

# Stock Optimization in Emergency Resupply Networks under Stuttering Poisson Demand

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We consider a network in which field stocking locations (FSLs) manage multiple parts according to an (S-1,S) policy. Demand processes for the parts are assumed to be independent stuttering Poisson processes. Regular replenishments to an FSL occur from a regional stocking location (RSL) that has an unlimited supply of each part type. Demand in excess of supply at an FSL is routed to an emergency stocking location (ESL), which also employs an (S-1,S) policy to manage its inventory. Demand in excess of supply at the ESL is back-ordered. Lead time from the ESL to each FSL is assumed to be negligible compared to the RSL-ESL resupply time. In companion papers we have shown how to approximate the joint probability distributions of units on hand, units in regular resupply, and units in emergency resupply. In this paper, we focus on the problem of determining the stock levels at the FSLs and ESL across all part numbers that minimize backorder, and emergency resupply costs subject to an inventory investment budget constraint. The problem is shown to be a non-convex integer programming problem, and we explore a collection of heuristics for solving the optimization problem.

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## 1. Introduction and Literature Review

### 1.1 Introduction

In this paper we study a system containing a regional stocking location (RSL), which serves two types of facilities: a set of  $N$  field service locations (FSL) and an emergency stocking location (ESL). Each location stocks multiple part types, which are used by technical service representatives who make visits to customer sites to repair equipment. In our model, we use a stuttering Poisson process to represent the demand process for each of  $I$  part types at each FSL. We have employed this model since the variance to mean ratio of the demand at the FSLs is greater than 1 in systems that we have examined. We assume the inventory control policy followed at each location is an (s-1,s) or order-up-to, policy. Again, this policy type is the one used in applications we have studied. The system we will examine works as follows.

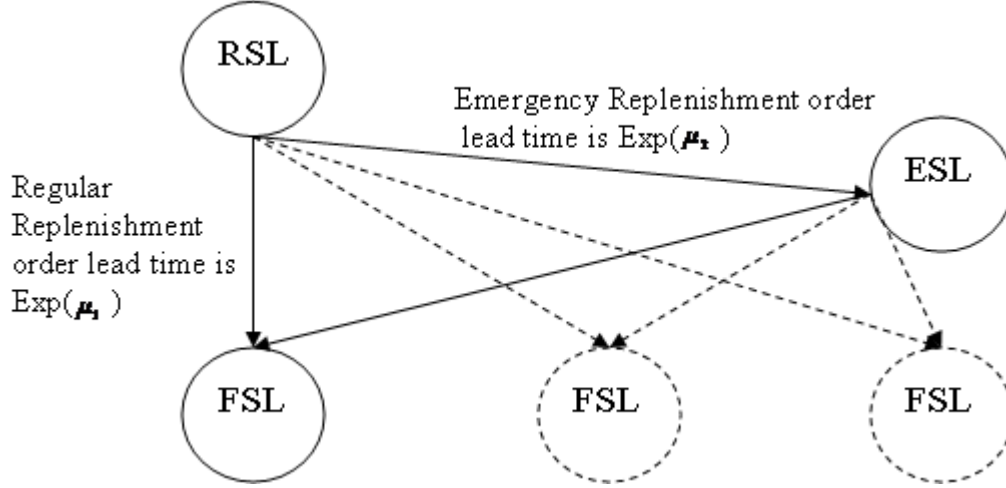


Figure 1: A System with Emergency Resupply

When a customer order occurs, if the on hand inventory at the FSL is sufficient to satisfy the entire customer demand, we fulfill this order from the FSL stock, and immediately place a regular replenishment order of the same order size on the RSL. Whenever a customer's demand exceeds the inventory on hand at an FSL, an emergency order is immediately placed on the ESL for an amount equal to the customer's order size. If the ESL does not have enough inventory on hand, the excess quantity becomes a backorder at the ESL. Upon receipt of a resupply request placed by a FSL for a given amount of stock, the ESL in turn places a replenishment order for the same amount on the RSL. Thus a customer's order may be satisfied by one of two different types of replenishment orders depending on whether or not the on hand inventory at the FSL is sufficient to satisfy the customer's demand. Additionally, we assume that the lead times from the RSL to the FSLs and the ESL are exponentially distributed. Figure 1 depicts the resupply system we have described.

To determine the optimal order-up-to levels at the FSLs and the ESL, we require the stationary distributions of the number of units in regular and emergency resupply. Chen.*et al.* (2011,2013) analyze the same system and propose both an exact and an approximate method for determining these distributions. Observe that an emergency order can be treated as a lost sale at the FSL since the FSL replenishment process from the RSL is equal to the amount of stock withdrawn for the FSL inventory, which corresponds to customer's demand can be satisfied totally from FSL inventory. Chen.*et al.* (2011) analyze such a lost sales system and derive the exact steady state distribution of the number of units in regular resupply of a field service location that employs an  $(s-1,s)$  inventory policy. Furthermore,

Chen.*et.al.* (2013) use a zero-truncated negative binomial distribution with an atom at zero to approximate the steady state distribution of the number of emergency ordered units outstanding in a system consisting of both FSLs and an ESL.

Our goal in this paper is to develop optimization algorithms for setting stock levels in the emergency resupply network that we have described. The exact and an approximate stationary distribution of the number of outstanding ordered units at the FSLs and ESL are used to construct the expected cost function corresponding to the service network system. The costs considered in the optimization model are emergency order penalty costs for each order placed by an FSL on the ESL, and backorder penalty costs charged at the ESL for each backordered unit per unit of time. Since each time an order arrives at the FSLs, a regular or an emergency replenishment order is placed at the RSL therefore the inventory in the FSLs and ESL system is kept at a constant level. We use a budget constraint instead of a holding cost to capture the cost associated with the inventory handled in the whole system. We show that the emergency order cost function at the FSL is non-increasing and observe its convexity in its targeted inventory level. The backorder costs charged at the ESL depend on the steady state distribution of the number of units in emergency resupply which in turn, is influenced by the target inventory levels at the FSLs. These backorder costs are non-increasing in the budget allocated to the ESL, for given FSL stock levels. Therefore, in our algorithm, we set the inventory levels at the FSLs first and then allocate all the remaining budget to the ESL. Using these ideas, we develop a simple bisection search method to set the desired stock levels.

The reminder of the paper is organized as follows. In section 2 we briefly review the results discussed in Chen. *et al.* (2013) and extend them to the case where there are multiple part types. In section 3, we formulate the objective function and in the following section 4, we build mathematical models and explain our optimization approach. In section 5, we develop algorithms to search for the optimal stock levels for a single part type. Experimental results are shown in section 6. Further discussion on the optimization algorithm for multi-part types is in section 7. Concluding comments are found in section 8.

## 1.2 Literature Review

Sherbrooke (1966) builds a mathematical model for the inventory control of recoverable or repairable items in a base-depot supply system. The model is well known as METRIC (Multi-Echelon Technique for Recoverable Item Control) and is extended and improved by

amount of subsequent papers such as Graves (1985) and Sherbrooke (1986). The METRIC model uses negative binomial distribution to approximate the stationary distribution of the number of units in resupply, which largely simplifies the computational complexity. Two-phase marginal analysis algorithm is used to determine the depot and base stock levels, which optimize the objective function under investment constraints. Alternatively, Fox and Landi (1970) propose a Lagrangian multiplier method for solving the one-constraint optimization problems as in METRIC by using one pass method to search for the suitable multipliers. Muckstadt (1978) suggests a simple approximation for the optimization problem and develops an easier algorithm to determine the stock levels compared with the previous two methods. More details are discussed and summarized in Muckstadt (2005,2010).

