STATISTICAL CHARACTERISTICS OF REPEATED BLOCK DESIGNS

by

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Abstract

For each set of parameters v, b, k, r, λ of a balanced incomplete block design, a class of repeated block balanced incomplete block designs results. Attention to date has centered on construction of the members of the class and on finding the minimum number of distinct blocks. The present work relates to characteristics which distinguish among members of a given class. It was found that the usual statistical characteristics would not distinguish among members of a class, but that effects related to block totals, for example fixed-ratio mixture treatment designs, can be used as distinguishing characteristics. Also, it is noted that the duals of the designs are different for members of the class.

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1. Introduction. Given that there are $\binom{v}{k} = b^*$ distinct blocks for a given block design, a repeated block design is one for which there are $d \le b$ distinct blocks, for which the j^{th} block, $j=1,2,\cdots,b^*$, is repeated $w_j = 0,1,\cdots$ times, and for which $\Sigma_1^{b^*} w_j = b$, the number of blocks in the design. Herein we consider only balanced incomplete block designs with repeated blocks, equal sized blocks, and equally replicated treatments. The balanced incomplete block designs have parameters v, b, d, k, r, and λ, where v is the number of treatments, b is the number of blocks, d is the number of distinct blocks, k is the block size, $k < v$, r is the number of times the i^{th} treatment, $i = 1,2,\cdots,v$, is repeated, and λ is the number of times any pair of treatments occurs together in the b blocks. Also, we shall restrict the occurrence of the i^{th} treatment in the j^{th} block to be $n_{ij} = 0$ or 1, or the binary design.

One may construct a class of repeated block designs for a given set of parameters v, b, k, r and λ (see Foody and Hedayat (1977)). This then brings up the problem of obtaining criteria which distinguish between members of the class. Our purpose here is to consider a number of criteria and to demonstrate which will distinguish members of the class from each other and which will not. In Section 2 we consider the estimation of treatment effect contrasts from the usual additive linear model and note that the class of repeated block designs do not have distinguishable standard properties. However, when we turn our attention to the estimation of block effect contrasts, we find some distinguishing characteristics and there are some invariance properties. This forms the content of Section 3. Finally, we consider the class of repeated block designs for $v = 7$, $b = 21$, $k = 3$, $r = 9$ and $\lambda = 3$ in Section 4.

2. Class of repeated block designs for estimating treatment effect contrasts. Let C be the class of repeated block designs with parameters v, b, r, k and λ. From standard design theory, the following theorem can easily be verified.

THEOREM 2.1. In the class C of repeated block designs, all members of the class have the following in common:

(i) intra- and inter-block estimates of the treatment effect contrasts and variances of the estimates,

(ii) the expected value of the blocks (eliminating treatment effects) mean square, and

(iii) A-, D-, and E- optimality criteria for estimating treatment effects.

In the case of analysis with recovery of inter-block information, some differences may arise when one considers the estimation of σ_β^2, the block effects variance. Let us consider a design of class C with d distinct blocks and let σ^2 be the intra-block error variance. Then the blocks eliminating treatments sum of squares can be partitioned further. Such a partitioning, along with the expected value of mean squares, is given in Table 2.1.

Table 2.1.

Partitioning of blocks (eliminating treatment) mean square and expectations of M.S.

Source	d.f.	M.S.	E(M.S)
Blocks (elim. treat.)	b-1	B_1	$\sigma^2 + (bk-v)\sigma_\beta^2/(b-1)$
Distinct blocks (elim. treat.)	d-1	B_2	$\sigma^2 + (dk-v)\sigma_\beta^2/(d-1)$
Repeated blocks	b-d	B_3	$\sigma^2 + k\sigma_\beta^2$

σ_β^2 can be estimated using the intra-block error mean square and either one of B_1, B_2, or B_3. The estimated σ_β^2 will give different accuracies for the members of the class C if either B_2 or B_3 is used. However, it is inefficient to use

B_2 or B_3 in estimating σ_β^2, and it is a common practice to use B_1 while estimating σ_β^2 and, in such cases, the members of the class C are indistinguishable as indicated in Theorem 2.1.(ii).

There are really no distinguishing criteria between members of class C in estimating treatment effects when individual treatment yields in each block are available and when only the treatment effects themselves are estimated.

