ILLUSTRATIVE CALCULATIONS OF SUMS OF SQUARES IN THE 2-WAY CROSSED CLASSIFICATION, UNBALANCED DATA, ALL CELLS FILLED

by

S. R. Searle
Cornell University, Ithaca, New York

BU-608-M May, 1977

Abstract

Hypothetical data are used to illustrate calculations of sums of squares in the 2-way crossed classification having unequal numbers of observations in the subclasses (unbalanced data) but all cells filled. Sums of squares are illustrated for the classical analyses of variance, for certain tests of hypotheses, and for certain restricted models that yield sums of squares usually found in other contexts.

1. Introduction

Full details are given in Searle [1971] of calculating the sums of squares in the classical analyses of variance of the 2-way crossed classification having unequal numbers of observations in the subclasses (unbalanced data). Description is also given there of the hypotheses tested when these sums of squares are used as numerators of F-statistics. In contrast to these sums of squares a calculation procedure for sums of squares for fitting reduced models, as used in some computer programs, is described and challenged in Searle [1972]. More recently, Hocking and co-workers [1, 6, 7] have explained and extended these calculations. Detailed numerical illustration of them is given in this paper, alongside that of the classical sums of squares.

Paper No. BU-608-M in the Biometrics Unit Mimeo Series, Department of Plant Breeding and Biometry, Cornell University, Ithaca, New York 14853.

The illustrations are based on a small set of simple, hypothetical data consisting of 8 observations in 2 rows and 3 columns. Because description of the procedures involved is available mainly in matrix terminology, and because most numerical examples (e.g., Kutner [1974] and Snee [1973]) are sufficiently complex as to benefit from using a computer, it is felt that a simple illustration involving easy-to-follow numbers will be useful for many readers. The illustrations demonstrate the meaning of some of the sums of squares in terms of hypotheses tested when the sums of squares are used as numerators of F-statistics; and others are explained by demonstrating their equivalence to some of the well-known classical sums of squares.

2. Model and Notation

The usual model for the 2-way classification with interaction is

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} \tag{1}$$

for $i = 1, \cdots, a$, $j = 1, \cdots, b$ and $k = 1, \cdots, n_{ij} \geq 0$. Detailed description is available in many places (e.g., Searle [1971], Chapter 7). When the data are written in vector form as

$$\underset{\sim}{y} = \underset{\sim\sim}{X}\underset{\sim}{b} + \underset{\sim}{e} \tag{2}$$

the normal equations for $\underset{\sim}{b}$ are

$$\underset{\sim}{X}'\underset{\sim\sim}{X}\underset{\sim}{b}^{\circ} = \underset{\sim}{X}'\underset{\sim}{y} \tag{3}$$

where $\underset{\sim}{b}^{\circ}$ is any solution

$$\underset{\sim}{b}^{\circ} = (\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{y} \tag{4}$$

with $(\underset{\sim}{X}'\underset{\sim}{X})^{-}$ being a generalized inverse of $\underset{\sim}{X}'\underset{\sim}{X}$ satisfying $\underset{\sim}{X}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{X} = \underset{\sim}{X}'\underset{\sim}{X}$.

The reduction in sums of squares due to fitting (2) is denoted by $R(\underset{\sim}{b})$ and can be calculated as

$$R(\underset{\sim}{b}) = \underset{\sim}{b}^{o\prime}\underset{\sim}{X}'\underset{\sim}{y} \tag{5}$$

$$= \left\{ \begin{array}{l} \text{inner product of solution vector} \\ \text{and vector of right-hand sides of} \\ \text{the normal equations (3).} \end{array} \right. \tag{6}$$

Considerable use is made of this algorithm which we henceforth refer to as the R-algorithm.

Suppose $\underset{\sim}{X}$ and $\underset{\sim}{b}$ of the model (2) are partitioned as in

$$\underset{\sim}{y} = [\underset{\sim}{X}_1 \quad \underset{\sim}{X}_2] \begin{bmatrix} \underset{\sim}{b}_1 \\ \underset{\sim}{b}_2 \end{bmatrix} + \underset{\sim}{e}. \tag{7}$$

Then

$$\underset{\sim}{y} = \underset{\sim}{X}_1\underset{\sim}{b}_1 + \underset{\sim}{\epsilon} \tag{8}$$

is a sub-model of (2) with reduction in sum of squares

$$R(\underset{\sim}{b}_1) = \tilde{\underset{\sim}{b}}_1'\underset{\sim}{X}_1'\underset{\sim}{y} = \underset{\sim}{y}'\underset{\sim}{X}_1(\underset{\sim}{X}_1\underset{\sim}{X}_1')^{-}\underset{\sim}{X}_1\underset{\sim}{y}$$

corresponding to (5). Further, the reduction due to fitting (2) over and above fitting (8) is defined as $R(\underset{\sim}{b}_2|\underset{\sim}{b}_1)$ given by

$$\begin{aligned} R(\underset{\sim}{b}_2|\underset{\sim}{b}_1) &= R(\underset{\sim}{b}_1,\underset{\sim}{b}_2) - R(\underset{\sim}{b}_1) \\ &= \underset{\sim}{b}^{o\prime}\underset{\sim}{X}'\underset{\sim}{y} - \tilde{\underset{\sim}{b}}_1\underset{\sim}{X}_1'\underset{\sim}{y} \\ &= \underset{\sim}{y}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{y} - \underset{\sim}{y}'\underset{\sim}{X}_1(\underset{\sim}{X}_1'\underset{\sim}{X}_1)^{-}\underset{\sim}{X}_1'\underset{\sim}{y} \ . \end{aligned} \tag{9}$$

Particular uses of (8) and (9) applied to (1) are well-known. For example, on rewriting (1) as

$$\underset{\sim}{y} = \mu\underset{\sim}{1} + \underset{\sim}{X}_1\underset{\sim}{\alpha} + \underset{\sim}{X}_2\underset{\sim}{\beta} + \underset{\sim}{X}_3\underset{\sim}{\gamma} + \underset{\sim}{e} \qquad (10)$$

where $\underset{\sim}{1}$ is a vector of 1's, and $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$ and $\underset{\sim}{\gamma}$ are the vectors of the α-, β- and γ-effects of (1), it is well-known that

$$R(\mu) = N\bar{y}^2_{...}$$

$$R(\underset{\sim}{\alpha}|\mu) = \sum_{i=1}^{a} y^2_{i..}/n_{i.} - R(\mu) \qquad (11)$$

and

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) = \sum_{i=1}^{a}\sum_{j=1}^{b} y^2_{ij.}/n_{ij}$$

where $y_{ij.} = \sum_{k=1}^{n_{ij}} y_{ijk}$, $y_{i..} = \sum_{j=1}^{b} y_{ij.}$, $y_{...} = \sum_{i=1}^{a} y_{i..}$, $n_{i.} = \sum_{j=1}^{b} n_{ij}$, and $N = n_{..} = \sum_{i=1}^{a} n_{i.}$. Searle [1971, Chapter 7] has full details.

3. Data and Classical Analyses

The data to be used for the illustrations are as follows:

Table 1: Observations

	j = 1	j = 2	j = 3
i = 1	7,9	6	2
i = 2	8	4,8	12

Table 2: Totals

	$y_{ij.}$			$y_{i..}$
	16	6	2	24
	8	12	12	32
$y_{.j.}$	24	18	14	$56 = y_{..}$

Table 3: Numbers

	n_{ij}			$n_{i.}$
	2	1	1	4
	1	2	1	4
$n_{.j}$	3	3	2	$8 = N$

Table 4: Means

	$\bar{y}_{ij.}$			$\bar{y}_{i..}$
	8	6	2	6
	8	6	12	8
$\bar{y}_{.j.}$	8	6	7	$7 = \bar{y}_{..}$

The easy sums of squares for the classical analyses of variance are then

$$R(\mu) = N\bar{y}^2_{...} = 8(7^2) = 392,$$

$$R(\mu,\underset{\sim}{\alpha}) = \sum_i n_{i.}\bar{y}^2_{i..} = 4(6^2) + 4(8^2) = 400,$$

$$R(\mu,\underset{\sim}{\beta}) = \sum_j n_{.j}\bar{y}^2_{.j.} = 3(8^2) + 3(6^2) + 2(7^2) = 398,$$

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) = \sum\sum_{ij} n_{ij}\bar{y}^2_{ij.} = 2(8^2) + 6^2 + 2^2 + 8^2 + 2(6^2) + 12^2 = 448,$$

and

$$\underset{\sim}{y}'\underset{\sim}{y} = \sum\sum\sum_{ijk} y^2_{ijk} = 7^2 + 9^2 + 6^2 + 2^2 + 8^2 + 4^2 + 8^2 + 12^2 = 458.$$

These lead to

$$R(\underset{\sim}{\alpha}|\mu) = 400 - 392 = 8,$$

$$R(\underset{\sim}{\beta}|\mu) = 398 - 392 = 6,$$

and

$$SSE = \underset{\sim}{y}'\underset{\sim}{y} - R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) = 458 - 448 = 10.$$

The other term to compute is $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta})$. We derive it most easily as

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = R(\mu,\underset{\sim}{\beta}) + \underset{\sim}{u}'\underset{\sim}{T}^{-1}\underset{\sim}{u} = 398 + \underset{\sim}{u}'\underset{\sim}{T}^{-1}\underset{\sim}{u}$$

as in equation (69), page 297 of Searle [1971]. In this case $\underset{\sim}{T}^{-1}$ has order 1 with

$$t_{11} = n_{1.} - \sum_{j=1}^{3} n^2_{1j}/n_{.j} = 4 - \left(\frac{2^2}{3} + \frac{1^2}{3} + \frac{1^2}{2}\right) = 1\frac{5}{6},$$

a check on which is provided by noting that

$$t_{12} = -\sum_{j=1}^{3} n_{1j}n_{2j}/n_{.j} = \left[\frac{2(1)}{3} + \frac{1(2)}{3} + \frac{1(1)}{2}\right] = -1\frac{5}{6},$$

so giving $t_{11} + t_{12} = 0$, as required. Also,

$$u_1 = y_{1..} - \sum_{j=1}^{3} n_{1j}\bar{y}_{.j.} = 24 - [2(8) + 1(6) + 1(7)] = -5$$

and

$$u_2 = y_{2..} - \sum_{j=1}^{3} n_{2j}\bar{y}_{.j.} = 32 - [1(8) + 2(6) + 1(7)] = 5$$

with $u_1 + u_2 = 0$, as expected. Then

$$\underset{\sim}{u}'\underset{\sim}{T}^{-1}\underset{\sim}{u} = (-5)(1\tfrac{5}{6})^{-1}(-5) = 13\tfrac{7}{11}$$

and so

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 398 + 13\tfrac{7}{11} = 411\tfrac{7}{11}.$$

A check on this result is to derive it in the alternative form

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = R(\mu,\underset{\sim}{\alpha}) + \underset{\sim}{r}'\underset{\sim}{C}^{-1}\underset{\sim}{r} = 400 + \underset{\sim}{r}'\underset{\sim}{C}^{-1}\underset{\sim}{r}$$

as in equation (63), page 293 of Searle [1971]. In this case $\underset{\sim}{C}^{-1}$ has order 2, coming from a 3 × 3 symmetric matrix whose elements are (loc. cit.)

$$c_{11} = 3 - \left(\frac{2^2}{4} + \frac{1^2}{4}\right) = 1\tfrac{3}{4} \qquad c_{12} = -\left[\frac{2(1)}{4} + \frac{1(2)}{4}\right] = -1 \qquad c_{13} = -\left[\frac{2(1)}{4} + \frac{1(1)}{4}\right] = -\tfrac{3}{4}$$

$$c_{22} = 3 - \left(\frac{2^2}{4} + \frac{1^2}{4}\right) = 1\tfrac{3}{4} \qquad c_{23} = -\left[\frac{1(1)}{4} + \frac{2(1)}{4}\right] = -\tfrac{3}{4}$$

$$c_{33} = 2 - \left(\frac{1^2 + 1^2}{4}\right) = 1\tfrac{1}{2}$$

and it is easily seen that the check $c_{j.} = 0$ is upheld for $j = 1, 2, 3$. Similarly

$$r_1 = 24 - [2(6) + 1(8)] = 4$$

$$r_2 = 18 - [1(6) + 2(8)] = -4$$

$$r_3 = 14 - [1(6) + 1(8)] = 0$$

with the check $r_1 + r_2 + r_3 = 0$ being satisfied. Hence, using r_1 and r_2 and the corresponding $c_{jj'}$'s

$$\underset{\sim}{r}'\underset{\sim}{C}^{-1}\underset{\sim}{r} = [4 \quad -4]\begin{bmatrix} 1\frac{3}{4} & -1 \\ -1 & 1\frac{3}{4} \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ -4 \end{bmatrix} = 11\frac{7}{11},$$

so giving

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 400 + 11\frac{7}{11} = 411\frac{7}{11}$$

as before.