One of the basic assumptions in these multi-echelon resupply networks is that each location has a single source of resupply. However, there are numerous examples in practice that locations share inventories among themselves (*lateral transshipment*) or obtain emergency orders from alternative sources (emergency stocking locations). Early studies consider emergency lateral transshipment include Gross (1963), Das (1975), Hoadly and Heyman (1977), Karmarkar and Patel (1977), Cohen *et al.* (1986), Dada (1985), Bowman(1986), and Slay (1986). Using the pooling idea of Cohen *et al.*, Lee (1987) extends the METRIC model so that the out-of-stock bases could get emergency lateral transshipment from other identical bases with inventories in the same group. If all bases in the group have zero inventory, the current demand is sent to the depot. Approximations for the system performance measures, such as backorder level and the number of emergency lateral transshipment, are derived and used to optimize the stocking levels with two-phase method. Axsäter (1990) applies alternative method to model the demand at the bases allowing non-identical bases and compares the results with Lee's when the bases are identical.

In contrast to the military base-depot model as METRIC, Grahovac and Chakravarty (2001) present a commercial supply chains allowing emergency orders and lateral transshipment. When the inventory at the retailers is below some point  $K$ , they place emergency orders from their upstream distribution center. An emergency transshipment is requested only when the distribution center runs out of stock and at least one retailer has more than  $K$  inventory on hand. With the guaranteed and expedited shipment delivery service, this model could prevent unnecessary lateral transshipment and complicate transaction.

There are more papers examining the effect of employing decision rules for making lateral transshipment, such as Dada (1992), Sherbrooke (1992), Evers (1997,1999), Alfredsson and

Verrijdt (1999), Kukreja *et al.* (2001), Muckstadt (2005), Vidgren (2005), Axsäter (2006) and Vliegen (2009). Another type of papers presents methods for optimizing the decisions concerning lateral transshipment. Examples are Das (1975), Robinson (1990), Tagaras and Cohen (1992), Archibald *et al.* (1997), Rudi *et al.* (2001), Minner *et al.* (2003), Wong *et al.* (2006), Olsson (2009), Kranenburg and van Houtum (2009), Wijk *et al.* (2009) and Reijnen *et al.* (2009). Paterson *et al.* (2009) provide an up-to-date review of the inventory models with lateral transshipment.

Different from the resupply networks considering both lateral transshipment and emergency orders from upstream or external supplier, the system considered in this paper contains special stocking location, the ESL, which is dedicated to satisfying emergency orders. Once the FSL is out of stock, it can only fulfill the arriving customer by placing emergency orders from the ESL. No lateral transshipment is allowed among the FSLs at any time. It could also be interpreted as that the inventory shared among the FSLs is stocked at the ESL and is consumed only when one FSL incurs shortage. The systems allowing emergency orders have also been widely studied such as the papers by Rosenshine and Obee (1976), Whittemore and Saunders (1977), Blumenfeld *et al.* (1985), Moinzadeh and Schmidt (1991), Johansen and Thorstenson (1998), Tagaras and Vlachos (2001), Chiang (2002), Axsäter (2007), etc. Refer to Chen *et al.* (2010) for an overview of these papers. Most of the literature focuses on the optimal inventory policy and the replenishment orders modeling with a single-echelon. The emergency orders are placed either from the same source of the regular replenishment or external supplier with infinite inventory. In this paper, the inventory at the ESL is limited and backorder costs at the ESL is included in the objective function. We develop optimization algorithms to determine the optimal stocking levels for this multi-item two-echelon network and investigate the advantage of the ESL under different variance-to-mean-ratio demand scenarios.

## 2. Model Review

To analyze the system we have described, we make a number of simplifying assumptions. As we have stated, we assume that the RSL has an infinite stock of multiple types of items on hand and the lead time from the ESL to the FSL is negligible. Assuming that the demand of the different types of items arrive independently, we are able to construct separable cost and customer service measures that depend on the steady state distributions of the number

of units in regular replenishment from the RSL to the FSL (i.e. in regular resupply) and the number of units in regular replenishment from the RSL to the ESL (i.e. in emergency resupply). We use the complete fill assumption at the FSLs instead of partial fill to maintain consistency among these papers. We could easily extend our results to the partial fill case.

Let  $\lambda^{(n,i)}$  denote the rate of customer arrivals at the  $n$ th FSL ( $n = 1, 2, \dots, N$ ) for type  $i$  part ( $i = 1, 2, \dots, I$ ). Let  $X^{(n,i)}$  denote the size of any customer's order, a positive, integer-valued, random variable. Let  $p_k^{(n,i)} \equiv P\{X^{(n,i)} = k\}$  and let  $\bar{P}_k^{(n,i)} \equiv P\{X^{(n,i)} > k\}$  for all  $k = 0, 1, 2, \dots$ . Since the arrival processes are stuttering Poisson processes,  $X^{(n,i)}$  is geometrically distributed, that is,  $p_k^{(n,i)} = (1 - p^{(n,i)})^{k-1} p^{(n,i)}$ , where  $p^{(n,i)} = p_1^{(n,i)}$  for  $k = 1, 2, \dots$  and  $p_0^{(n,i)} = 0$ . We can easily generalize our results to allow for zero-sized orders.

Let  $I_t^{(n,i)}$  denote the inventory of item  $i$  on hand at the  $n$ th FSL if  $n > 0$  or the ESL if  $n = 0$ , at time  $t$ ,  $t \geq 0$ , a non-negative integer-valued random variable. Recall that the system is managed according to an  $(S - 1, S)$  policy. Suppose a customer arrives the  $n$ th FSL at time  $t$  with demand for part type  $i$ , denoted by  $X_t^{(n,i)}$ . If  $X_t^{(n,i)} \leq I_t^{(n,i)}$  the demand is satisfied by the inventory at the  $n$ th FSL and triggers a regular replenishment order from the RSL to the  $n$ th FSL with size  $X_t^{(n,i)}$ ; otherwise, it is filled by the ESL and the ESL places a replenishment order from the RSL of size  $X_t^{(n,i)}$ . Our decision variables are the stock up to levels, denoted by  $S^{(n,i)}$ , of part type  $i$  at the  $n$ th FSL ( $n > 0$ ) or the ESL ( $n = 0$ ).

From Chen. *et al.* (2011), we have the following result:

Let  $f_{NB}(\cdot; m, p)$  denote the negative binomial probability distribution with parameters  $m$  and  $p$ :

$$f_{NB}(x; m, p) \equiv \binom{m+x-1}{x} p^m (1-p)^x \text{ for } x = 0, 1, 2, \dots$$

**Proposition 1.** *For a lost sales system with stuttering Poisson demand with complete fills and targeted inventory level  $S^{(n,i)}$ , the stationary distribution of the number of units of item  $i$  at the  $n$ th FSL on order, denoted by  $\pi_{s|S^{(n,i)}}$  ( $s = 0, 1, \dots, S^{(n,i)}$ ), is given by:*

$$\pi_{s|S^{(n,i)}} = \frac{\sum_{m=0}^s \left(\frac{\lambda^{(n,i)}}{\mu}\right)^m \frac{f_{NB}(s-m; m, p^{(n,i)})}{m!}}{G(S^{(n,i)})},$$

where  $S^{(n,i)}$  is the stock up to level,  $G(S) = \sum_{s=0}^S \sum_{m=0}^s \frac{(\frac{\lambda^{(n,i)}}{\mu})^m}{m!} f_{NB}(s-m; m, p^{(n,i)})$ , and  $f_{NB}(s-m; 0, p) = 1\{s=0\}$  when  $m=0$ . i.e. the truncated compound Poisson distribution.

The notation  $\pi_{s|S^{(n,i)}}$  emphasizes its dependence on the value of  $S^{(n,i)}$ .

