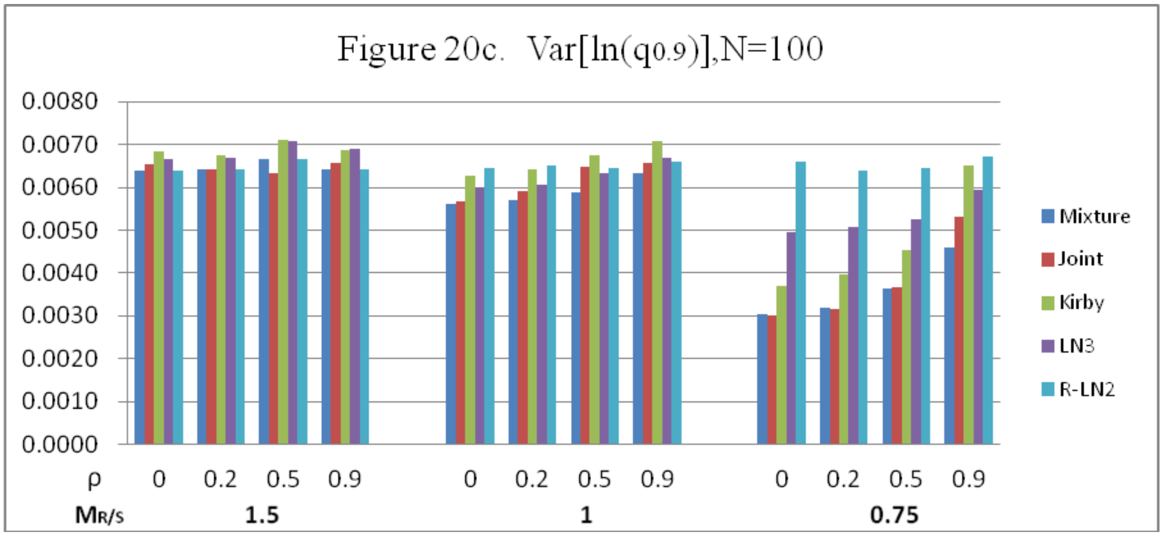


Figure 20c. Var[ln(q0.9)],N=100
0.0080
0.0070
0.0060
0.0050
0.0040
0.0030
0.0020
0.0010
0.0000
ρ 0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
Kirby
LN3
R-LN2

In general, the Kirby Method and One LN3 Method have larger Bias than the other methods, that is because distribution for R' and annual maxima has a larger skewnesses and thus is more difficult to fit with a 3-parameter distribution. By combining Bias and Variances, we can explain the MSE results. More details are provided in appendix.

CHAPTER 5

5 Summary and Conclusions

This thesis considers several models for representing the distribution of the annual maximum flood at a site when the available annual maximum flood series corresponds to two distinct annual flood series. The two series might correspond to late spring and summer rainfall versus winter and early spring snowmelt. Often a snowmelt population has a low Coefficient of Variation (CV) and thus provides a lower bound on the annual maximum, while the rainfall population has a higher CV and determines the upper tail of the annual maximum distribution.

1. The classic mixture model assumes that the annual maximums produced by both processes are independent, which is actually a special case for the Joint model. It has been recommended by many other authors (Cudworth, 1989, Stedinger et al., 1993; ASCE, 1996). Here 2-parameter lognormal distributions are used to model rainfall maxima, and the snowmelt maxima. When the values drawn from the populations are independent, the CDF of the annual maximum is then simply the product of the CDFs of the two populations. However, when the values drawn from each population are correlated, the Mixture Method will provide a biased estimator, especially when describing the lower tail of the mixed population, where the snowmelt events provide the lower bound of the mixed population. Nevertheless, because the rainfall population dominates the upper tail of the mixed population, when describing the upper tail, even large correlations between the two series may have little effect on the accuracy of the mixed population model. Actually, there is a critical probability above which flood risk is dominated by the rainfall events and cross-correlation has no impact. A method is provided to compute such thresholds.

2. When we do not have a record of the annual maxima for both series, but the rainfall and snowmelt floods that are also the annual maxima can be identified, a model provided by W. Kirby can be employed. The Kirby method does not assume that the two separate series are independent. Instead it estimates the probability an annual maximum flood is a snowmelt or rainfall event. This manuscript shows what the conditional Kirby distributions look like as a function of the parameters of the joint distribution model. The conditional distribution can be more complicated than the complete component series. In particular for our case wherein rainfall events are the largest floods, it is as if the rainfall series had a lower bound determined by the snowmelt distribution; as a result, the PDF peaks of the rainfall floods that are also the annual maxima moves to the right side of the PDFs of the complete rainfall series and is peakier than the PDFs of the complete data sets. The effect on the snowmelt distribution seems small, but we can still find that the PDFs of the snowmelt floods that are also the annual maxima are peakier than the PDFs of the complete snowmelt series.

3. In practice, a concern is which model should be used to get a reliable description of flood risk, given limited data. If data on both annual maximum series are available, and the two series are independent, then the mixture model is the natural choice: it is correct and uses all the data. Moreover, it was found to be relatively accurate for modest correlations between the two series. If the two series are cross-correlated by a substantial amount, then one can adopt the Joint Distribution model unless the focus is on the upper tail (above the critical probability), which is dominated by the rainfall events in our example.

4. The Monte Carlo analysis provided significant insight into the performance of the Kirby method. An advantage of the Kirby method is that works with only the annual maximum

series, which may be all a hydrologist has in practice. The mixture and Joint distribution methods cannot be used in those cases. However, in practice, the Kirby method has the disadvantage that it uses only the N annual maxima, and thus may provide less precise quantile estimators in many cases. This may mean that there is insufficient number of observations to estimate one of the two conditional distributions with any reliability. Here we assumed one needs at least 10 observations and did not consider samples with only 25 observations. The Monte Carlo analysis demonstrated that if one can, it is always better to use the Joint Distribution model if one has access to the two series.

5. The Kirby method was also compared with use of a single 3-parameter lognormal distribution to model that also used only the annual maximum series. As suggested by the theoretical analysis, the rainfall conditional distribution needed by the Kirby method was modeled using 3-parameter lognormal distributions because a single 2-parameter lognormal distribution would not be consistent with the data. The conditional snowmelt distribution needed by the Kirby method was modeled using 2-parameter lognormal distribution. Thus the choice was between use of one 3-parameter distribution with the whole annual maximum series, and using two lognormal distributions (LN3 for rainfall and LN2 for snowmelt) with each modeling a subset of the entire series. In general, the Kirby Method works a little better because the distribution for the annual maxima has a larger skewness than the distribution of the rainfall floods that are also the annual maxima, which makes the annual maxima more difficult to fit.

6. When the mixed population is completely dominated by one population, in our case rainfall floods, just modeling the dominant population would yield accurate results.

Overall, this thesis has considered the challenge of estimating flood risk when the annual maximum flood series is the maximum of two dominant annual maximum flood series, which in this presentation are considered to be snowmelt and rainfall events. In other applications they could be winter and spring, or winter and summer series that arise from different storm types. Or they could be winter storms and unusual tropical hurricanes whose seasons might overlap. When data on both series are available, the mixture model was found to be relatively accurate for modest correlation between the two series. If the mixed population is completely dominated by one population, just modeling the mixed population with a distribution that represents the dominant process would be sufficient. The Kirby Method or a Single Distribution Method that only uses the annual maximum series to develop a flood risk model was found to have several challenges when making developed of precise flood-risk estimates because they need to fit the mixed population with relatively more complicated distributions that have more parameters using fewer observations. Additional research should consider the case wherein the individual rainfall and snowmelt distributions need to be described by 3-parameter distributions such as the log-Pearson type 3 distribution (perhaps with regional skew) or a GEV distributions. A similar case to that considered here is when the individual series have Gumbel distributions, but the annual maxima might be described by a GEV distribution.

Appendix

Table 12-14 and Figure 21-23 report *MSE[ln($q_{0.63}$)]*, *Bias[ln($q_{0.63}$)]*, and *Var[ln($q_{0.63}$)]* for each combination, where 0.63 is the critical probability when $M_{R/S}$=1.5, P_R≈0.8. Note that we don't provide *MSE[ln($q_{0.63}$)]*, *Bias[ln($q_{0.63}$)]*, and *Var[ln($q_{0.63}$)]* of the Just-Rainfall Method in Figure 21-23 because in this case the Just-Rainfall Method would be just wrong. We don't provide *MSE[ln($q_{0.63}$)]*, *Bias[ln($q_{0.63}$)]*, and *Var[ln($q_{0.63}$)]* of the Kirby Method when N=25 in Figure 21a 22a 23a because of the highly probability that the sample size of the rainfall or snowmelt events that are also the annual maximum is smaller than 10.

Table 15-17 and Figure 24-26 report *MSE[ln($q_{0.88}$)]*, *Bias[ln($q_{0.88}$)]*, and *Var[ln($q_{0.88}$)]* for each combination, where 0.88 is the critical probability when $M_{R/S}$=1, P_R≈0.5. We don't provide *MSE[ln($q_{0.88}$)]*, *Bias[ln($q_{0.88}$)]*, and *Var[ln($q_{0.88}$)]* of the Kirby Method when N=25 in Figure 24a 25a 26a because of the highly probability that the sample size of the rainfall or snowmelt events that are also the annual maximum is smaller than 10.

Table 18-20 and Figure 27-29 report *MSE[ln($q_{0.96}$)]*, *Bias[ln($q_{0.96}$)]*, and *Var[ln($q_{0.96}$)]* for each combination, where 0.96 is the critical probability when $M_{R/S}$=0.75, P_R≈0.3. We do not provide *MSE[ln($q_{0.96}$)]*, *Bias[ln($q_{0.96}$)]*, and *Var[ln($q_{0.96}$)]* of the Kirby Method when N=25 in Figure 27a 28a 29a because of the highly probability that the sample size of the rainfall or snowmelt events that are also the annual maximum is smaller than 10.