Having checked $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta})$, we then have

$$R(\underset{\sim}{\alpha}|\mu,\underset{\sim}{\beta}) = 411\frac{7}{11} - 398 = 13\frac{7}{11},$$

$$R(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha}) = 411\frac{7}{11} - 400 = 11\frac{7}{11}$$

and

$$R(\underset{\sim}{\gamma}|\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta}) = 448 - 411\frac{7}{11} = 36\frac{4}{11},$$

and all of these values can be assembled into the two partitionings of the total sum of squares that are the basis of the classical analyses of variance.

Table 5: Partitionings of Total Sums of Squares

5a: Rows before Columns

Term	d.f.	Sum of Squares
$R(\mu)$	1	392
$R(\underset{\sim}{\alpha}\|\mu)$	1	8
$R(\underset{\sim}{\beta}\|\mu,\underset{\sim}{\alpha})$	2	$11\frac{7}{11}$
$R(\underset{\sim}{\gamma}\|\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta})$	2	$36\frac{4}{11}$
SSE	2	10
SST	8	458

5b: Columns before Rows

Term	d.f.	Sum of Squares
$R(\mu)$	1	392
$R(\underset{\sim}{\beta}\|\mu)$	2	6
$R(\underset{\sim}{\alpha}\|\mu,\underset{\sim}{\beta})$	1	$13\frac{7}{11}$
$R(\underset{\sim}{\gamma}\|\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta})$	2	$36\frac{4}{11}$
SSE	2	10
SST	8	458

4. Hypothesis Testing

4.1. General procedure

For hypothesis testing in the model (2), the general form of testable hypothesis is $H: \underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{m}$ where $\underset{\sim}{K}'\underset{\sim}{b}$ is estimable and $\underset{\sim}{K}'$ has full row rank, s say. Under normality assumptions, the numerator sum of squares for an F-statistic for testing H is

$$Q = (\underset{\sim}{K}'\underset{\sim}{b}^{o} - \underset{\sim}{m})'[\underset{\sim}{K}'(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{K}]^{-1}(\underset{\sim}{K}'\underset{\sim}{b}^{o} - \underset{\sim}{m}) \tag{12}$$

and the statistic is

$$F(H) = Q/s\hat{\sigma}^2,$$

having, under H, an F-distribution with s and $N - r(\underset{\sim}{X})$ degrees of freedom where

$$\hat{\sigma}^2 = \underset{\sim}{y}'[\underset{\sim}{I} - \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}']\underset{\sim}{y}/[N - r(\underset{\sim}{X})]$$

with $r(\underset{\sim}{X})$ being the rank of $\underset{\sim}{X}$. In the case of the 2-way crossed classification $r(\underset{\sim}{X}) = N'$, the number of cells having data in them; and for all cells filled this is, of course, $r(\underset{\sim}{X}) = N' = ab$.

4.2. Illustrations

In the illustration

$$\underset{\sim}{b}' = [\mu\ \alpha_1\ \alpha_2\ \beta_1\ \beta_2\ \beta_3\ \gamma_{11}\ \gamma_{12}\ \gamma_{13}\ \gamma_{21}\ \gamma_{22}\ \gamma_{23}] \tag{13}$$

and

$$\underset{\sim}{b}^{o}{}' = [0\ 0\ 0\ 0\ 0\ 0\ 8\ 6\ 2\ 8\ 6\ 12] \tag{14}$$

and, using the symbol $\underset{\sim}{G}$ for $(\underset{\sim}{X}'\underset{\sim}{X})^{-}$ which in this case is diagonal,

$$\underset{\sim}{G} = (\underset{\sim}{X}'\underset{\sim}{X})^{-} = \text{diag}\{0\ 0\ 0\ 0\ 0\ 0\ \tfrac{1}{2}\ 1\ 1\ 1\ \tfrac{1}{2}\ 1\}. \tag{15}$$

In illustrating the relationship of the sums of squares in Table 5 to testing hypotheses, we consider only hypotheses of the form $H: \underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{0}$ and so Q of (12) reduces to

$$Q = (\underset{\sim}{K}'\underset{\sim}{b}^{o})'(\underset{\sim}{K}'\underset{\sim\sim}{G}\underset{\sim}{K})^{-1}\underset{\sim}{K}'\underset{\sim}{b}^{o} \ . \tag{16}$$

By way of example we show that $R(\underset{\sim}{\alpha}|\mu)$ is the numerator sum of squares for testing

$$\begin{aligned} H: \alpha_1 + \tfrac{1}{4}[2(\beta_1 + \gamma_{11}) + (\beta_2 + \gamma_{12}) + (\beta_3 + \gamma_{13})] \\ = \alpha_2 + \tfrac{1}{4}[(\beta_1 + \gamma_{21}) + 2(\beta_2 + \gamma_{22}) + (\beta_3 + \gamma_{23})] \end{aligned} \tag{17}$$

as given by equation (100), page 307 of Searle [1971]. Expressed as $H: \underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{0}$, the matrix $\underset{\sim}{K}'$ for this is

$$\underset{\sim}{K}' = [0 \quad 1 \quad -1 \quad \tfrac{1}{4} \quad -\tfrac{1}{4} \quad 0 \quad \tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{4} \quad -\tfrac{1}{4} \quad -\tfrac{1}{2} \quad -\tfrac{1}{4}] \ . \tag{18}$$

Because in (14) the first 6 entries of $\underset{\sim}{b}^{o}$ are zero (for μ^{o}, $\underset{\sim}{\alpha}^{o}$ and $\underset{\sim}{\beta}^{o}$), and because corresponding entries in $\underset{\sim}{G}$ are also zero, using (14), (15) and (18) for (16) gives

$$\underset{\sim}{K}'\underset{\sim}{b}^{o} = \tfrac{1}{2}(8) + \tfrac{1}{4}(6) + \tfrac{1}{4}(2) - \tfrac{1}{4}(8) - \tfrac{1}{2}(6) - \tfrac{1}{4}(12) = -2 \tag{19}$$

and

$$\underset{\sim}{K}'\underset{\sim\sim}{G}\underset{\sim}{K} = (\tfrac{1}{2})^3 + (\tfrac{1}{2})^2 + (\tfrac{1}{4})^2 + (\tfrac{1}{4})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{4})^2 = \tfrac{1}{2} \ . \tag{20}$$

Hence in (16)

$$Q = (-2)(\tfrac{1}{2})^{-1}(-2) = 8 = R(\underset{\sim}{\alpha}|\mu) \ . \tag{21}$$

Thus it is, that using $R(\underset{\sim}{\alpha}|\mu) = 8$ of Table 5 in an F-statistic tests the hypothesis shown in (17).

Similar use of $R(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha})$ as a numerator sum of squares results in testing the hypothesis

$$H: \left(3-\frac{4+1}{4}\right)\beta_1+\left(2-\frac{4}{4}\right)\gamma_{11}+\left(1-\frac{1}{4}\right)\gamma_{21} = \left(\frac{2}{4}+\frac{2}{4}\right)\beta_2+\frac{2+1}{4}\beta_3+\frac{2}{4}\gamma_{12}+\frac{2}{4}\gamma_{13}+\frac{2}{4}\gamma_{22}+\frac{1}{4}\gamma_{23}$$

$$\left(3-\frac{1+4}{4}\right)\beta_2+\left(1-\frac{1}{4}\right)\gamma_{12}+\left(2-\frac{4}{4}\right)\gamma_{22} = \left(\frac{2}{4}+\frac{2}{4}\right)\beta_1+\frac{1+2}{4}\beta_3+\frac{2}{4}\gamma_{11}+\frac{1}{4}\gamma_{13}+\frac{2}{4}\gamma_{21}+\frac{2}{4}\gamma_{23} \qquad (22)$$

derived from equation (106), page 309 of Searle [1971]. Expressing this hypothesis as $\underset{\sim}{K}'\underset{\sim}{b} = \underset{\sim}{0}$, the part of $\underset{\sim}{K}'$ corresponding to the non-zero elements of $\underset{\sim}{b}^{o}$ and $\underset{\sim}{G}$ is

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & -2 & -2 & 3 & -2 & -1 \\ -2 & 3 & -1 & -2 & 4 & -2 \end{bmatrix} \qquad (23)$$

and so

$$\underset{\sim}{K}'\underset{\sim}{b}^{o} = \frac{1}{4}\begin{bmatrix} 4(8) - 2(6) - 2(2) + 3(8) - 2(6) - 1(12) \\ -2(8) + 3(6) - 1(2) - 2(8) + 4(6) - 2(12) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

and

$$\underset{\sim}{K}'\underset{\sim\sim}{G}\underset{\sim}{K} = \frac{1}{16}\begin{bmatrix} 2 & -2 & -2 & 3 & -1 & -1 \\ -1 & 3 & -1 & -2 & 2 & -2 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -2 & 3 \\ -2 & -1 \\ 3 & -2 \\ -2 & 4 \\ -1 & -2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & -4 \\ -4 & 7 \end{bmatrix}.$$

Hence

$$Q = [4 \quad -4]\left[\frac{1}{4}\begin{pmatrix} 7 & -4 \\ -4 & 7 \end{pmatrix}\right]^{-1}\begin{bmatrix} 4 \\ -4 \end{bmatrix} = 11\frac{7}{11} = R(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha}) ;$$

i.e., using $R(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha})$ tests the hypothesis in (22).

4.3. μ_{ij}-models

It is important to appreciate how much easier this calculation and interpretation of hypothesis testing is if, instead of the over-parameterized model based on μ, α's, β's and γ's, we use the model

$$y_{ijk} = \mu_{ij} + e_{ijk} . \tag{24}$$

Then the b.l.u.e. of the parameter vector $\underset{\sim}{\mu}' = [\mu_{11}\ \mu_{12}\ \mu_{13}\ \mu_{21}\ \mu_{22}\ \mu_{23}]$ is

$$\underset{\sim}{b}^{o\prime} = \hat{\underset{\sim}{\mu}}' = \bar{\underset{\sim}{y}}' = [8\ \ 6\ \ 2\ \ 8\ \ 6\ \ 12], \tag{25}$$

the vector of cell means. The corresponding $\underset{\sim}{G}$ is

$$\underset{\sim}{G} = \text{diag}\{\tfrac{1}{2}\ \ 1\ \ 1\ \ 1\ \ \tfrac{1}{2}\ \ 1\}, \tag{26}$$

just the non-zero part of the $\underset{\sim}{G}$ in (15) for the μ, α, β and γ model. Similarly, the hypothesis (17) is

$$\tfrac{1}{4}(2\mu_{11} + \mu_{12} + \mu_{13}) = \tfrac{1}{4}(\mu_{21} + 2\mu_{22} + \mu_{23})$$

for which $\underset{\sim}{K}'$ is

$$\underset{\sim}{K}' = [2\ \ 1\ \ 1\ \ -1\ \ -2\ \ -1] \tag{27}$$

corresponding to the last 6 elements of (18). Then from (25) and (27)

$$\underset{\sim}{K}'\underset{\sim}{b}^{o} = \underset{\sim}{K}'\hat{\underset{\sim}{\mu}} = 2(8) + 1(6) + 1(2) - 1(8) - 2(6) - 1(12) = -8$$

and (26) and (27) give

$$\underset{\sim}{K}'\underset{\sim}{G}\underset{\sim}{K} = 2 + 1 + 1 + 1 + 2 + 1 = 8$$

so that (16) gives

$$Q = -8(8)^{-1}(-8) = 8 = R(\underset{\sim}{\alpha}|\mu),$$

as before.