Chen. *et al.* (2013) show how to approximate the steady state distribution of the number of units of type  $i$  in emergency resupply by a zero-truncated negative binomial distribution with an atom at zero given the stock levels of each item at the FSLs:  $\vec{S}^{(i)} = (S^{(1,i)}, S^{(2,i)}, \dots, S^{(N,i)})$ . Denoting this approximate distribution function by  $f_z(\vec{S}^{(i)})$  for  $z = 0, 1, \dots$ , we have

$$f_z(\vec{S}^{(i)}) = \begin{cases} f_0, & \text{if } z = 0, \\ (1 - f_0) \left[ \frac{1}{1 - (p_E)^{r_E}} \binom{z + r_E - 1}{z} (p_E)^{r_E} (1 - p_E)^z \right], & \text{if } z > 0, \\ 0, & \text{otherwise.} \end{cases}$$

where  $f_0$ ,  $r_E$  and  $p_E$  depend on  $\vec{S}^{(i)}$ .

Next, we will use the steady state distributions  $\pi_{s|S^{(n,i)}}$  and  $f_z(\vec{S}^{(i)})$  to construct the cost function.

### 3. Objective Function Formulation

As we have mentioned, our objective function consists of the emergency order penalty cost and a backorders penalty cost at the ESL. Define  $\vec{S}_T^{(i)} = (\vec{S}^{(i)}, S^{(0,i)})$  and  $\vec{S}_T = (\vec{S}_T^{(1)}, \vec{S}_T^{(2)}, \dots, \vec{S}_T^{(I)})$ .

Let  $c_E^{(i)}$  denote the emergency order cost per backordering incident per order of part type  $i$  at the FSLs. The expected emergency order penalty at the  $n$ th FSL equals

$$\begin{aligned} c_{FSL}^{(n,i)}(S^{(n,i)}) &\equiv c_E^{(i)} \lambda^{(n,i)} \{ \sum_{s=0}^{S^{(n,i)}} P[X^{(n,i)} > S^{(n,i)} - s] \pi_{s|S^{(n,i)}} \} \\ &= c_E^{(i)} \lambda^{(n,i)} \sum_{s=0}^{S^{(n,i)}} (1 - p^{(n,i)})^{S^{(n,i)} - s} \pi_{s|S^{(n,i)}}, \end{aligned} \quad (1)$$

and the expected emergency order penalty at FSLs for part type  $i$  is given by

$$C_{FSL}^{(i)}(\vec{S}^{(i)}) = \sum_{n=1}^N c_{FSL}^{(n,i)}(S^{(n,i)}) \quad (2)$$

which does not depend on  $S^{(0,i)}$ . The total expected emergency order penalty at the FSLs is  $C_{FSL}(\vec{S}_T) \equiv \sum_{i=1}^I C_{FSL}^{(i)}(\vec{S}_T^{(i)}) = \sum_{i=1}^I \sum_{n=1}^N c_{FSL}^{(n,i)}(\vec{S}_T^{(n,i)})$ .

Let  $c_B^{(i)}$  denote the backorder cost per unit time per unit of part type  $i$  backordered at the ESL. The expected backorder costs for part type  $i$  given  $\vec{S}_T^{(i)}$  is denoted by

$$C_{ESL}^{(i)}(\vec{S}_T^{(i)}) \equiv c_B^{(i)} E[(z - S^{(0,i)}) | \vec{S}^{(i)}] = c_B^{(i)} \left[ \sum_{z=S^{(0,i)}}^{\infty} (z - S^{(0,i)}) f_z(\vec{S}^{(i)}) \right].$$

The total expected backorder cost is  $C_{ESL}(\vec{S}_T) \equiv \sum_{i=1}^I C_B^{(i)}(\vec{S}_T^{(i)})$ .

Therefore the total expected cost associated with part type  $i$  is

$$\begin{aligned} C^{(i)}(\vec{S}_T^{(i)}) &\equiv C_{FSL}^{(i)}(\vec{S}_T^{(i)}) + C_{ESL}^{(i)}(\vec{S}_T^{(i)}) \\ &= c_E^{(i)} \sum_{n=1}^N \lambda^{(n,i)} \sum_{s=0}^{S^{(n,i)}} (1 - p^{(n,i)})^{S^{(n,i)}-s} \pi_{s|S^{(n,i)}} + c_B^{(i)} \sum_{z=S^{(0,i)}}^{\infty} (z - S^{(0,i)}) f_z(\vec{S}_T^{(i)}), \end{aligned}$$

and the total cost for the system with multiple part types is

$$C(\vec{S}_T) = \sum_{i=1}^I C^{(i)}(\vec{S}_T^{(i)}) = \sum_{i=1}^I C_{FSL}^{(i)}(\vec{S}_T^{(i)}) + C_{ESL}^{(i)}(\vec{S}_T^{(i)}, S^{(0,i)}).$$

**Proposition 2.** *For any  $(n, i)$ , the function  $c_{FSL}^{(n,i)}(S^{(n,i)})$  is non-increasing in  $S^{(n,i)}$ .*

**Proof:** Without loss of generality, we drop the superscript  $(n, i)$ . The value  $c_{FSL}(S)$  is the expected emergency order cost given the target stock level  $S$ . For the FSL with target stock level  $S + 1$ , if the inventory policy of the FSL is changed and it is not allowed to use the last unit on hand at the FSL (Scenario 1), the resupply process is exactly the same as that of the FSL employing (S-1,S) inventory policy with target stock level  $S$  (Scenario 2). For any sample path of the arrival process, the emergency order cost is the same for both scenarios. Now given any sample path of the arrival process, if the spare unit is consumed at any point in time and never resupplied, the corresponding emergency order cost is non-increased. Furthermore, if the spare unit is resupplied later and could be consumed again, the corresponding emergency order cost for the same sample path should be not larger than the no resupply case. Therefore, given sample path of the arrival process, the emergency order cost for the FSL employing (S-1,S) inventory policy with target stock level  $S + 1$  is smaller than the cost for the Scenario 1 FSL in Scenario 1, or the Scenario 2 FSL, which employs (S-1,S) inventory policy with target stock level  $S$ . Thus, the expected emergency order cost  $c_{FSL}(S)$  is non-increasing in  $S$ . ■

In addition, empirical investigation of  $c_{FSL}(S)$  suggests that the function is convex in  $S$  over a wide range of parameters. Consequently, we use algorithms that exploit this apparent convexity. Should a case emerge in which this function is found to be non-convex, we recommend using the largest convex minorant of the true function.

**Proposition 3.** *For any  $i$ , the function  $C_{ESL}^{(i)}(\vec{S}_T^{(i)}, S^{(0,i)})$  is non-increasing and convex in  $S^{(0,i)}$  when  $\vec{S}_T^{(i)}$  is fixed.*

The result could be easily proved by using first order differences.



## 4. Mathematical Modeling

As seen in proposition 2 and 3, it seems that the system should set the inventory levels at the FSLs and ESL as high as possible to minimize the associated emergency order penalty costs and backorder penalty costs. However, in real life situations, limits often exist on the system investment in inventory over all part types due to the holding costs and capital limits. Given the investment limits, we then have to balance the stock levels of different part types to minimize the overall expected costs. Let  $B$  denote the fixed inventory investment budget. Since the cost functions are non-increasing, the optimal targeted inventory stock levels should sum up to  $B$ . Let  $Z^+$  be the state space of nonnegative integers. The mathematical model of the whole system is as follows:

$$\begin{aligned} \min_{\vec{S}_T} \quad & C(\vec{S}_T) \\ \text{s.t.} \quad & \sum_{i=1}^I \sum_{n=0}^N S^{(n,i)} = B, \\ & S^{(n,i)} \in Z^+ \text{ for } n = 0, 1, \dots, N; i = 0, 1, \dots, I. \end{aligned}$$

