LEAST SQUARES ESTIMATES AND SUMS OF SQUARES FOR A DOUBLE CHANGE-OVER DESIGN

BU-70-M

W. T. Federer

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For this analysis consider the double change-over design for 3 treatments in the following design.

	Square 1 column or sequence				Square 2 column or sequence			
Period	1	2	3	total	4	5	6	total
1	X_{111A}	X_{112B}	X_{113C}	$X_{11\cdot\cdot}$	X_{211A}	X_{212B}	X_{213C}	$X_{21\cdot\cdot}$
2	X_{121B}	X_{122C}	X_{123A}	$X_{12\cdot\cdot}$	X_{221C}	X_{222A}	X_{223B}	$X_{22\cdot\cdot}$
3	X_{131C}	X_{132A}	X_{133B}	$X_{13\cdot\cdot}$	X_{231B}	X_{232C}	X_{233A}	$X_{23\cdot\cdot}$
	$X_{1\cdot1\cdot} = s_1$	$X_{1\cdot2\cdot} = s_2$	$X_{1\cdot3\cdot} = s_3$	$X_{1\cdot\cdot\cdot}$	$X_{2\cdot1\cdot} = s_4$	$X_{2\cdot2\cdot} = s_5$	$X_{2\cdot3\cdot} = s_6$	$X_{2\cdot\cdot\cdot}$

The linear model, a modification of the one for several latin squares (see formula (XI-38), Federer, Experimental Design,) is:

$$X_{ghij} = \mu + \alpha_g + \beta_{gh} + \upsilon_{gi} + \delta_j + \rho_p + \epsilon_{ghij} \tag{1}$$

where μ = mean effect, α_g = effect of g'th square, β_{gh} = effect of h'th row in g'th square, υ_{gi} = effect of i'th column in g'th square, δ_j = direct of j'th treatment, ρ_p = residual effect of p'th treatment, and ϵ_{ghij} = random component. g = 1,2; h = 1,2,3 for each g; i = 1,2,3 for each g; j = A,B,C; p = A,B,C.

The residual sum of squares is

$$R = \sum_{ghij}\sum\sum\sum(X_{ghij} - \mu - \alpha_g - \beta_{gh} - \upsilon_{gi} - \delta_j - \rho_p)^2 \tag{2}$$

Partial differentiation of (2) with respect to the various parameters and equation of the results to zero leads to the following normal equations:

For $\hat{\mu}$:

$$18\hat{\mu} + 9\Sigma\hat{\alpha}_g + 3\Sigma\Sigma_{gh}\hat{\beta}_{gh} + 3\Sigma\Sigma\hat{\upsilon}_{gi} + 6\Sigma\hat{\delta}_j + 4\Sigma\hat{\rho}_p = X_{....} \tag{3}$$

For the $\hat{\alpha}_g$:

$$9(\hat{\mu} + \hat{\alpha}_1) + 3\Sigma\hat{\beta}_{1h} + 3\Sigma\hat{\upsilon}_{1i} + 3\Sigma\hat{\delta}_j + 2(\hat{\rho}_A + \hat{\rho}_\beta + \hat{\rho}_C) = X_{1...} \tag{4}$$

$$9(\hat{\mu} + \hat{\alpha}_2) + 3\Sigma\hat{\beta}_{2h} + 3\Sigma\hat{\upsilon}_{2i} + 3\Sigma\hat{\delta}_j + 2(\hat{\rho}_A + \hat{\rho}_B + \hat{\rho}_C) = X_{2...} \tag{5}$$

For the $\hat{\beta}_{gh}$:

$$3(\hat{\mu} + \hat{\beta}_{11} + \hat{\alpha}_1) + \Sigma\hat{\upsilon}_{1i} + \Sigma\hat{\delta}_j = X_{11..} \tag{6}$$

$$3(\hat{\mu} + \hat{\beta}_{12} + \hat{\alpha}_1) + \Sigma\hat{\upsilon}_{1i} + \Sigma\hat{\delta}_j + \Sigma\hat{\rho}_p = X_{12..} \tag{7}$$