4.4. Important comment

Displaying these hypotheses is not to be taken as promoting their use. Far from it. In particular, hypotheses like (17) and (22) have severe disadvantages (as indicated in Searle [1971], e.g., pages 303 and 317): they are not simple functions of just α's or just β's, and they are based on the n_{ij}'s of the data and hence are hypotheses that are dependent on the observed sample. Indeed, as hypotheses of practical interest they have little value whatever.

Their prime use and purpose is for connecting the sums of squares of Table 5 to the hypothesis testing concept. Having calculated such a table, it is an irresistible temptation to many data analysts to go on and use the sums of squares in F-statistics: assume normality, calculate mean squares, and calculate ratios of them to MSE = SSE/[N - r(X)]. Having done this, however, one then needs to know what hypothesis it is that each of the resulting F-statistics is testing. Many people, based on their knowledge of analysis of variance of balanced (equal subclass numbers) data, take it for granted, for example, that using $R(\underset{\sim}{\alpha}|\mu)$ of Table 5 in this manner tests $H: \alpha_1 = \alpha_2$. But it does not. It tests the messy hypothesis given in (17). And similar use of $R(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha})$ tests not $H: \beta_1 = \beta_2 = \beta_3$, but the hypothesis given in (22). And the only reason that such sums of squares test messy hypotheses like this rather than useful ones like "all rows equal" or "all columns equal" is that the data are unbalanced.

The appropriate use of hypothesis testing is, of course, that of setting up a hypothesis about nature, collecting data to test it, and carrying out the test. Statisticians have a strong conviction that, unfortunately, few data gatherers operate this way, at least formally. Often, it is only after data have been collected that hypotheses get carefully formulated, and sometimes only after analyses have been made. The preceding discussion illustrates how useless the hypotheses are corresponding to some of the F-statistics available in an analysis of variance like Table 5.

5. Normal Equations and Constraints on Solutions

5.1. The easiest procedure

The equations of the model (1) for the data of Table 1 are

$$\begin{bmatrix} 7 \\ 9 \\ 6 \\ 2 \\ 8 \\ 4 \\ 8 \\ 12 \end{bmatrix} = \underset{\sim}{y} = \underset{\sim\sim}{X}\underset{\sim}{b} + \underset{\sim}{e} = \left[\begin{array}{c|cc|ccc|cccccc} 1 & 1 & . & 1 & . & . & 1 & . & . & . & . & . \\ 1 & 1 & . & 1 & . & . & 1 & . & . & . & . & . \\ 1 & 1 & . & . & 1 & . & . & 1 & . & . & . & . \\ 1 & 1 & . & . & . & 1 & . & . & 1 & . & . & . \\ 1 & . & 1 & 1 & . & . & . & . & . & 1 & . & . \\ 1 & . & 1 & . & 1 & . & . & . & . & . & 1 & . \\ 1 & . & 1 & . & 1 & . & . & . & . & . & 1 & . \\ 1 & . & 1 & . & . & 1 & . & . & . & . & . & 1 \end{array}\right] \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix} + \underset{\sim}{e}. \qquad (28)$$

(Dots in a matrix represent zeros.)

The normal equations (3) resulting from this are

$$\left[\begin{array}{c|cc|ccc|cccccc} 8 & 4 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 2 & 1 \\ \hline 4 & 4 & . & 2 & 1 & 1 & 2 & 1 & 1 & . & . & . \\ 4 & . & 4 & 1 & 2 & 1 & . & . & . & 1 & 2 & 1 \\ \hline 3 & 2 & 1 & 3 & . & . & 2 & . & . & 1 & . & . \\ 3 & 1 & 2 & . & 3 & . & . & 1 & . & . & 2 & . \\ 2 & 1 & 1 & . & . & 2 & . & . & 1 & . & . & 1 \\ \hline 2 & 2 & . & 2 & . & . & 2 & . & . & . & . & . \\ 1 & 1 & . & . & 1 & . & . & 1 & . & . & . & . \\ 1 & 1 & . & . & . & 1 & . & . & 1 & . & . & . \\ 1 & . & 1 & 1 & . & . & . & . & . & 1 & . & . \\ 2 & . & 2 & . & 2 & . & . & . & . & . & 2 & . \\ 1 & . & 1 & . & . & 1 & . & . & . & . & . & 1 \end{array}\right] \begin{bmatrix} \mu^\circ \\ \alpha_1^\circ \\ \alpha_2^\circ \\ \beta_1^\circ \\ \beta_2^\circ \\ \beta_3^\circ \\ \gamma_{11}^\circ \\ \gamma_{12}^\circ \\ \gamma_{13}^\circ \\ \gamma_{21}^\circ \\ \gamma_{22}^\circ \\ \gamma_{23}^\circ \end{bmatrix} = \begin{bmatrix} 56 \\ 24 \\ 32 \\ 24 \\ 18 \\ 14 \\ 16 \\ 6 \\ 2 \\ 8 \\ 12 \\ 12 \end{bmatrix} \qquad (29)$$

Writing down model and normal equations like this is well-known and easy. We do it so as to have them available for what follows.

The superscript zero on the parameter symbols in (29) emphasizes that any solution to these equations suffices for estimating estimable functions and the residual variance. The easiest way to obtain a solution is to apply constraints to the solution which easily yield a solution, in this case

$$\mu^o = \alpha_1^o = \alpha_2^o = \beta_1^o = \beta_2^o = \beta_3^o = 0, \tag{30}$$

and obtain the solution

$$\underset{\sim}{b}^{o\prime} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 8 \quad 6 \quad 2 \quad 8 \quad 6 \quad 12]$$

as given in (14). Then, using the R-algorithm of (6),

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) = 8(16) + 6(6) + 2(2) + 8(8) + 6(12) + 12(12) = 448 \tag{31}$$

as obtained earlier.

Constraints other than (3) are, of course, permissible. (Searle [1971], pages 212-213, discusses general limitations on what can be used as constraints.) With all of them, best linear unbiased estimators of estimable functions are the same, of course, invariant to whatever constraints are used, even though the values of $\underset{\sim}{b}^o$ will be different. And with all of the constraints the sum of squares $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})$ will be the same too, namely 448 as in (31).

5.2. A sub-model

Consider the model

$$y_{ijk} = \mu + \alpha_i + \gamma_{ij} + e_{ijk}$$

obtained from (2) by deleting β's. It is indistinguishable from the model for the

2-way nested classification. The model equations are those of (28) but with β's deleted:

$$\begin{bmatrix} 7 \\ 9 \\ 6 \\ 2 \\ 8 \\ 4 \\ 8 \\ 12 \end{bmatrix} = \left[\begin{array}{c|cc|ccccc} 1 & 1 & . & 1 & . & . & . & . & . \\ 1 & 1 & . & 1 & . & . & . & . & . \\ 1 & 1 & . & . & 1 & . & . & . & . \\ 1 & 1 & . & . & . & 1 & . & . & . \\ 1 & . & 1 & . & . & . & 1 & . & . \\ 1 & . & 1 & . & . & . & . & 1 & . \\ 1 & . & 1 & . & . & . & . & 1 & . \\ 1 & . & 1 & . & . & . & . & . & 1 \end{array}\right] \left[\begin{array}{c} \mu \\ \hline \alpha_1 \\ \alpha_2 \\ \hline \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{array}\right] + \underset{\sim}{e} . \tag{32}$$

The corresponding normal equations are

$$\left[\begin{array}{c|cc|ccccc} 8 & 4 & 4 & 2 & 1 & 1 & 1 & 2 & 1 \\ \hline 4 & 4 & . & 2 & 1 & 1 & . & . & . \\ 4 & . & 4 & . & . & . & 1 & 2 & 1 \\ \hline 2 & 2 & . & 2 & . & . & . & . & . \\ 1 & 1 & . & . & 1 & . & . & . & . \\ 1 & 1 & . & . & . & 1 & . & . & . \\ 1 & . & 1 & . & . & . & 1 & . & . \\ 2 & . & 2 & . & . & . & . & 2 & . \\ 1 & . & 1 & . & . & . & . & . & 1 \end{array}\right] \left[\begin{array}{c} \mu^\circ \\ \hline \alpha_1^\circ \\ \alpha_2^\circ \\ \hline \gamma_{11}^\circ \\ \gamma_{12}^\circ \\ \gamma_{13}^\circ \\ \gamma_{21}^\circ \\ \gamma_{22}^\circ \\ \gamma_{23}^\circ \end{array}\right] = \left[\begin{array}{c} 56 \\ \hline 24 \\ 32 \\ \hline 16 \\ 6 \\ 2 \\ 8 \\ 12 \\ 12 \end{array}\right] . \tag{33}$$

These, too, can be obtained by a "delete β's" operation: from (29) delete the β's and the β-equations. We return to this operation later.

A solution to (33) is

$$\underset{\sim}{b}^{o\prime} = [0 \quad 0 \quad 0 \quad 8 \quad 6 \quad 2 \quad 8 \quad 6 \quad 12]. \tag{34}$$

Then the R-algorithm applied to (34) and the right-hand side of (33) gives

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma}) = 448 \qquad (35)$$

just as in (31).

Notice here from (31) and (35) that $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma}) = 448 - 448 = 0$. This is no coincidence. It is, in fact, an identity

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma}) \equiv 0 \qquad (36)$$

that is always true. It arises from the fact that the reductions in sums of squares for fitting either the 2-way crossed classification model or the 2-way nested classification model are $\Sigma\Sigma n_{ij}\bar{y}^2_{ij\cdot}$. Hence the identity (36) always holds. This was the basis for the complaint in Searle [1972], that it is wrong to calculate $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})$ in any way that gives a value other than zero. That statement is still correct – provided it is limited to unrestricted models, namely models that have no restrictions on the parameters.

The subtlety is that in restricted models the difference $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})$ can be calculated in a certain way as non-zero, and Hocking and co-workers [1, 6, 7] have shown, for at least 3 kinds of restrictions, what the meanings of these non-zero values are, in terms of the restricted model. We proceed with illustration of these 3 cases.

5.3. Constraints and restrictions

In mentioning both constraints on solutions (of the normal equations) and restrictions on models, the distinction between the two must be reiterated. They are not the same; and a constraint does not imply the corresponding restriction. For example, corresponding to $\mu^\circ = 0$ of (30) would be the restriction $\mu = 0$; but $\mu = 0$ is not implied by (30). Lengthy discussion of constraints and restrictions is given in Sections 5.6 and 5.7 of Searle [1971].

Constraints are only an arithmetical gimmick required for obtaining a solution to a set of less-than-full rank normal equations. Constraints do not apply to the parameters of the model; restrictions do. Restrictions are customarily used to make a model that is not of full rank become one that is of full rank; the normal equations are then likewise, and so require no constraints. Their solution will, of course, satisfy the same relationships as given by the restrictions. For example, in a 1-way classification restricted model $y_{ij} = \mu + \alpha_i + e_{ij}$ with the restriction $\Sigma\alpha_i = 0$, the solutions to the normal equations will satisfy $\Sigma\alpha_i^\circ = 0$. This is a consequence of the restricted model: it is not an artifact imposed on solutions in order to derive those solutions.

6. The Σ-restrictions

Restrictions on the model (1) that have traditionally been used for balanced data are

$$\sum_{i=1}^{a} \alpha_i = 0, \quad \sum_{j=1}^{b} \beta_j = 0, \quad \sum_{i=1}^{a} \gamma_{ij} = 0 \ \forall\ j, \quad \text{and} \quad \sum_{j=1}^{b} \gamma_{ij} = 0 \ \forall\ i\ . \tag{37}$$

To distinguish these from other restrictions that will be used, we refer to (37) as the Σ-restrictions. For unbalanced data the summations of the γ's are only for those γ_{ij}'s for which the corresponding n_{ij}'s are non-zero; i.e., only for the cells containing data. When all cells are filled they apply in the same manner as for balanced data.