Our approach is to minimize the cost of each product  $C^{(i)}(\vec{S}_T^{(i)})$  given a specified budget  $B^{(i)}$  for part type  $i$  first, and then to minimize the overall cost  $C(\vec{S}_T)$  by varying  $\vec{B} = (B^{(1)}, \dots, B^{(I)})$  where  $\sum_{i=1}^I B^{(i)} = B$ . Therefore, the model is changed into:

$$\begin{aligned} \min_{\vec{B}} \quad & \sum_{i=1}^I \min_{\{\sum_{n=0}^N S^{(n,i)} = B^{(i)}\}} C^{(i)}(\vec{S}_T^{(i)}) \\ \text{s.t.} \quad & \sum_{i=1}^I B^{(i)} = B, \\ & B^{(i)} \in Z^+ \text{ for } i = 0, 1, \dots, I. \end{aligned} \tag{3}$$

Hence, for each part type  $i$ , we have the subproblem  $(i)$ :

$$\begin{aligned} \min_{\vec{S}_T^{(i)}} \quad & C^{(i)}(\vec{S}_T^{(i)}) \\ \text{s.t.} \quad & \sum_{n=0}^N S^{(n,i)} = B^{(i)}, \\ & S^{(n,i)} \in Z^+ \text{ for } n = 0, 1, \dots, N, \end{aligned}$$

which is the same as

$$\begin{aligned}
G^*(B^{(i)}) &\equiv \min_{\vec{S}^{(i)}} C_{FSL}(\vec{S}^{(i)}) + C_{ESL}(\vec{S}_T^{(i)}) \\
\text{s.t.} \quad &\sum_{n=1}^N S^{(n,i)} + S^{(0,i)} = B^{(i)}, \\
&S^{(n,i)} \in Z^+ \text{ for } n = 0, 1, \dots, N.
\end{aligned} \tag{4}$$

Define the optimal minimizer as  $\vec{S}^*(B^{(i)})$ . Now, our problem (3) is equivalent to

$$\begin{aligned}
\min_{\vec{B}} \quad &\sum_{i=1}^I G^*(B^{(i)}) \\
\text{s.t.} \quad &\sum_{i=1}^I B^{(i)} = B, \\
&B^{(i)} \in Z^+ \text{ for } i = 0, 1, \dots, I.
\end{aligned} \tag{5}$$

Our goal is to construct the maximal convex minorant of function  $G^*(B^{(i)})$  for each part type  $i$  and then to use marginal analysis to solve the resulting optimization problem (5). After obtaining the optimal allocation of the budget among the  $I$  part types  $\vec{B}^* = (B^{*(1)}, \dots, B^{*(I)})$ , the optimal stock levels on the FSLs are the corresponding values of the  $\vec{S}^*(B^{*(i)})$  from (4) for the associated budgets  $B^{*(i)}$ .

#### 4.1 Optimize Order-up-to-Levels at Different Locations for a Single Item

In this section, we demonstrate a method to approximate the function  $G^*(B^{(i)})$  for a given part type  $i$  (problem(4)). To simplify notation, we drop the superscript  $(i)$  and use  $S_n$  instead of  $S^{(n,i)}$ . Let  $\vec{S} = (S_1, \dots, S_N)$  represent the stock levels for the FSLs. The optimization problem we wish to solve, (4), is rewritten as

$$\begin{aligned}
G^*(B) &\equiv \min_{\vec{S}} C_{FSL}(\vec{S}) + C_{ESL}(\vec{S}_T) \\
\text{s.t.} \quad &\sum_{n=1}^N S_n + S_0 = B, \\
&S_n \in Z^+ \text{ for } n = 0, \dots, N.
\end{aligned} \tag{6}$$

Let  $B_F$  denote the total inventory at the FSLs. The the optimal stock level at the ESL should be  $S_0 = B - B_F$ . That is, all of the remaining budget should be allocated to the ESL due to the non-increasing nature of the backorder cost function. The expected cost given

$B_F$  and  $B$  is denoted as  $J^*(B_F, B)$ :

$$\begin{aligned} J^*(B_F, B) &\equiv \min_{\vec{S}} C_{FSL}(\vec{S}) + C_{ESL}(\vec{S}, B - B_F) \\ \text{s.t.} \quad &\sum_{n=1}^N S_n = B_F \\ &S_n \in Z^+ \text{ for } n = 1, \dots, N. \end{aligned}$$

It follows that  $G^*(B) = \min_{B_F \leq B} J^*(B_F, B)$ .

Due to the time consuming step of matrix inversion used to analyze the behavior of the ordered units at the ESL (Chen, *et al.* 2013), it takes a much longer time to compute  $C_{ESL}(\vec{S}, B - B_F)$  than to compute  $C_{FSL}(\vec{S})$  given any  $\vec{S}$ . To mitigate this problem, we define the following alternative optimization problem focusing on the FSLs:

$$\begin{aligned} H_F^*(B_F) &\equiv \min_{\vec{S}} C_{FSL}(\vec{S}) \\ \text{s.t.} \quad &\sum_{n=1}^N S_n = B_F \\ &S_n \in Z^+ \text{ for } n = 1, \dots, N. \end{aligned} \tag{7}$$

Denote its optimal solution as  $\vec{S}^*(B_F)$ . Instead of solving problem (6), we then use

$$\tilde{G}^*(B) \equiv \min_{B_F \leq B} \tilde{J}^*(B_F, B) = \min_{B_F \leq B} H_F^*(B_F) + C_{ESL}(\vec{S}^*(B_F), B - B_F). \tag{8}$$

as an approximation to  $J^*(B_F, B)$  given any  $B$  and  $B_F \leq B$ .

Recall from (1) and (2) that

$$\begin{aligned} C_{FSL}(\vec{S}) &= \sum_{n=1}^N c_{FSL}^{(n)}(S_n) \\ &= c_E \sum_{n=1}^N \sum_{s=0}^{S_n} [\lambda^{(n)}(1 - p^{(n)})^{S_n - s}] \pi_{s|S_n}, \end{aligned} \tag{9}$$

where  $c_{FSL}^{(n)}(S_n)$  is non-increasing (proposition 2). As noted, numerical experiments suggest  $c_{FSL}^{(n)}(S_n)$  convex in  $S_n$ . Therefore, we use marginal analysis to solve the optimization problem (7) given any  $B_F$ . Denote  $B_F^*(B)$  as the optimal solution of problem (7), as found by marginal analysis.

Next we use a line search method such as bisection or golden section search to solve  $\tilde{G}^*(B) = \min_{B_F \leq B} \tilde{J}^*(B_F, B)$  as an approximation of  $G^*(B)$  given  $B$ . Denote the corresponding optimized inventory levels which depend on  $B$  by

$$\vec{S}_T(B_F^*(B)) = (\vec{S}(B_F^*(B)), B - B_F^*(B)).$$

Hence we find  $\tilde{G}^*(B)$  through a combination of marginal analysis over  $S_n$  nested within a line search over  $B_F$ .

It is conceivable that the optimal solution to problem (6) does not simultaneously optimize problem (8), i.e.  $G^*(B) \neq \tilde{G}^*(B)$ . To explore the difference, we compare the solutions to (8) with the best solutions to (6) found using a more comprehensive search algorithm. The particular search algorithm used as a benchmark is the Particle Swarm Pattern Search method (PSWARM), developed by Vaz and Vicente(2007,2009) for solving minimization problem subject to simple bounds (linear constraints) without the use of derivatives. We find that by solving problem (8), we are able to obtain solutions close to the benchmark results.

## 5. Heuristic Algorithms Given Inventory Investment for Single Item

In this section, we describe the methods sketched in the previous section.

- (H1) “**Nested Search**”: Experimentation strongly suggests that  $\tilde{J}^*(B_F, B)$  is unimodal in  $B_F$ . Consequently, we use a bisection search for  $B_F$  minimizing  $\tilde{J}^*(B_F, B)$  instead of computing  $C_{ESL}(\vec{S}, B - B_F)$  exhaustively for all possible values of  $B_F$ . This algorithm has two steps: Step one, use marginal analysis to determine  $H_F^*(B_F)$  and  $\vec{S}^*(B_F)$  for  $B_F = 0, \dots, B$ ; Step two, use bisection search to determine the optimal  $B_F^*(B)$  which minimizes  $\tilde{J}^*(B_F, B) = H_F^*(B_F) + C_{ESL}(\vec{S}, B - B_F)$  where  $B$  is known and fixed.