$$3(\hat{\mu} + \hat{\beta}_{13} + \hat{\alpha}_1) + \Sigma\hat{\upsilon}_{1i} + \Sigma\hat{\delta}_j + \Sigma\hat{\rho}_p = X_{13..} \tag{8}$$

$$3(\hat{\mu} + \hat{\alpha}_2 + \hat{\beta}_{21}) + \Sigma\hat{\upsilon}_{2i} + \Sigma\hat{\delta}_j = X_{21..} \tag{9}$$

$$3(\hat{\mu} + \hat{\alpha}_2 + \hat{\beta}_{22}) + \Sigma\hat{\upsilon}_{2i} + \Sigma\hat{\delta}_j + \Sigma\hat{\rho}_p = X_{22..} \tag{10}$$

$$3(\hat{\mu} + \hat{\alpha}_2 + \hat{\beta}_{23}) + \Sigma\hat{\upsilon}_{2i} + \Sigma\hat{\delta}_j + \Sigma\hat{\rho}_p = X_{23..} \tag{11}$$

For the $\hat{\upsilon}_{gi}$:

$$3(\hat{\mu} + \hat{\alpha}_1 + \hat{\upsilon}_{11}) + \Sigma\hat{\beta}_{1h} + \Sigma\hat{\delta}_j + \hat{\rho}_A + \hat{\rho}_B = X_{1.1.} \tag{12}$$

$$3(\hat{\mu} + \hat{\alpha}_1 + \hat{\upsilon}_{12}) + \Sigma\hat{\beta}_{1h} + \Sigma\hat{\delta}_j + \hat{\rho}_B + \hat{\rho}_C = X_{1.2.} \tag{13}$$

$$3(\hat{\mu} + \hat{\alpha}_1 + \hat{\upsilon}_{13}) + \Sigma\hat{\beta}_{1h} + \Sigma\hat{\delta}_j + \hat{\rho}_C + \hat{\rho}_A = X_{1.3.} \tag{14}$$

$$3(\hat{\mu} + \hat{\alpha}_2 + \hat{\upsilon}_{21}) + \Sigma\hat{\beta}_{2h} + \Sigma\hat{\delta}_j + \hat{\rho}_A + \hat{\rho}_C = X_{2.1.} \tag{15}$$

$$3(\hat{\mu} + \hat{\alpha}_2 + \hat{\upsilon}_{22}) + \Sigma\hat{\beta}_{2h} + \Sigma\hat{\delta}_j + \hat{\rho}_B + \hat{\rho}_A = X_{2.2.} \tag{16}$$

$$3(\hat{\mu} + \hat{\alpha}_2 + \hat{\upsilon}_{23}) + \Sigma\hat{\beta}_{2h} + \Sigma\hat{\delta}_j + \hat{\rho}_C + \hat{\rho}_B = X_{2.3.} \tag{17}$$

For the $\hat{\delta}_j$:

$$6(\hat{\mu} + \hat{\delta}_A) + 3\Sigma\hat{\alpha}_i + \Sigma\Sigma\hat{\beta}_{gh} + \Sigma\Sigma\hat{\upsilon}_{gi} + 2(\hat{\rho}_C + \hat{\rho}_B) = X_{...A} \tag{18}$$

$$6(\hat{\mu} + \hat{\delta}_B) + 3\Sigma\hat{\alpha}_i + \Sigma\Sigma\hat{\beta}_{gh} + \Sigma\Sigma\hat{\upsilon}_{gi} + 2(\hat{\rho}_A + \hat{\rho}_C) = X_{\dots B} \tag{19}$$

$$6(\hat{\mu} + \hat{\delta}_C) + 3\Sigma\hat{\alpha}_i + \Sigma\Sigma\hat{\beta}_{gh} + \Sigma\Sigma\hat{\upsilon}_{gi} + 2(\hat{\rho}_A + \hat{\rho}_B) = X_{\dots C} \tag{20}$$