With balanced data, the two parts of Table 5 are the same. And the effect of the Σ-restrictions is that $R(\underset{\sim}{\alpha}|\mu)$ can then be used to test H: α_i's all equal. With unbalanced data, the two parts of Table 5 are distinct; but, provided all cells are filled, the effect of the Σ-restrictions is that, by an easy manipulation of the normal equations, the sums of squares for the weighted squares of means analysis

can be derived. The manipulation involved is that described and complained of in Searle [1972]. At that time the computing procedure was known but its interpretation was not. It has since been provided by Hocking and colleagues [1, 6, 7]. It is illustrated here.

6.1. The restrictions

The Σ-restrictions (37) for the all cells filled data of Table 1 are as follows:

$$\alpha_1 + \alpha_2 = 0, \quad \text{implying} \quad \alpha_2 = -\alpha_1 , \tag{38}$$

$$\beta_1 + \beta_2 + \beta_3 = 0 \quad \text{implying} \quad \beta_3 = -\beta_1 - \beta_2 , \tag{39}$$

and

$$\left.\begin{aligned} \gamma_{11} + \gamma_{12} + \gamma_{13} &= 0 \\ \gamma_{21} + \gamma_{22} + \gamma_{23} &= 0 \\ \gamma_{11} + \gamma_{21} &= 0 \\ \gamma_{12} + \gamma_{22} &= 0 \\ \gamma_{13} + \gamma_{23} &= 0 \end{aligned}\right\} \quad \text{implying} \quad \begin{aligned} \gamma_{11} &= \gamma_{11} \\ \gamma_{12} &= \gamma_{12} \\ \gamma_{13} &= -\gamma_{11} - \gamma_{12} \\ \gamma_{21} &= -\gamma_{11} \\ \gamma_{22} &= -\gamma_{12} \\ \gamma_{23} &= \gamma_{11} + \gamma_{12} . \end{aligned} \tag{40}$$

In (40), the right-hand statements include the obvious $\gamma_{11} = \gamma_{11}$ and $\gamma_{12} = \gamma_{12}$. This is to emphasize that the set of restrictions, shown as the left-hand set of statements in (40), can be restated so that all the γ's are in terms of just γ_{11} and γ_{12}. For the general case of a rows and b columns and all cells filled, there will be a + b restrictions on the γ's, which can be restated so that all γ's are expressible in terms of just (a-1)(b-1) of them.

6.2. Model equations

The effect of the restrictions on changing the unrestricted model to the restricted model is seen by applying the restated restrictions of (38), (39) and

(40) to the model equations (28). The result is that the model equations for the restricted model are

$$\begin{bmatrix} 7 \\ 9 \\ 6 \\ 2 \\ 8 \\ 4 \\ 8 \\ 12 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c|c} 1 & 1 & 1 & . & 1 & . \\ 1 & 1 & 1 & . & 1 & . \\ 1 & 1 & . & 1 & . & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & . & -1 & . \\ 1 & -1 & . & 1 & . & -1 \\ 1 & -1 & . & 1 & . & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{array}\right] \begin{bmatrix} \mu \\ \alpha_1 \\ \beta_1 \\ \beta_2 \\ \gamma_{11} \\ \gamma_{12} \end{bmatrix} + \underset{\sim}{e} . \tag{41}$$

6.3. Normal equations

The normal equations resulting from (41) are

$$\left[\begin{array}{c|c|c|c|c|c} 8 & 0 & 1 & 1 & 1 & -1 \\ \hline 0 & 8 & 1 & -1 & 1 & 1 \\ \hline 1 & 1 & 5 & 2 & 1 & 0 \\ 1 & -1 & 2 & 5 & 0 & -1 \\ \hline 1 & 1 & 1 & 0 & 5 & 2 \\ -1 & 1 & 0 & -1 & 2 & 5 \end{array}\right] \begin{bmatrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\gamma}_{11} \\ \hat{\gamma}_{12} \end{bmatrix} = \begin{bmatrix} 56 \\ -8 \\ 10 \\ 4 \\ 18 \\ 4 \end{bmatrix} . \tag{42}$$

Because these are full rank equations with just one solution, that solution is denoted by $\hat{b}$. Its value is

$$\underset{\sim}{\hat{b}}' = \left[7 \quad -\frac{10}{6} \quad 1 \quad -1 \quad \frac{10}{6} \quad \frac{10}{6}\right] . \tag{43}$$

6.4. Reduction in sum of squares

The reduction in sum of squares for this restricted model will be denoted by $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$; the superscript asterisk designates that it is for a restricted model and the subscript Σ indicates that the restrictions are the Σ-restrictions.

Its value is calculated by the R-algorithm as

$$R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma = 7(56) - \frac{10}{6}(-8) + 1(10) - 1(4) + \frac{10}{6}(18) + \frac{10}{6}(4) = 448. \qquad (44)$$

The value of $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$ in (44) is the same as that of $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$ in (31). This is no coincidence. The normal equations (42) have been derived as those for the restricted model with Σ-restrictions. They could also be derived from the normal equations (29) for the unrestricted model by applying Σ-constraints corresponding to the Σ-restrictions (37) of the restricted model (e.g., $\sum_{i=1}^{a} \alpha_i^o = 0$). This will be true for any set of restrictions that reparameterizes the unrestricted model to a full rank restricted model; i.e.,

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) = R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) \qquad (45)$$

for any appropriate restrictions.

We are here dealing with restricted models rather than just constraints on solutions of normal equations for unrestricted models. For this reason we prefer to derive the normal equations for a restricted model from the basic model equations for the restricted model; e.g., derive (42) from (41). In doing so, it must be appreciated that the symbols for parameters in (41) do not represent the same parameters as they do in (28). Those model equations, (28), are for the unrestricted model whereas equations (42) are for the restricted model. This point would have been clearer had (38), (39) and (40) been written with symbols different from α_i, β_j and γ_{ij}, say α_i^*, β_j^* and γ_{ij}^*. Then the model equations (41) would have also been in terms of the starred symbols, their general form being

$$
\begin{aligned}
E(y_{11k}) &= \mu^* + \alpha_1^* + \beta_1^* + \gamma_{11}^* \\
E(y_{12k}) &= \mu^* + \alpha_1^* + \beta_2^* + \gamma_{12}^* \\
E(y_{13k}) &= \mu^* + \alpha_1^* - \beta_1^* - \beta_2^* - \gamma_{11}^* - \gamma_{12}^* \\
E(y_{21k}) &= \mu^* - \alpha_1^* + \beta_1^* - \gamma_{11}^* \\
E(y_{22k}) &= \mu^* - \alpha_1^* + \beta_2^* - \gamma_{12}^* \\
E(y_{23k}) &= \mu^* - \alpha_1^* - \beta_1^* - \beta_2^* + \gamma_{11}^* + \gamma_{12}^* \, .
\end{aligned}
\tag{46}
$$

The reduction in sum of squares would then be designated $R(\mu^*,\underset{\sim}{\alpha}^*,\underset{\sim}{\beta}^*,\underset{\sim}{\gamma}^*)$ rather than $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$ as in (44). The latter is used to avoid the profusion of symbols that would ensue when we come to deal with restrictions other than the Σ-restrictions. Our interest is in $R(\ )$-values, and $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$ is quite clear as to both the use of restrictions and their nature.

7. A Sub-model

Having explained the equality of $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$ and $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})$, one may ask why we gave such specific discussion to the notation $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$. It is because we now come to discuss a term that might be denoted $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})_\Sigma$, which does <u>not</u> equal $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})$; [and we recall from (36) that $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma}) = R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})$].

7.1. The sub-model with Σ-restrictions

We consider the same sub-model as used earlier, namely (1) with β_j's deleted:

$$y_{ijk} = \mu + \alpha_i + \gamma_{ij} + e_{ijk} \, .$$

But now we consider it in a restricted form, with Σ-restrictions. These restrictions, however, are not those of (38), (39) and (40) but are now, by virtue of the model being indistinguishable from a nested model, of the form

$$\alpha_1 + \alpha_2 = 0, \quad \text{implying} \quad \alpha_2 = -\alpha_1 ,$$

$$\gamma_{11} + \gamma_{12} + \gamma_{13} = 0, \quad \text{implying} \quad \gamma_{13} = -\gamma_{11} - \gamma_{12} \tag{47}$$

and

$$\gamma_{21} + \gamma_{22} + \gamma_{23} = 0, \quad \text{implying} \quad \gamma_{23} = -\gamma_{21} - \gamma_{22} .$$

The model equations for this restricted model come from adapting (32), those of the unrestricted model, and are

$$\begin{bmatrix} 7 \\ 9 \\ 6 \\ 2 \\ 8 \\ 4 \\ 8 \\ 12 \end{bmatrix} = \left[\begin{array}{c|cc|cc} 1 & 1 & 1 & . & . & . \\ 1 & 1 & 1 & . & . & . \\ 1 & 1 & . & 1 & . & . \\ 1 & 1 & -1 & -1 & . & . \\ 1 & -1 & . & . & 1 & . \\ 1 & -1 & . & . & . & 1 \\ 1 & -1 & . & . & . & 1 \\ 1 & -1 & . & . & -1 & -1 \end{array}\right] \begin{bmatrix} \mu \\ \alpha_1 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{21} \\ \gamma_{22} \end{bmatrix} + \underset{\sim}{e} . \tag{48}$$

and the corresponding normal equations are then

$$\left[\begin{array}{c|c|cc|cc} 8 & . & 1 & . & . & 1 \\ \hline . & 8 & 1 & . & . & -1 \\ \hline 1 & 1 & 3 & 1 & . & . \\ . & . & 1 & 2 & . & . \\ . & . & . & . & 2 & 1 \\ 1 & -1 & . & . & 1 & 3 \end{array}\right] \left[\begin{array}{c} \hat{\mu} \\ \hline \hat{\alpha}_1 \\ \hline \hat{\gamma}_{11} \\ \hat{\gamma}_{12} \\ \hat{\gamma}_{21} \\ \hat{\gamma}_{22} \end{array}\right] = \left[\begin{array}{c} 56 \\ \hline -8 \\ \hline 14 \\ 4 \\ -4 \\ 0 \end{array}\right] . \tag{49}$$

The solution is

$$\hat{\underset{\sim}{b}}' = \left[7 \quad -1\tfrac{2}{3} \quad 2\tfrac{2}{3} \quad \tfrac{2}{3} \quad -\tfrac{2}{3} \quad -2\tfrac{2}{3}\right] \tag{50}$$

and the corresponding reduction in sum of squares is

$$R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})_\Sigma = 7(56) - 1\tfrac{2}{3}(-8) + 2\tfrac{2}{3}(14) + \tfrac{2}{3}(4) - \tfrac{2}{3}(-4) - 2\tfrac{2}{3}(0) = 448,$$

the same as $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})$ of (35). Again, this is no coincidence; it is the same kind of equality for the sub-model as $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma = R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})$ is for the full model.

7.2. The "Delete β's" operation on normal equations

In the unrestricted model, deleting β's from the model equations (28) for the full model yields the model equations (32) for the sub-model. Similarly for normal equations: deleting both β's and β-equations from the normal equations (29) for the full model yields those for the sub-model, (3).