Table 12. MSE[ln($q_{0.63}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0124	0.0123		0.0124	0.0156
0.2	0.0125	0.0131		0.0128	0.0150

0.5	0.0131	0.0139		0.0139	0.0150
0.9	0.0139	0.0149		0.0159	0.0150
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0058	0.0059		0.0078	0.0325
0.2	0.0064	0.0065		0.0080	0.0278
0.5	0.0091	0.0075		0.0088	0.0216
0.9	0.0200	0.0105		0.0110	0.0157
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0036	0.0037		0.0049	0.1055
0.2	0.0044	0.0040		0.0052	0.0976
0.5	0.0068	0.0042		0.0052	0.0803
0.9	0.0172	0.0039		0.0057	0.0533
n=50	Mixture	Joint	Kirby	LN3	R-LN2

ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0062	0.0063	0.0074	0.0064	0.0076
0.2	0.0062	0.0066	0.0075	0.0065	0.0074
0.5	0.0068	0.0071	0.0078	0.0073	0.0077
0.9	0.0071	0.0076	0.0079	0.0084	0.0076
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0029	0.0028	0.0037	0.0043	0.0233
0.2	0.0035	0.0031	0.0042	0.0046	0.0198
0.5	0.0055	0.0039	0.0047	0.0049	0.0143
0.9	0.0152	0.0052	0.0077	0.0060	0.0081
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0017	0.0018	0.0021	0.0025	0.0956
0.2	0.0022	0.0020	0.0023	0.0027	0.0887
0.5	0.0042	0.0021	0.0025	0.0029	0.0738
0.9	0.0143	0.0019	0.0027	0.0034	0.0454
n=100	Mixture	Joint	Kirby	LN3	R-LN2

ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0031	0.0032	0.0040	0.0033	0.0039
0.2	0.0032	0.0033	0.0041	0.0034	0.0038
0.5	0.0035	0.0036	0.0044	0.0038	0.0039
0.9	0.0035	0.0038	0.0042	0.0043	0.0037
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0013	0.0014	0.0018	0.0024	0.0194
0.2	0.0017	0.0016	0.0021	0.0026	0.0165
0.5	0.0036	0.0020	0.0024	0.0029	0.0104
0.9	0.0137	0.0027	0.0040	0.0037	0.0041
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0009	0.0009	0.0011	0.0014	0.0928
0.2	0.0012	0.0010	0.0012	0.0015	0.0851
0.5	0.0031	0.0010	0.0013	0.0016	0.0694
0.9	0.0124	0.0010	0.0013	0.0020	0.0420

*Table 12 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

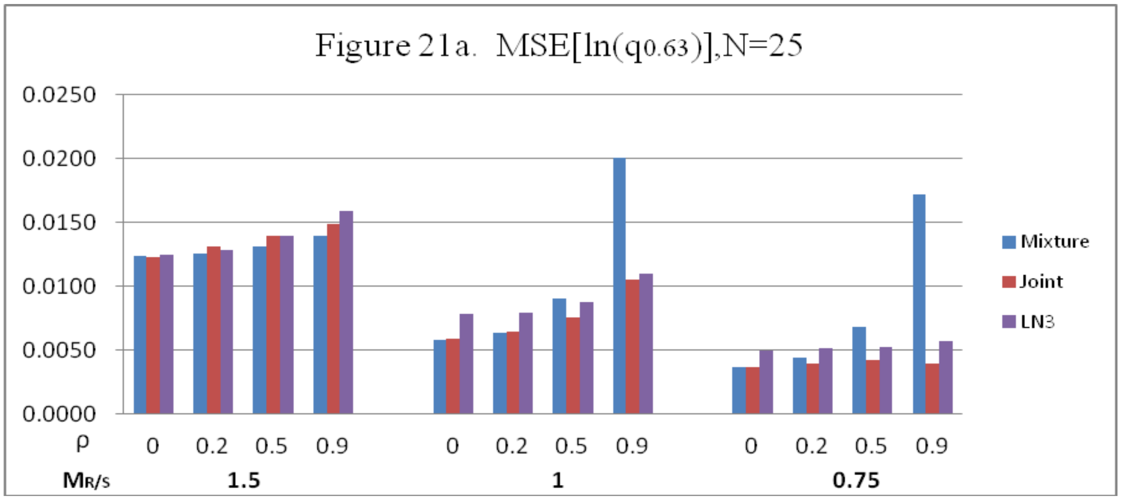


Figure 21a. MSE[ln(q0.63)],N=25
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
LN3

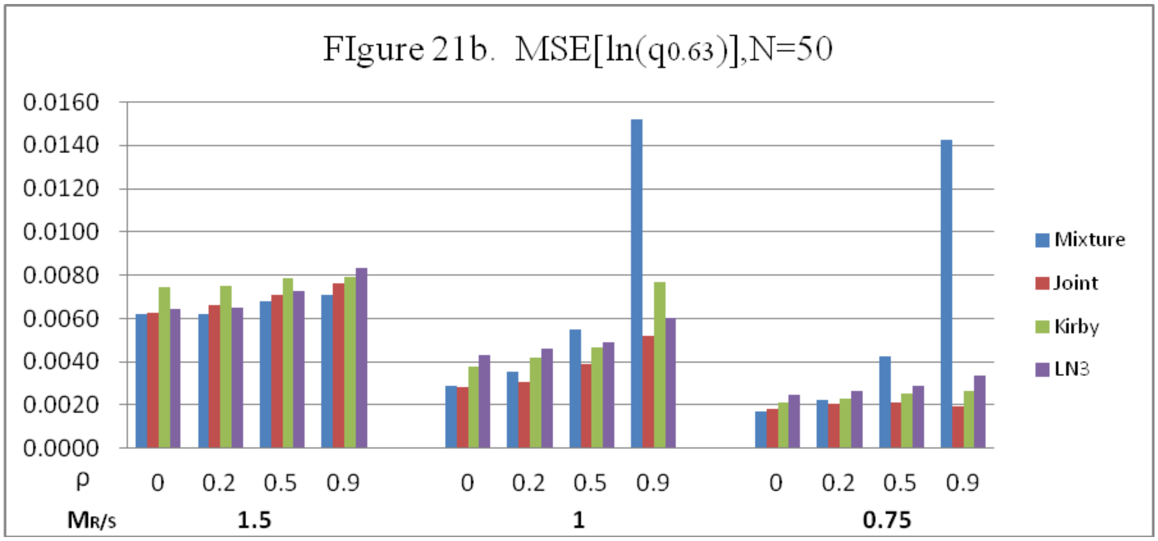


FIgure 21b. MSE[ln(q0.63)],N=50
0.0160
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3

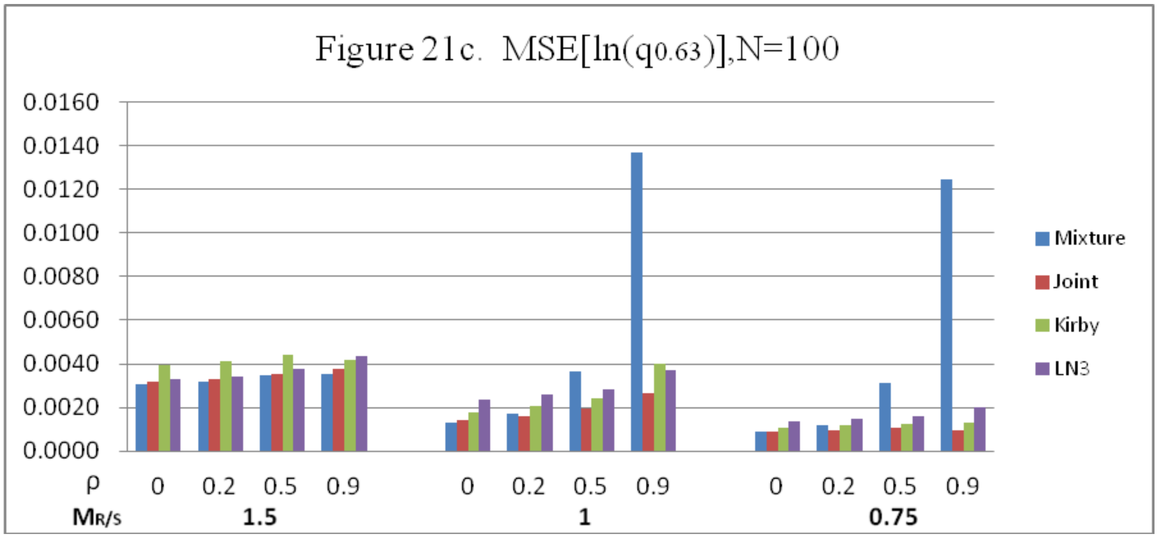


Figure 21c. MSE[ln(q0.63)],N=100
0.0160
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3

Table 13. Bias[ln($q_{0.63}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0019	0.0055		0.0081	-0.0142
0.2	0.0092	0.0022		0.0070	-0.0052
0.5	0.0092	0.0009		-0.0035	-0.0045
0.9	0.0092	-0.0017		-0.0187	-0.0025
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0031	0.0023		0.0171	-0.1286
0.2	0.0147	0.0023		0.0160	-0.1144
0.5	0.0441	0.0042		0.0182	-0.0834
0.9	0.1046	0.0037		0.0252	-0.0212
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0011	-0.0003		0.0051	-0.3010
0.2	0.0151	-0.0016		0.0046	-0.2863
0.5	0.0435	-0.0005		0.0070	-0.2561
0.9	0.1034	0.0000		0.0164	-0.1962
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0027	0.0027	0.0108	0.0127	-0.0097
0.2	0.0036	0.0020	0.0075	0.0067	-0.0085
0.5	0.0079	0.0002	0.0037	-0.0012	-0.0037
0.9	0.0102	-0.0008	-0.0101	-0.0153	-0.0002
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0020	0.0009	0.0056	0.0205	-0.1253
0.2	0.0166	0.0005	0.0058	0.0207	-0.1089
0.5	0.0429	0.0016	0.0057	0.0208	-0.0826
0.9	0.1032	0.0044	0.0170	0.0262	-0.0230

	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0010	0.0004	0.0029	0.0083	-0.2969
0.2	0.0146	-0.0005	0.0011	0.0070	-0.2849
0.5	0.0418	-0.0006	0.0013	0.0083	-0.2577
0.9	0.1043	0.0008	0.0072	0.0190	-0.1941
n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0009	0.0016	0.0067	0.0120	-0.0121
0.2	0.0035	0.0013	0.0081	0.0095	-0.0073
0.5	0.0073	-0.0005	0.0066	0.0008	-0.0032
0.9	0.0082	0.0004	-0.0005	-0.0165	-0.0017
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0009	0.0007	0.0034	0.0223	-0.1254
0.2	0.0142	0.0006	0.0029	0.0224	-0.1126
0.5	0.0433	0.0014	0.0035	0.0241	-0.0819
0.9	0.1068	0.0000	0.0127	0.0321	-0.0177
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0006	0.0002	0.0016	0.0092	-0.2985
0.2	0.0146	0.0000	0.0009	0.0097	-0.2854
0.5	0.0432	-0.0005	0.0000	0.0107	-0.2562
0.9	0.1038	0.0002	0.0001	0.0212	-0.1953

*Table 13 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

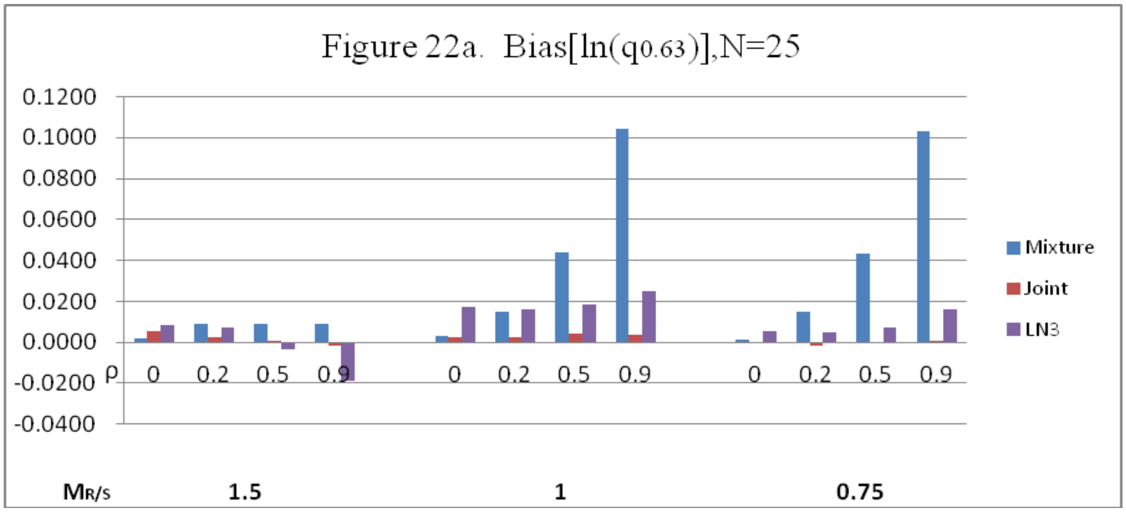


Figure 22a. Bias[ln(q0.63)],N=25
0.1200
0.1000
0.0800
0.0600
0.0400
0.0200
0.0000
-0.0200
-0.0400
ρ
0
0.2
0.5
0.9
Mixture
Joint
LN3
MR/S
1.5
1
0.75