For the $\hat{\rho}_p$:

$$X_{121B} + X_{133B} + X_{221C} + X_{232C} = 4(\hat{\mu} + \hat{\rho}_A) + 2\Sigma\hat{\alpha}_i + \hat{\beta}_{12} + \hat{\beta}_{13}$$
$$+ \hat{\beta}_{22} + \hat{\beta}_{23} + \hat{\upsilon}_{11} + \hat{\upsilon}_{13} + \hat{\upsilon}_{21} + \hat{\upsilon}_{22} + 2(\hat{\delta}_B + \hat{\delta}_C) = a \tag{21}$$

$$X_{131C} + X_{122C} + X_{222A} + X_{233A} = 4(\hat{\mu} + \hat{\rho}_B) + 2\Sigma\hat{\alpha}_i + \hat{\beta}_{13} + \hat{\beta}_{12}$$
$$+ \hat{\beta}_{22} + \hat{\beta}_{23} + \hat{\upsilon}_{11} + \hat{\upsilon}_{12} + \hat{\upsilon}_{22} + \hat{\upsilon}_{23} + 2(\hat{\delta}_A + \hat{\delta}_C) = b \tag{22}$$

$$X_{132A} + X_{123A} + X_{231B} + X_{223B} = 4(\hat{\mu} + \hat{\rho}_C) + 2\Sigma\hat{\alpha}_i + 2(\hat{\delta}_A + \hat{\delta}_B)$$
$$+ \hat{\beta}_{13} + \hat{\beta}_{12} + \hat{\beta}_{23} + \hat{\beta}_{22} + \hat{\upsilon}_{12} + \hat{\upsilon}_{13} + \hat{\upsilon}_{21} + \hat{\upsilon}_{23} = c \tag{23}$$

No unique solution for the unknowns in the above equations is possible. Therefore, additional equations are required for a solution. An additional set of equations, which would hold if this were the entire population, are added; these are

$$\Sigma\hat{\alpha}_g = 0; \tag{24}$$
$$\Sigma\hat{\beta}_{1h} = \Sigma\hat{\beta}_{2h} = 0; \tag{25}$$
$$\Sigma\hat{\upsilon}_{1h} = \Sigma\hat{\upsilon}_{2h} = 0; \tag{26}$$
$$\Sigma\hat{\delta}_j = 0; \tag{27}$$
$$\Sigma\hat{\rho}_p = 0; \tag{28}$$

where the hat (^) indicates that the sum of the estimates for a particular set of parameters is required to sum to zero. With the addition of equations (24) to (28) to the original set ((3) to (23)), a unique solution for the estimates is possible; thus,

$$\hat{\mu} = \bar{x} = X_{....}/18; \tag{29}$$

$$\hat{\alpha}_g = \bar{x}_{g...} - \bar{x} = \frac{X_{i...}}{9} - \bar{x}; \tag{30}$$

$$\hat{\beta}_{gh} = \frac{X_{gh..}}{3} - \bar{x}_{g...} \; . \tag{31}$$

The remaining equations do not come out so nicely. Perhaps the easiest thing to do first is to obtain the solution for the $\hat{\rho}_p$'s. From formula (21)

$$a = 4(\hat{\mu} + \hat{\rho}_A) + 0 + \left\{\tfrac{1}{3}(X_{12..} + X_{13..} + X_{22..} + X_{23..}) - 4\hat{\mu}\right\}$$
$$+ \left\{\tfrac{1}{3}(X_{1.1.} + X_{1.3.} + X_{2.1.} + X_{2.2.}) - 4\hat{\mu} - \tfrac{2}{3}\hat{\rho}_A\right\} + \tfrac{2}{6}\left\{X_{...B} + X_{...C}\right.$$
$$\left. - 2\hat{\rho}_A - 12\hat{\mu}\right\}, \tag{32}$$

or

$$\hat{\rho}_A = \tfrac{1}{8}\left\{3a + \frac{8X_{....}}{6} + s_2 + s_6 - X_{....} - (X_{....} - X_{11..} - X_{21..})\right.$$
$$\left. - (X_{....} - X_{...A})\right\}$$
$$= \tfrac{1}{8}\left\{3a + s_2 + s_6 + X_{...A} - \tfrac{2}{3}X_{...} - (X_{....} - X_{11..} - X_{21..})\right\}$$
$$= \tfrac{1}{8}\left\{3a + s_2 + s_6 + X_{...A} - \tfrac{1}{3}(2X_{...} + 3(a + b + c))\right\}, \tag{33}$$

since $X_{...} - X_{11..} - X_{21..} = a + b + c$.