*Step One: Marginal Analysis*

1. Start with  $B_F = 0$  and  $\vec{S} = (0, \dots, 0)$ . Define  $g(B_F) = \vec{S}$ .
2. For all  $n$ , compute  $\Delta c_{FSL}^{(n)}(S_n) = c_{FSL}^{(n)}(S_n + 1) - c_{FSL}^{(n)}(S_n)$  and let  $n^* = \operatorname{argmin}_n \Delta c_{FSL}^{(n)}(S_n)$ .
3. Update  $S_{n^*} = S_{n^*} + 1$ ,  $B_F = B_F + 1$  and  $g(B_F) = \vec{S}$ . If  $B_F < B$  then go back to 2. Otherwise, go to Step Two.

*Step Two: Bisection Search*

1. Choose initial interval  $[a, b]$  over which the minimum of  $g(B_F) = C_{FSL}(\vec{S}^*(B_F)) + C_{ESL}(\vec{S}^*(B_F), B - B_F)$  is to be found. For example,  $a = 0$  and  $b = B$ . Choose separation constant  $\varepsilon$  and stopping tolerance  $\theta$ .

2. If  $b-a < \theta$ , then go to step 4. Otherwise, let  $a' = (a+b)/2-\varepsilon$  and  $b' = (a+b)/2+\varepsilon$ . Round  $a', b'$  to their nearest integers.
3. If  $g(a') < g(b')$ , then update  $b = b'$ . Otherwise, update  $a = a'$ . Then go back to 2.
4. Identify  $B_F^*(B) = \operatorname{argmin}_{\{x=a, a', b, b'\}} g(x)$  and  $\tilde{G}^*(B) = g(B_F^*(B))$ .

Validate the method of approximating  $G^*(B)$  by  $\tilde{G}^*(B)$ , we use PSWARM to search over the state space

$$\mathfrak{S}(B) = \{\vec{S}_T \in \{Z^+, \dots, Z^+\}_{1 \times N+1} : \sum_{n=1}^N S_n + S_0 = B\}.$$

Refer to Vaz and Vicente(2007,2009) for more details of PSWARM. A high level view of the algorithm is as follows:

- (H2) “**PSWARM Search**”:

1. Pick an initial point  $\vec{S}_T$  in  $\mathfrak{S}(B)$ .
2. Use PSWARM algorithm to search over  $\mathfrak{S}(B)$  for the optimal solution  $\vec{S}_T^*(B) = (\vec{S}^*(B), S_0^*(B))$  which minimizes  $C_{FSL}(\vec{S}) + C_{ESL}(\vec{S}_T)$ .
3. Return  $G^*(B) = C_{FSL}(\vec{S}^*(B)) + C_{ESL}(\vec{S}_T^*(B))$ .

## 6. Experimental Results

We conduct experiments with five FSLs,  $N = 5$ , for a single part type in the system. We fix the lead time,  $\tau_F = \tau_E$ , equal to 1. The backorders cost parameter is  $C_B = 20$ , and the emergency order penalty,  $C_E$ , is one of the values 0.25, 1, 5 or 10.

To describe the arrival processes, let  $W_n(t)$  denote the cumulative unit arrivals during time  $t$  for the  $n$ th FSL. Let  $\mu_n$  denote the arrival process mean rate, which is equal to  $\frac{E(W_n(t))}{t} = \frac{\lambda_n}{p_n}$ . Let  $\sigma_n^2$  denote the arrival process variance rate, which is equal to  $\frac{Var(W_n(t))}{t} = \lambda_n(\frac{1-p_n}{p_n^2} + \frac{1}{p_n^2})$ . Let  $\rho_n$  denote the arrival process variance-to-mean ratio (VTMR), which is equal to

$$\frac{Var(W_n(t))/t}{E(W_n(t))/t} = \frac{\sigma_n^2}{\mu_n} = \frac{2-p_n}{p_n}.$$

We study cases both of identical FSLs and non-identical FSLs. After considering these cases, we explore the shape of the cost function  $\tilde{G}^*(B)$  and recommend using a convex minorant of this function for solving multiple item problems.

Table 1: The Optimal Inventory Levels and Expected Total Cost for the Identical Independent FSLs case when  $B = 80$  :

$\mu_n$	$C_E$	$\rho_n = \frac{\sigma_n^2}{\mu_n} = 1.01$			$\rho_n = \frac{\sigma_n^2}{\mu_n} = 2$			$\rho_n = \frac{\sigma_n^2}{\mu_n} = 5$		
		$\vec{S}^*(B)$	$S_0^*$	$G^*(B)$	$\vec{S}^*(B)$	$S_0^*$	$G^*(B)$	$\vec{S}^*(B)$	$S_0^*$	$G^*(B)$
1	0.25	[ 16 16 16 16 16 ]	0	0	[ 13 13 13 13 13 ]	15	0.000147	[ 10 10 10 10 10 ]	30	0.0207
	1	[ 16 16 16 16 16 ]	0	0	[ 14 14 13 13 13 ]	13	0.000416	[ 11 11 11 10 10 ]	27	0.0692
	5	[ 16 16 16 16 16 ]	0	0	[ 14 14 14 14 14 ]	10	0.00119	[ 12 12 11 11 11 ]	23	0.272
	10	[ 16 16 16 16 16 ]	0	0	[ 15 15 14 14 14 ]	8	0.00197	[ 12 12 12 12 12 ]	20	0.482
5	0.25	[ 16 15 15 15 15 ]	4	0.00111	[ 14 14 13 13 13 ]	13	0.0803	[ 6 6 6 5 5 ]	52	1.14
	1	[ 16 16 15 15 15 ]	3	0.00376	[ 14 14 14 14 14 ]	10	0.252	[ 8 8 8 8 8 ]	40	2.76
	5	[ 16 16 16 15 15 ]	2	0.0149	[ 15 15 14 14 14 ]	8	0.990	[ 11 11 10 10 10 ]	28	8.40
	10	[ 16 16 16 16 15 ]	1	0.0271	[ 15 15 15 15 14 ]	6	1.78	[ 12 11 11 11 11 ]	24	14.0
10	0.25	[ 14 13 13 13 13 ]	14	1.29	[ 8 8 7 7 7 ]	43	4.76	[ 0 0 0 0 0 ]	80	11.8
	1	[ 14 14 14 14 14 ]	10	3.55	[ 11 10 10 10 10 ]	29	11.4	[ 1 1 1 1 1 ]	75	24.0
	5	[ 15 15 15 15 15 ]	5	12.1	[ 13 13 13 13 12 ]	16	32.7	[ 6 6 6 6 5 ]	51	64.5
	10	[ 16 15 15 15 15 ]	4	21.2	[ 14 14 13 13 13 ]	13	52.8	[ 8 8 8 8 8 ]	40	98.6

## 6.1 Identical Independent FSLs

First, we conduct an experimental study assuming that the demand distribution is identical for all FSLs. Therefore  $\mu_n$  and  $\rho_n$  are common for  $n = 1, \dots, 5$ . We choose  $\mu_n$  to be equal to one of 1, 5, or 10 and  $\rho_n$  to be equal to one of 1.01, 2, or 5. The investment budget  $B$  is fixed and equal to 80. For this identical independent FSLs case, our heuristic algorithm (H1) solving problem (8) and the benchmark PSWARM algorithm (H2) solving problem (6) all lead to the same optimal solutions, i.e.  $G^*(B) = \tilde{G}^*(B)$ .