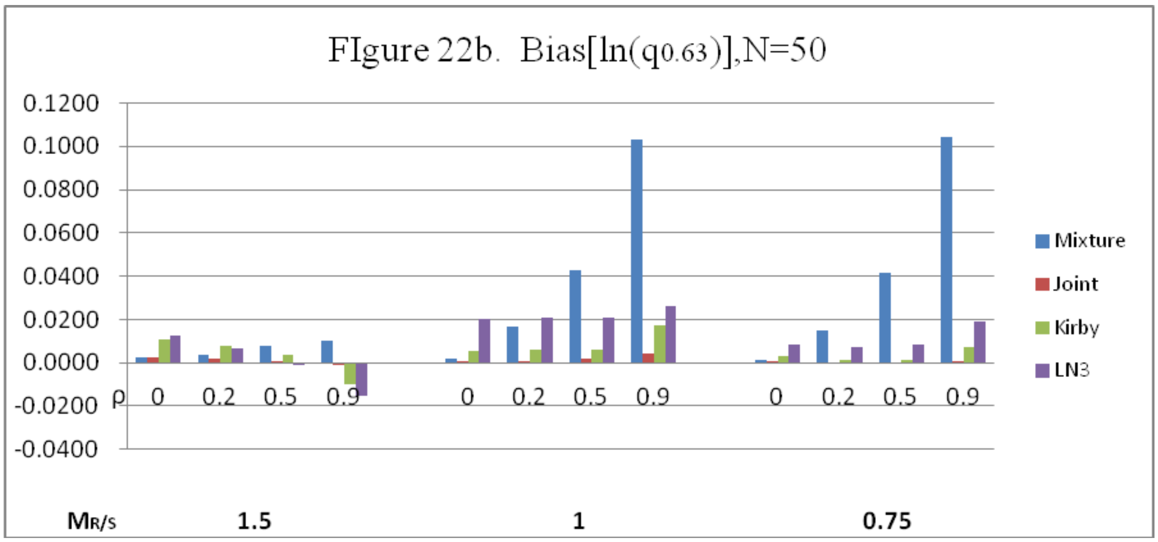


FIgure 22b. Bias[ln(q0.63)],N=50
0.1200
0.1000
0.0800
0.0600
0.0400
0.0200
0.0000
-0.0200
-0.0400
ρ
0
0.2
0.5
0.9
Mixture
Joint
Kirby
LN3
MR/S
1.5
1
0.75

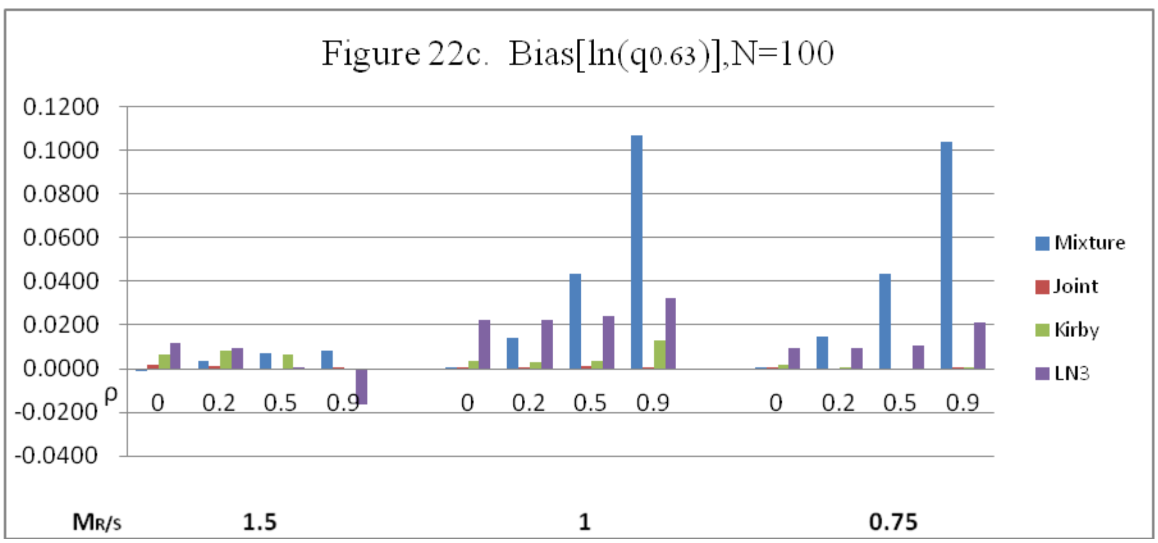


Figure 22c. Bias[ln(q0.63)],N=100
0.1200
0.1000
0.0800
0.0600
0.0400
0.0200
0.0000
-0.0200
-0.0400
ρ
0
0.2
0.5
0.9
Mixture
Joint
Kirby
LN3
MR/S
1.5
1
0.75

Table 14. Var[ln($q_{0.63}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0124	0.0123		0.0124	0.0154
0.2	0.0125	0.0131		0.0128	0.0149
0.5	0.0130	0.0139		0.0139	0.0150
0.9	0.0138	0.0149		0.0156	0.0150
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0058	0.0059		0.0075	0.0160
0.2	0.0062	0.0065		0.0077	0.0148
0.5	0.0071	0.0075		0.0084	0.0147
0.9	0.0091	0.0105		0.0104	0.0153
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0036	0.0037		0.0049	0.0149
0.2	0.0042	0.0040		0.0051	0.0156
0.5	0.0049	0.0042		0.0052	0.0148
0.9	0.0065	0.0039		0.0054	0.0148
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0062	0.0063	0.0073	0.0062	0.0075
0.2	0.0062	0.0066	0.0074	0.0064	0.0074
0.5	0.0067	0.0071	0.0078	0.0073	0.0077
0.9	0.0070	0.0076	0.0078	0.0081	0.0076
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0029	0.0028	0.0037	0.0039	0.0076
0.2	0.0032	0.0031	0.0041	0.0042	0.0079
0.5	0.0036	0.0039	0.0046	0.0044	0.0075
0.9	0.0046	0.0052	0.0074	0.0053	0.0076

	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0017	0.0018	0.0021	0.0024	0.0074
0.2	0.0020	0.0020	0.0023	0.0026	0.0075
0.5	0.0025	0.0021	0.0025	0.0028	0.0074
0.9	0.0034	0.0019	0.0026	0.0030	0.0077
n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0031	0.0032	0.0039	0.0032	0.0037
0.2	0.0032	0.0033	0.0040	0.0033	0.0037
0.5	0.0034	0.0036	0.0044	0.0038	0.0039
0.9	0.0034	0.0038	0.0042	0.0041	0.0037
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0013	0.0014	0.0018	0.0019	0.0037
0.2	0.0015	0.0016	0.0021	0.0021	0.0038
0.5	0.0018	0.0020	0.0024	0.0023	0.0037
0.9	0.0023	0.0027	0.0038	0.0027	0.0038
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0009	0.0009	0.0011	0.0013	0.0037
0.2	0.0010	0.0010	0.0012	0.0014	0.0037
0.5	0.0013	0.0010	0.0013	0.0015	0.0038
0.9	0.0017	0.0010	0.0013	0.0016	0.0039

*Table 14 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

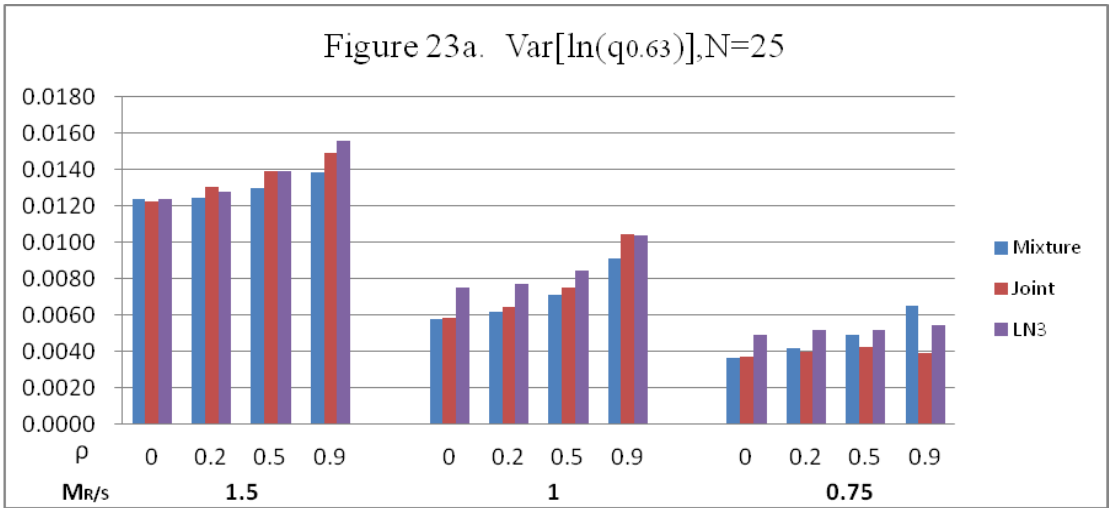


Figure 23a. Var[ln(q0.63)],N=25
0.0180
0.0160
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
LN3

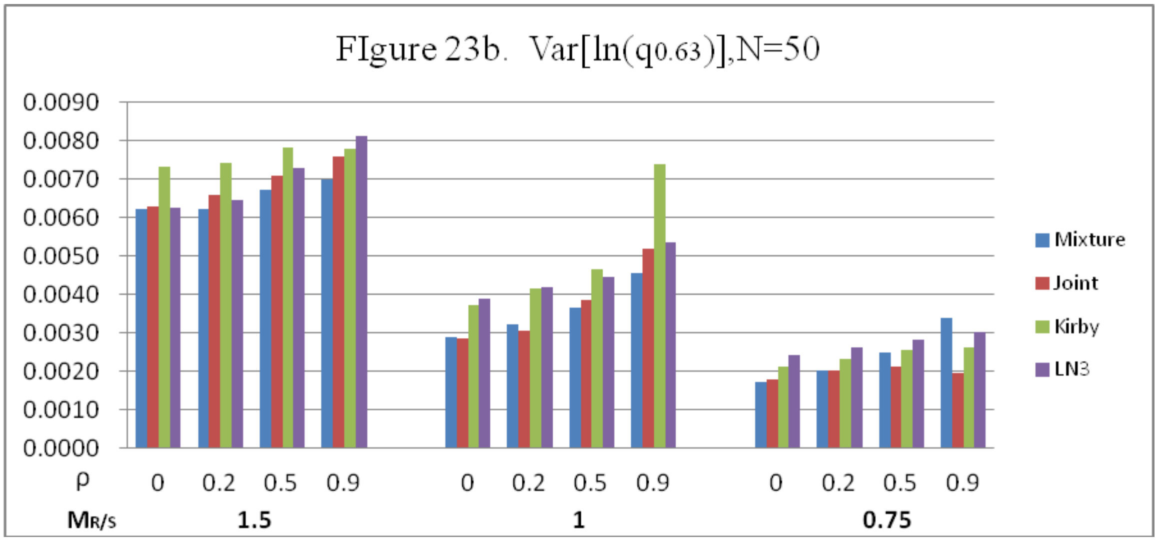


FIgure 23b. Var[ln(q0.63)],N=50
0.0090
0.0080
0.0070
0.0060
0.0050
0.0040
0.0030
0.0020
0.0010
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3