From formula (22)

$$b = 4\hat{\mu} + 4\hat{\rho}_B - \hat{\beta}_{11} - \hat{\beta}_{21} - \hat{\upsilon}_{13} - \hat{\upsilon}_{21} - 2\hat{\delta}_B$$
$$= \frac{4X_{....}}{18} + 4\hat{\rho}_B - \tfrac{1}{3}\left\{X_{11..} - 3\bar{x} + X_{21..} - 3\bar{x}\right\} - \tfrac{1}{3}\left\{X_{1.3.}\right.$$
$$\left. + X_{2.1.} - 6\bar{x} + 2\hat{\rho}_\beta\right\} - \tfrac{2}{6}\left\{X_{...B} - 6\bar{x} + 2\hat{\rho}_B\right)$$
$$= X_{....}\left\{\frac{4}{18} + \frac{2}{18} + \frac{2}{18} + \frac{2}{18}\right\}$$
$$- \tfrac{1}{3}\left\{X_{11..} + X_{21..}\right\} - \tfrac{1}{3}(s_3 + s_4) + \frac{X_{...B}}{3} + \hat{\rho}_B\left\{4 - \tfrac{2}{3} - \tfrac{2}{3}\right\}$$

$$\therefore \hat{\rho}_B = \frac{1}{8}\left\{3b + s_3 + s_4 + X_{\cdot\cdot\cdot B} - \frac{10}{6} X_{\cdot\cdot\cdot\cdot} + X_{11\cdot\cdot} + X_{21\cdot\cdot}\right\}$$

$$= \frac{1}{8}\left\{3b + s_3 + s_4 + X_{\cdot\cdot\cdot B} - \frac{(2X_{\cdot\cdot\cdot\cdot} + 3(a + b + c)}{3}\right\} \qquad (34)$$

since $X_{\cdot\cdot\cdot\cdot} - X_{11\cdot\cdot} - X_{21\cdot\cdot} = a + b + c.$

$$\hat{\rho}_C = \frac{3}{24}\left\{3c + s_1 + s_5 + X_{\cdot\cdot\cdot C} - \frac{1}{3}(2X_{\cdot\cdot\cdot\cdot} + 3(a + b + c)\right\}. \qquad (35)$$

Also,

$$\hat{\delta}_A = \frac{1}{6}[X_{\cdot\cdot\cdot A} - 6\bar{x} + 2(\frac{3}{24})\left\{3a + s_2 + s_6 + X_{\cdot\cdot\cdot A}\right.$$

$$\left. - \frac{1}{3}(2X_{\cdot\cdot\cdot\cdot} + 3(a + b + c)\right\}]$$

$$= \frac{1}{6}[X_{\cdot\cdot\cdot A}(1 + \frac{1}{4}) + \frac{1}{4}(3a + s_2 + s_6) - \frac{6X_{\cdot\cdot\cdot\cdot}}{4(3)} - \frac{1}{4}(a + b + c)]$$

$$= \frac{1}{24}[5X_{\cdot\cdot\cdot A} + 2a - b - c + s_2 + s_6 - \frac{2}{3}X_{\cdot\cdot\cdot\cdot}] - \bar{x}, \qquad (36)$$

$$\hat{\delta}_B = \frac{1}{24}[5X_{\cdot\cdot\cdot B} + 2b - a - c + s_3 + s_4 - \frac{2}{3}X_{\cdot\cdot\cdot\cdot}] - \bar{x}, \qquad (37)$$

and

$$\hat{\delta}_C = \frac{1}{24}[5X_{\cdot\cdot\cdot C} + 2c - a - b + s_1 + s_5 - \frac{2}{3}X_{\cdot\cdot\cdot\cdot}] - \bar{x} \qquad (38)$$