Table 1 shows the optimal solutions of stock levels,  $\vec{S}_T = [\vec{S}, S_0]$ , and the optimized cost,  $G^*(B) = \tilde{G}^*(B)$ . From the experimental results, we see that the optimal stock levels of the FSLs are not necessary identical but the stock level differences are at most one. For each combination of  $(\mu_n, \rho_n)$ , we observe that as the emergency order penalty  $C_E$  increases, the optimal stock levels at the FSLs increases. This means less inventory is kept at the ESL. Furthermore, the increase in  $C_E$  also causes the total expected cost  $G^*(B)$  to increase.

When the demand mean,  $\mu_n$  is held constant but the variance-to-mean ratio  $\rho_n$  increases, the total expected cost  $G^*(B)$  increases and more inventory is stocked at the ESL.

When the variance-to-mean ratio,  $\rho_n$ , is held constant but the mean,  $\mu_n$ , increases, the total expected cost  $G^*(B)$  also increases and more inventory is stocked at the ESL as well.

Table 2: Parameters of the Non-Identical Demand Arrival Processes at the FSLs :

$n$	1	2	3	4	5
Demand Type	LM	LH	MM	HL	HM
$\mu_n$	1	1	5	10	10
$\sigma_n^2$	2	5	10	10.1	20
$\rho_n = \frac{\sigma_n^2}{\mu_n}$	2	5	2	1.01	2
$\lambda_n$	0.667	0.333	3.33	9.95	6.67
$p_n$	0.667	0.333	0.667	0.995	0.667

## 6.2 Non-Identical Independent FSLs

We conduct another experiment study assuming that the five FSLs have different arrival process as shown in Table 2. The parameters are chosen to represent the different scenarios that might be encountered in real life: low mean demand with medium variance (LM), low mean demand with high variance (LH), medium mean demand with medium variance (MM), high mean demand with low variance (HL), and high mean demand with medium variance (HM).

In Table 3, we show the optimal solutions of stock levels  $\vec{S}_T = [\vec{S}, S_0]$  and the optimal cost  $\tilde{G}^*(B)$  solved by algorithm (H1) and  $G^*(B)$  solved by algorithm (H2). The investment budget  $B$  is equal to either 30, 50 or 80.

For this case of non-identical FSLs, the optimal solution of problem (6) is different from but close to that of (8) for most cases. The relative error,  $\frac{\tilde{G}^*(B) - G^*(B)}{G^*(B)}$  is bounded by 5% and less than 1% for most cases. The approximation is especially good when the budget  $B$  is small. Combined with the results of the identical independent FSLs, it suggests that  $\tilde{G}^*(B)$  is a reasonable approximation of  $G^*(B)$  and therefore the bisection search algorithm (H2) is recommended to save computing time.

When the investment budget  $B$  is low and equal to 30, no inventory is kept at the FSLs with low demand rate. Instead, inventories are concentrated at the FSLs with high mean and at the ESL. The FSL with high mean and medium variance receives less stock allocation

Table 3: The Optimal Inventory Levels and Expected Total Cost for the Non-Identical Independent FSLs case:

$B$	$C_E$	Algorithms (H1)			Algorithm (H2)			Relative Error
		$\tilde{S}^*(B)$	$S_0^*$	$\tilde{G}^*(B)$	$\tilde{S}^*(B)$	$S_0^*$	$G^*(B)$	
30	0.25	[ 0 0 0 0 0 ]	30	36.2	[ 0 0 0 0 0 ]	30	36.2	0 %
	1	[ 0 0 0 4 0 ]	26	50.9	[ 0 0 0 4 0 ]	26	50.9	0 %
	5	[ 0 0 1 10 5 ]	14	100	[ 0 0 1 9 6 ]	14	99.6	0.564 %
	10	[ 0 0 3 11 7 ]	9	139.9	[ 0 0 3 10 8 ]	9	139.6	0.196 %
50	0.25	[ 0 0 4 12 9 ]	25	2.10	[ 0 0 3 12 12 ]	23	2.05	2.80 %
	1	[ 0 0 6 14 13 ]	17	4.84	[ 0 0 6 13 14 ]	17	4.82	0.404 %
	5	[ 1 0 8 15 15 ]	11	14.0	[ 1 0 8 15 15 ]	11	14.0	0 %
	10	[ 2 0 8 16 16 ]	8	23.1	[ 2 0 9 15 16 ]	8	22.9	0.908 %
80	0.25	[ 4 4 13 19 21 ]	19	0.111	[ 4 3 13 19 22 ]	19	0.106	4.36 %
	1	[ 4 4 13 19 22 ]	18	0.345	[ 5 3 13 20 23 ]	16	0.332	3.77 %
	5	[ 5 6 14 20 23 ]	12	1.22	[ 5 5 14 20 23 ]	13	1.21	0.958 %
	10	[ 5 7 15 20 23 ]	10	2.09	[ 5 7 14 20 23 ]	11	2.09	0.188 %

than the FSL with high mean and low variance.

When the investment budget  $B$  increases to 50, the optimal stock levels for low mean demand are still very small. In particular, the optimized solution still chooses to keep zero inventory at the FSL which has a low mean and high variance. In addition, with increased budget, less inventory is stocked at the ESL.

When the investment budget  $B$  increases to 80, the optimal stock levels increases significantly for all the FSLs. The FSL with a high mean demand stocks more if its variance is higher. This suggests that if the investment budget is large enough, the optimal stock levels are positively correlated with both the mean and variance of the demand. As a general rule, our intuition is that when the investment budget  $B$  is large enough, all the inventory should be kept at the FSLs to prevent out-of-stock/emergency orders and no inventory is needed at the ESL. At the same time, the FSLs with higher variance need to stock more than FSLs with the same mean level but low variance.



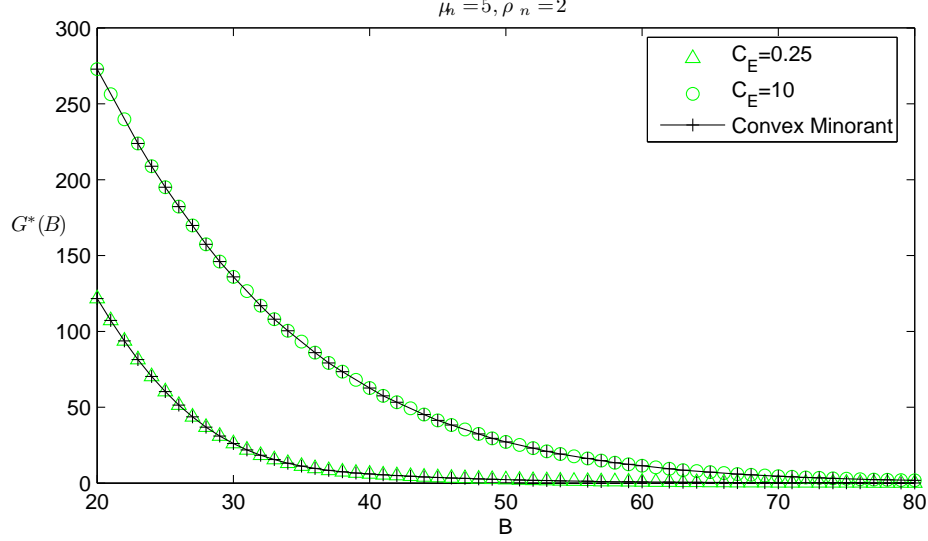


Figure 2: Cost Curve with Largest Minimal Convex Minorant for Identical Independent FSLs Case

### 6.3 Inventory Investment Budget Cost Curve

In this section, we explore the shape of the cost function  $\tilde{G}^*(B)$  and recommend using a convex minorant of this function for advanced work.