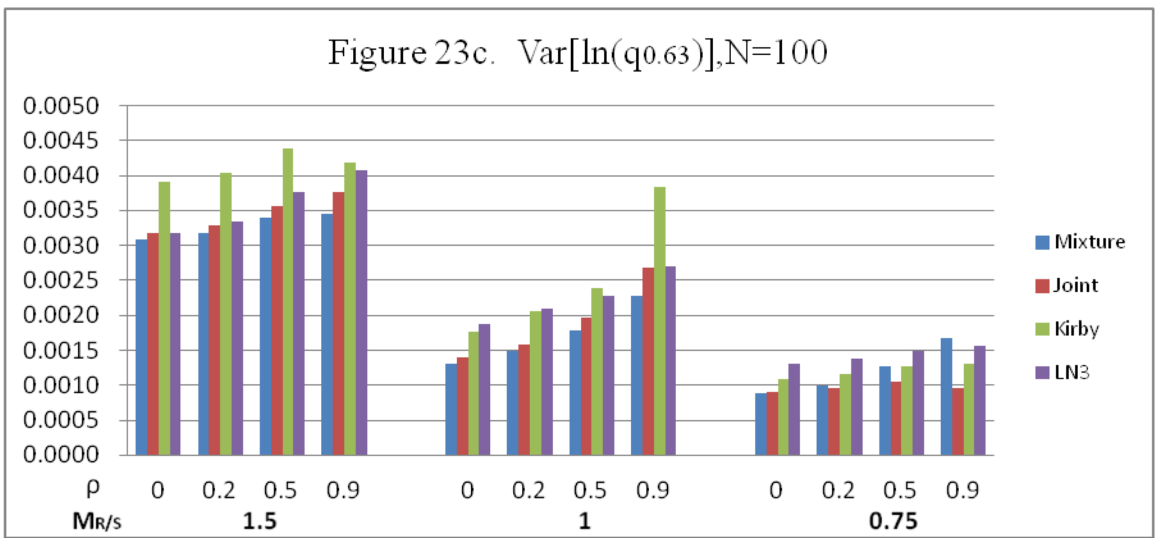


Figure 23c. Var[ln(q0.63)],N=100
0.0050
0.0045
0.0040
0.0035
0.0030
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3

Table 15. MSE[ln($q_{0.88}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0245	0.0243		0.0259	0.0248
0.2	0.0242	0.0248		0.0253	0.0244
0.5	0.0246	0.0239		0.0264	0.0247
0.9	0.0247	0.0242		0.0269	0.0247
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0195	0.0193		0.0184	0.0255
0.2	0.0189	0.0199		0.0179	0.0239
0.5	0.0203	0.0218		0.0196	0.0242
0.9	0.0223	0.0242		0.0243	0.0244
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0108	0.0107		0.0135	0.0323
0.2	0.0117	0.0108		0.0138	0.0321
0.5	0.0130	0.0123		0.0137	0.0280
0.9	0.0204	0.0166		0.0163	0.0251

n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0121	0.0123	0.0128	0.0131	0.0121
0.2	0.0118	0.0121	0.0123	0.0129	0.0118
0.5	0.0123	0.0121	0.0129	0.0137	0.0123
0.9	0.0124	0.0125	0.0135	0.0139	0.0124
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0099	0.0098	0.0108	0.0098	0.0122
0.2	0.0103	0.0099	0.0111	0.0101	0.0124
0.5	0.0103	0.0112	0.0115	0.0102	0.0119
0.9	0.0114	0.0120	0.0131	0.0128	0.0124

	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0050	0.0051	0.0061	0.0081	0.0186
0.2	0.0055	0.0056	0.0066	0.0083	0.0183
0.5	0.0067	0.0062	0.0071	0.0085	0.0164
0.9	0.0126	0.0085	0.0101	0.0101	0.0128
n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0059	0.0061	0.0063	0.0068	0.0060
0.2	0.0060	0.0060	0.0062	0.0068	0.0060
0.5	0.0062	0.0059	0.0066	0.0072	0.0062
0.9	0.0060	0.0061	0.0068	0.0072	0.0060
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0049	0.0049	0.0054	0.0053	0.0061
0.2	0.0050	0.0052	0.0056	0.0053	0.0062
0.5	0.0052	0.0058	0.0060	0.0054	0.0060
0.9	0.0058	0.0061	0.0065	0.0064	0.0061
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0025	0.0025	0.0030	0.0056	0.0130
0.2	0.0027	0.0026	0.0033	0.0056	0.0121
0.5	0.0036	0.0030	0.0037	0.0057	0.0098
0.9	0.0083	0.0044	0.0057	0.0067	0.0066

*Table 15 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

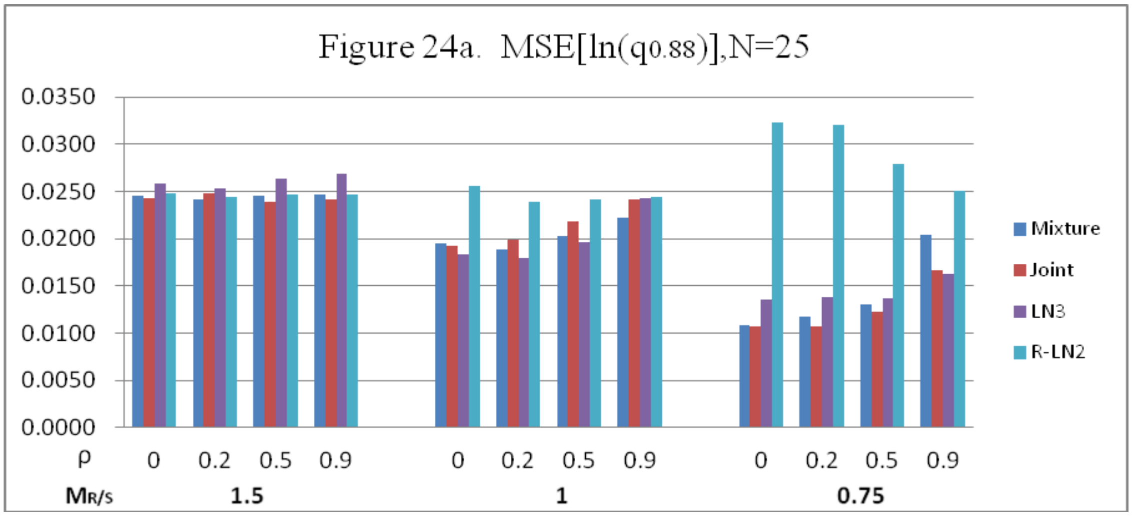


Figure 24a. MSE[ln(q0.88)],N=25
0.0350
0.0300
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
ρ 0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
LN3
R-LN2

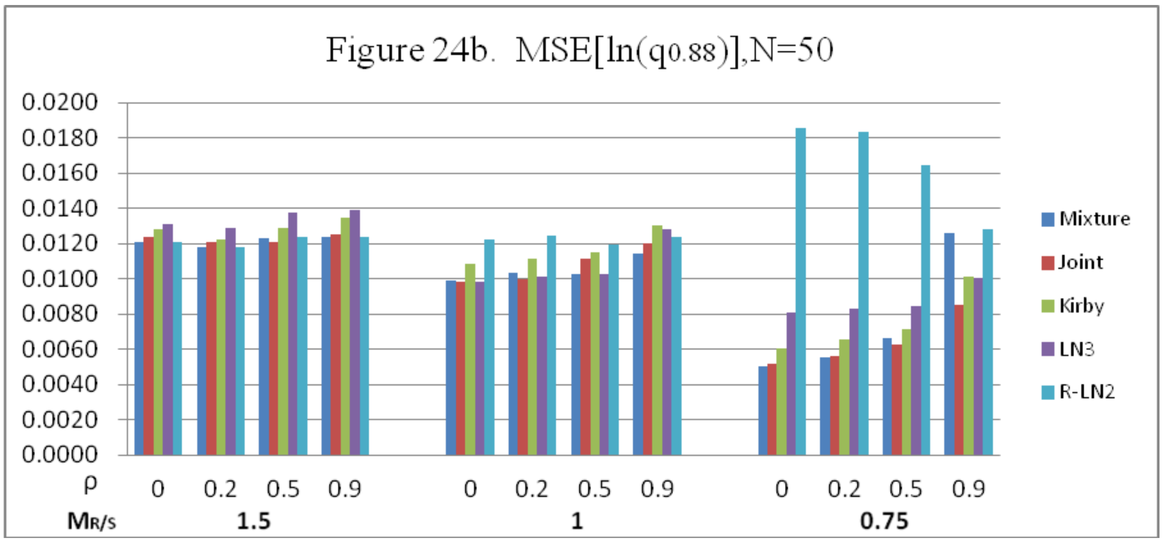


Figure 24b. MSE[ln(q0.88)],N=50
0.0200
0.0180
0.0160
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
ρ 0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
Kirby
LN3
R-LN2

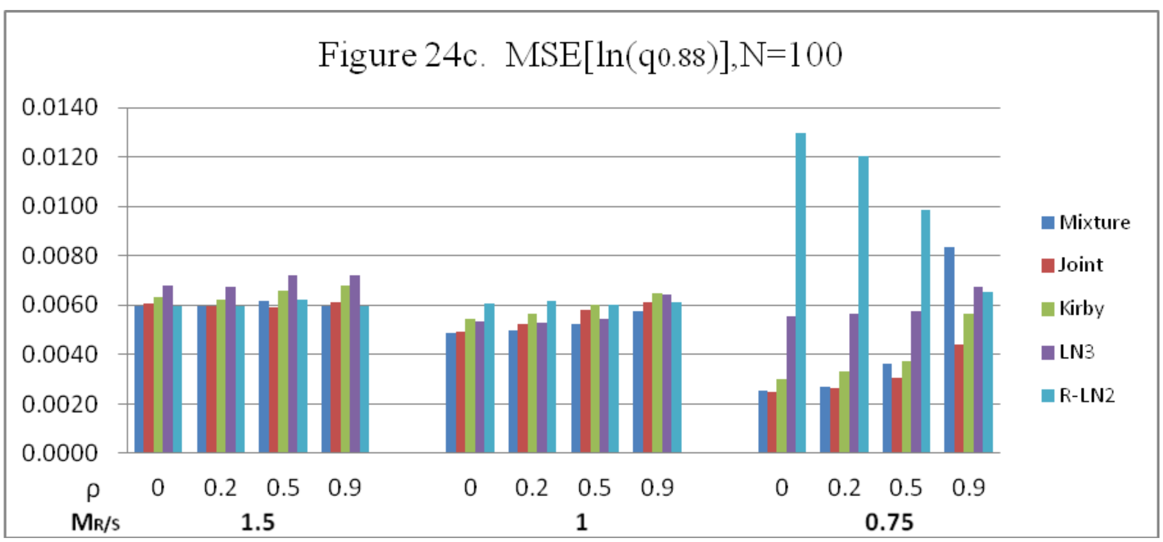


Figure 24c. MSE[ln(q0.88)],N=100
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
ρ 0 0.2 0.5 0.9 0 0.2 0.5 0.9 0 0.2 0.5 0.9
MR/S 1.5 1 0.75
Mixture
Joint
Kirby
LN3
R-LN2