The estimates of column effects are:

$$\hat{\upsilon}_{11} = \frac{1}{3}\left\{X_{1\cdot1\cdot} + \frac{1}{8}[3c + s_1 + s_5 + X_{\cdot\cdot\cdot C} - \frac{2}{3}X_{\cdot\cdot\cdot\cdot} - (a+b+c)]\right\} - \bar{x}_{1\cdot\cdot\cdot}$$

$$= \frac{1}{24}\left\{8X_{1\cdot1\cdot} + 3c + s_1 + s_5 + X_{\cdot\cdot\cdot C} + X_{11\cdot\cdot} + X_{21\cdot\cdot}\right\}$$

$$- \bar{x}_{1\cdot\cdot\cdot} - \frac{5}{4}\bar{x}; \qquad (39)$$

$$\hat{\upsilon}_{12} = \frac{1}{3}\left\{X_{1\cdot2\cdot} + \frac{1}{8}[3a + s_2 + s_6 + X_{\cdot\cdot\cdot A} - \frac{1}{3}(2X_{\cdot\cdot\cdot} + 3(a+b+c)]\right\} - \bar{x}_{1\cdot\cdot\cdot}$$

$$= \frac{1}{24}\left\{8X_{1\cdot2\cdot} + 3a + s_2 + s_6 + X_{\cdot\cdot\cdot A} + X_{11\cdot\cdot}\ X_{21\cdot\cdot}\right\} - \bar{x}_{1\cdot\cdot\cdot} - \frac{5}{4}\bar{x}; \qquad (40)$$

$$\hat{\upsilon}_{13} = \frac{1}{3}\left\{X_{1\cdot 3\cdot} + \frac{1}{8}[\,3b + s_3 + s_4 + X_{\cdot\cdot\cdot B} - \frac{1}{3}(2X_{\cdot\cdot\cdot\cdot} + 3(a+b+c))\,]\right\} - \bar{x}_{1\cdot\cdot\cdot}$$

$$= \frac{1}{24}\left\{8X_{1\cdot 3\cdot} + 3b + s_3 + s_4 + X_{\cdot\cdot\cdot B} + X_{11\cdot\cdot} + X_{21\cdot\cdot}\right\} - \bar{x}_{1\cdot\cdot\cdot} - \frac{5}{4}\bar{x}\ . \quad (41)$$

(As a partial check equation (39) + eq. (40) + eq. (41) = zero)

$$\hat{\upsilon}_{21} = \frac{1}{24}\left\{8X_{2\cdot 1\cdot} + 3b + s_3 + s_4 + X_{\cdot\cdot\cdot B} + X_{11\cdot\cdot} + X_{21\cdot\cdot}\right\} - \bar{x}_{2\cdot\cdot\cdot} - \frac{5}{4}\bar{x}; \quad (42)$$

$$\hat{\upsilon}_{22} = \frac{1}{24}\left\{8X_{2\cdot 2\cdot} + 3c + s_1 + s_5 + X_{\cdot\cdot\cdot C} + X_{11\cdot\cdot} + X_{21\cdot\cdot}\right\} - \bar{x}_{2\cdot\cdot\cdot} - \frac{5}{4}\bar{x}; \quad (43)$$

$$\hat{\upsilon}_{23} = \frac{1}{24}\left\{8X_{2\cdot 3\cdot} + 3a + s_2 + s_6 + X_{\cdot\cdot\cdot A} + X_{11\cdot\cdot} + X_{21\cdot\cdot}\right\} - \bar{x}_{2\cdot\cdot\cdot} - \frac{5}{4}\bar{x}. \quad (44)$$

(As a partial check eq. (42) + eq. (43) + eq. (44) = zero)

If the residual effects $\hat{\rho}_p$ are set equal to zero then the estimates of the column parameters are:

$$\hat{\upsilon}_{gi}'' = \bar{x}_{gi\cdot\cdot} - \bar{x}_{g\cdot\cdot\cdot} = \frac{X_{g\cdot i\cdot}}{3} - \bar{x}_{g\cdot\cdot\cdot} \quad (45)$$