We use bisection search algorithm, (H1) to investigate the shape of cost function  $\tilde{G}^*(B)$ . Figure 2 and 3 are the plots of  $\tilde{G}^*(B)$  (light color with different shapes) and its corresponding convex minorant (black line with cross), named as  $\hat{G}^*(B)$ , for the identical and non-identical FSLs. The plots show that  $\tilde{G}^*(B)$  is nearly convex and the corresponding largest convex minorant  $\hat{G}^*(B)$  provides a very nice approximation of  $\tilde{G}^*(B)$ .

## 7. Algorithms Optimizing Stock up to Levels for Multiple Items

In this section, we return to the original system with  $I$  part types and propose heuristics to search for the optimal solution to problem (6). Define an alternative optimization problem

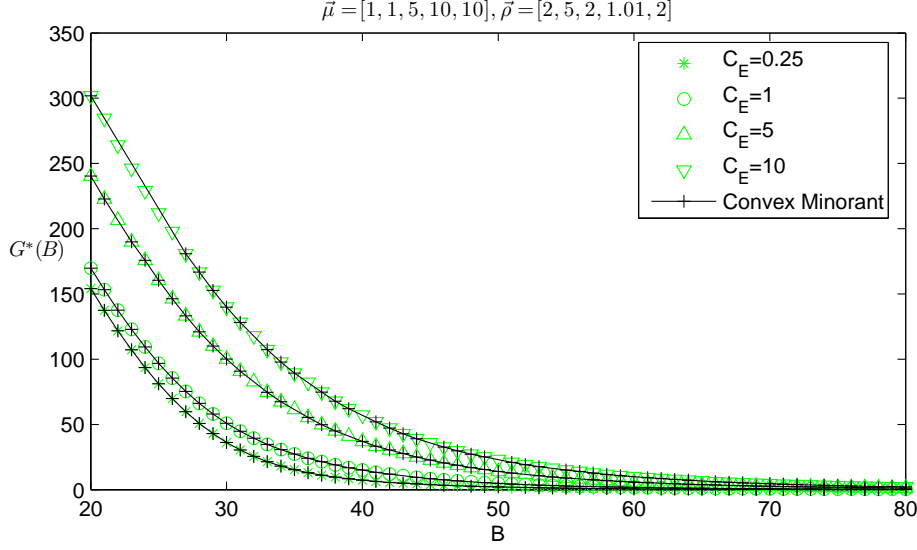


Figure 3: Cost Curve with Largest Minimal Convex Minorant for Non-Identical Independent FSLs Case

as follows:

$$\begin{aligned}
 \min_{\vec{B}} \quad & \sum_{i=1}^I \hat{G}^*(B^{(i)}) \\
 \text{s.t.} \quad & \sum_{i=1}^I B^{(i)} = B, \\
 & B^{(i)} \in Z^+ \text{ for } i = 0, 1, \dots, I.
 \end{aligned} \tag{10}$$

where  $\hat{G}^*(B^{(i)})$  is the convex minorant introduced in the previous section.

As discussed in section 4 and subsection 6.3,  $\hat{G}^*(B^{(i)})$  provides a good approximation for  $G^*(B^{(i)})$ . It is anticipated that the optimal solution of problem (10) will provide a good approximation for  $\vec{B}^*$ , the optimal solution of problem (6).

We propose the three main elements for the algorithm to solve optimization problem (10) as follows:

- **Convex Minorant Local Construction:** For each part type  $i = 1, \dots, I$ , we use bisection search algorithm (H2) to compute the value of  $\tilde{G}^*(B^{(i)})$  for  $B^{(i)} \in [0, U^{(i)}]$ , where  $[0, U^{(i)}]$  is the region containing the optimal solution  $B^{(i)*}$ . Initially,  $U^{(i)}$  is chosen to be a value much smaller than  $B$ .
- **Marginal Analysis Search:** Compute the convex minorant  $\hat{G}^*(B^{(i)})$  for  $\tilde{G}^*(B^{(i)})$  and use a marginal analysis algorithm to search for the optimal solution of problem (10).

- **Upper Bound Update:** If the solution  $B^{(i)}$  to the marginal analysis results in  $B^{(i)} = U^{(i)}$  for some FSL, then estimate another larger upper bound  $U'^{(i)}$  and use bisection search to compute  $\tilde{G}^*(B^{(i)})$  for  $B^{(i)} \in [U^{(i)}, U'^{(i)}]$ . Update  $\hat{G}^*(B^{(i)})$  over  $[0, U'^{(i)}]$ . Continue with the marginal analysis search for problem (10).

## 8. Conclusions

We develop optimization algorithms for setting stock levels in a resupply network with both field service locations (FSL) and an emergency stocking location(ESL). We proposed a bisection search algorithm to determine the stock levels at the FSLs and ESL given inventory investment for single item. Since the problem is a problem with a potentially non-convex objective, we use PSWARM as a benchmark to validate the bisection search algorithm.

From the empirical results, we find that when the inventory investment budget is small, as the VTMR of demand at the FSL increases, the optimal solutions incline to stock less at the FSL. While the VTMR is small, the demand rate is the dominating factor in deciding the stock levels at the FSLs. When the budget is small, the ESL plays an important role to stock the inventory shared among the FSLs. However, when the inventory investment budget becomes large enough, the optimal solutions incline to stock more at the FSLs which have higher demand rate and higher variance. Less inventory is kept at the ESL since the emergency orders and its associated costs are mainly reduced by the large amount inventory stocked at the FSLs.

On the other hand, as the emergency penalty cost per order decreases, meaning that the shortage of the FSLs incurs less penalty, the optimal stock levels of the FSLs decrease and it is optimal to stock more at the ESL. This also supports a strategy that when there is little emergency penalty cost, we incline to stock everything at one location, the ESL, to pool the variability of demand at each FSL. Besides, given a fixed inventory investment budget, the expected cost always increases as the mean and the variance of the demand increases.

We conclude that the benefit of the ESL becomes significant when the inventory investment budget is small and the VTMR of the demand is large. Besides, it is recommended to increase the investment budget to control the system cost when the mean and the variance of the demand at the FSLs increase.

After understanding the cost function given any inventory investment budget for single item, we propose the main elements of an algorithm to solve the optimal stock levels for the

multi-item and multi-location problem. These elements include using the convex minorant of the cost function for each item and applying marginal analysis over the total inventory investment budget.

## References

- [1] Alfredsson, P. and J. Verrijdt. Modeling Emergency Supply Flexibility in a Two-Echelon Inventory System. *Management Science*, 45 (1999), No. 10, 1416-1431.
- [2] Audet, C. and J.E. Dennis. Analysis of generalized pattern searches. *SIAM J. Optim.* 13. 889-903 (2003)
- [3] Axsäer, S.. Modeling emergency lateral transshipments in inventory systems. *Management Science*, 36 (1990), 1329-1338.
- [4] Axsäter, S. A New Decision Rule for Lateral Transshipments in Inventory Systems, *Management Science*, Vol. 49, No. 9, (Sep. 2003), pp. 1168-1179.
- [5] Axsäter, S.. *Inventory Control*, 2nd ed., Springer, New York, 2006.
- [6] Axsäter, S.. A heuristic for triggering emergency orders in an inventory system. *European Journal of Operational Research*, 176 (2007) 880C891.
- [7] Archibald, T.W., S.A. Sassen, and L.C. Thomas. An optimal policy for a two depot inventory problem with stock transfer. *Management Science*, 43 (1997), 173C183.
- [8] Chen, J., P. Jackson, and J. Muckstadt. Exact Analysis of a Lost Sales Model under Stuttering Poisson Demand. Technical Note, *Operations Research* Vol 59 (2011), No. 1 249-253. Technical Report No. 1474.
- [9] Chen, J., P. Jackson, and J. Muckstadt. An Emergency Resupply Network Model under Stuttering Poisson Demand. (2013) Submitted. Technical Report No. 1475, School of Operations Research and Information Engineering, Cornell University.
- [10] Chiang, C.. Ordering Policies for Inventory Systems with Expediting. *The Journal of the Operational Research Society*, Vol 53 (2002), No.11, 1291-1295.
- [11] Cohen, M. A., P. R. Kleindorfer and H. L. Lee, Optimal Stocking for Low Usage Items in Multi-Echelon Inventory Systems, *Naval Research Logistics Quarterly*, 33 (1986), 17-38.