Table 16. Bias[ln($q_{0.88}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0090	-0.0058		-0.0559	-0.0096
0.2	-0.0036	-0.0068		-0.0490	-0.0041
0.5	-0.0084	-0.0078		-0.0520	-0.0088
0.9	-0.0090	-0.0079		-0.0480	-0.0091
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0016	0.0034		-0.0132	-0.0193
0.2	0.0024	0.0035		-0.0162	-0.0160
0.5	0.0063	0.0012		-0.0241	-0.0100
0.9	0.0021	-0.0096		-0.0562	-0.0096
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0098	0.0081		0.0173	-0.0881
0.2	0.0151	0.0052		0.0166	-0.0828
0.5	0.0266	0.0068		0.0144	-0.0662
0.9	0.0664	0.0037		0.0159	-0.0211
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0025	-0.0029	-0.0205	-0.0370	-0.0027
0.2	-0.0048	-0.0033	-0.0225	-0.0388	-0.0050
0.5	-0.0032	-0.0032	-0.0239	-0.0373	-0.0033
0.9	-0.0042	-0.0042	-0.0336	-0.0370	-0.0043
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0037	0.0009	-0.0032	0.0063	-0.0103
0.2	0.0059	0.0010	-0.0019	0.0039	-0.0073
0.5	0.0051	-0.0001	-0.0058	-0.0084	-0.0070
0.9	0.0023	-0.0020	-0.0184	-0.0412	-0.0078

	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0066	0.0056	0.0033	0.0313	-0.0821
0.2	0.0104	0.0032	0.0025	0.0288	-0.0784
0.5	0.0217	0.0035	0.0032	0.0263	-0.0662
0.9	0.0672	0.0030	0.0256	0.0319	-0.0171

n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0044	-0.0012	-0.0136	-0.0300	-0.0044
0.2	-0.0023	-0.0013	-0.0114	-0.0282	-0.0024
0.5	-0.0024	-0.0030	-0.0130	-0.0293	-0.0024
0.9	-0.0030	-0.0009	-0.0243	-0.0312	-0.0030
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0010	0.0015	0.0015	0.0160	-0.0105
0.2	0.0002	0.0011	-0.0005	0.0102	-0.0112
0.5	0.0054	0.0018	0.0022	0.0032	-0.0050
0.9	0.0079	-0.0035	-0.0038	-0.0233	-0.0011
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0029	0.0031	0.0028	0.0381	-0.0828
0.2	0.0076	0.0020	0.0039	0.0376	-0.0783
0.5	0.0224	0.0004	0.0030	0.0367	-0.0621
0.9	0.0659	0.0019	0.0145	0.0420	-0.0175

*Table 16 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

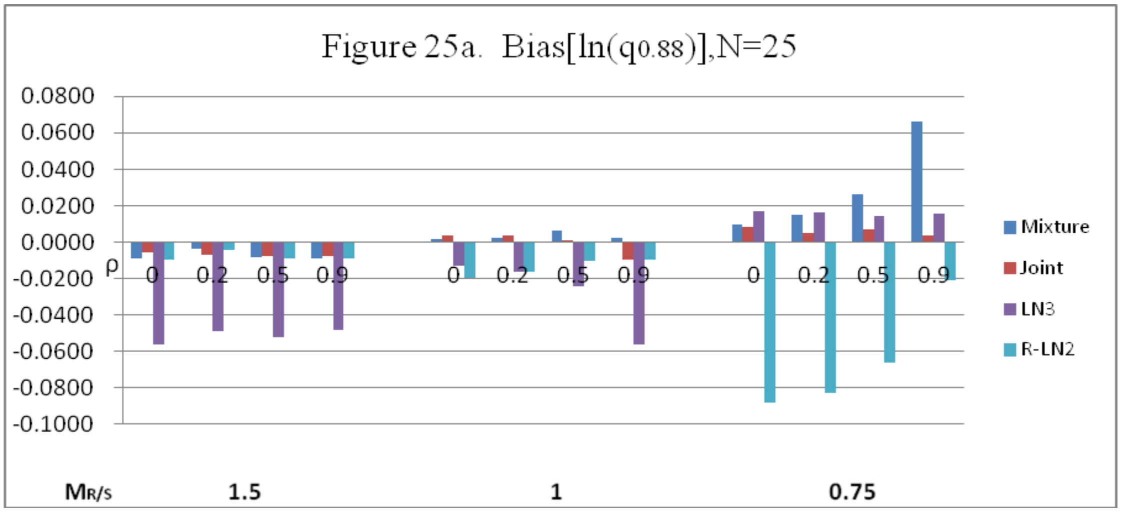


Figure 25a. Bias[ln(q0.88)],N=25
0.0800
0.0600
0.0400
0.0200
0.0000
-0.0200
-0.0400
-0.0600
-0.0800
-0.1000
ρ
0
0.2
0.5
0.9
Mixture
Joint
LN3
R-LN2
MR/S
1.5
1
0.75

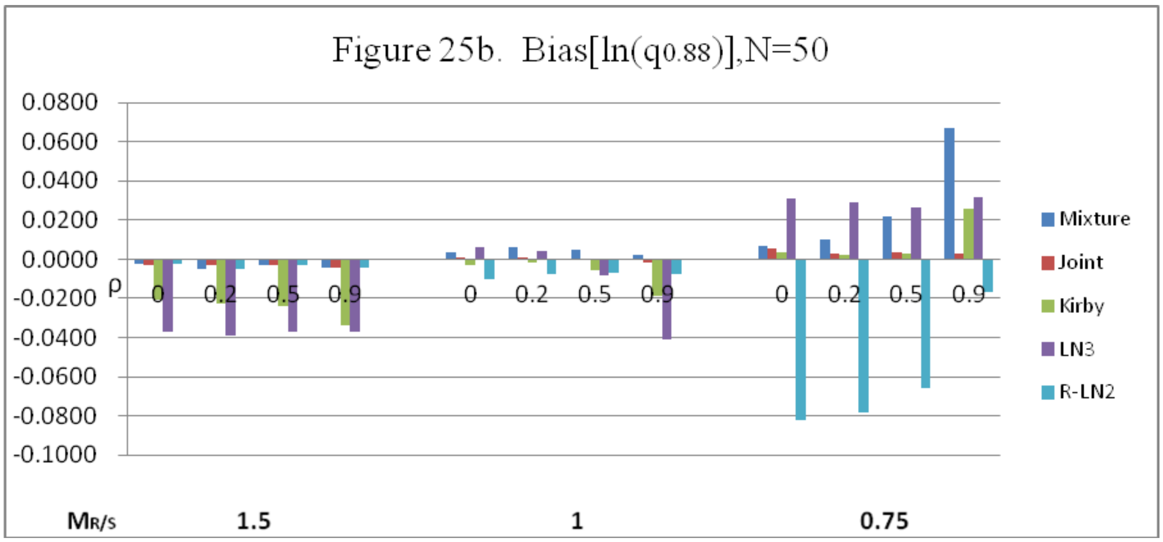


Figure 25b. Bias[ln(q0.88)],N=50
0.0800
0.0600
0.0400
0.0200
0.0000
-0.0200
-0.0400
-0.0600
-0.0800
-0.1000
ρ
0
0.2
0.5
0.9
Mixture
Joint
Kirby
LN3
R-LN2
MR/S
1.5
1
0.75

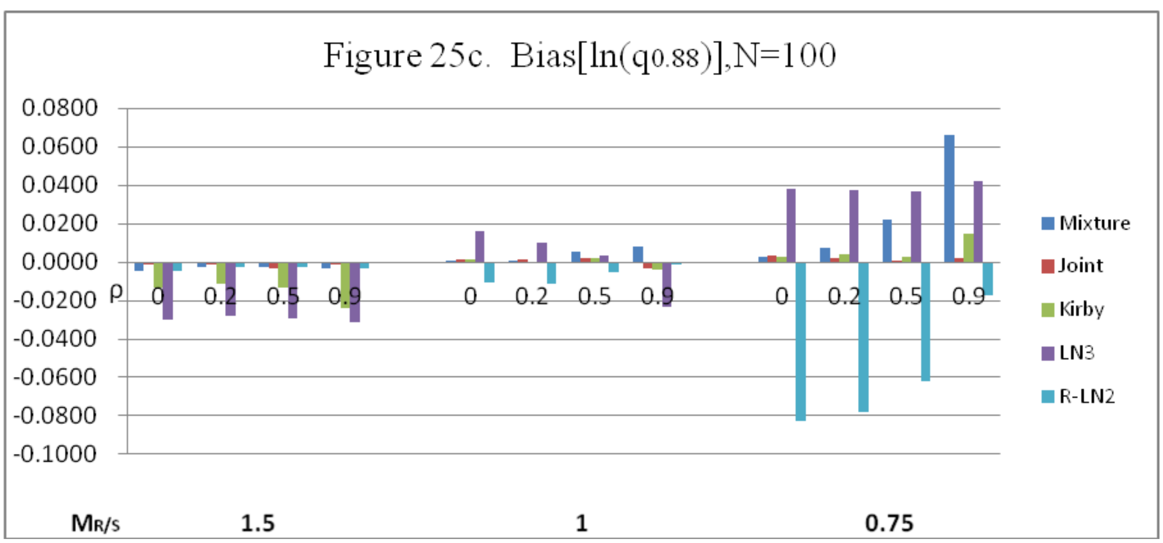


Figure 25c. Bias[ln(q0.88)],N=100
0.0800
0.0600
0.0400
0.0200
0.0000
-0.0200
-0.0400
-0.0600
-0.0800
-0.1000
ρ
0
0.2
0.5
0.9
Mixture
Joint
Kirby
LN3
R-LN2
MR/S
1.5
1
0.75

Table 17. Var[ln($q_{0.88}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0244	0.0242		0.0228	0.0247
0.2	0.0242	0.0247		0.0229	0.0244
0.5	0.0245	0.0238		0.0237	0.0247
0.9	0.0246	0.0241		0.0246	0.0247
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0195	0.0193		0.0182	0.0252
0.2	0.0189	0.0199		0.0177	0.0237
0.5	0.0202	0.0218		0.0190	0.0241
0.9	0.0223	0.0241		0.0211	0.0243
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0107	0.0106		0.0132	0.0245
0.2	0.0115	0.0107		0.0135	0.0252
0.5	0.0123	0.0123		0.0135	0.0236
0.9	0.0160	0.0166		0.0161	0.0246
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0121	0.0123	0.0124	0.0117	0.0121
0.2	0.0118	0.0121	0.0118	0.0113	0.0118
0.5	0.0123	0.0121	0.0123	0.0123	0.0123
0.9	0.0124	0.0125	0.0123	0.0125	0.0124
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0099	0.0098	0.0108	0.0098	0.0121
0.2	0.0103	0.0099	0.0111	0.0101	0.0124
0.5	0.0102	0.0112	0.0115	0.0102	0.0119
0.9	0.0114	0.0120	0.0127	0.0111	0.0123