To estimate the residual effects of treatments ignoring direct effects of the treatments, it is assumed that $\hat{\delta}_A = \hat{\delta}_B = \hat{\delta}_C = 0$. This assumption changes formulae (21) to (23), in that the $\hat{\delta}_j$ are each set equal to zero, but does not affect the formulae (3) to (17). With these revised equations, the estimates of residual effects ignoring direct effects are ($\hat{\beta}_{gh}' = \hat{\beta}_{gh}$ since the row and square effects are orthogonal to the direct effects of the treatments; $\hat{\upsilon}_{gi}'$ are obtained from formulae (39) to (44) with $\hat{\rho}_p'$ substituted for $\hat{\rho}_p$ (formula (45)):

$$4\hat{\rho}_A' = [\,a - 4\bar{x} - \hat{\beta}_{12} - \hat{\beta}_{13} - \hat{\beta}_{22} - \hat{\beta}_{23} - \hat{\upsilon}_{11}' - \hat{\upsilon}_{13}' - \hat{\upsilon}_{21}' - \hat{\upsilon}_{22}'\,],$$

$$= [\,a - \frac{4X_{\cdot\cdot\cdot\cdot}}{18} + \hat{\beta}_{11} + \hat{\beta}_{21} + \hat{\upsilon}_{12} + \hat{\upsilon}_{23}\,] \quad (46)$$

$$\therefore \hat{\rho}_A' = \frac{3}{10}[a - \frac{4X_{....}}{18} + \frac{1}{3}(X_{11..} + X_{21..} - 6(\frac{X_{....}}{18}))$$

$$+ \frac{1}{3}\left\{s_2 + s_6 - 6(\frac{X_{....}}{18})\right\}]$$

$$= \frac{3}{10}[a - \frac{X_{....}}{18}(4 + 2 + 2) + \frac{1}{3}(X_{11..} + X_{21..}) + \frac{1}{3}(s_2 + s_6)]$$

$$= \frac{1}{10}[3a + s_2 + s_6 - \frac{X_{....} + 3(a + b + c)}{3}], \quad (47)$$

since $X_{....} - X_{11..} - X_{21...} = a + b + c.$

Likewise,

$$\hat{\rho}_B' = \frac{1}{10}[3b + s_3 + s_4 - \frac{(X_{....} + 3(a + b + c))}{3}] \quad (48)$$

and

$$\hat{\rho}_C' = \frac{1}{10}[3c + s_1 + s_5 - \frac{(X_{....} + 3(a + b + c))}{3}] . \quad (49)$$

The direct effects ignoring residual effects are estimated to be (from formulae (18) to (20) in which $\hat{\rho}_p = 0$):

$$\hat{\delta}_j' = \frac{X_{...j}}{6} - \bar{x} = \bar{x}_{...j} - \bar{x} \quad (50)$$

The various sums of squares are computed as follows:

<u>Total (17 d.f.):</u>

$$\Sigma\Sigma\Sigma\Sigma X_{ghij}^2 - X_{....}^2/18. \quad (51)$$

<u>Rows within squares (2 + 2 = 4 d.f.):</u>

$$\sum_{g=1}^{2}\left\{\sum_{h=1}^{3}\frac{X_{gh..}^2}{3} - X_{g...}^2/9\right\}. \quad (52)$$

Sequences (ignoring residual effects) = squares + columns within squares (ignoring residual effects)(5 = 1 + 2 + 2 d.f.):

$$\sum_{g=1}^{2} \frac{X_{g\cdots}^2}{9} - \frac{X_{\cdots\cdot}^2}{18} + \sum_{g=1}^{2} \left\{ \sum_{i=1}^{3} \frac{X_{g\cdot i\cdot}^2}{3} - \frac{X_{g\cdots}^2}{9} \right\}. \tag{53}$$

Direct effects (ignoring residual effects)(2 d.f.):

$$\sum_j \frac{X_{\cdots j}^2}{6} - \frac{X_{\cdots\cdot}^2}{18} \tag{54}$$

Residual effects (eliminating column and direct effects)(2 d.f.):

$$\hat{\mu}X_{\cdots\cdot} + \Sigma\hat{\alpha}_g X_{g\cdots} + \Sigma\Sigma\hat{\beta}_{gh}X_{gh\cdot\cdot} + \Sigma\Sigma\hat{\upsilon}_{gi}X_{g\cdot i\cdot}$$

$$+ \Sigma\hat{\delta}_j X_{\cdots j} + (\hat{\rho}_A a + \hat{\rho}_B b + \hat{\rho}_C c) - \hat{\mu}'X_{\cdots\cdot}$$