3. <u>Class of repeated block designs for estimating block effect contrasts.</u> Let N be the incidence matrix of the design. The coefficient matrix D in estimating the block effects vector D is given by

$$D = kI_b - \frac{1}{r} N'N \quad , \tag{3.1}$$

where I_b is the identity matrix of order b. The eigen values of D can easily be verified to be $\phi_0 = 0$, $\phi_1 = \frac{\lambda v}{r}$ and $\phi_2 = k$ with respective multiplicities $\alpha_0 = 1$, $\alpha_1 = v-1$ and $\alpha_2 = b-v$. Noting that if $\underline{\xi}_1, \underline{\xi}_2, \cdots, \underline{\xi}_{v-1}$ is a complete set of ortho-normal eigen vectors corresponding to the root $r-\lambda$ of NN', then $\underline{\eta}_1, \underline{\eta}_2, \cdots, \underline{\eta}_{v-1}$ is a complete set of ortho-normal eigen vectors of $N'N$ corresponding to the root $r-\lambda$, where

$$\underline{\eta}_i = \frac{1}{\sqrt{r-\lambda}} N'\underline{\xi}_i \quad , \qquad i=1,2,\cdots,v-1 \quad , \tag{3.2}$$

the orthogonal idempotent corresponding to the root ϕ_1 of D is

$$A_1 = \sum_{i=1}^{v-1} \underline{\eta}_i \underline{\eta}_i' = \frac{1}{(r-\lambda)} (N'N - \frac{k^2}{v} J_{b,b}) \quad , \tag{3.3}$$

where $J_{b,b}$ is a b X b matrix with unit entries everywhere. The indempotent corresponding to the root ϕ_0 of D is $A_0 = \frac{1}{b} J_{b,b}$, and hence the orthogonal, idempotent corresponding to the root ϕ_2 of D is

$$(3.4) \qquad A_2 = I_v - A_0 - A_1 \quad .$$

Using

$$(3.5) \qquad \begin{aligned} D^- &= \frac{1}{\phi_1} A_1 + \frac{1}{\phi_2} A_2 \\ &= \frac{1}{\lambda vk} N'N + \frac{1}{k} I_b - \frac{\lambda v + rk}{\lambda v^2 r} J_{b,b} \end{aligned}$$

as a generalized inverse of D in solving the normal equations for $\hat{\underline{\beta}}$ and finding the variances of estimated contrasts of block effects, we observe that the estimated block effects contrasts may differ between members of the class C, as well as the variances of the estimated block effect contrasts. In fact, if S_i and S_j are any two blocks of the design which have μ treatments in common, the (i,j) element of N'N is μ and

$$(3.6) \qquad V(\hat{\beta}_i - \hat{\beta}_j) = 2\sigma^2(\lambda v + k - \mu)/\lambda vk \quad .$$

This is a distinguishing characteristic of the members of the class C which we state in the form of the following theorem.

THEOREM 3.1. Members of class C differ in estimating block effect contrasts. If the i^{th} and j^{th} blocks have μ treatments in common, the variance of the estimated elementary contrast of the i^{th} and j^{th} block effects is as given in (3.6).

We note that μ can take at most k+1 values, namely, $0,1,\cdots,k$, and hence there are at most k+1 possible variances for the estimated elementary block effect contrasts. The frequency of times a particular variance arises is determined by the number of pairs of blocks which provide the required inter-section number in formula (3.6). Even when the distribution of the variances of eliminating block effect contrasts are different, interestingly enough, the following holds.

THEOREM 3.2. *For members of the class C,*

(i) *the average variance of all estimated block effects elementary contrasts is constant, and*

(ii) *the variance of all variances of estimated block effects elementary contrasts is constant.*

PROOF. Let $D^- = (d^{ij})$. Then

$$\sum_{j=1}^{b} d^{ij} = 0 \quad , \qquad \text{for every } i \quad . \tag{3.7}$$

We have analagous to equation (4.3.4) of Raghavarao (1971) that the average variance of all estimated elementary contrasts of block effects is equal to

$$\frac{2\sigma^2}{(b-1)}\, tr(D^-) = \frac{2\sigma^2}{(b-1)}\, \frac{(\lambda vb+bk-\lambda v-rk)}{\lambda vk} \quad , \tag{3.8}$$

which is a constant for all members of class C. This proves (i).

To prove (ii), it is sufficient to prove that $\sum_{\substack{i,j=1 \\ i<j}}^{b} \{V(\hat{\beta}_i-\hat{\beta}_j)\}^2$ is constant for members of C. We have, after some straight forward and algebraic manipulations,

$$\begin{aligned} \sum_{\substack{i,j=1 \\ i<j}}^{b} \{V(\hat{\beta}_i-\hat{\beta}_j)\}^2 &= \sigma^4 \sum_{\substack{i,j=1 \\ i<j}}^{b} (d^{ii}+d^{jj}-2d^{ij})^2 \\ &= \sigma^4 \left\{b \sum_{i=1}^{b} (d^{ii})^2 + 2tr(D^-)^2 + \left(tr(D^-)\right)^2\right\} \\ &= \frac{\sigma^4}{\lambda^2 v^2 k^2} \{(bk+\lambda vb-\lambda v-rk)^2 + 2(r^2k^2(v-1) \\ &\qquad + (b-v)\lambda^2 v^2) + (rk(v-1) + (b-v)\lambda v)^2\} \ , \end{aligned} \tag{3.9}$$

which is again constant for all members of class C.

The members of the class C differ in the estimability of types of treatment effects from block totals and the degrees of freedom, b-d, for an error mean square for treatment effects estimated from block totals, and such is the case in statistical designs for mixtures (see Federer (1979) and Raghavarao and Federer (1979)), where the proportions in the mixtures are fixed a priori.