Suppose this "delete β's" operation is carried over from unrestricted models to restricted models with Σ-restrictions. The model equations for the restricted full model are (41); and, by inspection, it is clear that deleting β's from those does not yield the model equations for the restricted sub-model, namely (48). Despite this observation, suppose that the "delete β" operation is carried out on the normal equations based on (41), namely (42). The result is

$$\left[\begin{array}{c:c:cc} 8 & 0 & 1 & -1 \\ \hdashline 0 & 8 & 1 & 1 \\ \hdashline 1 & 1 & 5 & 2 \\ -1 & 1 & 2 & 5 \end{array}\right] \left[\begin{array}{c} \hat{\mu} \\ \hdashline \hat{\alpha}_1 \\ \hdashline \hat{\gamma}_{11} \\ \hat{\gamma}_{12} \end{array}\right] = \left[\begin{array}{c} 56 \\ \hdashline -8 \\ \hdashline 18 \\ 4 \end{array}\right] . \qquad (51)$$

Although (42) are the normal equations for the restricted full model, (51) are not those for the restricted sub-model which, we have seen, are (49). Nevertheless, this is the operation that is carried out in some computer programs, ostensibly with the objective of using the R-algorithm to calculate what might be called $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})$ – a sum of squares for fitting a model with β's deleted. But since we know that $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})$ is the same as $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) = \Sigma\Sigma n_{ij}\bar{y}^2_{ij\cdot}$, and also because the "delete β's" operation has been carried out not on a model (which is the proper place for deriving a model without β's) but on normal equations, we introduce the

symbol $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\not\beta},\underset{\sim}{\gamma})_\Sigma$ to denote the result of using the R-algorithm on (51). Solving (51) gives

$$\hat{\underset{\sim}{b}}' = \left[7 \quad -1\tfrac{4}{9} \quad 1\tfrac{7}{9} \quad 1\tfrac{7}{9}\right] \tag{52}$$

and then

$$R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\not\beta},\underset{\sim}{\gamma})_\Sigma = 7(56) - 1\tfrac{4}{9}(-8) + 1\tfrac{7}{9}(18) + 1\tfrac{7}{9}(4) = 442\tfrac{2}{3} . \tag{53}$$

Two questions about (53) immediately arise: (i) what is the significance of $\underset{\sim}{\not\beta}$ in the symbol $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\not\beta},\underset{\sim}{\gamma})_\Sigma$, and (ii) what is the meaning, if any, of the calculated value?

The $\underset{\sim}{\not\beta}$ symbol in $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\not\beta},\underset{\sim}{\gamma})_\Sigma$ indicates that β's and β-equations have been deleted from the normal equations of the restricted full model (involving μ, α's, β's and γ's); and after solving the resultant equations, the R-algorithm yields what is designated as $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\not\beta},\underset{\sim}{\gamma})_\Sigma$.

The meaning of $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\not\beta},\underset{\sim}{\gamma})_\Sigma$ is closely related to the weighted squares of means analysis. Indeed, as Hocking and co-workers [1, 6, 7] tell us,

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\not\beta},\underset{\sim}{\gamma})_\Sigma = SSB_w, \tag{54}$$

where SSB_w is the sum of squares due to columns in the weighted squares of means analysis (Searle [1971], p. 361-373). Since $R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) = R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$, we can write (54) as

$$SSB_w = R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\not\beta},\underset{\sim}{\gamma})_\Sigma \tag{55}$$

which might also be symbolized as $R^*(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})_\Sigma$, <u>provided</u> one remembers that it is expressed as the difference given in (55) and not as $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})_\Sigma$, which we know is always zero.

The important thing is that $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$ and $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})_\Sigma$ are not the same. $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$ comes from using the R-algorithm on equations obtained by deleting β's and β-equations from the normal equations of the restricted full model with Σ-restrictions. But $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\gamma})_\Sigma$ comes from the normal equations of the restricted sub-model.

7.3. The weighted squares of means analysis

Expressions for the sums of squares for this analysis are given in Table 8.18 of Searle [1971]. Applied to our illustration they give

$$v_1 = v_2 = \left[\tfrac{1}{4}(\tfrac{1}{2} + \tfrac{1}{1})\right]^{-1} = \frac{8}{3} \quad \text{and} \quad v_3 = \left[\tfrac{1}{4}(\tfrac{1}{1} + \tfrac{1}{1})\right]^{-1} = 2 \ .$$

$$\bar{x}_{[2]} = \frac{(8/3)8 + (8/3)6 + 2(7)}{8/3 + 8/3 + 2} = \frac{154}{22} = 7$$

and

$$SSB_w = \frac{8}{3}(5 - 7)^2 + \frac{8}{3}(6 - 7)^2 + 2(7 - 7)^2 = 5\tfrac{1}{3}.$$

Confirmation of (54) is now available: from (31) and (53)

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) = 448 - 442\tfrac{2}{3} = 5\tfrac{1}{3} = SSB_w,$$

so illustrating (54).

7.4. Conclusion

For data having all cells filled, the following algorithm is a calculation technique for obtaining the sums of squares for the weighted squares of means analysis:

(i) Form the normal equations for the restricted full model with Σ-restrictions.

(ii) Delete β's and the β-equations from (i).

(iii) Solve the resulting equations, and calculate $R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma$ using the R-algorithm.

(iv) $\Sigma\Sigma y_{ij.}^2/n_{ij} - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) = SSB_w$.

7.5. Hypothesis testing

The hypothesis tested using SSB_w as a numerator sum of squares is, analogous to (68) of Searle [1971], p. 371, H: $\beta_j + \bar{\gamma}_{.j}$ all equal. This is for the unrestricted model. For the restricted model with Σ-restrictions (and all cells filled) it is H: β_j all equal.

7.6. Another example

As a second example we illustrate derivation of SSA_w by this procedure:

(i) The normal equations for the restricted full model are (42).

(ii) Deleting the (sole) α and the α-equation leaves

$$\left[\begin{array}{c|cc|cc} 8 & 1 & 1 & 1 & -1 \\ \hline 1 & 5 & 2 & 1 & 0 \\ 1 & 2 & 5 & 0 & -1 \\ \hline 1 & 1 & 0 & 5 & 2 \\ -1 & 0 & -1 & 2 & 5 \end{array}\right] \left[\begin{array}{c} \hat{\mu} \\ \hline \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hline \hat{\gamma}_{11} \\ \hat{\gamma}_{12} \end{array}\right] = \left[\begin{array}{c} 56 \\ \hline 10 \\ 4 \\ \hline 18 \\ 4 \end{array}\right] . \qquad (56)$$

(iii) The solution is

$$\hat{\underset{\sim}{b}}' = [7 \quad \tfrac{1}{2} \quad -\tfrac{1}{2} \quad 1\tfrac{1}{2} \quad 1\tfrac{1}{2}]$$

and

$$R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma = 7(56) + \tfrac{1}{2}(10) - \tfrac{1}{2}(4) + 1\tfrac{1}{2}(18) + 1\tfrac{1}{2}(4) = 428.$$

(iv) Hence

$$SSA_w = R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) = 448 - 428 = 20.$$

Confirmation comes from noting that for the weighted squares of means analysis

$$w_1 = w_2 = \left[\frac{1}{9}\left(\frac{1}{2} + \frac{1}{1} + \frac{1}{1}\right)\right]^{-1} = \frac{18}{5}$$

and

$$\bar{x}_{[1]} = \frac{(18/5)(16/3) + (18/5)(26/3)}{18/5 + 18/5} = \frac{42}{2(3)} = 7.$$

Hence

$$SSA_w = \frac{18}{5}\left(\frac{16}{3} - 7\right)^2 + \frac{18}{5}\left(\frac{26}{3} - 7\right)^2 = 20,$$

as already obtained.

8. The 0-restrictions

The Σ-restrictions are popular, originating in their very reasonable and useful application in balanced data analysis. In unrestricted models, the Σ-constraints on solutions (constraints corresponding to the Σ-restrictions) are also widely used. Other useful constraints are those which put some elements of the solution vector equal to zero (e.g., Searle [1971], p. 213). We consider here the consequences of using such restrictions, which we call 0-restrictions, in the same way that we have already used the Σ-restrictions.

The particular set of 0-restrictions considered by Speed and Hocking [1976] is

$$\left.\begin{array}{ll} \alpha_1 = 0 & \gamma_{1j} = 0 \ \forall\ j \\ \beta_1 = 0 & \gamma_{i1} = 0 \ \forall\ i \end{array}\right\} \ 0_{11}\text{-restrictions}. \tag{57}$$

A generalization of these is:

for an arbitrarily chosen k and t

$$\left.\begin{array}{ll} \alpha_k = 0 & \gamma_{kj} = 0 \ \forall\ j \\ \beta_t = 0 & \gamma_{it} = 0 \ \forall\ i \end{array}\right\} \ 0_{kt}\text{-restrictions}. \tag{58}$$

We call these the 0_{kt}-restrictions. Then Speed and Hocking indicate that using

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_{0_{kt}} \tag{59}$$

as the numerator sum of squares in an F-statistic tests

$$H: \beta_j + \gamma_{kj} \text{ equal } \forall\ j, \text{ in the unrestricted model.} \tag{60}$$

In view of the restrictions (58) this is equivalent to a test of

$$H: \beta_j \text{ equal } \forall\ j, \text{ in the restricted model.}$$

We demonstrate (58), (59) and (60) using the illustration.

Suppose (58) is

$$\left.\begin{array}{llll} \alpha_1 = 0 & \gamma_{11} = 0 & \gamma_{12} = 0 & \gamma_{13} = 0 \\ \beta_1 = 0 & \gamma_{21} = 0 & & \end{array}\right\} 0_{11} \tag{61}$$

equivalent to (57). The effect of these on the model equations (28) of the unrestricted model is to eliminate columns of the $\underset{\sim}{X}$ matrix corresponding to the elements equated to zero in (61). The effect on the normal equations (29) is deletion of the corresponding rows and columns, thus giving the normal equations for the restricted model as

$$\left[\begin{array}{c|c|cc|cc} 8 & 4 & 3 & 2 & 2 & 1 \\ \hline 4 & 4 & 2 & 1 & 2 & 1 \\ \hline 3 & 2 & 3 & 0 & 2 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 \\ \hline 2 & 2 & 2 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \hat{\mu} \\ \hline \hat{\alpha}_2 \\ \hline \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hline \hat{\gamma}_{22} \\ \hat{\gamma}_{23} \end{array}\right] = \left[\begin{array}{c} 56 \\ \hline 32 \\ \hline 18 \\ 14 \\ \hline 12 \\ 12 \end{array}\right]. \tag{62}$$

Solution of these is

$$\hat{\underset{\sim}{b}}' = [8 \quad 0 \quad -2 \quad -6 \quad 0 \quad 10]$$

and use of the R-algorithm gives

$$R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_{0_{11}} = 8(56) + 0(32) - 2(18) - 6(14) + 0(12) + 12(10)$$
$$= 448 = R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}),$$

as is to be expected.

Deleting β's from (62) gives equations

$$\left[\begin{array}{c|c|c|c} 8 & 4 & 2 & 1 \\ \hline 4 & 4 & 2 & 1 \\ \hline 2 & 2 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \hat{\mu} \\ \hline \hat{\alpha}_2 \\ \hline \hat{\gamma}_{22} \\ \hat{\gamma}_{23} \end{array}\right] = \left[\begin{array}{c} 56 \\ \hline 32 \\ \hline 12 \\ 12 \end{array}\right].$$

Solution of these is

$$\hat{\underset{\sim}{b}}' = [6 \quad 2 \quad -2 \quad 4]$$

and the R-algorithm yields

$$R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_{0_{11}} = 6(56) + 2(32) - 2(12) + 4(12) = 424,$$

so giving (59) as

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_{0_{11}} = 448 - 424 = 24. \qquad (63)$$

The hypothesis (60) is

$$H: \beta_1 + \gamma_{11} = \beta_2 + \gamma_{12} = \beta_3 + \gamma_{13}$$

which, for $\underset{\sim}{b}$ of (13), can be written as

$$H: \underset{\sim}{K}'\underset{\sim}{b} = 0 \quad \text{for} \quad \underset{\sim}{K}' = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, for (16)

$$\underset{\sim}{K}'\underset{\sim}{b}^{o} = \begin{bmatrix} 8 - 6 \\ 8 - 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

and

$$\underset{\sim}{K}'\underset{\sim\sim}{G}\underset{\sim}{K} = \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix},$$

so that (16) is

$$Q = [2 \quad 6] \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = 2(4)[1 \quad 3] \frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 24.$$

Thus Q has the same value as (63), and so (59) and (60) are confirmed for this example.