$$- \Sigma\hat{\alpha}_g' X_{g\cdots} - \Sigma\Sigma\hat{\beta}_{gh}'X_{gh\cdot\cdot} - \Sigma\Sigma\hat{\upsilon}_{gi}'X_{g\cdot i\cdot}$$

$$- \Sigma\hat{\delta}_j' X_{\cdots j} = \Sigma\Sigma X_{g\cdot i\cdot}(\hat{\upsilon}_{gi} - \hat{\upsilon}_{gi}')$$

$$+ \Sigma X_{\cdots j}(\hat{\delta}_j - \hat{\delta}_j') + (\hat{\rho}_A a + \hat{\rho}_B b + \hat{\rho}_C c)$$

$$= \Sigma X_{g\cdot i\cdot} \left[\text{eq.}(39) \text{ to } (44) - \text{eq.}(45) \right] + \Sigma X_{\cdots j}(\text{eq.}(36) \text{ to } (38) - \text{eq.}(50))$$

$$+ (\hat{\rho}_A a + \hat{\rho}_B b + \hat{\rho}_C c)$$

$$= \tfrac{1}{3}(X_{1\cdot 1} + X_{2\cdot 2})\hat{\rho}_C + \tfrac{1}{3}(X_{1\cdot 2} + X_{2\cdot 3})\hat{\rho}_A + \tfrac{1}{3}(X_{1\cdot 3\cdot} + X_{2\cdot 1\cdot})\hat{\rho}_B$$

$$+ X_{\cdots A}(\tfrac{1}{3}\hat{\rho}_A) + X_{\cdots B}(\tfrac{1}{3}\hat{\rho}_B) + X_{\cdots C}(\tfrac{1}{3}\hat{\rho}_C) + a\hat{\rho}_A + b\hat{\rho}_B + c\hat{\rho}_C$$

$$= \tfrac{1}{3}\hat{\rho}_A(s_2 + s_6 + X_{\cdots A} + 3a) + \tfrac{1}{3}\hat{\rho}_B(s_3 + s_4 + X_{\cdots B} + 3b)$$

$$+ \tfrac{1}{3}\hat{\rho}_C(s_1 + s_5 + X_{\cdots C} + 3c)$$

$$= \frac{(3a + s_2 + s_6 + X_{\cdots A})^2}{24} + \frac{3b + s_3 + s_4 + X_{\cdots B})^2}{24}$$

$$+ \frac{(3c + s_1 + s_5 + X_{\cdots C})}{24} - \frac{1}{72}(2X_{\cdots\cdot} + 3(a+b+c))^2 \tag{55}$$

The sum of squares for residual effects eliminating columns but ignoring direct effects is:

$$\hat{\mu}X_{....} + \Sigma\hat{\upsilon}_{gi}{'}X_{g\cdot i\cdot} + a\hat{\rho}_A{'} + b\hat{\rho}_\beta{'} + \hat{\rho}_C{'} - \Sigma X_{g\cdot i\cdot}{}^2/3$$

$$= \frac{1}{3}\left\{\hat{\rho}_A(3a + s_2 + s_6) + \hat{\rho}_B(3b + s_3 + s_4) + \hat{\rho}_C(3a + s_1 + s_5)\right\}$$

$$= \frac{1}{30}\left\{(3a + s_2 + s_6)^2 + (3b + s_3 + s_4)^2 + (3c + s_1 + s_5)^2\right\}$$

$$- \frac{(X_{....} + 3(a+b+c))^2}{90} \qquad (56)$$

The sum of squares due to direct effects eliminating residual effects is equal to

$$\hat{\mu}X_{....} + \Sigma\hat{\alpha}_i X_{i...} + \Sigma\Sigma\hat{\beta}_{gh}X_{gh..} + \Sigma\Sigma\hat{\upsilon}_{gi}X_{g\cdot i\cdot}$$

$$+ \Sigma\hat{\delta}_j X_{...j} + a\hat{\rho}_A + b\hat{\rho}_B + c\hat{\rho}_C - \hat{\mu}X_{....}$$

$$- \Sigma\hat{\alpha}_i X_{i...} - \Sigma\Sigma\hat{\beta}_{gh}X_{gh..} - \Sigma\Sigma\hat{\upsilon}_{gi}{'}X_{g\cdot i\cdot}$$