- [12] Dada, M., A Two-Echelon Inventory System with Priority Shipments, *Management Science*, 1140–1153 (1992).
- [13] Dada, M.. A Two-Echelon Inventory System with Priority Shipments, *Management Science*. 38 (1992), 1140-1153.
- [14] Davis, C., Theory of positive linear dependence. *Am. J. Math* 76, 733-746 (1954)
- [15] Das, C.. Supply and Redistribution Rules for Two-Location Inventory Systems: One Period Analysis, *Management Science*, 21 (1975), 765-779.
- [16] Evers, P.R., Hidden benefits of emergency transshipments. *Journal of Business Logistics*. 18(2):55-77,1997.
- [17] Evers, P.R., Filling customer orders from multiple locations: A comparison of pooling methods. *Journal of Business Logistics*, 20(1): 121-140, 1999.
- [18] Fox, B.L. and D.M. Landi. Searching for the multiplier in one constraint optimization problems. *Operations Research*, 18(2):253-262, March-April 1970.
- [19] Grahovac, J. and A. Chakravarty. Sharing and Lateral Transshipment of Inventory in a Supply Chain with Expensive, Low-Demand Items. *Management Science*, 47 (2001), 579-594.
- [20] Graves, S.C. A multi-echelon inventory model for a repairable item with one for one replenishment. *Management Science*, 31(10): 1247-1256, October 1985.
- [21] Gross, D., *Centralized Inventory Control in Multilocation Supply Systems in Multistage Inventor’J Models and Techniques*, H. E. Scarf, D. M. Gilford and M. W. Shelly (Eds.), Stanford University Press, Stanford, CA, 1963, 47-84.
- [22] Hoadley, B. AND D. P. Heyman, A Two-echelon Inventory Model with Purchases, Dispositions, Shipments, Returns and Transshipments, *Naval Research Logistics Quarterly*, 24 (1977), 1-19.
- [23] Johansen, S.G. and A. Thorstenson. An inventory model with Poisson demands and emergency orders. *International Journal of Production Economics*, 56C57 (1998), 275C289.

- [24] Karmarkar, U. S., and N. R. Patel, The One-Period, N-Location Distribution Problem, *Naval Research Logistics Quarterly*, 24 (1977), 559-575.
- [25] Kranenburg, A. and G. van Houtum . A new partial pooling structure for spare parts networks. *International Journal of Production Economics*, 108 (2009), 221-227.
- [26] Kukreja, A., D.M. Miller, and C.P. Schmidt. Stocking decisions for low-usage items in a multilocation inventory system. *Management Science*, 47 (2001), 1371C1383.
- [27] Kolda, T.G., R.M. Lewis, and V. Torczon. Optimization by direct search: new perspectives on some classical and modern methods. *SIAM Rev.* 45, 385-482(2003)
- [28] Lee, H.L. A multi-echelon inventory model for repairable items with emergency lateral transshipments. *Management Science*, 33(10):1302-1216, Oct 1987.
- [29] Minner, S., D.J. Robb, and E.A. Silver, An improved heuristic for deciding on emergency transshipments. *European Journal of Operational Research*, 148 (2003), 384C400.
- [30] Moinzadeh, K., and C.P. Schmidt. An (S-1,S) Inventory System with Emergency Orders. *Operations Research*, 39 (1991), No. 2. pp. 308-321.
- [31] Muckstadt, J.A. Some approximations in multi item multi echelon inventory systems for recoverable item. *Naval Research Logistics Quarterly*, 25(3),:377-394, September 1978.
- [32] Muckstadt, J.. 2005. *Analysis and Algorithms for Service Parts Supply Chains*, Springer.
- [33] Muckstadt, J. and A. Sapra. 2010. *Principles of Inventory Management: When You Are Down to Four, Order More*, Springer.
- [34] Olsson, F.. Optimal Policies for Inventory Systems with Lateral Transshipments, *Int. J. Production Economics*, 118 (2009), 175-184.
- [35] Paterson C., G. Kiesmüller, R. Teunter, and K. Glazebrook, Inventory models with lateral transshipments: A review, Beta Working 2009, Paper 287, Eindhoven University of Technology.
- [36] Reijnen, I.C., T. Tan, and G.J. van Houtum. Inventory Planning for spare parts networks with delivery time requirements. Working paper, Eindhoven University of Technology. 2009.

- [37] Robinson, L. W.. Optimal and Approximate Policies in Multiperiod, Multilocation Inventory Models with Transshipments. *Operations Research*, 38 (1990), 278-295.
- [38] Rosenshine, M. and D. Obee. Analysis of a Standing Order Inventory System with Emergency Orders. *Operations Research*, 24 (1976), 1143-1155.
- [39] Rudi, N., S. Kapur, and D. F. Pyke. A Two-Location Inventory Model with Transshipment and Local Decision Making, *Management Science*, 47 (2001), 1668-1680.
- [40] Sherbrooke, C.C. Multiechelon Inventory Systems with Lateral Supply. *Naval Research Logistics*, 39 (1992), 29-40.
- [41] Sherbrooke, C.C. METRIC A multi echelon technique for recoverable item control. *Operations Research*, 16(1): 122-141, 1968.
- [42] Sherbrooke, C.C. VARI-METRIC: Improved approximations for multi indenture, multi-echelon availability models. *Operations Research*, 34(2):311-319, March-April 1986.
- [43] Sherbrooke, C.C. Multi-echelon inventory systems with lateral supply. *Naval Research Logistics*, 39:29-40, 1992.
- [44] Tagaras, G. and M.A. Cohen, Pooling in two-location inventory systems with non-negligible replenishment lead times. *Management Science*, 38 (1992), 1067C1083.
- [45] Tagaras, G. and D. Vlachos. A Periodic Review Inventory System with Emergency Replenishments. *Management Science*, 47 (2001), 415-429.
- [46] Teunter, R. and D. Vlachos. An Inventory System with Periodic Regular Review and Flexible Emergency Review. *IIE Transactions*, 33 (2001), 625-635.
- [47] Vaz, A.I.F. and L.N. Vicente, A particle swarm pattern search method for bound constrained global optimization, *Journal of Global Optimization*, 39 (2007) 197-219.
- [48] Vaz, A.I.F. and L.N. Vicente, PSwarm: A hybrid solver for linearly constrained global derivative-free optimization. Submitted (draft). <http://www.norg.uminho.pt/aivaz/pswarm/>
- [49] Van Den Berghand, F. An analysis of particle swarm optimizers. Ph.D. thesis, Faculty of Natural and Agricultural Science, University of Pretoria (2001).

- [50] Vidgren, S.. Evaluation of a Class of Lateral Transshipment Policies, Lund University, 2005.
- [51] Vliegen, I.. *Integrated planning for service tools and spare parts for capital goods*, Ipskamp Drukkers, 2009.
- [52] Wijk A. van, I. Adan, and G. van Houtum. Optimal lateral transshipment policy for a two location inventory problem. Working paper, Eindhoven University of Technology. 2009.
- [53] Wong, H., G.J. Van Houtum, D. Cattrysse, and D. Van Oudheusden, Multi-item spare parts systems with lateral transshipment and waiting time constraints. *European Journal of Operational Research*, 171 (2006), 1071C1093.
- [54] Whittmore, A.S. and S. Saunders, Optimal Inventory under Stochastic Demand with Two Supply Options. *SLAM J. Appl. Math.*, 32 (1977), 293-305.