The reader can verify that $R^{*}(\mu, \underset{\sim}{\alpha}, \underset{\sim}{\beta}, \underset{\sim}{\gamma})_{O_{11}} = 448$ for the illustration, and confirm its value through considering the hypothesis $H: \alpha_i + \gamma_{i1}$ equal for all i, corresponding to (60).

9. The W-restrictions

The following set of restrictions, which we call the W-restrictions,

$$\left.\begin{array}{ll} \sum_{i=1}^{a} n_{i\cdot}\alpha_i = 0 & \sum_{i=1}^{a} n_{ij}(\alpha_i + \gamma_{ij}) = 0 \ \forall\ j \\ \sum_{j=1}^{b} n_{\cdot j}\beta_j = 0 & \sum_{j=1}^{b} n_{ij}(\beta_j + \gamma_{ij}) = 0 \ \forall\ i \end{array}\right\} \text{ W-restrictions} \qquad (64)$$

is also considered by Hocking et al. [1, 6, 7]. They indicate that

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^{*}(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_W = R(\underset{\sim}{\beta}|\mu) \qquad (65)$$

of the classical analysis of variance in Table 5. This we now illustrate.

The W-restrictions for the illustrative data are as follows:

$$4\alpha_1 + 4\alpha_2 = 0, \quad \text{implying} \quad \alpha_2 = -\alpha_1 \qquad (66)$$

$$3\beta_1 + 3\beta_2 + 2\beta_3 = 0, \quad \text{implying} \quad \beta_3 = -1\tfrac{1}{2}(\beta_1 + \beta_2) \qquad (67)$$

and

$$\begin{aligned} 2(\alpha_1 + \gamma_{11}) + \alpha_2 + \gamma_{21} &= 0 \\ \alpha_1 + \gamma_{12} + 2(\alpha_2 + \gamma_{22}) &= 0 \\ \alpha_1 + \gamma_{13} + \alpha_2 + \gamma_{23} &= 0 \\ 2(\beta_1 + \gamma_{11}) + \beta_2 + \gamma_{12} + \beta_3 + \gamma_{13} &= 0 \\ \beta_1 + \gamma_{21} + 2(\beta_2 + \gamma_{22}) + \beta_3 + \gamma_{23} &= 0 \end{aligned}$$

all of which imply, in the nature of (40),

$$\begin{aligned} \gamma_{11} &= \gamma_{11} \\ \gamma_{12} &= \gamma_{12} \\ \gamma_{13} &= -(\tfrac{1}{2}\beta_1 - \tfrac{1}{2}\beta_2 + 2\gamma_{11} + \gamma_{12}) \\ \gamma_{21} &= -(\alpha_1 + 2\gamma_{11}) \\ \gamma_{22} &= \tfrac{1}{2}(\alpha_1 - \gamma_{12}) \end{aligned} \qquad (68)$$

and

$$\gamma_{23} = \tfrac{1}{2}\beta_1 - \tfrac{1}{2}\beta_2 + 2\gamma_{11} + \gamma_{12} .$$

Applying (66), (67) and (68) to the model equations (28) of the unrestricted model yields the model equations for this restricted model, with W-restrictions, as

$$\left[\begin{array}{c|c|cc|cc} 1 & 1 & 1 & . & 1 & . \\ 1 & 1 & 1 & . & 1 & . \\ 1 & 1 & . & 1 & . & 1 \\ 1 & 1 & -2 & -1 & -2 & -1 \\ 1 & -2 & 1 & . & -2 & . \\ 1 & -\tfrac{1}{2} & . & 1 & . & -\tfrac{1}{2} \\ 1 & -\tfrac{1}{2} & . & 1 & . & -\tfrac{1}{2} \\ 1 & -1 & -1 & -2 & 2 & 1 \end{array}\right] \left[\begin{array}{c} \mu \\ \alpha_1 \\ \beta_1 \\ \beta_2 \\ \gamma_{11} \\ \gamma_{12} \end{array}\right] = \left[\begin{array}{c} 7 \\ 9 \\ 6 \\ 2 \\ 8 \\ 4 \\ 8 \\ 12 \end{array}\right]. \tag{69}$$

The corresponding normal equations are then

$$\left[\begin{array}{c|c|cc|cc} 8 & . & . & . & . & . \\ \hline . & 9\tfrac{1}{2} & -1 & 1 & 2 & -\tfrac{1}{2} \\ \hline . & -1 & 8 & 4 & 2 & 1 \\ . & 1 & 4 & 8 & -2 & -1 \\ \hline . & 2 & 2 & -2 & 14 & 4 \\ . & -\tfrac{1}{2} & 1 & -1 & 4 & 3\tfrac{1}{2} \end{array}\right] \left[\begin{array}{c} \hat{\mu} \\ \hline \hat{\alpha}_1 \\ \hline \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hline \hat{\gamma}_{11} \\ \hat{\gamma}_{12} \end{array}\right] = \left[\begin{array}{c} 56 \\ \hline -10 \\ \hline 8 \\ -8 \\ \hline 20 \\ 10 \end{array}\right] \tag{70}$$

with solution

$$\hat{\underset{\sim}{b}}' = [7 \quad -1 \quad 1 \quad -1 \quad 1 \quad 1];$$

and the R-algorithm gives

$$R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_W = 7(56) + 10 + 8 + 8 + 20 + 10 = 448 = R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}),$$

as is expected.

To demonstrate (65) delete β's and β-equations from (70) leaving

$$\begin{bmatrix} 8 & . & . & . \\ . & 9\frac{1}{8} & 2 & -\frac{1}{8} \\ . & 2 & 14 & 4 \\ . & -\frac{1}{8} & 4 & 3\frac{1}{8} \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \hat{\gamma}_{11} \\ \hat{\gamma}_{12} \end{bmatrix} = \begin{bmatrix} 56 \\ -10 \\ 20 \\ 10 \end{bmatrix}.$$

The solution to these equations is

$$\hat{\underset{\sim}{b}}' = [7 \quad -1\tfrac{1}{4} \quad 1\tfrac{1}{4} \quad 1\tfrac{1}{4}]$$

and the R-algorithm gives

$$R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_W = 7(56) - 1\tfrac{1}{4}(-10) + 1\tfrac{1}{4}(20) + 1\tfrac{1}{4}(10) = 442.$$

Hence

$$R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_W = 448 - 442 = 6 = R(\underset{\sim}{\beta}|\mu),$$

so illustrating (65). The reader can use (70) to verify $R(\underset{\sim}{\alpha}|\mu)$ in the same manner.

10. Appendix: A Summary of Hypotheses

The sums of squares that have been discussed provide tests of a variety of hypotheses. Although these hypotheses are detailed in several places, different notations and labels can make cross-referencing confusing. Descriptive names for the hypotheses, equivalent expressions for each of them, and for the corresponding sums of squares, are therefore now listed, together with references coded as follows:

HS : Hocking and Speed [1975]

K : Kutner [1974]

S : Searle [1971]

S' : Searle [1977], this paper

SH : Speed and Hocking [1976]

SHH: Speed, Hocking and Hackney [1977].

The hypotheses are stated in terms of the two models

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}$$

and

$$y_{ijk} = \mu_{ij} + e_{ijk} ,$$

and numerator sum of squares are denoted by Q.

1. "Rows equal"

Hypothesis

$$\bar{\mu}_{1.} = \bar{\mu}_{2.} = \cdots = \bar{\mu}_{a.} \qquad \text{[(A) in K]}$$

1/
$$\begin{cases} \alpha_i + \bar{\gamma}_{i.} = \alpha_{i'} + \bar{\gamma}_{i'.} & \forall\ i \neq i' \\ \alpha_i + \bar{\gamma}_{i.} \text{ equal } \forall\ i \end{cases} \qquad \begin{matrix} \text{[H1 in SH]} \\ \text{[(68), p. 371, in S]} \end{matrix}$$

$$\begin{cases} \bar{\mu}_{i.} = \bar{\mu}_{i'.} & \forall\ i \neq i' \\ \bar{\mu}_{i.} \text{ equal } \forall\ i \end{cases} \qquad \begin{bmatrix} H_A \text{ of (2.26) in HS} \\ \text{H1 in SHH} \end{bmatrix}$$

Sum of squares

$$Q = R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma \qquad \text{[Sec. 7 in S']}$$

$$= SSA_w \qquad \begin{bmatrix} \text{p. 371 in S} \\ \text{Table 3 in SHH} \end{bmatrix}$$

$$= R(\underset{\sim}{\alpha}^* | \mu^*,\underset{\sim}{\beta}^*,\underset{\sim}{\gamma}^*)^{(1)} \qquad \text{[Table 2 in SH]}$$

2. "Columns equal"

Hypothesis

$$\bar{\mu}_{.1} = \bar{\mu}_{.2} = \cdots = \bar{\mu}_{.b} \qquad \text{[(B) in K]}$$

2/
$$\begin{cases} \beta_j + \bar{\gamma}_{.j} = \beta_{j'} + \bar{\gamma}_{.j'} & \forall\ j \neq j' \\ \beta_j + \bar{\gamma}_{.j} \text{ equal } \forall\ j \end{cases} \qquad \text{[H2 in SH]}$$

$$\begin{cases} \bar{\mu}_{.j} = \bar{\mu}_{.j'} & \forall\ j \neq j' \\ \bar{\mu}_{.j} \text{ equal } \forall\ j \end{cases} \qquad \begin{bmatrix} H_B \text{ of (2.26) in HS} \\ \text{H2 in SHH} \end{bmatrix}$$

Sum of squares

$$Q = R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_\Sigma \qquad \text{[Sec. 7 in S']}$$

$$= SSB_w \qquad \begin{bmatrix} \text{p. 370 in S} \\ \text{Table 3 in SHH} \end{bmatrix}$$

$$= R(\underset{\sim}{\beta}^* | \mu^*,\underset{\sim}{\alpha}^*,\underset{\sim}{\gamma}^*)^{(1)} \qquad \text{[Table 2 in SH]}$$

1/ In the restricted model with Σ-restrictions the hypothesis is H: α_i all equal.

2/ In the restricted model with Σ-restrictions the hypothesis is H: β_j all equal.

3. "Weighted rows equal"

Hypothesis

$$\left\{\begin{array}{ll}
\frac{1}{n_{i\cdot}} \sum_{j=1}^{b} n_{ij}\mu_{ij} \quad \text{equal } \forall\ i & \text{[p. 307 in S]} \\
\frac{\sum_{j=1}^{b} n_{1j}\mu_{1j}}{n_{1\cdot}} = \frac{\sum_{j=1}^{b} n_{2j}\mu_{2j}}{n_{2\cdot}} = \cdots = \frac{\sum_{j} n_{aj}\mu_{aj}}{n_{a\cdot}} & \text{[(C) in K]} \\
\sum_{j=1}^{b} \frac{n_{ij}}{n_{i\cdot}} \mu_{ij} = \sum_{j=1}^{b} \frac{n_{i'j}}{n_{i'\cdot}} \mu_{i'j} \quad \forall\ i \neq i' & \text{[H_{A^*} of (2.30) in HS]} \\
\sum_{j=1}^{b} n_{ij}\mu_{ij}/n_{i\cdot} = \sum_{j=1}^{b} n_{i'j}\mu_{i'j}/n_{i'\cdot} \quad \forall\ i \neq i' & \text{[H5 in SHH]}
\end{array}\right.$$

3/

$$\left\{\begin{array}{ll}
\alpha_i + \frac{1}{n_{i\cdot}} \sum_{j=1}^{b} n_{ij}(\beta_j + \gamma_{ij}) \quad \text{equal } \forall\ i & \text{[(100), p. 307 in S]} \\
\alpha_i + \sum_{j=1}^{b} \frac{n_{ij}\beta_j}{n_{i\cdot}} + \sum_{j=1}^{b} \frac{n_{ij}\gamma_{ij}}{n_{i\cdot}} & \\
\quad = \alpha_{i'} + \sum_{j=1}^{b} \frac{n_{i'j}\beta_j}{n_{i'\cdot}} + \sum_{j=1}^{b} \frac{n_{i'j}\gamma_{i'j}}{n_{i'\cdot}} \quad \forall\ i \neq i' & \text{[H3 in SH]}
\end{array}\right.$$

Sum of squares

$$Q = R(\underset{\sim}{\alpha}|\mu) \qquad \left[\begin{array}{l}\text{p. 307 in S} \\ \text{Table 4 in SHH}\end{array}\right]$$

$$= R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^*(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_W \qquad \text{[Sec. 9 in S']}$$

$$= R(\underset{\sim}{\alpha}^*|\mu^*) \qquad \text{[Table 2 in SH]}$$

$$= R(\underset{\sim}{\alpha}^*|\mu^*,\underset{\sim}{\beta}^*,\underset{\sim}{\gamma}^*)^{(3)} \qquad \text{[Table 2 in SH]}$$

3/ In the restricted model with W-restrictions the hypothesis is H: α_i all equal.