For those working in combinatorics, it is noted that the approach outlined in this section is based on a consideration of the properties of dual designs. Each member of C produces a different dual design.

4. An example of replicated block designs for $v = 7$, $b = 21$, $r = 9$, $k = 3$, $\lambda = 3$.

The total class of repeated block designs for $v = 7$, $b = 21$, $k = 3$, $r = 9$, $\lambda = 3$ which are balanced incomplete block designs for which $n_{ij} = 0$ or 1, is given in Table 4.1. Any other designs may be shown to be isomorphic to the designs given.

Table 4.1.

BIB(7,21,9,3,3)

Block composition	Number of distinct blocks (frequency of occurrence)									
	7	11	13	14	15	17	18	19	20	21
1 2 3	3	3	3	2	3	2	2	3	2	1
1 2 4	-	-	-	1	-	1	-	-	-	1
1 2 5	-	-	-	-	-	-	-	-	-	1
1 2 6	-	-	-	-	-	-	-	-	1	-
1 2 7	-	-	-	-	-	-	1	-	-	-
1 3 4	-	-	-	-	-	-	-	-	1	-
1 3 5	-	-	-	-	-	-	-	-	-	-
1 3 6	-	-	-	1	-	1	1	-	-	1
1 3 7	-	-	-	-	-	-	-	-	-	1
1 4 5	3	3	-	2	1	2	2	1	1	1
1 4 6	-	-	1	-	1	-	-	1	-	-
1 4 7	-	-	2	-	1	-	1	1	1	1
1 5 6	-	-	2	-	1	-	1	1	1	1
1 5 7	-	-	1	1	1	1	-	1	1	-
1 6 7	3	3	-	2	1	2	1	1	1	1
2 3 4	-	-	-	-	-	-	-	-	-	1
2 3 5	-	-	-	-	-	1	1	-	-	-
2 3 6	-	-	-	-	-	-	-	-	-	-
2 3 7	-	-	-	1	-	-	-	-	1	1

Table 4.1. (Cont'd)

Block composition	Number of distinct blocks (frequency of occurrence) 7	11	13	14	15	17	18	19	20	21
2 4 5	-	-	1	-	1	-	-	1	1	-
2 4 6	3	2	2	2	2	1	2	1	1	1
2 4 7	-	1	-	-	-	1	1	1	1	-
2 5 6	-	1	-	1	-	1	1	1	1	1
2 5 7	3	2	2	2	2	1	1	1	1	1
2 6 7	-	-	1	-	1	1	-	1	-	1
3 4 5	-	-	2	1	1	-	1	1	1	1
3 4 6	-	1	-	-	-	1	1	1	1	1
3 4 7	3	2	1	2	2	2	1	1	-	-
3 5 6	3	2	1	2	2	1	-	1	1	1
3 5 7	-	1	-	-	-	1	1	1	1	1
3 6 7	-	-	2	-	1	-	1	1	1	-
4 5 6	-	-	-	-	-	1	-	-	-	-
4 5 7	-	-	-	-	-	-	-	-	-	1
4 6 7	-	-	-	1	-	-	-	-	1	1
5 6 7	-	-	-	-	-	-	1	-	-	-

There are four different variances among block means adjusted for treatment effects possible for this class of designs. These are of the form 2(a-b), 2(a-c) = 2(a-b/2), 2a, and 2(a+c). The frequency of occurrence of the various variances are given in Table 4.2 where it may be noted that they differ for various values of d in most cases. It should be noted for d = 14 and d = 15, that the frequencies are identical. The same is true for the pair of designs d = 18 and d = 19. Thus, the frequency of occurrence of the various types of variances may be used to distinguish among most members of the class. The frequency of variances 2(a-b) decreases from 21 to zero as the number of distinct blocks increases from 7 to 21. The reverse situation holds for the variance 2(a+c). The frequency of the variance 2(a-c) is three times the frequency 2(a+c) for any given number of distinct blocks. The frequency of the variance 2a is simply $\binom{21}{2}$ = 210 minus the sum of frequencies of the other three types of variance.

Table 4.2.

Frequency of occurrence of variance of differences between block mean adjusted for treatment effects.

Number of distinct blocks	Variance* 2(a-b)	2(a-c)	2a	2(a+c)	Average variance
7	21	0	189	0	2a-b/5
11	13	24	165	8	2a-b/5
13	9	36	153	12	2a-b/5
14	7	42	147	14	2a-b/5
15	7	42	147	14	2a-b/5
17	4	51	138	17	2a-b/5
18	3	54	135	18	2a-b/5
19	3	54	135	18	2a-b/5
20	1	60	129	20	2a-b/5
21	0	63	126	21	2a-b/5

* $a = \frac{24}{63}\sigma^2$, $b = \frac{2}{63}\sigma^2$, $c = \frac{1}{63}\sigma^2$

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