	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0050	0.0051	0.0061	0.0071	0.0118
0.2	0.0054	0.0056	0.0066	0.0075	0.0122
0.5	0.0062	0.0062	0.0071	0.0078	0.0120
0.9	0.0081	0.0085	0.0095	0.0090	0.0125

n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0059	0.0061	0.0062	0.0059	0.0059
0.2	0.0059	0.0060	0.0061	0.0060	0.0060
0.5	0.0062	0.0059	0.0064	0.0064	0.0062
0.9	0.0060	0.0061	0.0062	0.0062	0.0060
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0049	0.0049	0.0054	0.0051	0.0060
0.2	0.0050	0.0052	0.0056	0.0052	0.0060
0.5	0.0052	0.0058	0.0060	0.0054	0.0060
0.9	0.0057	0.0061	0.0065	0.0059	0.0061
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0025	0.0025	0.0030	0.0041	0.0061
0.2	0.0027	0.0026	0.0033	0.0042	0.0059
0.5	0.0031	0.0030	0.0037	0.0044	0.0060
0.9	0.0040	0.0044	0.0055	0.0050	0.0062

*Table 17 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

Figure 26a. Var[ln(q0.88)],N=25

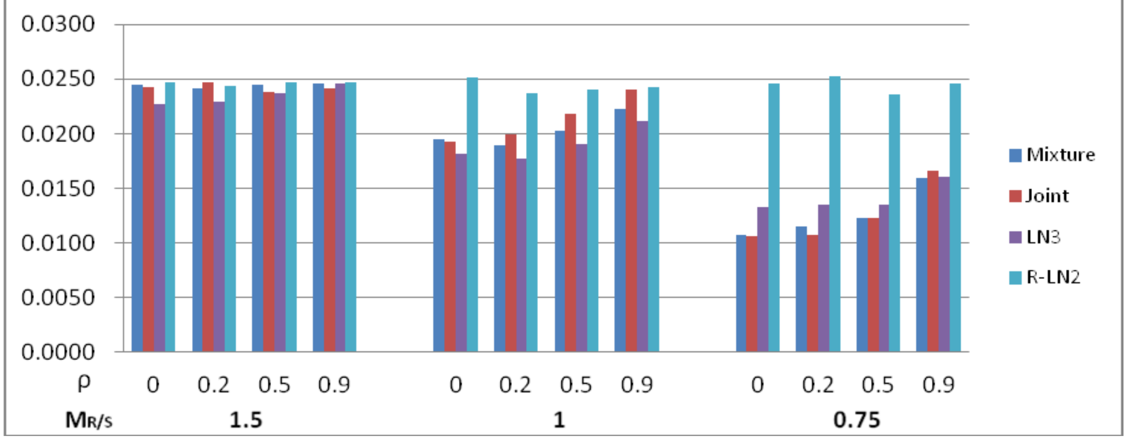


Figure 26b. Var[ln(q0.88)],N=50

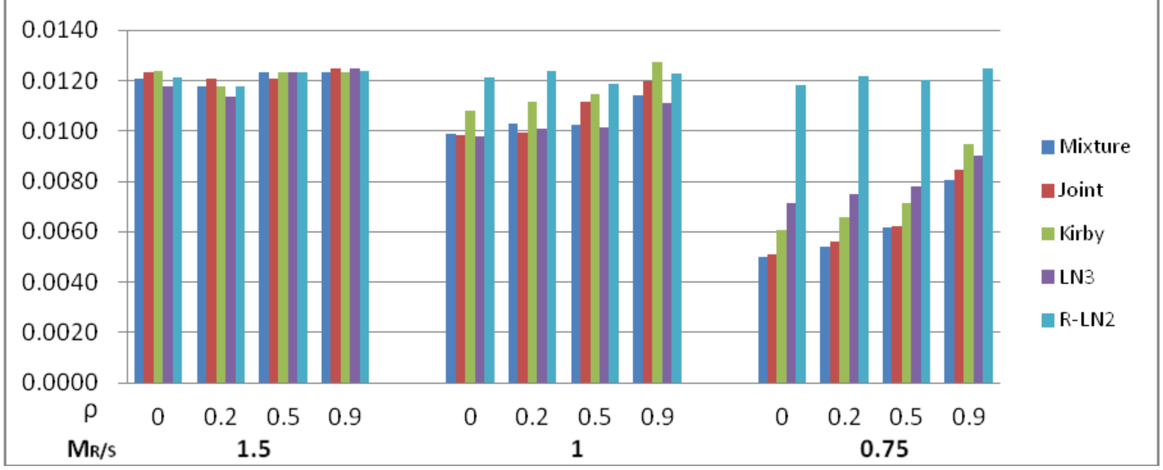


Figure 26c. Var[ln(q0.88)],N=100

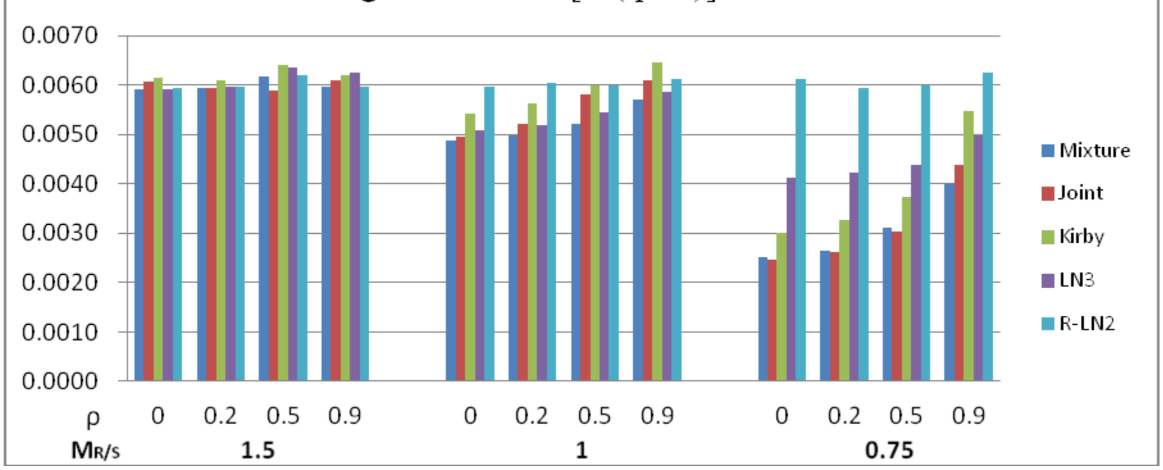


Table 18. MSE[ln($q_{0.96}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0373	0.0370		0.0450	0.0373
0.2	0.0368	0.0378		0.0430	0.0369
0.5	0.0375	0.0361		0.0432	0.0376
0.9	0.0375	0.0366		0.0405	0.0375
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0353	0.0352		0.0428	0.0377
0.2	0.0341	0.0350		0.0411	0.0360
0.5	0.0355	0.0369		0.0429	0.0367
0.9	0.0363	0.0367		0.0461	0.0365
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0264	0.0260		0.0309	0.0375
0.2	0.0275	0.0259		0.0311	0.0382
0.5	0.0274	0.0287		0.0313	0.0356
0.9	0.0327	0.0346		0.0403	0.0375
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0183	0.0187	0.0236	0.0243	0.0183
0.2	0.0179	0.0181	0.0223	0.0230	0.0179
0.5	0.0185	0.0182	0.0230	0.0233	0.0185
0.9	0.0186	0.0189	0.0222	0.0219	0.0186
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0178	0.0177	0.0230	0.0235	0.0182
0.2	0.0180	0.0175	0.0229	0.0237	0.0184
0.5	0.0175	0.0182	0.0230	0.0240	0.0179
0.9	0.0186	0.0181	0.0248	0.0258	0.0186

	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0134	0.0135	0.0172	0.0177	0.0179
0.2	0.0139	0.0143	0.0182	0.0184	0.0185
0.5	0.0145	0.0150	0.0189	0.0188	0.0185
0.9	0.0166	0.0176	0.0234	0.0228	0.0187

n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0089	0.0091	0.0123	0.0128	0.0089
0.2	0.0089	0.0090	0.0121	0.0125	0.0089
0.5	0.0093	0.0088	0.0126	0.0127	0.0093
0.9	0.0089	0.0092	0.0124	0.0117	0.0089
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0089	0.0089	0.0123	0.0129	0.0091
0.2	0.0089	0.0090	0.0122	0.0131	0.0091
0.5	0.0090	0.0094	0.0125	0.0134	0.0090
0.9	0.0092	0.0092	0.0128	0.0137	0.0092
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0071	0.0070	0.0097	0.0108	0.0094
0.2	0.0070	0.0071	0.0098	0.0109	0.0091
0.5	0.0074	0.0077	0.0106	0.0111	0.0090
0.9	0.0084	0.0092	0.0125	0.0130	0.0094

*Table 18 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

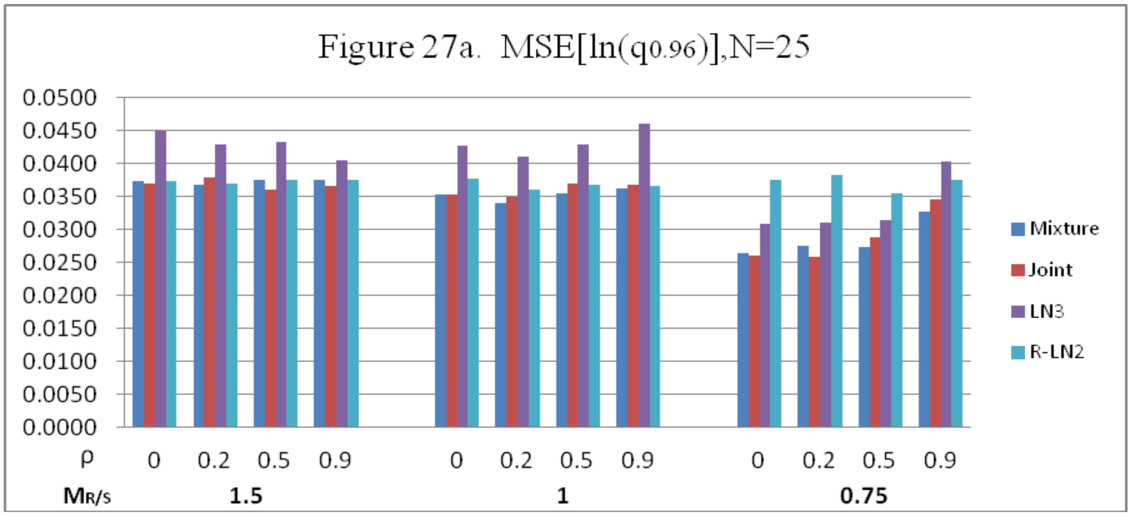


Figure 27a. MSE[ln(q0.96)],N=25
0.0500
0.0450
0.0400
0.0350
0.0300
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
LN3
R-LN2