4. "Weighted columns equal"

Hypothesis

$$\begin{cases} \dfrac{1}{n_{\cdot j}} \sum_{i=1}^{a} n_{ij}\mu_{ij} \text{ equal } \forall\ j & \text{[see p. 307 in S]} \\[2ex] \dfrac{\sum_{i=1}^{a} n_{i1}\mu_{i1}}{n_{\cdot 1}} = \dfrac{\sum_{i=1}^{a} n_{i2}\mu_{i2}}{n_{\cdot 2}} = \cdots = \dfrac{\sum_{i}^{} n_{ib}\mu_{ib}}{n_{\cdot b}} & \text{[(D) in K]} \\[2ex] \sum_{i=1}^{a} \dfrac{n_{ij}}{n_{\cdot j}} \mu_{ij} = \sum_{i=1}^{a} \dfrac{n_{ij'}}{n_{\cdot j'}} \mu_{ij'} \quad \forall\ j \neq j' & \text{[}H_{B^*}\text{ of (2.30) in HS]} \\[2ex] \sum_{i=1}^{a} n_{ij}\mu_{ij}/n_{\cdot j} = \sum_{i=1}^{a} n_{ij'}\mu_{ij'}/n_{\cdot j'} \quad \forall\ j \neq j' & \text{[H6 in SHH]} \end{cases}$$

4/
$$\begin{cases} \beta_j + \dfrac{1}{n_{\cdot j}} \sum_{i=1}^{a} n_{ij}(\alpha_i + \gamma_{ij}) \text{ equal } \forall\ j & \text{[p. 308 in S]} \\[2ex] \beta_j + \sum_{i=1}^{a} \dfrac{n_{ij}\alpha_i}{n_{\cdot j}} + \sum_{i=1}^{a} \dfrac{n_{ij}\gamma_{ij}}{n_{\cdot j}} & \\[2ex] \quad = \beta_{j'} + \sum_{i=1}^{a} \dfrac{n_{ij'}\alpha_i}{n_{\cdot j'}} + \sum_{i=1}^{a} \dfrac{n_{ij'}\gamma_{ij'}}{n_{\cdot j'}} \quad \forall\ j \neq j' & \text{[H4 in SH]} \end{cases}$$

Sum of squares

$$Q = R(\underset{\sim}{\beta}|\mu) \qquad \left[\begin{array}{l}\text{p. 308 in S} \\ \text{Table 4 in SHH}\end{array}\right]$$

$$= R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^{*}(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_{W} \qquad \text{[Sec. 9 in S']}$$

$$= R(\underset{\sim}{\beta}^{*}|\mu^{*}) \qquad \text{[Table 2 in SH]}$$

$$= R(\underset{\sim}{\beta}^{*}|\mu^{*},\underset{\sim}{\alpha}^{*},\underset{\sim}{\gamma}^{*})^{(3)} \qquad \text{[Table 2 in SH]}$$

4/ In the restricted model with W-restrictions the hypothesis is H: β_j all equal.

5. "Rows adjusted for columns" equal

Hypothesis

$$\left\{\begin{array}{l}
\left(n_{i.} - \sum_{j=1}^{b} n_{ij}^2/n_{.j}\right)\alpha_i - \sum_{i'\neq i}^{a}\left(\sum_{j=1}^{b} n_{ij}n_{i'j}/n_{.j}\right)\alpha_{i'} \\
\qquad + \sum_{j=1}^{b}(n_{ij} - n_{ij}^2/n_{.j})\gamma_{ij} - \sum_{i'\neq i}^{a}\sum_{j=1}^{b}(n_{ij}n_{i'j}/n_{.j})\gamma_{i'j} = 0 \\
\qquad\qquad \text{for } i = 1, 2, \cdots, a-1. \quad [(107), \text{p. 310 in S}] \\
\left(n_{i.} - \sum_{j=1}^{b} n_{ij}^2/n_{.j}\right)\alpha_i + \sum_{j=1}^{b}(n_{ij} - n_{ij}^2/n_{.j})\gamma_{ij} \\
\qquad = \sum_{i'\neq i}^{a}\left(\sum_{j=1}^{b} n_{ij}n_{i'j}/n_{.j}\right)\alpha_{i'} + \sum_{i'\neq i}^{a}\sum_{j=1}^{b}(n_{ij}n_{i'j}/n_{.j})\gamma_{i'j} \\
\qquad\qquad \text{for } i = 1, 2, \cdots, a-1. \quad [\text{H5 in SH}]
\end{array}\right.$$

$$\left\{\begin{array}{ll}
\sum_{j=1}^{b}\left(n_{ij} - \frac{n_{ij}^2}{n_{.j}}\right)\mu_{ij} - \sum_{i'\neq i}^{a}\sum_{j=1}^{b}\frac{n_{ij}n_{i'j}}{n_{.j}}\mu_{i'j} = 0 \ \forall\, i & [H_A^{**} \text{ of (2.31) in HS}] \\
\sum_{j=1}^{b} n_{ij}\mu_{ij} = \sum_{i'=1}^{a}\sum_{j=1}^{b}\frac{n_{ij}n_{i'j}\mu_{i'j}}{n_{.j}} \ \forall\, i & [\text{H3 of Table 1 in SHH}]
\end{array}\right.$$

Sum of squares

$$Q = R(\underset{\sim}{\alpha}|\mu,\underset{\sim}{\beta}) \qquad \left[\begin{array}{l}\text{p. 310 in S} \\ \text{Table 4 in SHH}\end{array}\right]$$

$$= R(\underset{\sim}{\alpha}^*|\mu^*,\underset{\sim}{\beta}^*) \qquad [\text{Table 2 in SH}]$$

6. "Columns adjusted for rows" equal

Hypothesis

$$\left\{\begin{array}{l}
\left(n_{\cdot j} - \sum_{i=1}^{a} n_{ij}^2/n_{i\cdot}\right)\beta_j - \sum_{j'\neq j}^{b}\left(\sum_{i=1}^{a} n_{ij}n_{ij'}/n_{i\cdot}\right)\beta_{j'} \\
\qquad + \sum_{i=1}^{a}(n_{ij} - n_{ij}^2/n_{i\cdot})\gamma_{ij} - \sum_{j'\neq j}^{b}\sum_{i=1}^{a}(n_{ij}n_{ij'}/n_{i\cdot})\gamma_{ij'} = 0 \\
\qquad\qquad \text{for } j = 1, 2, \cdots, b-1. \quad [(106), \text{ p. } 309 \text{ in S}] \\
\left(n_{\cdot j} - \sum_{i=1}^{a} n_{ij}^2/n_{i\cdot}\right)\beta_j + \sum_{i=1}^{a}(n_{ij} - n_{ij}^2/n_{i\cdot})\gamma_{ij} \\
\qquad = \sum_{j'\neq j}^{b}\left(\sum_{i=1}^{a} n_{ij}n_{ij'}/n_{i\cdot}\right)\beta_{j'} + \sum_{j'\neq j}^{b}\sum_{i=1}^{a}(n_{ij}n_{ij'}/n_{i\cdot})\gamma_{ij'} \\
\qquad\qquad \text{for } j = 1, 2, \cdots, b-1. \quad [\text{H6 in SH}]
\end{array}\right.$$

5/

$$\left\{\begin{array}{ll}
\sum_{i=1}^{a}\left(n_{ij} - \frac{n_{ij}^2}{n_{i\cdot}}\right)\mu_{ij} - \sum_{j'\neq j}^{b}\sum_{i=1}^{a}\frac{n_{ij}n_{ij'}}{n_{i\cdot}}\mu_{ij'} = 0 \ \forall\, j & [H_{B}^{**} \text{ of (2.31) in HS}] \\
\sum_{i=1}^{a} n_{ij}\mu_{ij} = \sum_{j'=1}^{b}\sum_{i=1}^{a}\frac{n_{ij}n_{ij'}\mu_{ij'}}{n_{i\cdot}} \ \forall\, j & [H7 \text{ of Table 1 in SHH}]
\end{array}\right.$$

Sum of squares

$$Q = R(\underset{\sim}{\beta}|\mu,\underset{\sim}{\alpha}) \qquad \left[\begin{array}{l}\text{p. 309 in S} \\ \text{Table 4 in SHH}\end{array}\right]$$

$$= R(\underset{\sim}{\beta}^*|\mu^*,\underset{\sim}{\alpha}^*) \qquad [\text{Table 2 in SH}]$$

5/ There are two typographical errors in H_B^{**} of (2.31) in HS: the i of i=1 of the first summation is erroneously j, and the j of $j'\neq j$ of the second summation is erroneously i.

7. "Interactions"

Hypothesis

6/ $\left\{\begin{array}{l}\text{Any column vector consisting of } s\text{-}a\text{-}b\text{+}1 \\ \text{linear independent functions of the} \\ \theta_{ij,i'j'} = \gamma_{ij} - \gamma_{i'j} - \gamma_{ij'} + \gamma_{i'j'}, \text{ where such} \\ \text{functions are either estimable } \theta\text{'s or} \\ \text{estimable sums or differences of } \theta\text{'s.}\end{array}\right\} = \underset{\sim}{0}$ [(110), p. 311 of S]

$$\mu_{ij} - \mu_{i'j} - \mu_{ij'} + \mu_{i'j'} = 0 \quad \forall\, i \neq i',\ j = j'$$ [(E) in K; H7 in SH; H9 in SHH; H_{AB} in HS]

Sum of squares

$$Q = R(\underset{\sim}{\gamma}|\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta})$$ [p. 311 in S; Tables 3,4,5 in SHH]

$$= R(\underset{\sim}{\gamma}^*|\mu^*,\underset{\sim}{\alpha}^*,\underset{\sim}{\beta}^*)$$ [Table 2 in SH]

6/ For all cells filled, $s = N' = (a-1)(b-1)$.

8. "Rows over a specified column"

Hypothesis: For an arbitrary t:

$$\underline{7/} \quad \alpha_k + \gamma_{kt} = \alpha_i + \gamma_{it} \quad \forall\, i \neq k \qquad \text{[H8 in SH]}$$

$$\mu_{it} = \mu_{i't} \quad \forall\, i \neq i' \qquad \text{[H4 in SHH]}$$

Sum of squares

$$Q = R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^{*}(\mu,\underset{\sim}{\not\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma})_{O_{kt}} \qquad \text{[Sec. 8 in S']}$$

$$= R(\underset{\sim}{\alpha}^{*}|\mu^{*},\underset{\sim}{\beta}^{*},\underset{\sim}{\gamma}^{*})^{(2)} \qquad \begin{bmatrix}\text{Table 2 in SH}\\ \text{Table 5 in SHH}\end{bmatrix}$$

9. "Columns over a specified row"

Hypothesis: For an arbitrary k:

$$\underline{8/} \quad \beta_t + \gamma_{kt} = \beta_j + \gamma_{kj} \quad \forall\, j \neq t \qquad \text{[H9 in SH]}$$

$$\mu_{kj} = \mu_{kj'} \quad \forall\, j \neq j' \qquad \text{[H8 in SHH]}$$

Sum of squares

$$Q = R(\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\gamma}) - R^{*}(\mu,\underset{\sim}{\alpha},\underset{\sim}{\not\beta},\underset{\sim}{\gamma})_{O_{kt}} \qquad \text{[Sec. 8 in S']}$$

$$= R(\underset{\sim}{\beta}^{*}|\mu^{*},\underset{\sim}{\alpha}^{*},\underset{\sim}{\gamma}^{*})^{(2)} \qquad \begin{bmatrix}\text{Table 2 in SH}\\ \text{Table 5 in SHH}\end{bmatrix}$$

7/ In the restricted model with O_{kt}-restrictions the hypothesis is H: α_i all equal.