$$- (a\hat{\rho}_A{'} + b\hat{\rho}_B{'} + c\hat{\rho}_C{'})$$

$$= \frac{1}{3}(\hat{\rho}_A - \hat{\rho}_A{'})(s_2 + s_6 + 3a) \quad = \frac{\hat{\rho}_A{'}}{40}$$

$$+ \frac{1}{3}(\hat{\rho}_B - \hat{\rho}_B{'})(s_3 + s_4 + 3b)$$

$$+ \frac{1}{3}(\hat{\rho}_C - \hat{\rho}_C{'})(s_1 + s_5 + 3c) + \Sigma\hat{\delta}_j X_{...j}$$

$$= \frac{1}{120}(2a - b - c + s_2 + s_6 + 5X_{...A} - \tfrac{2}{3}X_{....})(s_2 + s_6 + 3a)$$

$$+ \frac{1}{120}(2b - a - c + s_3 + s_4 + 5X_{...B} - \tfrac{2}{3}X_{....})(s_3 + s_4 + 3b)$$

$$+ \frac{1}{120}(2c - a - b + s_1 + s_5 + 5X_{...C} - \tfrac{2}{3}X_{....})(s_1 + s_5 + 3c)$$

$$+ \Sigma\hat{\delta}_j X_{...j} - \frac{4X_{....}}{360}(X_{....} + 3(a + b + c))$$

$$= \frac{1}{5}(\hat{\delta}_A + \bar{x})(s_2 + s_6 + 3a + 5X_{\cdot\cdot\cdot A})$$

$$+ \frac{1}{5}(\hat{\delta}_B + \bar{x})(s_3 + s_4 + 3b + 5X_{\cdot\cdot\cdot B})$$

$$+ \frac{1}{5}(\hat{\delta}_C + \bar{x})(s_1 + s_5 + 3c + 5X_{\cdot\cdot\cdot C})$$

$$- \frac{X_{\cdot\cdot\cdot\cdot}{}^2}{(18)} - \frac{4}{360}X_{\cdot\cdot\cdot\cdot}(X_{\cdot\cdot\cdot\cdot} + 3(a + b + c))$$

$$= \frac{1}{5}(\hat{\delta}_A + \bar{x})(s_2 + s_6 + 3a + 5X_{\cdot\cdot\cdot A} - \frac{(2X_{\cdot\cdot\cdot\cdot} + 3(a + b + c))}{3})$$

$$+ \frac{1}{5}(\hat{\delta}_B + \bar{x})(s_3 + s_4 + 3b + 5X_{\cdot\cdot\cdot B} - \frac{(2X_{\cdot\cdot\cdot\cdot} + 3(a + b + c))}{3})$$

$$+ \frac{1}{5}(\hat{\delta}_C + \bar{x})(s_1 + s_5 + 3c + 5X_{\cdot\cdot\cdot C} - \frac{(2X_{\cdot\cdot\cdot\cdot} + 3(a + b + c))}{3})$$

$$- \frac{(4X_{\cdot\cdot\cdot\cdot})^2}{360}$$

$$= \frac{1}{5}\Sigma(\hat{\delta}_j + \bar{x})^2 - (4X_{\cdot\cdot\cdot\cdot})^2/360$$

$$= \frac{1}{120}\Big\{ (5X_{\cdot\cdot\cdot A} + 2a - b - c + s_2 + s_6 - \frac{2}{3}X_{\cdot\cdot\cdot\cdot})^2$$

$$+ (5X_{\cdot\cdot\cdot B} + 2b - a - c + s_3 + s_4 - \frac{2}{3}X_{\cdot\cdot\cdot\cdot})^2$$

$$+ (5X_{\cdot\cdot\cdot C} + 2c - a - b + s_1 + s_5 - \frac{2}{3}X_{\cdot\cdot\cdot\cdot})^2 \Big\}$$

$$- (4X_{\cdot\cdot\cdot\cdot})^2/360$$

$$= \frac{1}{5}\Sigma\hat{\delta}_j{}^2 - 71X_{\cdot\cdot\cdot\cdot}{}^2/1620 \qquad (57)$$