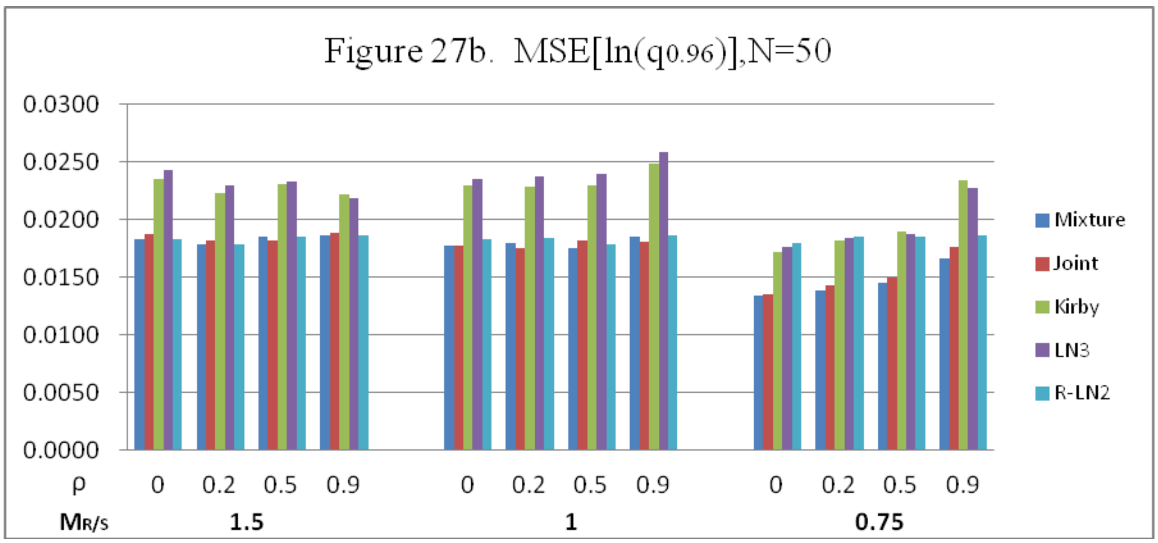


Figure 27b. MSE[ln(q0.96)],N=50
0.0300
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3
R-LN2

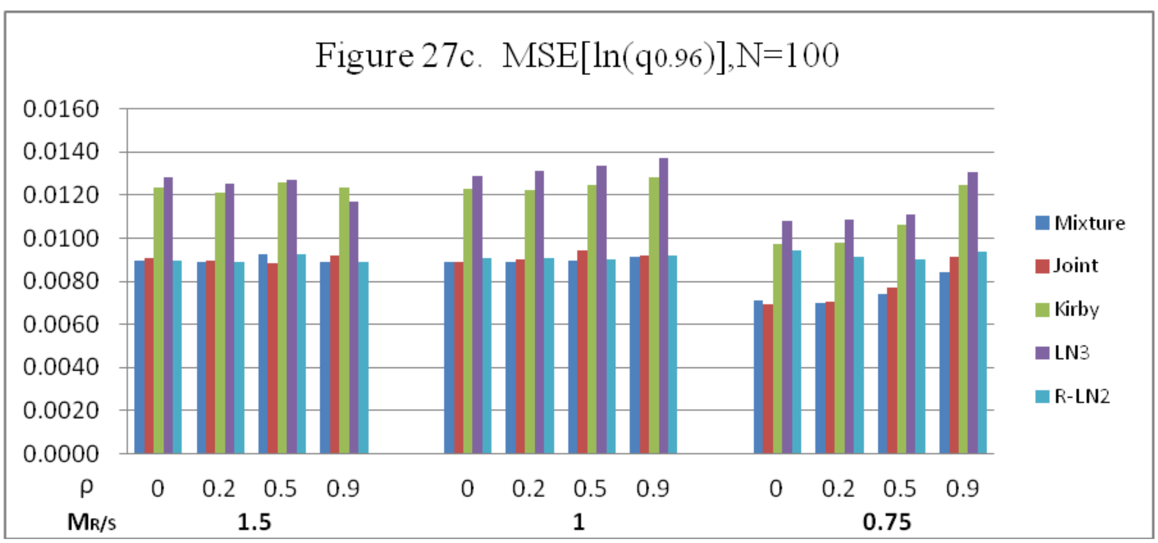


Figure 27c. MSE[ln(q0.96)],N=100
0.0160
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3
R-LN2

Table 19. Bias[ln($q_{0.96}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0127	-0.0103		-0.0745	-0.0128
0.2	-0.0076	-0.0105		-0.0626	-0.0077
0.5	-0.0131	-0.0122		-0.0615	-0.0131
0.9	-0.0137	-0.0121		-0.0485	-0.0137
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	-0.0118	-0.0076		-0.0722	-0.0161
0.2	-0.0095	-0.0061		-0.0706	-0.0129
0.5	-0.0075	-0.0074		-0.0713	-0.0100
0.9	-0.0133	-0.0143		-0.0853	-0.0140
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0117	0.0090		-0.0266	-0.0197
0.2	0.0116	0.0051		-0.0264	-0.0196
0.5	0.0106	0.0052		-0.0344	-0.0153
0.9	0.0081	-0.0103		-0.0611	-0.0092
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0041	-0.0050	-0.0350	-0.0469	-0.0041
0.2	-0.0069	-0.0054	-0.0363	-0.0469	-0.0069
0.5	-0.0045	-0.0047	-0.0335	-0.0401	-0.0045
0.9	-0.0071	-0.0065	-0.0336	-0.0332	-0.0071
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	-0.0017	-0.0061	-0.0304	-0.0432	-0.0031
0.2	-0.0007	-0.0045	-0.0260	-0.0405	-0.0020
0.5	-0.0046	-0.0045	-0.0312	-0.0485	-0.0055
0.9	-0.0094	-0.0041	-0.0499	-0.0629	-0.0098

	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0085	0.0077	-0.0110	-0.0065	-0.0122
0.2	0.0074	0.0040	-0.0107	-0.0091	-0.0132
0.5	0.0045	0.0029	-0.0137	-0.0169	-0.0142
0.9	0.0099	-0.0055	-0.0287	-0.0375	-0.0037
n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	-0.0055	-0.0021	-0.0246	-0.0360	-0.0055
0.2	-0.0034	-0.0022	-0.0225	-0.0327	-0.0034
0.5	-0.0033	-0.0044	-0.0225	-0.0293	-0.0033
0.9	-0.0039	-0.0019	-0.0264	-0.0229	-0.0039
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	-0.0027	-0.0016	-0.0179	-0.0297	-0.0035
0.2	-0.0053	-0.0018	-0.0209	-0.0345	-0.0060
0.5	-0.0020	0.0003	-0.0163	-0.0330	-0.0025
0.9	-0.0017	-0.0044	-0.0239	-0.0384	-0.0020
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0036	0.0044	-0.0063	0.0030	-0.0125
0.2	0.0031	0.0024	-0.0033	0.0023	-0.0126
0.5	0.0059	-0.0009	-0.0036	-0.0020	-0.0083
0.9	0.0085	-0.0019	-0.0118	-0.0238	-0.0036

*Table 19 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

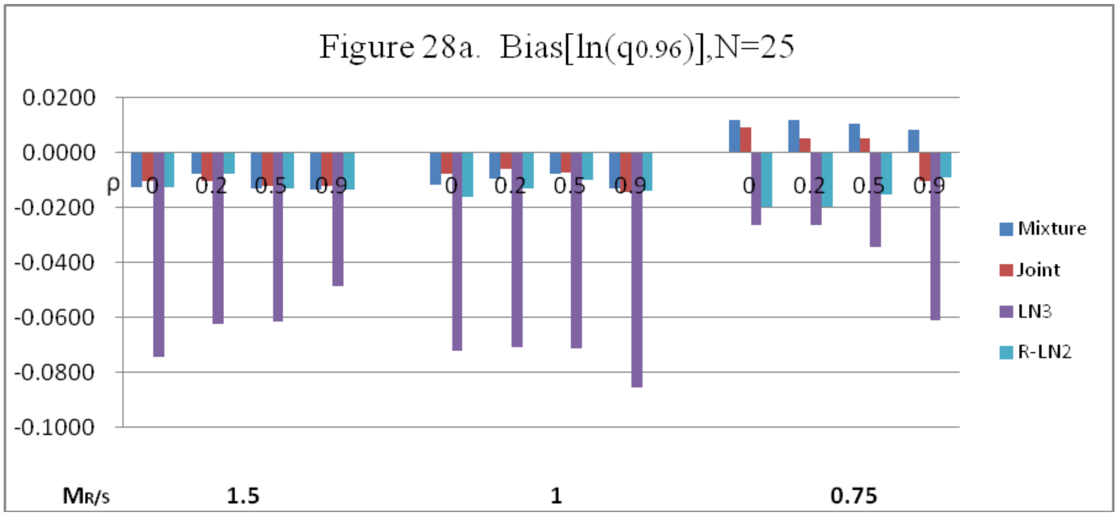


Figure 28a. Bias[ln(q0.96)],N=25
0.0200
0.0000
-0.0200
-0.0400
-0.0600
-0.0800
-0.1000
ρ
0
0.2
0.5
0.9
Mixture
Joint
LN3
R-LN2
MR/S
1.5
1
0.75

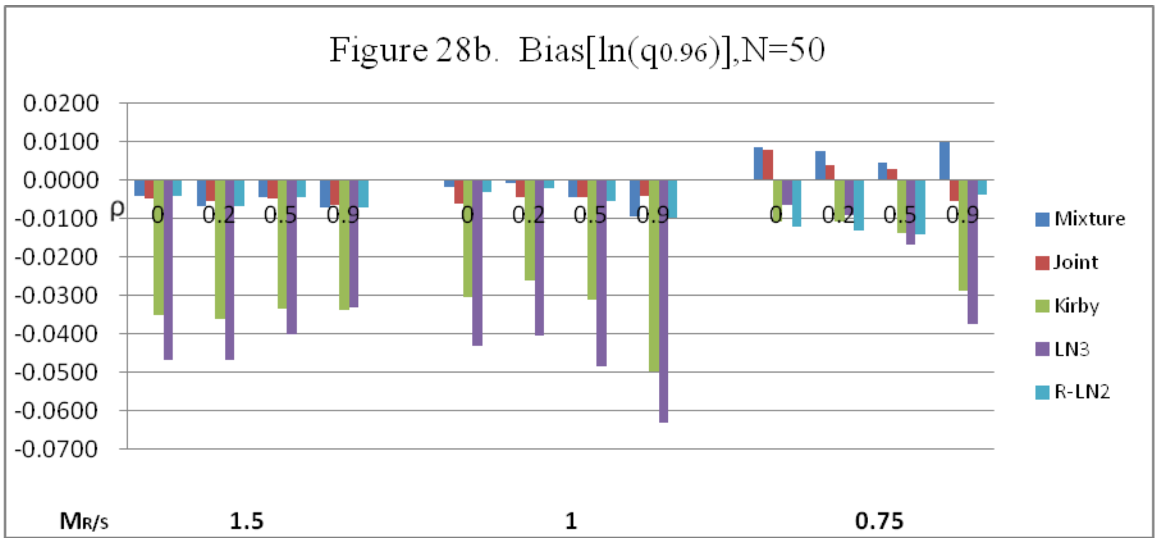


Figure 28b. Bias[ln(q0.96)],N=50
0.0200
0.0100
0.0000
-0.0100
-0.0200
-0.0300
-0.0400
-0.0500
-0.0600
-0.0700
ρ
0
0.2
0.5
0.9
Mixture
Joint
Kirby
LN3
R-LN2
MR/S
1.5
1
0.75