8/ In the restricted model with O_{kt}-restrictions the hypothesis is H: β_j all equal.

The references given in the preceding list are summarized in the following table:

Table 6: Hypotheses and Their Labels in a Variety of References

Hypothesis	Reference					
	S'	HS	K	S	SH	SHH
Rows equal	1	H_A	A	p. 371	H1	H1
Columns equal	2	H_B	B	p. 370	H2	H2
Weighted rows equal	3	H_{A*}	C	p. 307	H3	H5
Weighted columns equal	4	H_{B*}	D	p. 308	H4	H6
Rows adjusted, equal	5	H_{A**}	-	p. 310	H5	H3
Columns adjusted, equal	6	H_{B**}	-	p. 309	H6	H7
Interactions	7	H_{AB}	E	p. 311	H7	H9
Rows over column t, equal	8	-	-	-	H8	H4
Columns over row k, equal	9	-	-	-	H9	H8

11. Addendum: Hypotheses of "no interaction"

It is sometimes said that hypotheses of the form

$$H: \gamma_{ij} - \gamma_{i'j} - \gamma_{ij'} + \gamma_{i'j'} \tag{A1}$$

are testing "no interaction". Or equivalently, that restrictions of the form

$$\gamma_{ij} - \gamma_{i'j} - \gamma_{ij'} + \gamma_{i'j'} = 0 \tag{A2}$$

reduce an interaction model to a no interaction model. These kinds of statements are true in the case of all cells filled; but when some cells are empty, the situation is changed. Restrictions differing from (A2) have to be used. We illustrate with 2 examples, one of filled cells, the other with some empty cells.

11.1. The 2 x 2 case with all cells filled

Represent this case by the grid

*	*
*	*

(A3)

where an asterisk represents the occurrence of data in a cell. Now consider the interaction model

$$E\, y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} . \tag{A4}$$

For data of the form (A3) we have

$$\begin{aligned} E\, y_{11k} &= \mu + \alpha_1 + \beta_1 + \gamma_{11} \\ E\, y_{12k} &= \mu + \alpha_1 + \beta_2 + \gamma_{12} \\ E\, y_{21k} &= \mu + \alpha_2 + \beta_1 + \gamma_{21} \\ E\, y_{22k} &= \mu + \alpha_2 + \beta_2 + \gamma_{22} \end{aligned} \tag{A5}$$

Consider the effect on this model of restrictions like (A2), of which there is only one in this case, namely

$$\gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} = 0 \ . \tag{A6}$$

From it we can write

$$\gamma_{11} = \gamma_{12} + \gamma_{21} - \gamma_{22} \ . \tag{A7}$$

On substituting (A7) into the first equation of (A5), adding and subtracting γ_{22} in the third equation of (A5), and then re-arranging terms, we can re-write those equations as

$$\begin{aligned} E\, y_{11k} &= \mu + (\alpha_1 + \gamma_{12}) + (\beta_1 + \gamma_{21} - \gamma_{22}) \\ E\, y_{12k} &= \mu + (\alpha_1 + \gamma_{12}) + \beta_2 \\ E\, y_{21k} &= \mu + (\alpha_2 + \gamma_{22}) + (\beta_1 + \gamma_{21} - \gamma_{22}) \\ E\, y_{22k} &= \mu + (\alpha_2 + \gamma_{22}) + \beta_2 \end{aligned} \ . \tag{A8}$$

On defining

$$\begin{aligned} \alpha_1^* &= \alpha_1 + \gamma_{12} & \beta_1^* &= \beta_1 + \gamma_{21} - \gamma_{22} \\ \alpha_2^* &= \alpha_2 + \gamma_{22} & \beta_2^* &= \beta_2 \end{aligned} \tag{A9}$$

equations (A8) then represent a no-interaction model in terms of the starred parameters of (A9). In other words, the restriction (A6) does reduce the interaction model (A5) to a no interaction model.

Note that (A7) is only one example of how (A6) may be used, leading via (A8) to definition of new parameters in accord with (A9). Another example of using (A6) is to rewrite it as

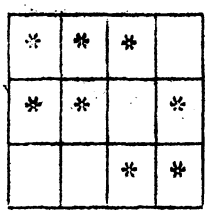
$$\gamma_{12} = \gamma_{11} - \gamma_{21} + \gamma_{22} \ . \tag{A10}$$

This leads, as is shown below, to a definition of new parameters different from (A9).

This illustration of 2 rows and 2 columns extends directly to any numbers of rows and columns when all cells are filled. But it does not extend in quite the same manner when there are empty cells.

11.2. An empty cells case

Suppose there are data in 8 cells of a 3 rows and 4 columns case as follows:



(A11)

The model equations are:

$$\begin{aligned} E\, y_{11k} &= \mu + \alpha_1 + \beta_1 + \gamma_{11} \\ E\, y_{12k} &= \mu + \alpha_1 + \beta_2 + \gamma_{12} \\ E\, y_{21k} &= \mu + \alpha_2 + \beta_1 + \gamma_{21} \\ E\, y_{22k} &= \mu + \alpha_2 + \beta_2 + \gamma_{22} \end{aligned} \tag{A12}$$

and

$$\begin{aligned} E\, y_{13k} &= \mu + \alpha_1 + \beta_3 + \gamma_{13} \\ E\, y_{24k} &= \mu + \alpha_2 + \beta_4 + \gamma_{24} \\ E\, y_{33k} &= \mu + \alpha_3 + \beta_3 + \gamma_{33} \\ E\, y_{34k} &= \mu + \alpha_3 + \beta_4 + \gamma_{34} \end{aligned} \tag{A13}$$

Restrictions of the form (A2) can be applied only in one way, namely

$$\gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} = 0 \tag{A14}$$

as in (A6). By the nature of the pattern of filled cells in (A11) no other form

of (A2) can be used. However, recall from Sec. 10.7, or from Searle [1971, p. 311], the hypotheses that can be tested using $R(\underset{\sim}{\gamma}|\mu,\underset{\sim}{\alpha},\underset{\sim}{\beta})$. They are not only of the form (A1) but also of the form

$$H: \theta_{ij,i'j'} \pm \theta_{rs,r's'} = 0 \qquad \text{(A15)}$$

where

$$\theta_{ij,i'j'} \equiv \gamma_{ij} - \gamma_{i'j} - \gamma_{ij'} + \gamma_{i'j'}$$

and where, in (A15), each $\theta_{ij,i'j'}$ and $\theta_{rs,r's'}$ may not be estimable but their sum (or difference) is. This is just the case with the data pattern (A11). Neither

$$\gamma_{11} - \gamma_{13} - \gamma_{21} + \gamma_{23} \quad (= \theta_{11,23})$$

nor

$$\gamma_{23} - \gamma_{24} - \gamma_{33} + \gamma_{34} \quad (= \theta_{23,34})$$

are estimable (because cell 2,3 has no data), but their difference is. Using this difference as a restriction

$$\gamma_{11} - \gamma_{13} - \gamma_{21} + \gamma_{23} - (\gamma_{23} - \gamma_{24} - \gamma_{33} + \gamma_{34}) = 0 \qquad \text{(A16)}$$

together with (A14) reduces the model equations (A12) to be those of a no interaction model.

First, (A14) used in the form (A10) reduces (A12) to

$$\begin{aligned}
E\, y_{11k} &= \mu + (\alpha_1 + \gamma_{11}) + \beta_1 \\
E\, y_{12k} &= \mu + (\alpha_1 + \gamma_{11}) + (\beta_2 - \gamma_{21} + \gamma_{22}) \\
E\, y_{21k} &= \mu + (\alpha_2 + \gamma_{21}) + \beta_1 \\
E\, y_{22k} &= \mu + (\alpha_2 + \gamma_{21}) + (\beta_2 - \gamma_{21} + \gamma_{22})
\end{aligned} \qquad \text{(A17)}$$

in the same manner as equations (A8) were derived using (A6). We can then rewrite

(A13), using two of the parenthesized terms of (A17), as

$$\begin{aligned} E\, y_{13k} &= \mu + (\alpha_1 + \gamma_{11}) + \beta_3 + \gamma_{13} - \gamma_{11} \\ E\, y_{24k} &= \mu + (\alpha_2 + \gamma_{21}) + \beta_4 + \gamma_{24} - \gamma_{21} \\ E\, y_{33k} &= \mu + \alpha_3 + \beta_3 + \gamma_{33} \\ E\, y_{34k} &= \mu + \alpha_3 + \beta_4 + \gamma_{34} \end{aligned} \quad . \tag{A18}$$

Finally, on rewriting (A16) as

$$\gamma_{13} - \gamma_{11} = \gamma_{24} - \gamma_{21} + \gamma_{33} - \gamma_{34}$$

we can rewrite (A18) as

$$\begin{aligned} E\, y_{13k} &= \mu + (\alpha_1 + \gamma_{11}) + (\beta_3 + \gamma_{24} - \gamma_{21} + \gamma_{33} - \gamma_{34}) \\ E\, y_{24k} &= \mu + (\alpha_2 + \gamma_{21}) + (\beta_4 + \gamma_{24} - \gamma_{21}) \\ E\, y_{33k} &= \mu + (\alpha_3 - \gamma_{24} + \gamma_{21} + \gamma_{34}) + (\beta_3 + \gamma_{24} - \gamma_{21} + \gamma_{33} - \gamma_{34}) \\ E\, y_{34k} &= \mu + (\alpha_3 - \gamma_{24} + \gamma_{21} + \gamma_{34}) + (\beta_4 + \gamma_{24} - \gamma_{21}) \end{aligned} \quad . \tag{A19}$$

Then, on defining

$$\begin{aligned} \alpha_1^* &= \alpha_1 + \gamma_{11} & \beta_1^* &= \beta_1 \\ \alpha_2^* &= \alpha_2 + \gamma_{21} & \beta_2^* &= \beta_2 - \gamma_{21} + \gamma_{22} \\ \alpha_3^* &= \alpha_3 - \gamma_{24} + \gamma_{21} + \gamma_{34} & \beta_3^* &= \beta_3 + \gamma_{24} - \gamma_{21} + \gamma_{33} - \gamma_{34} \\ & & \beta_4^* &= \beta_4 + \gamma_{24} - \gamma_{21} \end{aligned} \tag{A20}$$

(A17) and (A19) represent a no-interaction reformulation of the interaction model equations (A12) and (A13).

The emphasis here is that with data having empty cells it is not just restrictions of the form (A2), exemplified by (A6) and (A14), that reduce the interaction model to the no-interaction model; restrictions like (A16) are also needed.

There is, of course, no unique way of deriving a reparameterization like (A20). Indeed (A16) is not the only restriction of that nature which could be used. For example,

$$\gamma_{12} - \gamma_{13} - \gamma_{22} + \gamma_{23} - (\gamma_{23} - \gamma_{24} - \gamma_{33} + \gamma_{34}) = 0$$

is another, and so is

$$\gamma_{12} - \gamma_{14} - \gamma_{22} + \gamma_{24} - (\gamma_{13} - \gamma_{14} - \gamma_{33} + \gamma_{34}) = 0.$$

Each of these will lead to a reparameterization different from (A20), although similar in nature.

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Note: This manuscript was prepared prior to the author's sighting the following paper:

Hocking, R. R., Hackney, O. P., and Speed, F. M. [1978]. The analysis of linear models with unbalanced data. Contribution to Contributions to Survey Sampling and Applied Statistics, Festschrift for H. O. Hartley, Ed. H. A. David, Academic Press.