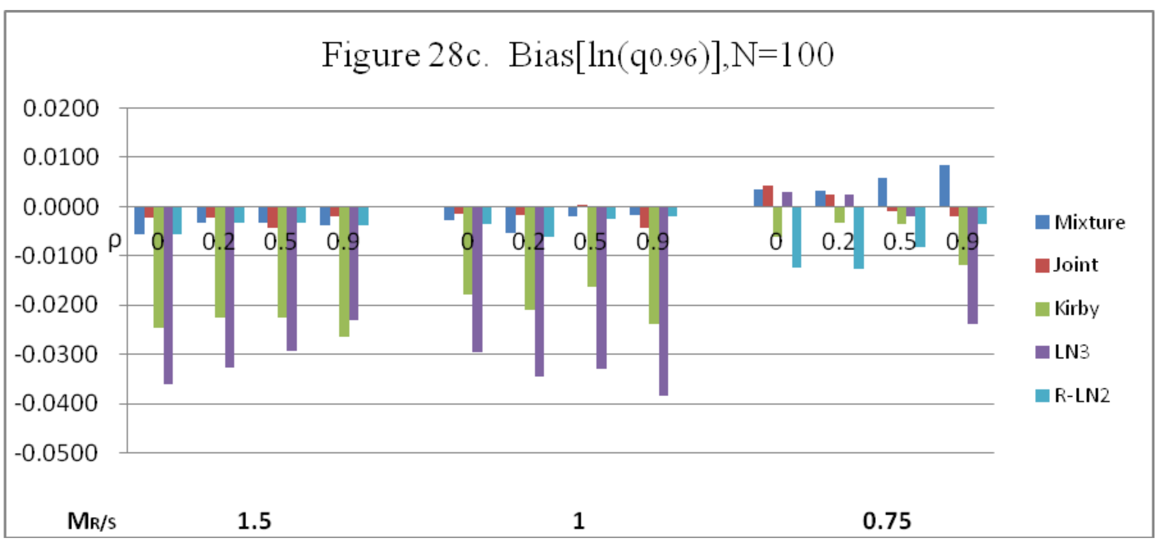


Figure 28c. Bias[ln(q0.96)],N=100
0.0200
0.0100
0.0000
-0.0100
-0.0200
-0.0300
-0.0400
-0.0500
ρ
0
0.2
0.5
0.9
Mixture
Joint
Kirby
LN3
R-LN2
MR/S
1.5
1
0.75

Table 20. Var[ln($q_{0.96}$)]

n=25	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0371	0.0369		0.0394	0.0371
0.2	0.0368	0.0377		0.0391	0.0368
0.5	0.0374	0.0360		0.0394	0.0374
0.9	0.0373	0.0364		0.0381	0.0373
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0351	0.0352		0.0375	0.0374
0.2	0.0340	0.0350		0.0361	0.0358
0.5	0.0354	0.0368		0.0379	0.0366
0.9	0.0361	0.0365		0.0388	0.0363
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0263	0.0260		0.0302	0.0371
0.2	0.0273	0.0259		0.0304	0.0378
0.5	0.0273	0.0287		0.0301	0.0353
0.9	0.0327	0.0345		0.0365	0.0374
n=50	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0183	0.0187	0.0223	0.0221	0.0183
0.2	0.0178	0.0181	0.0210	0.0208	0.0178
0.5	0.0184	0.0182	0.0219	0.0217	0.0184
0.9	0.0186	0.0188	0.0210	0.0208	0.0186
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0178	0.0177	0.0221	0.0217	0.0182
0.2	0.0180	0.0175	0.0222	0.0221	0.0184
0.5	0.0175	0.0182	0.0220	0.0216	0.0178
0.9	0.0185	0.0181	0.0223	0.0219	0.0185

	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0133	0.0134	0.0171	0.0176	0.0178
0.2	0.0138	0.0143	0.0181	0.0183	0.0183
0.5	0.0145	0.0150	0.0188	0.0185	0.0183
0.9	0.0165	0.0176	0.0226	0.0214	0.0186
n=100	Mixture	Joint	Kirby	LN3	R-LN2
ρ	$M_{R/S}$=1.5, P_R≈0.8, P_C=0.63				
0	0.0089	0.0091	0.0117	0.0115	0.0089
0.2	0.0089	0.0090	0.0116	0.0115	0.0089
0.5	0.0093	0.0088	0.0121	0.0118	0.0093
0.9	0.0089	0.0092	0.0117	0.0112	0.0089
	$M_{R/S}$=1, P_R=0.5, P_C=0.88				
0	0.0089	0.0089	0.0120	0.0120	0.0090
0.2	0.0089	0.0090	0.0118	0.0119	0.0090
0.5	0.0089	0.0094	0.0122	0.0123	0.0090
0.9	0.0092	0.0091	0.0123	0.0122	0.0092
	$M_{R/S}$=0.75, P_R≈0.3, P_C=0.96				
0	0.0071	0.0069	0.0097	0.0108	0.0093
0.2	0.0070	0.0070	0.0098	0.0109	0.0090
0.5	0.0074	0.0077	0.0106	0.0111	0.0090
0.9	0.0083	0.0091	0.0123	0.0125	0.0094

*Table 20 is based on 5000 replicates of each combination; a 90% confidence interval for true values of MSE is less than ±5%

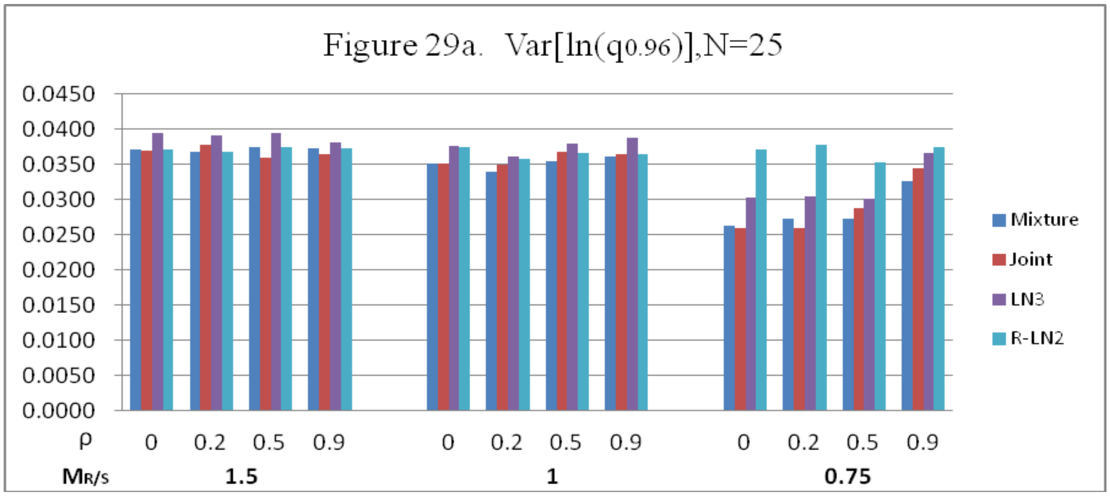


Figure 29a. Var[ln(q0.96)],N=25
0.0450
0.0400
0.0350
0.0300
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
LN3
R-LN2

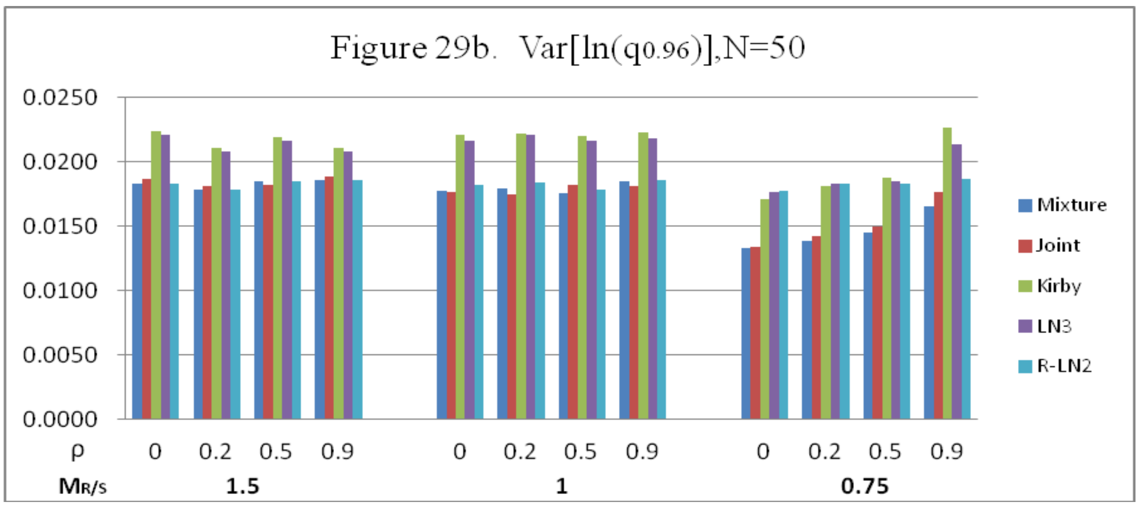


Figure 29b. Var[ln(q0.96)],N=50
0.0250
0.0200
0.0150
0.0100
0.0050
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3
R-LN2

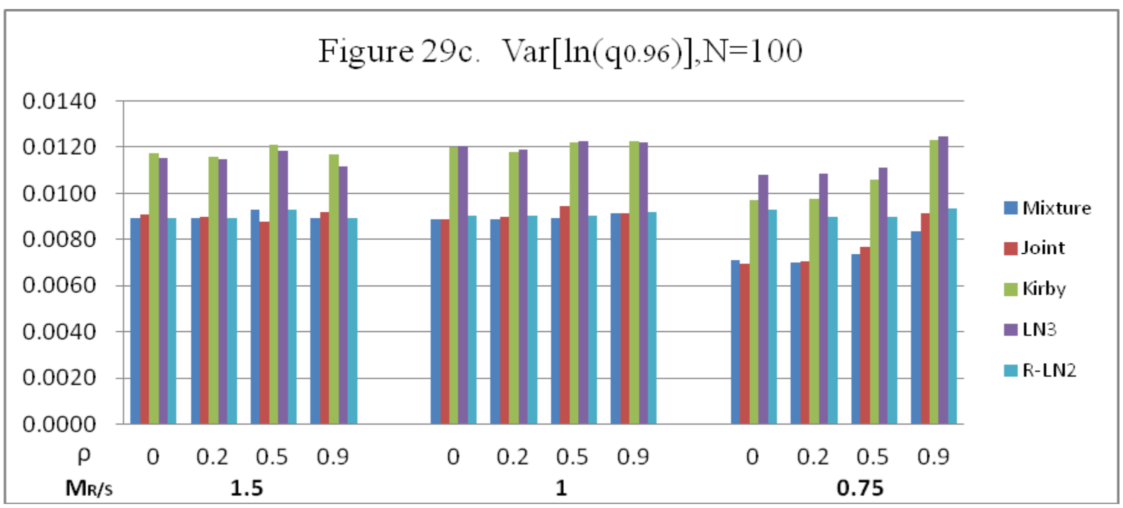


Figure 29c. Var[ln(q0.96)],N=100
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000
ρ
0
0.2
0.5
0.9
MR/S
1.5
1
0.75
Mixture
Joint
Kirby
LN3
R-LN2

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